Annali di Matematica (2017) 196:687–716
DOI 10.1007/s10231-016-0591-6

Sobolev spaces of isometric immersions of arbitrary dimension and co-dimension

Robert L. Jerrard[1] · **Mohammad Reza Pakzad**[2]

Received: 10 November 2015 / Accepted: 1 July 2016 / Published online: 21 July 2016

Abstract We prove the $C^1_{\rm loc}$ regularity and developability of $W^{2,p}_{\rm loc}$ isometric immersions of n-dimensional flat domains into $\mathbb{R}^{n+k}$ where $p \geq \min\{2k, n\}$. We also prove similar rigidity and regularity results for scalar functions of n variables for which the rank of the Hessian matrix is *a.e.* bounded by some $k < n$, again assuming $W^{2,p}_{\rm loc}$ regularity for $p \geq \min\{2k, n\}$. In particular, this includes results about the degenerate Monge–Ampère equation, $\det D^2 u = 0$, corresponding to the case $k = n - 1$.

Keywords Rigidity of isometric immersions · Developability · Monge–Ampere equations · Sobolev spaces of mappings · Geometric measure theory

Mathematics Subject Classification 35B99 · 53C24 · 58D10 · 30L05

Contents

✉ Mohammad Reza Pakzad
pakzad@pitt.edu

Robert L. Jerrard
rjerrard@math.toronto.edu

1 Department of Mathematics, University of Toronto, Toronto, ON, Canada

2 Department of Mathematics, University of Pittsburgh, Pittsburgh, PA, USA

1 Introduction

1.1 Background

The question of rigidity vs. flexibility of isometric immersions has been studied in differential geometry since the end of nineteenth century. It was already known, as established by Darboux, among others, that smooth surfaces in the three-dimensional space which are isometric to a piece of plane are *developable*, i.e. they are locally foliated as a ruled surface by straight segments aligned at each point in one of the principal directions. New developments in the mid-twentieth century highlighted the very fact that this *rigidity* statement relies strongly on the regularity of the surface. In particular, it followed from the results of Nash [29] and Kuiper [21] that there exist many C^1 isometric embeddings of a given flat n-dimensional domain into $\mathbb{R}^{n+1}$ (and hence into $\mathbb{R}^{n+k}$ for any $k \geq 1$) with arbitrarily small upper bound on the diameter of the image, a property which rules out the developability of the image. On the other hand, the developability of co-dimension one isometric immersions of flat n-dimensional domains was essentially established by Chern and Lashof [5, Lemma 2] and Hartman and Nirenberg [13, Lemma 2], who also provided more detailed results in the case $n = 2$ of surfaces. In [35], a generalized developability result for C^2 isometric immersions of a Euclidean domain $\Omega \subset \mathbb{R}^n$ into Euclidean spaces $\mathbb{R}^{n+k}$, $k < n$ was established.

A natural question arises, which consists in asking what would be the critical regularity threshold at which the distinction between *rigidity* and *flexibility* á la Nash and Kuiper is withheld. The most straightforward path would be to discuss this question for Hölder regular isometries of class $C^{1,\alpha}$, $0 < \alpha < 1$. Some progress is made in this direction, but the problem of the critical value of α is still open. While a careful analysis of the iteration methods of Nash and Kuiper has lead to flexibility results for surfaces for $\alpha < 1/13$ [2], and then for $\alpha < 1/7$ [6], it has only been established that $C^{1,\alpha}$ isometric immersions of two- dimensional flat domains into the three-dimensional space are rigid if $\alpha > 2/3$ [2,3,6]. In a different but related vein, Pogorelov showed that C^1 surfaces with total zero curvature are developable [32, Chapter II] and [31, Chapter IX]. If one only assumes Hölder regularity, it seems that there is no consensus on what the critical exponent should be, as it has been conjectured to be $\alpha = 1/3$, $1/2$ or $2/3$.

