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Periodic solutions of an asymptotically linear Dirac equation

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Abstract Using the variational method, we investigate periodic solutions of a Dirac equation with asymptotically nonlinearity. The variational setting is established and the existence and multiplicity of periodic solutions are obtained.

Keywords Dirac equation · Periodic solutions · Variational method · Asymptotically linear

Mathematics Subject Classification 35Q40 · 49J35

1 Introduction and main results

Let us consider the following (stationary) Dirac equation

$$-i\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + V(x)u = G_u(x,u) \tag{1.1}$$

for $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, where $\partial_k=\partial/\partial x_k$, $a>0$ is a constant, $\alpha_1,\alpha_2,\alpha_3$ and β are 4×4 Pauli-Dirac matrices:

$$\beta=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\qquad \alpha_k=\begin{pmatrix} 0 & \sigma_k\\ \sigma_k & 0\end{pmatrix},\quad k=1,2,3$$

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with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This equation arises when one seeks for the standing wave solutions of the nonlinear Dirac equation (see [25])

$$-i\hbar\partial_t\psi = ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi - M(x)\psi + F_\psi(x,\psi). \tag{1.2}$$

Assuming that $F(x, e^{i\theta}\psi) = F(x,\psi)$ for all $\theta \in [0, 2\pi]$, a standing wave solution of (1.2) is a solution of the form $\psi(t,x) = e^{\frac{i\mu t}{\hbar}}u(x)$. It is clear that $\psi(t,x)$ solves (1.2) if and only if $u(x)$ solves (1.1) with $a = mc/\hbar$, $V(x) = M(x)/c\hbar + \mu I_4/\hbar$ and $G(x,u) = F(x,u)/c\hbar$.

For notational convenience, denoting

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) \text{ and } \alpha\cdot\nabla = \sum_{k=1}^{3}\alpha_k\partial_k,$$

we rewrite the Eq. (1.1) as

$$-i\alpha\cdot\nabla u + a\beta u + V(x)u = G_u(x,u). \tag{D_V}$$

There are many papers studying the existence and multiplicity of standing wave of the equations under different assumptions on the potentials V and G, see, [3,8–11,14–18,21,23] and their references. Recall that, mathematically, the conditions that the potential functions depend periodically on x is used for describing a class of self-interaction of quantum electrodynamics in, e.g. [1,2,4,5,19,20,24,26] for Schrödinger equations and [3] for Dirac equations. Note that if the potentials are periodic in x one may also study the existence and multiplicity of periodic solutions. Naturally, a periodic solution of (D_V) may be referred as a standing periodic wave of (1.2). In recently paper [12], we have investigated periodic solutions of (D_V) in both cases that the nonlinearity $G_u(x,u)$ is of superlinear and subcritical growth as $|u| \to \infty$. The case of concave and convex has been researched in the paper [13].

In the present paper, we are interested in the case that $G(x,u)$ is asymptotically quadratic at 0 and ∞ and obtain the existence and multiplicity results of periodic solutions.

We make the following periodicity hypothesis on $V(x)$ and $G(x,u)$:

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$, and $V(x)$ is 1-periodic in x_k, $k = 1, 2, 3$.
(G_0) $G \in C^1(\mathbb{R}^3 \times \mathbb{C}^4, [0,\infty))$, and $G(x,u)$ is 1-periodic in x_k, $k = 1, 2, 3$.

We are looking for periodic solutions of (D_V): $u(x+z) = u(x)$ for any $z \in \mathbb{Z}^3$.

Setting $Q = [0,1]\times[0,1]\times[0,1]$, if u is a solution of (D_V), its energy will be denoted by

$$\Phi(u) = \int_Q \left[\frac{1}{2}(-i\alpha\cdot\nabla u + a\beta u + V(x)u)\cdot u - G(x,u)\right]dx, \tag{1.3}$$

where (here and in the following) by $v \cdot w$ we denote the scalar product in $\mathbb{C}^4$ of v and w.

In order to state our results, let $A_0 = -i\alpha\cdot\nabla + a\beta$ and $A_V = A_0 + V$ denote the self-adjoint operators acting in $L^2(Q, \mathbb{C}^4)$. Let $\{\lambda_j\}_{j\in\mathbb{Z}}$ denote the sequence of all eigenvalues of A_V counted by multiplicity:

$$\ldots \le \lambda_{-2} \le \lambda_{-1} < \lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \ldots,$$

and let $\{e_j\}_{j\in\mathbb{Z}}$ be the associated sequence of eigenvectors of A_V:

$$A_V e_j = \lambda_j e_j, \quad |e_j|_{L^2} = 1, \quad j = \pm 1, \pm 2, \ldots. \tag{1.4}$$

Remark 1.1 We can find out all eigenvalues and the associated eigenfunctions of A_0. Let

$$z = (k_1, k_2, k_3) \in \mathbb{N}^3, \quad x = (x_1, x_2, x_3) \in Q, \quad zx = k_1x_1 + k_2x_2 + k_3x_3,$$

and $|z| = \sqrt{k_1^2 + k_2^2 + k_3^2}$. Note that

$$A_0 = \begin{pmatrix} aI & -i(\sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3) \\ -i(\sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3) & -aI \end{pmatrix}$$

and

$$-i(\sigma_1\partial_1 e^{2\pi zxi} + \sigma_2\partial_2 e^{2\pi zxi} + \sigma_3\partial_3 e^{2\pi zxi}) = 2\pi e^{2\pi zxi} W,$$

where $W = \begin{pmatrix} k_3 & k_1 - ik_2 \\ k_1 + ik_2 & -k_3 \end{pmatrix}$. Setting $D = \begin{pmatrix} aI & 2\pi W \\ 2\pi W & -aI \end{pmatrix}$, one can verify that if $\lambda \neq 0$ is a eigenvalue of the matrix D and $\mathbf{v}$ is a eigenvector corresponding to λ, then λ must be a eigenvalue of A_0 and $e^{2\pi zxi}\mathbf{v}$ is a eigenfunction corresponding to λ. By $|\lambda I - D| = 0$ we obtain

$$\begin{aligned} &\begin{vmatrix} (\lambda - a)I & -2\pi W \\ -2\pi W & (\lambda + a)I \end{vmatrix} \\ &= \begin{vmatrix} (\lambda - a) & 0 & -2\pi k_3 & -2\pi(k_1 - ik_2) \\ 0 & (\lambda - a) & -2\pi(k_1 + ik_2) & 2\pi k_3 \\ -2\pi k_3 & -2\pi(k_1 - ik_2) & (\lambda + a) & 0 \\ -2\pi(k_1 + ik_2) & 2\pi k_3 & 0 & (\lambda + a) \end{vmatrix} \\ &= (\lambda^2 - a^2 - 4\pi^2|z|^2)^2 = 0, \end{aligned}$$

and therefore

$$\lambda = \pm\sqrt{a^2 + 4\pi^2|z|^2}.$$

For $\mathbf{v} = (c_1, c_2, c_3, c_4)$, in virtue of $D\mathbf{v}^T = \lambda\mathbf{v}^T$ we get

$$\begin{cases} 2\pi k_3 c_3 + 2\pi(k_1 - ik_2)c_4 = (\lambda - a)c_1, \\ 2\pi(k_1 + ik_2)c_3 - 2\pi k_3 c_4 = (\lambda - a)c_2, \end{cases}$$

and so

$$\begin{cases} \mathbf{v}_\lambda^{(1)} = (2\pi|z|^2,\ 0,\ (\lambda - a)k_3,\ (\lambda - a)(k_1 + ik_2)), \\ \mathbf{v}_\lambda^{(2)} = (0,\ 2\pi|z|^2,\ (\lambda - a)(k_1 - ik_2),\ (a - \lambda)k_3). \end{cases}$$

Put

$$\bar{\mathbf{e}}_1 = (1, 0, 0, 0), \bar{\mathbf{e}}_2 = (0, 1, 0, 0), \bar{\mathbf{e}}_3 = (0, 0, 1, 0), \bar{\mathbf{e}}_4 = (0, 0, 0, 1),$$

then

$$\begin{aligned} \varphi_\lambda^{(1)}(x) &:= e^{2\pi izx}[2\pi|z|^2\bar{\mathbf{e}}_1 + (\lambda - a)k_3\bar{\mathbf{e}}_3 + (\lambda - a)(k_1 + ik_2)\bar{\mathbf{e}}_4], \\ \varphi_\lambda^{(2)}(x) &:= e^{2\pi izx}[2\pi|z|^2\bar{\mathbf{e}}_2 + (\lambda - a)(k_1 - ik_2)\bar{\mathbf{e}}_3 - (\lambda - a)k_3\bar{\mathbf{e}}_4] \end{aligned} \tag{1.5}$$

satisfy $A_0\varphi_\lambda^{(j)} = \lambda\varphi_\lambda^{(j)}$, $j = 1, 2$.

We will use the following hypotheses:

(G_1) there is $b_0 \geq 0$ such that and $G_u(x,u) - b_0 u = o(|u|)$ as $u \to 0$ uniformly in $x \in Q$;
(G_2) there is $b_\infty > 0$ satisfying $G_u(x,u) - b_\infty u = o(|u|)$ as $|u| \to \infty$ uniformly in $x \in Q$;
(G_3) either (i) $b_\infty \notin \sigma(A_V)$ or (ii) $G_u(x,u) - b_\infty u$ is bounded and $G(x,u) - \frac{1}{2} b_\infty |u|^2 \to \infty$ as $|u| \to \infty$ uniformly in $x \in Q$.

Set

$$G^0(x,u) := G(x,u) - \frac{1}{2} b_0 |u|^2, \ G^\infty(x,u) := G(x,u) - \frac{1}{2} b_\infty |u|^2,$$

and define

$$b_0^+ := \min[\sigma(A_V) \cap (b_0, \infty)], \quad b_\infty^- := \max[\sigma(A_V) \cap (b_0, b_\infty)].$$

The first result reads as follows.

Theorem 1.2 *Let (V), (G_0) and $(G_1) - (G_3)$ be satisfied and $b_\infty > b_0^+$. Then*

(a) if $G^0(x,u) \geq 0$, then (D_V) has at least one nontrivial periodic solution in $H^1(Q, \mathbb{C}^4)$;
(b) if G is even in u, then (D_V) has at least $d(b_0, b_\infty)$ pairs of periodic solutions, where $d(b_0, b_\infty)$ denotes the dimensionality of the eigenspace associated to $\sigma(A_V) \cap (b_0, b_\infty)$.

If $b_0 \equiv 0$, then $b_0^+ = \lambda_1$, we have

Corollary 1.3 *Assume that (V), (G_0) and $(G_1) - (G_3)$ hold with $b_0 = 0$. If $b_\infty > \lambda_1$, then (D_V) has at least one nontrivial periodic solution in $H^1(Q, \mathbb{C}^4)$. If G is in addition even in u, then (D_V) has at least $d(0, b_\infty)$ pairs of periodic solutions.*

If $V(x) \equiv 0$, that is, $A_V = A_0$, then the equation (D_V) becomes the following

$$-i\alpha \cdot \nabla u + a\beta u = G_u(x,u). \tag{D_0}$$

We write $\{\mu_j\}$ the sequence of all eigenvalues of A_0 according to the size of order, not by multiplicity:

$$\ldots < \mu_{-2} < \mu_{-1} < \mu_0 = 0 < \mu_1 < \mu_2 < \ldots.$$

Let $\sharp_{\mu_k}$ define the multiplicity of μ_k, and $\lambda_j^{(\mu_k)}$ the eigenvalues such that $\lambda_j^{(\mu_k)} = \mu_k, \ j = 1, \ldots, \sharp_{\mu_k}$.

Let $N[j]$ denote the number of $z \in \mathbb{N}^3$ corresponding to $|z|^2 = j$. For $0 \leq |z|^2 \leq 10$, we have:

$$N[0] = N[3] = 1; \ N[j] = 3, \ j = 1, 2, 4, 6, 8;$$

$$N[k] = 6, \ k = 5, 9, 10; \quad N[7] = 0,$$

then by Remark 1.1,

$$\mu_j = \sqrt{a^2 + 4(j-1)\pi^2}, \ 1 \leq j \leq 7; \ \mu_k = \sqrt{a^2 + 4k\pi^2}, \ k = 8, 9, 10,$$

and

$$\sharp_{\mu_1} = \sharp_{\mu_4} = 1; \ \sharp_{\mu_j} = 3, j = 2, 3, 5, 7, 8; \ \sharp_{\mu_k} = 6, k = 6, 9, 10.$$

Accordingly, we see

$$\lambda_1^{(\mu_1)} = \mu_1 = a,\ \lambda_1^{(\mu_2)} = \lambda_2^{(\mu_2)} = \lambda_3^{(\mu_2)} = \sqrt{a^2+4\pi^2},$$
$$\lambda_1^{(\mu_3)} = \lambda_2^{(\mu_3)} = \lambda_3^{(\mu_3)} = \sqrt{a^2+8\pi^2},\ \lambda_8^{(\mu_4)} = \sqrt{a^2+12\pi^2},$$
$$\lambda_1^{(\mu_5)} = \lambda_2^{(\mu_5)} = \lambda_3^{(\mu_5)} = \sqrt{a^2+16\pi^2},\ \lambda_1^{(\mu_6)} = \cdots = \lambda_6^{(\mu_6)} = \sqrt{a^2+20\pi^2},$$
$$\lambda_1^{(\mu_7)} = \lambda_2^{(\mu_7)} = \lambda_3^{(\mu_7)} = \sqrt{a^2+24\pi^2},\ \lambda_1^{(\mu_8)} = \lambda_2^{(\mu_8)} = \lambda_3^{(\mu_8)} = \sqrt{a^2+32\pi^2},$$
$$\lambda_1^{(\mu_9)} = \cdots = \lambda_6^{(\mu_9)} = \sqrt{a^2+36\pi^2},\ \lambda_1^{(\mu_{10})} = \cdots = \lambda_6^{(\mu_{10})} = \sqrt{a^2+40\pi^2}.$$

By (1.5), we can list the first 10 eigenvalues λ_j and eigenfunctions e_j corresponding to λ_j as follows:

$$\begin{aligned}
&\lambda_1 = \lambda_2 = \mu_1 = a \ \text{ with } \ z = (0,\ 0,\ 0),\\
&e_1 = (1,\ 0,\ 0,\ 0), \quad e_2 = (0,\ 1,\ 0,\ 0);\\
&\lambda_3 = \lambda_4 = \mu_2 = \sqrt{a^2+4\pi^2} \ \text{ with } \ z = (1,0,0),\\
&e_3 = \Delta_1 \mathrm{e}^{2\pi x_1 i}(2\pi,\ 0,\ 0,\ \mu_2 - a),\\
&e_4 = \Delta_1 \mathrm{e}^{2\pi x_1 i}(0,\ 2\pi,\ \mu_2 - a,\ 0);\\
&\lambda_5 = \lambda_6 = \mu_2 \ \text{ with } \ z = (0,1,0),\\
&e_5 = \Delta_1 \mathrm{e}^{2\pi x_2 i}(2\pi,\ 0,\ 0,\ (\mu_2 - a)i),\\
&e_6 = \Delta_1 \mathrm{e}^{2\pi x_2 i}(0,\ 2\pi,\ (a - \mu_2)i,\ 0);\\
&\lambda_7 = \lambda_8 = \mu_2 \ \text{ with } \ z = (0,\ 0,\ 1),\\
&e_7 = \Delta_1 \mathrm{e}^{2\pi x_3 i}(2\pi,\ 0,\ \mu_2 - a, 0),\\
&e_8 = \Delta_1 \mathrm{e}^{2\pi x_3 i}(0,\ 2\pi,\ 0,\ a - \mu_2);\\
&\lambda_9 = \lambda_{10} = \mu_3 = \sqrt{a^2+8\pi^2} \ \text{ with } \ z = (1,\ 1,\ 0),\\
&e_9 = \Delta_2 \mathrm{e}^{2\pi(x_1+x_2)i}(4\pi,\ 0,\ (\mu_3 - a)(1+i),\ 0),\\
&e_{10} = \Delta_2 \mathrm{e}^{2\pi(x_1+x_2)i}(0,\ 4\pi,\ 0,\ (\mu_3 - a)(1-i)),
\end{aligned}$$

where $\Delta_1 = \frac{1}{\sqrt{4\pi^2+(\mu_2-a)^2}}$, $\Delta_2 = \frac{1}{\sqrt{16\pi^2+2(\mu_3-a)^2}}$.

Now we have a special consequence corresponding to the equation (D_0).

Corollary 1.4 *Let* (G_0) *and* $(G_1) - (G_3)$ *be satisfied with* $b_0 = 0$. *Then* (D_0) *has at least one nontrivial periodic solution in* $H^1(Q, \mathbb{C}^4)$, *provided* $b_\infty > a$. *If moreover* G *is in even in* u *and* $b_\infty^- = \mu_k$ *for some positive integer* k, *then* (D_0) *has at least* $l := 2(\sharp_{\mu_1} + \cdots + \sharp_{\mu_k})$ *pairs of periodic solutions.*

A more general result can be obtained if (G_1) is replaced by

(G_1') there is $b_0 \in C(Q, [0, \infty))$ such that $b_0(x)$ is 1-period with $b_0(x) \geq 0$ and $G_u(x, u) - b_0(x)u = o(|u|)$ as $|u| \to \infty$ uniformly in $x \in Q$,

(G_2) is replaced by

(G_2') there is $b_\infty \in C(Q, (0, \infty))$ such that $b_\infty(x)$ is 1-period and $G_u(x, u) - b_\infty(x)u = o(|u|)$ as $|u| \to \infty$ uniformly in $x \in Q$,

and (G_3) is replaced by

(G_3') either (i) $0 \notin \sigma(A_V - b_\infty)$ or (ii) $\hat{G}(x,u) := \frac{1}{2}\hat{G}_u(x,u)u - G(x,u) \geq 0$ and $\hat{G}(x,u) \to \infty$ as $|u| \to \infty$ uniformly in $x \in Q$.