One could also consider other function spaces which lie somewhat below C^2. In particular, Sobolev isometries arise in the study of nonlinear elastic thin films. Kirchhoff's plate model put forward in the nineteenth century [20] consists in minimizing the L^2 norm of the second fundamental form of isometric immersions of a 2d domain into $\mathbb{R}^3$ under suitable forces or boundary conditions. In other words, using the modern terminology, the space of admissible maps for this model is that of $W^{2,2}$ isometric immersions (See also [10,23]).

Quite strong results are known about regularity and rigidity of *co-dimension 1* isometric immersions, as summarized in the following

Theorem 1 *Let $U \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$ be an isometric immersion, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Then, $U \in C^{1,1/2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n+1})$. Moreover, for every $x \in \Omega$, either DU is constant in a neighbourhood of x, or there exists a unique $(n-1)$-dimensional hyperplane $\mathbb{P} \ni x$ of $\mathbb{R}^n$ such that DU is constant on the connected component of x in $\mathbb{P} \cap \Omega$.*

This was proved in by Liu and Pakzad [25] and followed earlier results [30] of the second author that established the $n = 2$ case of Theorem 1, drawing on work of Kirchheim in [19] on $W^{2,\infty}$ solutions to degenerate Monge–Ampère equations, discussed below.

The result is optimal which is the sense that it fails for $W^{2,p}$ isometries with $p < 2$.

Remark 1.1 In [28], it was established for $n = 2$ that the C^1 regularity can be extended up to the boundary if the domain is of class $C^{1,\alpha}$. This does not hold true anymore for merely C^1 regular domains.

Isometric immersions of flat domains are closely related to the degenerate Monge–Ampère equation

$$\det(D^2u) = 0 \quad \text{a.e.} \quad in \quad \Omega, \tag{1.1}$$

or more generally to the Hessian rank inequality

$$\text{rank}(D^2u) \le k \quad a.e. \quad in\ \Omega. \tag{1.2}$$

This is equivalent to the degenerate Monge–Ampère equation when $k = n - 1$, but for $k < n - 1$ is a stronger condition. As we recall in Sect. 2, it is satisfied by the components U^m of an isometric immersion $U : \Omega \to \mathbb{R}^{n+k}$ of co-dimension k (see Proposition 2.1), and many rigidity properties of isometric immersions can be deduced solely from the weaker condition (1.2).

In order to discuss Sobolev solutions with lower regularity than the assumptions of the above theorem, it is helpful to study distributional and measure theoretic variants of condition (1.1) including (in two-dimensional domains)

$$\text{Det}(D^2u) := -\frac{1}{2}\text{curl}^T \text{curl}(Du \otimes Du) = 0 \tag{1.3}$$

for $u \in H^1(\Omega)$; or

$$\int_\Omega \phi_{x_1}(x, Du)u_{x_k x_2} - \phi_{x_2}(x, Du)u_{x_k x_1}\, \mathrm{d}x = 0 \quad \text{for all } \phi \in C_c^\infty(\Omega \times \mathbb{R}^2) \text{ and } k = 1, 2 \tag{1.4}$$

for $u \in W^{2,1}(\Omega)$. Both of these imply (1.1) if $u \in W^{2,2}_{\text{loc}}(\Omega)$. It turns out that (1.1), even in the weak form (1.4), is strong enough to imply rigidity, as shown in the following result.

Theorem 2 *Let Ω be a bounded, open subset of $\mathbb{R}^2$.*

If $u \in W^{2,2}_{\text{loc}}(\Omega)$ and $\det D^2 u = 0$ a.e. in Ω, then $u \in C^1(\Omega)$, and for every point $x \in \Omega$, there exists either a neighbourhood of x, or a segment passing through x and joining $\partial\Omega$ at both ends, on which Du is constant.

More generally, the same conclusions hold if we merely assume that $u \in W^{2,1}(\Omega)$ and u satisfies (1.4).