Theorem 1.5 *Suppose that* $(V), (G_0), (G_1') - (G_3')$ *are satisfied and* $q_\infty > q_0^+$, *where* $q_\infty := \min_{x\in Q} b_\infty(x)$, $q_0^+ := \min[\sigma(A_V) \cap (q_0, \infty)]$ *and* $q_0 := \max_{x\in Q} b_0(x)$. *Then*

(a) if $G(x,u) - \frac{1}{2}q_0|u|^2 \geq 0$, *then* (D_V) *has at least one nontrivial periodic solution in* $H^1(Q, \mathbb{C}^4)$*;*
(b) if G *is even in* u, *then* (D_V) *has at least* $d(q_0, q_\infty)$ *pairs of periodic solutions.*

This paper is organized as follows. In Sect. 2, we state the variational setting and establish a deformation theorem and abstract critical point theorems under the Cerami condition ($(C)_c$-condition). The proofs of the main results are given in Sect. 3.

2 Variational setting and abstract critical point theorems

To prove our main results, some preliminaries are first in order.

In what follows by $|\cdot|_q$ we denote the usual L^q-norm, and $(\cdot,\cdot)_2$ the usual L^2-inner product. Let

$$L_T^q(Q) := \{u \in L_{loc}^q(\mathbb{R}^3, \mathbb{C}^4) : u(x + \hat{e}_i) = u(x)\ a.e.,\quad i = 1, 2, 3\},$$

where $\hat{e}_1 = (1,0,0), \hat{e}_2 = (0,1,0), \hat{e}_3 = (0,0,1)$. Let $A_0 = -i\alpha\cdot\nabla + a\beta$, $A_V = A_0 + V$ denote the self-adjoint operators on $L^2(Q, \mathbb{C}^4)$ with domain

$$\begin{aligned}\mathcal{D}(A_V) = \mathcal{D}(A_0) &= H_T^1(Q)\\ &:= \{u \in H_{loc}^1(\mathbb{R}^3, \mathbb{C}^4) : u(x + \hat{e}_i) = u(x)\ a.e.,\ i = 1, 2, 3\}.\end{aligned}$$

Set $E := \mathcal{D}(|A_V|^{\frac{1}{2}})$ which is a Hilbert space with the inner product and norm, for $u = \sum_{j\in\mathbb{Z}} a_j e_j$ and $v = \sum_{j\in\mathbb{Z}} b_j e_j \in E$,

$$(u, v) = \sum_{j\neq 0} |\lambda_j| a_j \cdot b_j + (u^0, v^0)_2 \quad\text{and}\quad \|u\|^2 = \sum_{j\neq 0} |\lambda_j||a_j|^2 + |u^0|_2^2, \tag{2.1}$$

here $\{e_j\}_{j\in\mathbb{Z}}$ are the eigenvectors of A_V.

Then we have an orthogonal decomposition $E = E^- \oplus E^0 \oplus E^+$ with $E^- := \text{span}\{e_j : j < 0\}$, $E^+ := \text{span}\{e_j : j > 0\}$, and $E^0 := \ker(A_V)$. Note that if $0 \notin \sigma(A_V)$ then $E^0 = \{0\}$.

The functional Φ defined by (1.3) can be rewritten by

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \int_Q G(x,u)$$

for $u = u^- + u^0 + u^+ \in E$. Then $\Phi \in C^1(E, \mathbb{R})$ and critical points of Φ are solutions of (D_V).

First we have the following (see [8,11])

Lemma 2.1 $E = H^{1/2}(Q, \mathbb{C}^4)$ *with equivalent norms, hence* E *embeds compactly into* $L_T^s(Q)$ *for all* $s \in [1,3)$. *In particular there is a constant* $a_s > 0$ *such that*

$$|u|_s \leq a_s\|u\| \quad \textit{for all} \quad u \in E. \tag{2.2}$$

We also use the following result, the proof is similar to that of Proposition B.10 in [22].

Lemma 2.2 *Assume that*

(i) $G \in C^1(Q \times \mathbb{C}^4, \mathbb{R})$*, and*
(ii) *there are* $k_1, k_2 > 0$ *such that*

$$|G_u(x,u)| \le k_1 + k_2|u|^s, \quad \forall (x,u) \in Q \times \mathbb{C}^4,$$

where $0 \le s < 3$.

Then

$$\psi(u) := \int_Q G(x,u) \tag{2.3}$$

is weakly continuous and $\psi' \in C(E, \mathbb{R})$ *is compact.*

Recall that a sequence $\{u_j\}$ in E is said to be a $(C)_c$-sequence of Φ, if $\Phi(u_j) \to c$ and $(1 + \|u_j\|)\Phi'(u_j) \to 0$ as $j \to \infty$. We say that Φ satisfies the $(C)_c$-condition if any $(C)_c$-sequence possesses a convergent subsequence ([6]).

Let X be a Banach space, and

$$\Phi_a^b := \Phi_a \cap \Phi^b, \;\; \Phi_a := \{u \in X : \Phi(u) \ge a\}, \;\; \Phi^b := \{u \in X : \Phi(u) \le b\}.$$

We first establish a deformation theorem which plays an important role in the multiplicity for (D_V).

Theorem 2.3 *Let* $\Phi \in C^1(X, \mathbb{R})$ *and satisfy the* $(C)_c$*-condition,* $K_c = \{u \in X : \Phi(u) = c$ *and* $\Phi'(u) = 0\}$*. If* $\bar{\varepsilon} > 0$ *and* $\mathcal{O}$ *is any neighborhood of* K_c*, then there exists an* $\varepsilon \in (0, \bar{\varepsilon})$ *and a deformation* $\eta \in C([0,1] \times X, X)$ *such that*

1° $\eta(0,u) = u$ *for all* $u \in X$.
2° $\eta(t,u) = u$ *for all* $t \in [0,1]$ *if* $u \notin \Phi_{c-\varepsilon}^{c+\varepsilon}$.
3° $\eta(t,\cdot) : X \to X$ *is homeomorphism for* $t \in [0,1]$.
4° $\Phi(\eta(\cdot,u))$ *is nonincreasing on* $[0,1]$ *for* $u \in E$.
5° $\eta(1, \Phi^{c+\varepsilon} \setminus \mathcal{O}) \subset \Phi^{c-\varepsilon}$.
6° *If* $K_c = \emptyset$, $\eta(1, \Phi^{c+\varepsilon}) \subset \Phi^{c-\varepsilon}$.
7° *If* $\Phi(u)$ *is even in* u, $\eta(t,u)$ *is odd in* u.

Proof By the $(C)_c$-condition, K_c is compact. Set $U_\delta = \{u \in X : d(u, K_c) < \delta\}$. Choosing δ suitably small ($\delta < 1$), $U_\delta \subset \mathcal{O}$. Therefore it suffices to prove 5° with $\mathcal{O}$ replaced by U_δ. Note that $U_\delta = \emptyset$ when $K_c = \emptyset$, and so we get 6° instead.

Let $M > 0$ such that $\|u\| \le M$ for all $u \in U_\delta$.

One can easy to verify that there are $\hat{\varepsilon} > 0$ and $\alpha > 0$ such that

$$(1 + \|u\|)\|\Phi'(u)\| \ge \alpha, \quad \text{for all } \; u \in \Phi_{c-\hat{\varepsilon}}^{c+\hat{\varepsilon}} \setminus U_{\delta/2}. \tag{2.4}$$

We may assume that

$$0 < \hat{\varepsilon} < \frac{3\delta}{8(1+M)} \min\left\{\bar{\varepsilon}, \alpha^2, \frac{1}{4}\right\}. \tag{2.5}$$

Let $\tilde{X} := \{u \in X \mid \Phi'(u) \ne 0\}$ and $V : \tilde{X} \to X$ be a pseudo gradient such that V is odd if Φ is even (see [22]). Choosing any $\varepsilon \in (0, \hat{\varepsilon})$, define

$$h(s) = \begin{cases} 1, & \text{if } 0 \le s \le 1, \\ \frac{1}{s}, & \text{if } s > 1, \end{cases}$$

$$f(u)=\frac{d(u,X\setminus\Phi_{c-\hat{\varepsilon}}^{c+\hat{\varepsilon}})}{d(u,X\setminus\Phi_{c-\hat{\varepsilon}}^{c+\hat{\varepsilon}})+d(u,\Phi_{c-\varepsilon}^{c+\varepsilon})},\ g(u)=\frac{d(u,U_{\delta/8})}{d(u,U_{\delta/8})+d(u,X\setminus U_{\delta/4})}.$$

Then

$$f|_{\Phi_{c-\varepsilon}^{c+\varepsilon}}=g|_{X\setminus U_{\delta/4}}=1,\ \ f|_{X\setminus\Phi_{c-\hat{\varepsilon}}^{c+\hat{\varepsilon}}}=g|_{U_{\delta/8}}=0.$$

Let

$$W(u)=\begin{cases}-f(u)g(u)h((1+\|u\|)\|V(u)\|)(1+\|u\|)^2V(u), & u\in\tilde{X},\\ 0, & \text{otherwise.}\end{cases}$$

It is easy to verify that

$$\|W(u)\|\le 1+\|u\| \qquad \text{for all } u. \tag{2.6}$$

Then by construction, W is locally Lipschitz continuous on X and W is odd if Φ is even.

Now we consider the Cauchy problem:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t}=W(\eta),\ \ \eta(0,u)=u. \tag{2.7}$$

By virtue of the locally Lipschitz continuity of W and (2.6), the basic existence uniqueness theorem for ordinary differentia equations implies that for each $u\in X$, (2.7) has a unique solution $\eta(t,u)$ defined for $t\in[0,\infty)$, and $\eta\in C([0,1]\times X,X)$. (2.7) implies that 1° holds. Since $f(u)=0$ on $X\setminus\Phi_{c-\hat{\varepsilon}}^{c+\hat{\varepsilon}}$, so 2° is true. The semigroup property for solutions of (2.7) gives 3°. The oddness of W when Φ is even yields 7°.

If $W(u)\neq 0,\ u\in\tilde{X}$ so $V(u)$ is defined as is $V(\eta(t,u))$ and

$$\begin{aligned}\frac{\mathrm{d}\Phi(\eta(t,u))}{\mathrm{d}t} &= (\Phi'(\eta(t,u)),W(\eta(t,u)))\\ &= -f(\eta)g(\eta)h((1+\|\eta\|)\|V(\eta)\|)(1+\|\eta\|)^2(\Phi'(\eta),V(\eta))\\ &\le -f(\eta)g(\eta)h((1+\|\eta\|)\|V(\eta)\|)(1+\|\eta\|)^2\|\Phi'(\eta)\|^2\le 0.\end{aligned} \tag{2.8}$$

It follows that 4° holds.

Finally, we verify $\eta(1,\Phi^{c+\varepsilon}\setminus U_\delta)\subset\Phi^{c-\varepsilon}$. Let $u\in\Phi^{c+\varepsilon}\setminus U_\delta$, then $\Phi(\eta(t,u))\le c+\varepsilon$ by 4° and 1°. We need only prove that there exists $t_0\in[0,1]$ such that $\Phi(\eta(t_0,u))\le c-\varepsilon$, then 4° gives $\Phi(\eta(1,u))\le c-\varepsilon$.

If otherwise, then $\Phi(\eta(t,u))>c-\varepsilon$ for all $t\in[0,1]$, and thus $\eta(t,u)\in\Phi_{c-\varepsilon}^{c+\varepsilon}$, which implies

$$\Phi(\eta(0,u))-\Phi(\eta(t,u))\le 2\varepsilon<2\hat{\varepsilon},\ \ \forall t\in[0,1]. \tag{2.9}$$

If $\eta(t,u)\in X\setminus U_{\delta/2}$ for all $t\in[0,1]$, we see $\eta(t,u)\in\Phi_{c-\varepsilon}^{c+\varepsilon}\setminus U_{\delta/2}$. This shows $f(\eta(t,u))=g(\eta(t,u))=1$ and by (2.4),

$$(1+\|\eta(t,u)\|)\|\Phi'(\eta(t,u))\|\ge\alpha,\ \ \forall t\in[0,1]. \tag{2.10}$$

This yields

$$\begin{aligned}\frac{\mathrm{d}\Phi(\eta(t,u))}{\mathrm{d}t} &= -h((1+\|\eta\|)\|V(\eta)\|)(1+\|\eta\|)^2(\Phi'(\eta),V(\eta))\\ &\le -h((1+\|\eta\|)\|V(\eta)\|)(1+\|\eta\|)^2\|\Phi'(\eta)\|^2,\ \ \forall t\in[0,1].\end{aligned} \tag{2.11}$$

If $(1+\|\eta\|)\|V(\eta)\|\le 1$, then $h((1+\|\eta\|)\|V(\eta)\|)=1$. It follows from (2.10) and (2.11) that

$$\frac{\mathrm{d}\Phi(\eta(t,u))}{\mathrm{d}t}\le-\alpha^2. \tag{2.12}$$

If $(1+\|\eta\|)\|V(\eta)\| > 1$, then

$$h((1+\|\eta\|)\|V(\eta)\|) = [(1+\|\eta\|)\|V(\eta)\|]^{-1},$$

so (2.11) and the property of $V(\cdot)$ imply

$$\frac{\mathrm{d}\Phi(\eta(t,u))}{\mathrm{d}t} \le -(1+\|\eta\|)\|V(\eta)\| \left[\frac{\|\Phi'(\eta)\|}{\|V(\eta)\|}\right]^2 \le -\frac{1}{4}. \tag{2.13}$$

Consequently, by (2.12) and (2.13) we have

$$\frac{\mathrm{d}\Phi(\eta(t,u))}{\mathrm{d}t} \le -\min\left\{\alpha^2, \frac{1}{4}\right\} \quad \text{for all } t \in [0,1]. \tag{2.14}$$

Integrating (2.14) and combing the result with (2.9) gives

$$\begin{aligned} 2\hat{\varepsilon} &\ge \Phi(\eta(0,u)) - \Phi(\eta(1,u)) \\ &= \int_0^1 -\frac{\mathrm{d}\Phi(\eta(t,u))}{\mathrm{d}t} \ge \min\left\{\alpha^2, \frac{1}{4}\right\}, \end{aligned} \tag{2.15}$$

this is contrary to (2.5). Consequently, we infer that there is $\bar{t} \in [0,1]$ such that $\eta(\bar{t},u) \in U_{\delta/2}$. Obviously, $\bar{t} > 0$ since $\eta(0,u) = u \notin U_\delta$. The continuity of $\eta(t,u)$ guarantees that there are $s_1, s_2 \in [0,1]$ with $s_1 \ne s_2$ such that $\eta(s_1,u) \in \partial U_{\delta/4}$, $\eta(s_1,u) \in \partial U_\delta$ and $\eta(t,u) \in U_\delta \setminus \overline{U}_{\delta/4}$ for all $t \in (s_1,s_2)$ or $t \in (s_2,s_1)$, where $\overline{B}$ denotes the closure of B. This yields

$$\|\eta(s_1,u) - \eta(s_2,u)\| \ge 3\delta/4. \tag{2.16}$$

By (2.6) we see $\|W(u)\| \le 1+M$ for all $u \in U_\delta$, and so

$$\|\eta(s_2,u) - \eta(s_1,u)\| \le (1+M)|s_2 - s_1|$$

which together with (2.16) shows

$$|s_2 - s_1| \ge \frac{3\delta}{4(1+M)}.$$

We may assume that $s_1 < s_2$.

On the other hand, similarly to (2.15) we get that

$$\begin{aligned} 2\hat{\varepsilon} &\ge \Phi(\eta(s_1,u)) - \Phi(\eta(s_2,u)) \\ &= \int_{s_1}^{s_2} -\frac{\mathrm{d}\Phi(\eta(t,u))}{\mathrm{d}t} \\ &\ge \min\left\{\alpha^2, \frac{1}{4}\right\}(s_2 - s_1) \\ &\ge \frac{3\delta}{4(1+M)} \min\left\{\alpha^2, \frac{1}{4}\right\}. \end{aligned}$$

This, however, leads to a contradiction. The proof is complete. □

Remark 2.4 In paper [12] (or [13]), we established a deformation theorem under the $(C)_c$-condition. However, it is difficult to use for the multiplicity. Therefore, Theorem 2.3 improves the corresponding result in [12].

In order to study the functional Φ, we need certain abstract critical point theorems. In the following, we suppose that E is a real Hilbert space with $E = X \oplus Y$.