Theorem 2 was established for $u \in W^{2,2}_{\text{loc}}(\Omega)$ by the second author in [30], see also Kirchheim [19]. The final assertion of the theorem, concerning $W^{2,1}$ functions, is in fact a special case of a more general result from [18] that applies in the (larger) class of Monge–Ampère functions, introduced by Fu [11] and developed in [17,18]. If one considers not the distributional condition (1.4) but just the pointwise Monge–Ampère equation (1.1), then the $W^{2,2}$ hypothesis of [30] is optimal. Indeed, conic solutions to (1.1) exist if the regularity is assumed to be only $W^{2,p}$ for $p < 2$ (see Example 1 below). One could even construct more sophisticated solutions by gluing these conic singularities in a suitable manner, using Vitali's covering theorem (Example 2). Furthermore, Liu and Malý [24] have established the existence of strictly convex $W^{2,p}$ solutions to (1.1) (but not to 1.3) when $p < 2$. In the meantime, it is known [9] that for $p < 2$, $W^{2,p}$ solutions to (1.3) exist which are not C^1 and fail to satisfy the developability statement of Theorem 2 at a given point in the domain.

Finally, Lewicka and the second author have recently proved in [22] that the conclusions of Theorem 2 hold for $C^{1,\alpha}$ solutions to (1.3) provided $\alpha > 2/3$, but not if $\alpha < 1/7$.

What interests us in this paper are regularity and rigidity results in the manner of Theorems 1 and 2 for arbitrary $1 \le k < n$, under Sobolev regularity assumptions. We note that the case $k = 0$ is trivial and that there is no rigidity whenever $k \ge n$, see, for example, [35].

The proof in [25] of Theorem 1 was based on induction on the dimension of slices of the domain and careful and detailed geometric arguments, applying the $W^{2,2}_{\mathrm{loc}}$ case of Theorem 2 to two-dimensional slices. These methods cannot be adapted to the solutions of (1.2) even for $k = 1$, since one loses some natural advantages when working with (1.2) rather than with the isometries themselves as done in [25]: the solution u is no more Lipschitz and being just a scalar function, one loses the extra information derived from the length preserving properties of isometries. On the other hand, contrary to the case of $k = 1$, regularity and developability of the Sobolev solutions to (1.2) do not directly lead to the same results for the corresponding isometries (see [30]).

Hence, the problems of regularity and developability of Sobolev isometric immersions of co-dimension higher than 1 and also of the developability of Sobolev solutions to (1.2) for $k > 1$ are more involved and could not be tackled through the methods discussed in [25,30]. In this paper, we adapt methods of geometric measure theory, applied by the first author in [17,18] to the class of Monge–Ampère functions, to overcome the above obstacles for $k > 1$ and tackle both of the isometry and rank problems for Sobolev regular solutions simultaneously.

Remark 1.2 It was proved furthermore in [30] that any $W^{2,2}$ isometry on a convex 2d domain can be approximated in strong norm by smooth isometries. The convexity assumption can be weakened to e.g. piecewise C^1 regularity of the boundary, see also [14–16]. A generalization of these results to the co-dimension one case was obtained in [25]. It could be expected that the results of this paper could help in proving similar density statements in higher co-dimensions, but that would be more technically challenging than the previous cases.

1.2 Main results

We first introduce a few fundamental definitions.

Definition 1.3 Let $\Omega \subset \mathbb{R}^n$ be an open set and $j \in \{1, \dots, n\}$. We say the set $P \subset \Omega$ is a *j-plane in Ω* whenever P is the connected component of the intersection of Ω and a j-dimensional affine subspace $\mathbb{P}$ of $\mathbb{R}^n$. We will generally write P to denote a j-plane in Ω for some subset $\Omega \subset \mathbb{R}^n$ and $\mathbb{P}$ to denote a complete j-plane.

Definition 1.4 Let $n \in \mathbb{N}$, $n > 1$, Ω be an open subset of $\mathbb{R}^n$. We say a mapping $w \in C^0(\Omega, \mathbb{R}^\ell)$ is $(n-k)$-flatly foliated whenever $0 \le k < n$ is an integer and there exists disjoint subsets F_j, $j = 0, \dots, k$ of Ω, such that the following properties hold:

(i) $\Omega = \bigcup_{j=0}^{k} F_j$,
(ii) For all $j \in \{0, \dots, k\}$, $\Omega_j := \bigcup_{m=0}^{j} F_m$ is open,
(iii) For all $j \in \{0, \dots, k\}$ and every $x \in F_j$, there exists at least one $(n-j)$-plane P in Ω_j such that $x \in P$ and w is constant on P.

We say a mapping is flatly foliated when it is $(n-k)$-foliated for some integer k.

Remark 1.5 Note that for all $j \in \{0, \dots, k\}$, $F_j = \Omega_j \setminus \Omega_{j-1}$. Hence, a straightforward conclusion of the above definition is that F_j is closed in Ω_j for all $j \in \{0, \dots, k\}$.

Definition 1.6 Let $n, N \in \mathbb{N}$, $n > 1$, $N \geq 1$, and let Ω be an open subset of $\mathbb{R}^n$. We say a mapping $y \in C^1(\Omega, \mathbb{R}^N)$ is $(n-k)$-developable whenever $Dy : \Omega \to \mathbb{R}^{N\times n} \cong \mathbb{R}^{nN}$ is $(n-k)$-flatly foliated. We say a mapping is developable when it is $(n-k)$-developable for an integer $k \in \{0, 1, \dots, n-1\}$.

We will later introduce weaker versions of the notions defined in Definitions 1.4 and 1.6for mappings which are not necessarily of the required regularity.

The following two theorems sum up the main contribution of this paper. The first theorem concerns Sobolev isometric immersions of Euclidean domains and extends Theorem 1 to arbitrary co-dimension.

Theorem 3 *Let $k \in \{1, \dots, n-1\}$. Assume that Ω is a bounded, open subset of $\mathbb{R}^n$ and that $U \in W^{2,p}_{\text{loc}}(\Omega; \mathbb{R}^{n+k})$ is an isometric immersion, so that U satisfies*

$$U_{x^i} \cdot U_{x^j} = \delta_{ij} \quad a.e. in\ \Omega, \quad \forall i, j \in \{1, \dots, n\}.$$

If $p \geq \min\{2k, n\}$, then $U \in C^1(\Omega; \mathbb{R}^{n+k})$, and U is $(n-k)$-developable.

The next theorem is a similar statement concerning scalar functions and generalizes to arbitrary n and k those parts of Theorem 2 that concern the (pointwise) degnerate Monge–Ampère equation (1.1) . This result is new whenever $n > 2$, even for $k = 1$.

Theorem 4 *Assume that Ω is a bounded, open subset of $\mathbb{R}^n$ and that $u : \Omega \to \mathbb{R}$ satisfies*

$$u \in W^{2,p}_{\text{loc}}(\Omega) \textit{ with } p \geq \min\{2k, n\} \qquad rank(D^2u) \leq k\ a.e \tag{1.5}$$

for some $k \in \{1, \dots, n-1\}$. Then, $u \in C^1(\Omega)$ and u is $(n-k)$-developable.

Remark 1.7 One interesting feature of these results is that the Sobolev regularity $W^{2,p}$ can be much below the required $W^{2,n+\varepsilon}$ for obtaining C^1 regularity by Sobolev embedding theorems. The argument used in [30, Lemma 2.1] to show the continuity of the derivatives of the given Sobolev isometry is no more generalizable to our case. In [30], the C^1 regularity is shown as a first step towards the proof of developability. Here, on the other hand, we first show a weaker version of developability for the mapping and use it to show the C^1 regularity.

Remark 1.8 In Example 1 below, we show that if $u \in W^{2,p}(\Omega)$ satisfies $\text{rank}(D^2u) \leq k\ a.e$, and if $p < k+1$, then u may fail to be C^1. Also, Liu and Malý [24] have established the existence of strictly convex $W^{2,p} \cap C^{1,\alpha}$ solutions for $0 < \alpha < 1$ to the above rank condition when $p < k+1$. These examples, in particular, imply that the condition $p \geq \min\{2k, n\}$ in Theorem 4 cannot be weakened if $k = 1$ or $k = n-1$. We believe, however, that it can be weakened if $k \in \{2, \dots, n-2\}$. Indeed, it seems likely that the conclusions of the theorem continue to hold under the assumption that

$$u \in W^{2,p}_{\text{loc}}(\Omega) \text{ with } p \geq k+1 \qquad \text{rank}(D^2u) \leq k\ a.e. \tag{1.6}$$