Theorem 2.5 *Let $e \in X \setminus \{0\}$ and $\Omega = \{u = se + v : \|u\| < R, s > 0, v \in Y\}$. Suppose that*

(Φ_1) $\Phi \in C^1(E, \mathbb{R})$, *satisfies the $(C)_c$-condition for any $c \in \mathbb{R}$;*
(Φ_2) *there is a $r \in (0, R)$ such that $\rho := \inf \Phi(X \cap \partial B_r) > \omega := \sup \Phi(\partial\Omega)$, where $\partial\Omega$ refers to the boundary of Ω relative to span$\{e\} \oplus Y$, and $B_r = \{u \in E : \|u\| < r\}$.*

Then Φ has a critical value $c \geq \rho$, with

$$c = \inf_{h\in\Gamma} \sup_{u\in\Omega} \Phi(h(u)),$$

here

$$\Gamma = \{h \in C(E, E) : h|_{\partial\Omega} = id,\ \Phi(h(u)) \leq \Phi(u) \text{ for } u \in \overline{\Omega}\}. \tag{2.17}$$

Proof Put $S = X \cap \partial B_r$. We first show that for any $h \in \Gamma$, $h(\Omega) \cap S \neq \emptyset$. We may assume $\|e\| = 1$. Chose $\hat{e} \in Y$ with $\|\hat{e}\| = 1$, and write $F := \text{span}\{e, \hat{e}\}$, $\Omega_F := F \cap \Omega$. Let $\overline{\Omega}_F$, $\partial\Omega_F$ denote the closure and bound of Ω in F, respectively, P the project of E onto Y. For $u \in \overline{\Omega}_F$, $t \in [0, 1]$, define

$$H(t, u) = t[\|(\text{id} - P)h(u)\|e + Ph(u)] + (1 - t)u.$$

Then $H : [0, 1] \times \overline{\Omega}_F \to E$ is continuous. Obviously H is a compact operator. Since $h|_{\partial\Omega} = \text{id}$, if $u \in \partial\Omega_F$,

$$H(t, u) = t[\|u - Pu\|e + Pu] + (1 - t)u = u,$$

i.e., $H(t, \cdot)|_{\partial\Omega_F} = \text{id}$ for $t \in [0, 1]$. In particular $H(t, u) \neq re$ for $t \in [0, 1]$, $u \in \partial\Omega_F$. By the property of Brouwer degree, we have

$$\deg(H(1, \cdot), \Omega_F, re) = \deg(H(0, \cdot), \Omega_F, re) = \deg(\text{id}, \Omega_F, re) = 1$$

which implies that there exists $u \in \Omega_F$ such that $H(1, u) = re \in S$. We find $Ph(u) = 0$, $\|h(u)\| = r$, i.e. $h(u) \in S$, and therefore $c \geq \rho$.

Next we prove there is a sequence $\{u_j\}$ in Ω such that

$$(1 + \|u_j\|)\|\Phi'(u_j)\| \to 0 \ \text{ for } j \to \infty. \tag{2.18}$$

Indeed otherwise there exist $\alpha_0 > 0$ and $\varepsilon_0 > 0$ such that

$$(1 + \|u\|)\|\Phi'(u)\| \geq \alpha_0 \quad \text{for all } u \in \Omega \cap \Phi_{c-\varepsilon_0}^{c+\varepsilon_0}.$$

Set $\bar{\varepsilon} = \min\{\frac{1}{2}(\rho-\omega), \varepsilon_0\}$. There is an $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0, 1]\times E, E)$ given by Theorem 2.3 such that $1° - 4°$ and $6°$ are satisfied. Chose $h \in \Gamma$ such that $\sup \Phi(h(\Omega)) \leq c + \varepsilon$. Consequently

$$h(\Omega) \subset \Phi^{c+\varepsilon}. \tag{2.19}$$

Let $g(u) := \eta(1, h(u))$, then $g \in C(E, E)$. It follows from 3° and 1° that

$$\Phi(g(u)) = \Phi(\eta(1, h(u))) \leq \Phi(\eta(0, h(u))) = \Phi(h(u)) \leq \Phi(u)$$

for all $u \in \overline{\Omega}$. For $u \in \partial\Omega$, (Φ_2) shows

$$\Phi(u) \leq \omega < \rho - \bar{\varepsilon} \leq c - \bar{\varepsilon} \leq c - \varepsilon$$

which, by 2°, implies $\eta(1, u) = u$, and so

$$g(u) = \eta(1, h(u)) = \eta(1, u) = u.$$

Thus $g \in \Gamma$. (2.19) and 6° yield $g(\Omega) = \eta(1, h(\Omega)) \subset \Phi^{c-\varepsilon}$ which leads to the contradiction $c \le \sup \Phi(g(\Omega)) \le c - \varepsilon$.

Now we find that there is a sequence $\{u_j\}$ in Ω satisfying (2.18). Since Φ satisfies (Φ_1) (the $(C)_c$-condition), there exists a convergent subsequence $\{u_{j_k}\}$ of $\{u_j\}$ such that $u_{j_k} \to \bar{u}$. The conclusion follows by $\Phi \in C^1(E, E)$. □

Remark 2.6 In [[22], Theorem 5.3], under the conditions that Y is finite dimensional and Φ satisfies the (PS)-condition, the same result was proved. Clearly, the conditions of Theorem 2.5 are weaker than that of Theorem 5.3.

Next, we consider a kind of pseudo-index (see [7]). Let Σ denote the class of closed subsets of E symmetric with respect to the origin, and $\gamma : \Sigma \to \mathbb{N} \cup \{\infty\}$ the $\mathbb{Z}_2$ genus map (see [22]). Let $\Phi \in C(E, \mathbb{R})$, $J = (\sigma, \infty)$,

$$\mathcal{H} = \{h \in C(E, E) : \ h \text{ is a homeomorphism and is odd}\},$$

$$\mathcal{M}_J = \{g \in \mathcal{H} : \ g|_{\Phi^{-1}(\mathbb{R}\setminus J)} = \text{id and } \Phi(g(u)) \le \Phi(u) \text{ for } u \in E\},$$

and $\Lambda_* = \{h \in \mathcal{M}_J : \ h(B_1 Y) \subset \Phi^{-1}(J) \cup B_r Y\}$.

Now we define the pseudo-index (Σ, i^*) relative to $\mathcal{M}_J$ for the genus γ as follows

$$i^*(A) = \inf_{h \in \Lambda_*} \gamma(A \cap h(S_1 Y)).$$

One can verify the following

Lemma 2.7 *Let $\Sigma^* = \Sigma$, then (Σ^*, i^*) satisfies all properties for pseudo-index ([7]):*

$(P1)$ $\Sigma^* \subset \Sigma$, $\overline{A \setminus B} \in \Sigma^*$ and $\overline{g(A)} \in \Sigma^*$ for all $A \in \Sigma^*$, $B \in \Sigma$ and $g \in \mathcal{M}_J$;
$(P2)$ $A \subset B$ implies $i^*(A) \le i^*(B)$ for all $A, B \in \Sigma^*$;
$(P3)$ $i^*(\overline{A \setminus B}) \ge i^*(A) - \gamma(B)$ for all $A \in \Sigma^*$ and $B \in \Sigma$;
$(P4)$ $i^*(\overline{g(A)}) \ge i^*(A)$ for all $A \in \Sigma^*$ and $g \in \mathcal{M}_J$.

Now, we give a abstract critical point theorem as follows.

Theorem 2.8 *Assume that Φ is even and satisfies (Φ_1). If*

(Φ_3) there exists $r > 0$ with $\rho := \inf \Phi(S_r Y) > \Phi(0) = 0$, where $S_r := \partial B_r$, $AB := A \cap B$;
(Φ_4) there exists a finite dimensional subspace $Y_0 \subset Y$ and $R > r$ such that for $E_* := X \oplus Y_0$, $M_* = \sup \Phi(E_*) < +\infty$ and $\sigma := \sup \Phi(E_* \setminus B_R) < \rho$,

then Φ possesses at least m distinct pairs of critical points, where $m = \dim Y_0$.

Proof Let

$$\Sigma_k = \{A \in \Sigma : i^*(A) \ge k\}, \quad k = 1, 2, \dots, m.$$

Define

$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} \Phi(u), \quad k = 1, 2, \dots, m. \tag{2.20}$$

We first show $\Sigma_k \ne \emptyset$. Set $\tilde{A} := B_R E_*$. (Φ_4) implies $\Phi^{-1}(J) \subset (E \setminus E_*) \cup B_R$, and hence

$$\tilde{A} \supset Y_0 \cap (\Phi^{-1}(J) \cup B_R Y) \supset Y_0 \cap h(B_1 Y)$$

for each $h \in \Lambda_*$, which yields

$$\tilde{A} \cap h(S_1 Y) \supset Y_0 \cap h(S_1 Y) \supset \partial(Y_0 \cap h(B_1 Y)),$$

and we get

$$\gamma(\tilde{A} \cap h(S_1 Y)) \geq \gamma(\partial(Y_0 \cap h(B_1 Y))) \geq m.$$

Consequently, $\Sigma_k \neq \emptyset$, and $c_m \leq M_*$ by (Φ_4). For any $A \in \Sigma_k$, by $h := r\mathrm{id} \in \Lambda_*$ one has

$$\gamma(A \cap S_r Y) = \gamma(A \cap h(S_1 Y) \geq i^*(A) \geq k$$

which yields $c_k \geq \rho$ by (Φ_3). Noting that $\Sigma_1 \supset \Sigma_2 \supset \cdots \supset \Sigma_m$, we have

$$\sigma < \rho \leq c_1 \leq c_2 \leq \cdots \leq c_m \leq M_*.$$

It is obvious that $K_c := \{u \in X : \Phi(u) = c \text{ and } \Phi'(u) = 0\} \in \Sigma$, and K_c is compact by the $(C)_c$-condition.

Finally, we claim:

(P^*) If $1 \leq j,\ j + l \leq m$, and $c_j = \cdots = c_{j+l} \equiv c$, then $\gamma(K_c) \geq l + 1$.

If $\gamma(K_c) \leq l$, then there is a $\delta > 0$ such that $\gamma(U_\delta(K_c)) = \gamma(K_c) \leq l$. Invoking Theorem 2.3 with $\mathcal{O} = U_\delta(K_c)$ and $\bar{\varepsilon} = \frac{\rho-\sigma}{2}$, there are $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0, 1] \times E, E)$ such that $\eta(1, \cdot)$ satisfies the properties $1° - 7°$ and

$$\eta(1, \Phi^{c+\varepsilon} \setminus \mathcal{O}) \subset \Phi^{c-\varepsilon}. \tag{2.21}$$

Choose $\hat{A} \in \Sigma_{j+l}$ such that $\sup \Phi(\hat{A}) \leq c + \varepsilon$, and hence

$$\hat{A} \subset \Phi^{c+\varepsilon}. \tag{2.22}$$

By $(P3)$ one has

$$i^*(\overline{\hat{A} \setminus \mathcal{O}}) \geq i^*(\hat{A}) - \gamma(\mathcal{O}) \geq j + l - l = j. \tag{2.23}$$

Using $3°$ and $7°$ we get $\eta(1, \cdot) \in \mathcal{H}$. $4°$ gives $\Phi(\eta(1, u)) \leq \Phi(u)$ for all $u \in E$. Since $\sigma < c - \varepsilon$, we have $\Phi^{-1}(\mathbb{R} \setminus J) \subset E \setminus \Phi_{c-\varepsilon}^{c+\varepsilon}$, and $2°$ implies $\eta(1, \cdot)|_{\Phi^{-1}(\mathbb{R}\setminus J)} = \mathrm{id}$. Therefore $\eta(1, \cdot) \in \mathcal{M}_J$. Set $A_* := \eta(1, \overline{\hat{A} \setminus \mathcal{O}}) \in \Sigma$. It follows from $(P4)$ and (2.23) that

$$i^*(A_*) = i^*\left(\overline{\eta(1, \overline{\hat{A} \setminus \mathcal{O}})}\right) \geq i^*\left(\overline{\hat{A} \setminus \mathcal{O}}\right) \geq j,$$

and thus $A_* \in \Sigma_j$. Combing with (2.21), (2.22) and (2.20) we see

$$c \leq \sup \Phi(A_*) \leq c - \varepsilon < c,$$

a contradiction. Therefore, the conclusion (P^*) is valid and the proof is complete. □

3 The proof of the main results

Throughout this section, we suppose that (V) and (G_0) are satisfied.

Observe that, (G_2) implies that for any $\varepsilon > 0$ there is $R_\varepsilon > 0$ such that

$$|G_u(x, u) - b_\infty u| \leq \varepsilon|u| \text{ whenever } |u| \geq R_\varepsilon, \tag{3.1}$$

hence

$$|G_u(x, u)u - b_\infty|u|^2| \leq |G_u(x, u) - b_\infty u||u| \leq \varepsilon|u|^2$$

or

$$(b_\infty - \varepsilon)|u|^2 \le G_u(x,u)u \le (b_\infty + \varepsilon)|u|^2 \quad \text{for all } |u| \ge R_\varepsilon.$$

Fixed $s_0 \in (0, 1)$, in virtue of $G(x,u) \ge 0$ we get

$$\begin{aligned} G(x,u) &= G(x, s_0 u) + \int_{s_0}^1 G_u(x, su) \cdot u ds \\ &\ge \int_{s_0}^1 \frac{1}{s} G_u(x, su) su ds \\ &\ge \frac{1}{2}(b_\infty - \varepsilon)(1 - s_0^2)|u|^2 \end{aligned}$$

for all $|u| \ge \frac{1}{s_0} R_\varepsilon$, and so

$$G(x,u) \ge \frac{1}{2}(b_\infty - \varepsilon)(1 - s_0^2)|u|^2 - C_{s_0} \quad \text{for all } (x,u). \tag{3.2}$$

First, we have the following lemma.

Lemma 3.1 *Suppose that* (G_1) *and* (G_2) *hold and* $\{u_j\}$ *is a bounded* $(C)_c$*-sequence of* Φ. *Then there exists a critical point* u *of* Φ *such that* $\Phi(u) = c$ *and after passing to a subsequence,* $u_j \to u$ *strongly in* E.

Proof By Lemma 2.1, without loss of generality, we may assume that

$$u_n \rightharpoonup u \text{ in } E \text{ and } u_u \to u \text{ in } L_T^s(Q) \text{ for } s \in [1, 3). \tag{3.3}$$

Plainly, u is a critical point of Φ. (G_1) and (G_2) yield that

$$|G_u(x,u)| \le C_1|u| \text{ for all } (x,u) \tag{3.4}$$

which shows that ψ' is continuous and compact by Lemma 2.2, where ψ is defined by (2.3). It follows from the representation of Φ', together with (3.3), the facts $\Phi'(u) = 0$ and $\Phi'(u_n) \to 0$, and the compactness of ψ', that

$$\begin{aligned} \|u_n^+ - u^+\|^2 &= (\Phi'(u_n) - \Phi'(u), u_n^+ - u^+) \\ &\quad + (\psi'(u_n) - \psi'(u), u_n^+ - u^+) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Similarly, $\|u_n^- - u^-\| \to 0$ as $n \to \infty$. It is clear that $\{u_j^0\}$ has a convergent subsequence since E^0 is finite dimensional. We have thus proved the lemma. □

Lemma 3.2 *If* $b_\infty > \lambda_1$ *and* (G_3) *holds, then any* $(C)_c$*-sequence of* Φ *is bounded.*

Proof Let $\{u_j\} \subset E$ be such that $\Phi(u_j) \to c$ and $(1 + \|u_j\|)\Phi'(u_j) \to 0$.

Defining

$$\tilde{E}^+ := \left\{ u \in E : u = \sum_{\lambda_j > b_\infty} a_j e_j \right\},$$

$$\tilde{E}^0 := \left\{ u \in E : u = \sum_{\lambda_j = b_\infty} a_j e_j \right\},$$

$$\tilde{E}^- := \left\{ u \in E : u = \sum_{\lambda_j < b_\infty, \lambda_j \ne 0} a_j e_j + u^0, u^0 \in E^0 \right\},$$

we have $E = \tilde{E}^+ \oplus \tilde{E}^0 \oplus \tilde{E}^-$ and write $u = \tilde{u}^+ + \tilde{u}^0 + \tilde{u}^-$ for $u \in E$ corresponding to this decomposition. Clearly, $\tilde{E}^0 = \{0\}$ if $\mathrm{b}_\infty \notin \sigma(A_V)$.

Let $P^\pm : E \to E^\pm$ be the orthogonal projections. One can see

$$\begin{aligned}(\Phi'(u), \tilde{u}^+) &= \|\tilde{u}^+\|^2 - b_\infty|\tilde{u}^+|_2^2 - \int_Q G_u^\infty(x,u)\tilde{u}^+,\\ (\Phi'(u), \tilde{u}^-) &= (P^+u, \tilde{u}^-) - (P^-u, \tilde{u}^-)\\ &\quad -b_\infty|\tilde{u}^-|_2^2 - \int_Q G_u^\infty(x,u)\tilde{u}^-.\end{aligned} \tag{3.5}$$

For $u = \sum_{j\in\mathbb{Z}, j\neq 0} a_j e_j + u^0 \in E$ ($u^0 \in E^0$), we have

$$\tilde{u}^+ = \sum_{\lambda_j > b_\infty} a_j e_j, \quad \tilde{u}^- = \sum_{\lambda_j < b_\infty, \lambda_j \neq 0} a_j e_j + u^0.$$