1.3 Some examples

Example 1 For any $k < n$ and $1 \leq p < k+1$, there exists $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ and rank $(D^2u) \leq k$ a.e. but such that the conclusions of the theorem fail. Indeed, consider u of the form

$$u(x^1, \dots, x^n) = u_0(x^1, \dots, x^{k+1}) \quad \text{for } u_0 \in C^2_{\text{loc}}(\mathbb{R}^{k+1} \setminus \{0\}) \text{ homogeneous of degree 1.}$$

One easily checks that $u \in \cup_{p<k+1} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$, and it is clear that Du is not continuous on the set $\{x \in \mathbb{R}^n : x^1 = \ldots, x^{k+1} = 0\}$, unless it is constant.

One could generalize the above example by gluing conic singularities in the following manner:

Example 2 By Vitali's covering theorem, we choose a covering $\mathcal{B} := \{B(a_i, r_i)\}_{i\in\mathbb{N}}$ of $\mathbb{R}^{k+1}$ of non-overlapping balls so that $\mathbb{R}^{k+1} \setminus \bigcup_{i\in\mathbb{N}} B(a_i, r_i)$ is of Lebesgue measure zero. We define $v_0 : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ by

$$v_0(x) := \begin{cases} a_i + r_i(x - a_i)/|x - a_i| & \text{if } x \in B(a_i, r_i), \\ x & \text{otherwise.} \end{cases}$$

It can be easily verified that $v_0 \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^{k+1})$ for all $1 \le p < k + 1$ and that $v_0 = Du_0$ for a scalar function. Let $u(x^1, \ldots, x^n) := u_0(x^1, \ldots, x^{k+1})$. Then, $u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for $1 \le p < k + 1$, rank $(D^2u) \le k$, but Du is not continuous on the set $\{a_i\}_{i\in\mathbb{N}} \times \mathbb{R}^{n-k-1}$.

One might naively hope that for every $k < n$, the set $\{x \in \Omega : \mathrm{rank}(D^2u) = k\}$ is foliated by $n - k$-planes on which Du is constant. This is not at all the case.

Example 3 Consider $u : (0, 1)^2 \to \mathbb{R}$ of the form $u(x, y) = F(x)$ where $F' = f : (0, 1) \to \mathbb{R}$ is a *strictly increasing* Lipschitz continuous function such that $\{x \in (0, 1) : f'(x) = 0\}$ has positive measure. For example, fix an open dense set $O \subset (0, 1)$ whose complement has positive measure, and let $f(x) := \mathcal{L}^1((0, x) \cap O)$, so that f is Lipschitz continuous and

$$f'(x) = \begin{cases} 1 & \text{for } a.e.\ x \in O \\ 0 & \text{for } a.e.\ x \notin O. \end{cases}$$

For a function of this form, we have $u \in W^{2,\infty}$, with

$$Du(x, y) = (f(x), 0), \qquad D^2u(x, y) = \begin{pmatrix} f'(x) & 0 \\ 0 & 0 \end{pmatrix} \ a.e.$$

so that rank$(D^2u) \le 1$ a.e. and rank$(D^2u) = 0$ on a dense set of positive measure. However, there is no two-dimensional set on which Du is locally constant; rather, for every $\xi \in \mathrm{Im}(Du)$, where $\mathrm{Im}(\cdot)$ denotes the image, $Du^{-1}\{\xi\}$ is the line segment $f^{-1}\{\xi\} \times (0, 1)$.

Example 4 Consider again $u : (0, 1)^2 \to \mathbb{R}$ of the form $u(x, y) = F(x)$, where $F' = f$ and $f(x) := \mathcal{L}^1((0, x) \setminus O)$, where O is as in Example 3 above. Then, f is Lipschitz continuous and

$$f'(x) = \begin{cases} 0 & \text{for } a.e.\ x \in O \\ 1 & \text{for } a.e.\ x \notin O. \end{cases}$$

Then, in the notation of Definition 1.4 below, $\Omega = \Omega_1$, and $\Omega_0 = O \times (0, 1)$. Thus, Ω_0 is a dense subset of Ω_1, and $F_1 = \Omega_1 \setminus \Omega_0$ is nowhere dense in Ω_1.