By (2.1) one finds

$$\begin{aligned}\|\tilde{u}^+\|^2 - b_\infty|\tilde{u}^+|_2^2 &= \sum_{\lambda_j > b_\infty} \lambda_j |a_j|^2 - b_\infty \sum_{\lambda_j > b_\infty} |a_j|^2\\ &\geq \left(1 - \frac{b_\infty}{\lambda'}\right)\|\tilde{u}^+\|^2,\end{aligned} \tag{3.6}$$

where $\lambda' := \min(\sigma(A_V) \cap (b_\infty, \infty))$. Since $b_\infty > \lambda_1, \sigma(A_V) \cap (0, b_\infty) \neq \emptyset$. Setting $\lambda'' := \max(\sigma(A_V) \cap (0, b_\infty))$, we obtain

$$\begin{aligned}&(P^+u, \tilde{u}^-) - (P^-u, \tilde{u}^-) - b_\infty|\tilde{u}^-|_2^2\\ &= \sum_{0<\lambda_j<b_\infty} \lambda_j|a_j|^2 - \sum_{\lambda_j<0} |\lambda_j||a_j|^2 - b_\infty \sum_{\lambda_j<b_\infty, \lambda_j\neq 0} |a_j|^2 - b_\infty|u^0|_2^2\\ &\leq \|\tilde{u}^-\|^2 - 2\sum_{\lambda_j<0} |\lambda_j||a_j|^2 - \frac{b_\infty}{\lambda''} \sum_{0<\lambda_j<b_\infty} \lambda_j|a_j|^2 - (1 + b_\infty)|u^0|_2^2,\end{aligned}$$

and therefore

$$-(P^+u, \tilde{u}^-) + (P^-u, \tilde{u}^-) + b_\infty|\tilde{u}^-|_2^2 \geq (w - 1)\|\tilde{u}^-\|^2, \tag{3.7}$$

here $w := \min\{1 + b_\infty, 2, \frac{b_\infty}{\lambda''}\}$. For $\delta > 0$ small, it follows from (3.1) that

$$|G_u^\infty(x,u)| < \delta|u| + C_\delta, \text{ for all } (x,u). \tag{3.8}$$

Putting $u_j = \tilde{u}_j^+ + \tilde{u}_j^- + \tilde{u}_j^0$, by (3.5) we know

$$\begin{aligned}&\|\tilde{u}_j^+\|^2 - b_\infty|\tilde{u}_j^+|_2^2 = (\Phi'(u_j), \tilde{u}_j^+) + \int_Q G_u^\infty(x,u_j)\tilde{u}_j^+,\\ &-(P^+u_j, \tilde{u}_j^-) + (P^-u_j, \tilde{u}_j^-) + b_\infty|\tilde{u}_j^-|_2^2\\ &\qquad = -(\Phi'(u_j), \tilde{u}_j^-) - \int_Q G_u^\infty(x,u_j)\tilde{u}_j^-.\end{aligned} \tag{3.9}$$

(3.6)–(3.9) and (2.2) yield

$$\begin{aligned}\xi\|\tilde{u}_j^+ + \tilde{u}_j^-\|^2 &\leq \|\Phi'(u_j)\|\|\tilde{u}_j^+ + \tilde{u}_j^-\|\\ &\quad + \delta C'\|u_j\|\|\tilde{u}_j^+ + \tilde{u}_j^-\| + C'_\delta\|\tilde{u}_j^+ + \tilde{u}_j^-\|\end{aligned} \tag{3.10}$$

with $\xi = \min\{1 - \frac{b_\infty}{\lambda'}, w - 1\}$.

If (i) of (G_3) holds, then $u_j = \tilde{u}_j^+ + \tilde{u}_j^-$. (3.10) implies that

$$\xi \|u_j\| \le \|\Phi'(u_j)\| + \delta C'\|u_j\| + C'_\delta,$$

and so $\{u_j\}$ is bounded.

Next let (ii) of (G_3) be satisfied. (3.6), (3.7) and (3.9) yield that $\{\tilde{u}_j^+ + \tilde{u}_j^-\}$ is bounded. We claim that $\{\tilde{u}_j^0\}$ is bounded.

Assume by contradiction that $\|\tilde{u}_j^0\| \to \infty$ as $j \to \infty$. Since $\tilde{E}^0$ is finite dimensional, we have: along a subsequence, there exists $Q_0 \subset Q$ satisfying $|Q_0| > 0$ such that $|\tilde{u}_j^0(x)| \to \infty$ as $j \to \infty$ uniformly in $x \in Q_0$. Here, we write $|W|$ for the Lebesgue measure of $W \subset \mathbb{R}^3$. It follows from the hypotheses that $G^\infty(x, \tilde{u}_j^0) \to \infty$ as $j \to \infty$ uniformly in $x \in Q_0$, and thus

$$\begin{aligned} G^\infty(x, u_j) &= G^\infty(x, \tilde{u}_j^0) + \int_0^1 G_u^\infty(x, s(u_j - \tilde{u}_j^0))(\tilde{u}_j^+ + \tilde{u}_j^-)ds \\ &\ge G^\infty(x, \tilde{u}_j^0) - K_1\|\tilde{u}_j^+ + \tilde{u}_j^-\| \to \infty \end{aligned} \tag{3.11}$$

as $j \to \infty$ uniformly in $x \in Q_0$.

By (3.2) and $G^\infty(x, u) \to \infty$ as $|u| \to \infty$ we obtain that there exists $m_0 > 0$ such that

$$G^\infty(x, u) \ge -m_0 \ \text{ for all } \ (x, u). \tag{3.12}$$

Noting that

$$\begin{aligned} &\|u^+\|^2 - \|u^-\|^2 - b_\infty |u|_2^2 \\ &\quad = \|\tilde{u}^+\|^2 + \sum_{0<\lambda_j<b_\infty} \lambda_j |a_j|^2 - \sum_{\lambda_j<0} |\lambda_j||a_j|^2 - b_\infty |\tilde{u}^+ + \tilde{u}^-|_2^2, \end{aligned}$$

we get by (2.2)

$$\left| \|u^+\|^2 - \|u^-\|^2 - b_\infty |u|_2^2 \right| \le (1 + a_2 b_\infty)(\|\tilde{u}^+\|^2 + \|\tilde{u}^-\|^2). \tag{3.13}$$

On account of (3.13), (3.12) and (3.11) we see that

$$\begin{aligned} |\Phi(u_j)| &= \left| \frac{1}{2}(\|u_j^+\|^2 - \|u_j^-\|^2 - b_\infty |u_j|_2^2) - \int_Q G^\infty(x, u_j) \right| \\ &\ge \left| \int_Q G^\infty(x, u_j) \right| - \frac{1}{2}(1 + a_2 b_\infty)(\|\tilde{u}_j^+\|^2 + \|\tilde{u}_j^-\|^2) \\ &\ge \int_{Q_0} G^\infty(x, u_j) - m_0 - \frac{1}{2}(1 + a_2 b_\infty)(\|\tilde{u}_j^+\|^2 + \|\tilde{u}_j^-\|^2) \to \infty \end{aligned}$$

as $j \to \infty$, a contradiction. Consequently $\{x_j^0\}$ is bounded and the proof is complete. □

We need to introduce another orthogonal decomposition: $E = \hat{E}^+ \oplus \hat{E}^0 \oplus \hat{E}^-$, where

$$\begin{cases} \hat{E}^+ := \left\{ u \in E : \ u = \sum_{\lambda_j > b_0} a_j e_j \right\}, \\ \hat{E}^0 := \left\{ u \in E : \ u = \sum_{\lambda_j = b_0} a_j e_j \right\}, \\ \hat{E}^- := \left\{ u \in E : \ u = \sum_{\lambda_j < b_0, \lambda_j \neq 0} a_j e_j + u^0, u^0 \in E^0 \right\}, \end{cases} \tag{3.14}$$

One can verify that there is $\xi_0 \in (0, 1)$ such that

$$\begin{aligned} &\|\hat{u}^+\|^2 - b_0|\hat{u}^+|_2^2 \geq \xi_0\|\hat{u}^+\|^2, \\ &(P^+u, \hat{u}^-) - (P^-u, \hat{u}^-) - b_0|\hat{u}^-|_2^2 \leq -\xi_0\|\hat{u}^-\|^2 \end{aligned} \tag{3.15}$$

for any $u = \hat{u}^+ + \hat{u}^0 + \hat{u}^- \in E$, the proof is similar to that of (3.6) and (3.7).

Lemma 3.3 *Suppose that* (G_1) *and* (G_2) *hold, then there exist* $r > 0$ *and* $\rho > 0$ *such that* $\inf \Phi(\hat{E}^+ \cap B_r) \geq 0$ *and* $\inf \Phi(\hat{E}^+ \cap \partial B_r) \geq \rho$.

Proof Choosing $q \in (2, 3)$, we have that, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$G^0(x, u) \leq \varepsilon|u|^2 + C_\varepsilon|u|^q, \quad \text{for all } (x, u).$$

This implies

$$\begin{aligned} \Phi(\hat{u}^+) &= \frac{1}{2}(\|\hat{u}^+\|^2 - b_0|\hat{u}^+|_2^2) - \int_Q G^0(x, \hat{u}^+) \\ &\geq \frac{1}{2}\xi_0\|\hat{u}^+\|^2 - \varepsilon C_1'\|\hat{u}^+\|^2 - C_2'C_\varepsilon\|\hat{u}^+\|^p \end{aligned}$$

via (3.15) for $\hat{u} \in \hat{E}^+$, which follows that the conclusion is valid. □

Lemma 3.4 *Let* (G_2) *be satisfied. If* $b_\infty > b_0^+$, *then for any* $n \in \mathbb{N}$ *with* $b_\infty^- = \lambda_n$, *there exists* $R_n > r$ *such that* $\sup \Phi(E_n \setminus B_{R_n}) < 0$ *and* $\sup \Phi(E_n) < \infty$, *where* r *is as in Lemma 3.3,* $E_n := E^- \oplus E^0 \oplus span\{e_1, ..., e_n\}$.