More generally, given $0 \le j < k \le n$, one can write down examples in the same spirit defined on the unit cube in $\mathbb{R}^n$, such that Ω_j is dense in Ω_k.

Example 5 Fix a C^2 map $\mathbf{v} : \mathbb{R} \to \mathbb{R}^2$ such that $\mathbf{v}(0) = 0$, $v'(z) \neq 0$ for $z \neq 0$, and $\lim_{z\to 0} \frac{\mathbf{v}'}{|\mathbf{v}'|}$ does not exist. For example, we may take $\mathbf{v}(z) = (z^5 \cos(1/z), z^5 \sin(1/z))$.

Now set $\Omega = (-1, 1)^3$, and let $u(x, y, z) = (x, y)\cdot\mathbf{v}(z)$. Then, we can write $Du(x, y, z) = (\mathbf{v}(z), (x, y)\cdot\mathbf{v}'(z))$. Thus, level sets of Du are the plane $z = 0$, together with the line segments

$$\{x, y, z) : z = z_0, (x, y)\cdot\mathbf{v}'(z_0) = c\}, \qquad z_0 \neq 0, c \in \mathbb{R}.$$

It is also easy to check that u is C^2, $\mathrm{rank}(D^2u) = 2$ if $z \neq 0$ and $\mathrm{rank}(D^2u) = 0$ if $z = 0$.

(Note also, $\tilde{u} := u + z^2$ has all the same properties as u described above, except that $\mathrm{rank}(D^2u) = 1$ when $z = 0$.)

This example shows that (in notation to be introduced later) $\bar{\Omega}^k$ may contain planes of dimension greater than $n-k$ on which Du is a.e. constant. By contrast, the previous example shows that it may also happen that $\bar{\Omega}^k \setminus \Omega^k$ is foliated by planes of dimension $n-k$.

Also, we can see from this example that the $(n-k)$-planes that locally foliate Ω^k may oscillate wildly as one approaches points in $\bar{\Omega}^k$ at which $\mathrm{rank}(D^2u) < k$.

1.4 Remarks on notation and an outline of proofs

Throughout the paper, we will often simply write "measurable", "almost everywhere", without specifying the Hausdorff measure at use, when the latter is clear from the context. Many of our arguments take place in a product space $\Omega \times \mathbb{R}^\ell$, where $\Omega \subset \mathbb{R}^n$ and ℓ is a positive integer. In this setting, we will think of Ω and $\mathbb{R}^\ell$ as "horizontal" and "vertical", respectively, and we will use subscripts h and v accordingly. For example, we will write p_h, p_v to designate projections of $\Omega \times \mathbb{R}^\ell$ onto the horizontal and vertical factors, respectively:

$$p_h(x, \xi) := x, \qquad\qquad p_v(x, \xi) := \xi. \tag{1.7}$$

If $w \in L^p(\Omega)$ for some $p < \infty$, then a Lebesgue point of w will mean a point x such that

$$\lim_{r\to 0} \fint_{B_r(x)} |w(y) - w(x)|^p \,\mathrm{d}y := \lim_{r\to 0} \frac{1}{\mathcal{L}^n(B_r(x))} \int_{B_r(x)} |w(y) - w(x)|^p \, d = 0. \tag{1.8}$$

Thus, we always understand "Lebesgue point" in an L^p sense. We assume that every function w appearing in this paper is precisely represented. Thus, w always equals its Lebesgue value at every point where the Lebesgue value exists. If $u \in W^{2,p}(\Omega)$, there is a set E such that $\mathrm{Cap}_p(E) = 0$ and every point of $\Omega \setminus E$ is a Lebesgue point of Du. The capacity estimate implies that $\mathcal{H}^{n-p+\varepsilon}(E) = 0$ for every $\varepsilon > 0$. These facts can be found, for example, in Ziemer [36], Theorem 3.3.3 and 2.6.16, respectively, or in [7].