Proof It will suffice to show that for $u \in E_n$

$$\Phi(u) \to -\infty \quad \text{as} \quad \|u\| \to \infty. \tag{3.16}$$

Choose $s_0 \in \left(0, \sqrt{1 - \frac{b_\infty^-}{b_\infty}}\right)$ in (3.2). Noting that $u^+ = \sum_{j=1}^n s_j e_j$ for $u \in E_n$, by (3.2), for $\varepsilon = \frac{1}{2}(b_\infty - \frac{b_\infty^-}{1-s_0^2})$, we find

$$\begin{aligned} 2\Phi(u) &= \|u^+\|^2 - \|u^-\|^2 - 2\int_Q G(x, u) \\ &\leq \|u^+\|^2 - \|u^-\|^2 - \alpha_0(|u^+|_2^2 + |u^0|_2^2 + |u^-|_2^2) + 2C_{s_0}, \end{aligned} \tag{3.17}$$

where $\alpha_0 := (b_\infty - \varepsilon)(1 - s_0^2) > b_\infty^-$. Since

$$\alpha_0|u^+|_2^2 - \|u^+\|^2 \geq (\alpha_0 - \lambda_n)\sum_{j=1}^n |s_j|^2 \geq \frac{\alpha_0 - b_\infty^-}{\lambda_1}\|u^+\|_2^2,$$

by (3.17) we get

$$2\Phi(u) \leq -\frac{\alpha_0 - b_\infty^-}{\lambda_1}\|u^+\|^2 - C_2(\|u^-\|^2 + \|u^0\|^2) + 2C_{s_0}$$
$$\leq -\hat{C}\|u\|^2 + 2C_{s_0}$$

which implies that (3.16) is valid and $\sup \Phi(E_n) < \infty$. □

As a consequence, we have

Lemma 3.5 *Under the conditions of Lemma 3.4, if $G^0(x,u) \geq 0$, then there is $R_0 > r$ such that* $\sup \Phi(\partial\Omega) \leq 0$, *where*

$$\Omega := \{u = \hat{u}^- + \hat{u}^0 + se_m : \hat{u}^- + \hat{u}^0 \in \hat{E}^- \oplus \hat{E}^0, s > 0, \|u\| < R_0\}$$

with $A_V e_m = b_0^+ e_m$, and $\partial\Omega$ refers to the boundary of Ω relative to span$\{e_m\} \oplus \hat{E}^- \oplus \hat{E}^0$.

Proof Since $\hat{E}^- \oplus \hat{E}^0 \oplus \mathbb{R}^+ e_m \subset E_m$ and $\lambda_m = b_0^+ \leq b_\infty^-$, by Lemma 3.4 we find that $\Phi(u) < 0$ for $u = \hat{u}^- + \hat{u}^0 + se_m$ with $\|u\| = R_0$ and $s > 0$ when $R_0 > r$ large.

Let $u = \hat{u}^- + \hat{u}^0$ with $\|u\| \leq R_0$. By $G^0(x,u) \geq 0$ and (3.15) one has

$$2\Phi(u) = (P^+u, \hat{u}^-) - (P^-u, \hat{u}^-) - b_0|\hat{u}^-|_2^2 - 2\int_Q G^0(x,u)$$
$$\leq -\xi_0\|\hat{u}^-\|^2 - 2\int_Q G^0(x,u) \leq 0$$

which yields that the result is valid. □

Now, with the above arguments, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 (Existence) Let us verify the conditions of Theorem 2.5. Let $X = \hat{E}^+$, $Y = \hat{E}^- \oplus \hat{E}^0$, $r > 0$ be from Lemma 3.3. Lemma 3.1 and 3.2 imply that (Φ_1) is true. Lemma 3.3 yields $\inf \Phi(X \cap \partial B_r) \geq \rho$, and Lemma 3.5 gives $\Phi|_{\partial\Omega} < \sigma_0$ for $\sigma_0 \in (0,\rho)$. Therefore (Φ_2) holds. It follows from Theorem 2.5 that Φ possesses a critical value $c \geq \rho$, with

$$c = \inf_{h\in\Gamma} \sup_{u\in\Omega} \Phi(h(1,u)),$$

where Γ is defined as (2.17).

Next, we proceed to prove the multiplicity. Since G is even in u, Φ is even. Using Lemma 3.3 we know that the condition (Φ_3) holds with $X = \hat{E}^- \oplus Y^0$ and $Y = \hat{E}^+$. Let span$\{e_m, \ldots, e_n\}$ be the eigenspace associated to $\sigma(A_V) \cap (b_0, b_\infty)$, and λ_j the eigenvalue corresponding to e_j (i.e., $A_V e_j = \lambda_j e_j$), $j = m, \ldots, n$, then $b_0^+ = \lambda_m$, $b_\infty^- = \lambda_n$ and $d(b_0, b_\infty) = n - m$. It follow from Lemma 3.4 that Φ satisfies (Φ_4) with $Y_0 =$ span$\{e_m, \ldots, e_n\}$, $R = R_n$, $M_* = M_n$ and $\sigma \in (0,\rho)$. Therefore, Φ has at least $n - m$ pairs of nontrivial critical points by Theorem 2.8. □

We are now in a position to give the proof of Theorem 1.5.

Proof The main difference to the proof of Theorem 1.2 lies in the boundedness of the (C)c-sequences.

Claim 1. Any (C)c-sequence is bounded.

Let $\{u_j\} \subset E$ be such that

$$\Phi(u_j) \to c,\ (1 + \|u_j\|)\Phi'(u_j) \to 0 \text{ as } j \to \infty.$$

We then have

$$\int_Q \hat{G}(x,u_j) = \Phi(u_j) - \frac{1}{2}\Phi'(u_j)\cdot u_j \le C_0. \tag{3.18}$$

Assume by contradiction that $\|u_j\| \to \infty$. Then the normalized sequence $v_j = u_j/\|u_j\|$ satisfies (up to a subsequence) $v_j \rightharpoonup v$ in E. Lemma 2.1 guarantees $v_j \to v$ in $L^s_T(Q)$ for $s \in [1,3)$ and $|v_j|_s \le a_s$ for all $s \in [1,3]$. We write $\tilde{u}_j = u_j^- + u_j^+$, $\tilde{v}_j = v_j^- + v_j^+$. Then

$$\Phi'(u_j)(u_j^+ - u_j^-) = \|u_j\|^2\left(\|\tilde{v}_j\|^2 - \int_Q \frac{G_u(x,u_j)(v_j^+ - v_j^-)|v_j|}{|u_j|}\right),$$

and therefore

$$o(1) = \|\tilde{v}_j\|^2 - \int_Q \frac{G_u(x,u_j)(v_j^+ - v_j^-)|v_j|}{|u_j|}. \tag{3.19}$$

We distinguish the two cases: $v = 0$ or $v \neq 0$.

let $v = 0$. (G_1') and (G_2') yield that (3.4) is true, this implies

$$\int_Q \frac{|G_u(x,u_j)|}{|u_j|}|v_j^+ - v_j^-|\,|v_j| \le C_1|v_j|_2^2$$

which jointly with (3.19) shows $\|\tilde{v}_j\|^2 \to 0$, and so $|\tilde{v}_j|_2 \to 0$. $|v_j|_2 \to 0$ yields $|v_j^0|_2 \to 0$. We obtain $1 = \|v_j\| = \|\tilde{v}_j\| + |v_j^0|_2 \to 0$, a contradiction.

Assume $v \neq 0$. First let (i) of (G_3) hold. Since $|u_j(x)| = |v_j(x)|\,\|u_j\| \to \infty$, by (3.4) and Lebesgue dominated convergence theorem we obtain

$$\int_Q \frac{G_u(x,u_j)v_j\varphi}{|u_j|} \to \int_Q b_\infty(x)v\varphi$$

for any $\varphi \in C^\infty[Q,\mathbb{C}^4]$, hence $A_V v = b_\infty v$, which contradicts $0 \notin \sigma(A_V - b_\infty)$.

Suppose that (ii) of (G_3) is satisfied. $v_j \to v$ in $L^s_T(Q)$ guarantees (up to a subsequence) $v_j(x) \to v(x)$ a.e. on Q. Since $v \neq 0$, there exists $Q_0 \subset Q$ with $|Q_0| > 0$ such that

$$v_j(x) \to v(x) \ \text{ as } \ j \to \infty \ \text{ uniformly on } \ Q_0$$

and $|v_j(x)| \ge \varepsilon_0 > 0$ for large j. Observe that $|u_j(x)| = \|u_j\||v_(x)| \ge \varepsilon_0\|u_j\| \to \infty$ for $x \in Q_0$. By (ii) of (G_3) we have

$$\int_{Q_0} \hat{G}(x,u_j) \to \infty,$$

which contradicts (3.18).

Next we have

Claim 2. The conclusions of Lemmas 3.3–3.5 are true where (G_1) and (G_2) are replaced by (G_1') and (G_2') respectively, and b_0 is replaced by q_0 in (3.14).

Since (3.2) and (3.4) are satisfied, where b_∞ is replaced by q_∞, one can prove as before.

Finally, repeating the arguments of the proof of Theorem 1.2, we obtain the desired results. □

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