To describe the proof, it is useful to introduce several weaker versions of the notions of flatly foliated, defined above.

Definition 1.9 Let $n \in \mathbb{N}$, $n > 1$, Ω be an open subset of $\mathbb{R}^n$. We say a measurable mapping $w : \Omega \to \mathbb{R}^\ell$ is densely weakly $(n-k)$-flatly foliated whenever there exist some $k \in \{0, 1, \ldots, n-1\}$ and disjoint subsets F_j, $j = 0, \ldots, k$ of Ω, such that

$$\Omega = \cup_{j=0}^k F_j, \tag{1.9}$$

and in addition, the following properties hold for every j:

$$\Omega_j := \cup_{m=0}^j F_m \text{ is open}, \tag{1.10}$$

and

$$\begin{aligned}&\text{for every } x \text{ in some dense subset of } F_j, \text{ there exists at least one } n-j\text{-plane } P \text{ in } \Omega_j \\ &\text{such that } x \in P \text{ and } w \text{ is } \mathcal{H}^{n-j} \text{a.e. constant on } P.\end{aligned} \tag{1.11}$$

Definition 1.10 Let $n \in \mathbb{N}$, $n > 1$, Ω be an open subset of $\mathbb{R}^n$. We say a measurable mapping $w : \Omega \to \mathbb{R}^\ell$ is pointwise weakly $(n-k)$-flatly foliated whenever there exist some $k \in \{0, 1, \ldots, n-1\}$ and disjoint subsets F_j, $j = 0, \ldots, k$ of Ω, such that (1.9) and (1.10) hold, and

$$\begin{aligned}&\text{for every } x \in F_j, \text{ there exists at least one } n-j\text{-plane } P \text{ in } \Omega_j\\ &\quad\text{such that } x \in P \text{ and } w \text{ is } \mathcal{H}^{n-j}\text{a.e. } constant\ on\ P.\end{aligned} \tag{1.12}$$

Remark 1.11 The definitions require that the values of w are well defined for $\mathcal{H}^{n-j}$ a.e. points on the given $n-j$-planes in Ω. As noted above, this is the case if we assume that e.g. $w \in W^{1,k+1}_{\text{loc}}(\Omega, \mathbb{R}^\ell)$ and w is precisely represented, since in that case the set of points that fail to be Lebesgue points of w has dimension less than $n-k$.

We start in Sect. 2 by showing that if $U \in W^{2,2}(\Omega; \mathbb{R}^{n+k})$ is an isometric immersion for $\Omega \subset \mathbb{R}^n$, then $w = DU$ satisfies

$$\text{rank}(Dw) \le k \ \ a.e. \text{ in } \Omega.$$

This is a classical fact for smooth maps. As a consequence, both of our main results reduce to the study of maps $w : \Omega \to \mathbb{R}^\ell$ for some ℓ, such that

$$\text{rank}(Dw(x)) \le k \ \ a.e. \text{ in } \Omega, \qquad w = (Du^1, \ldots, Du^q) \text{ for some } q \ge 1. \tag{1.13}$$

A main challenge we must address is to find a way to extract information from the hypotheses (1.13) under conditions of low regularity. We carry this out making extensive use of the machinery of geometric measure theory, including in particular some results from Giaquinta, Modica and Souček [12], Fu [11] and the first author [18] about the related topics of Cartesian maps and Monge–Ampère functions.

To explain the role of geometric measure theory, we first outline the basic argument on a formal level. Towards that end, consider a *smooth* map $w = (Du^1, \ldots, Du^q)$ such that $\text{rank}(Dw) = k$ everywhere, and further suppose that

- image(w) is a smooth embedded k-dimensional submanifold $\Gamma_v \subset \mathbb{R}^n$, where Im$(w)$ denotes the image of w, and
- for every $\xi \in \Gamma_v$, $\Gamma_h(\xi) := w^{-1}\{\xi\}$ is a smooth $(n-k)$-dimensional submanifold of Ω.

These assumptions are far stronger than one can reasonably expect, but in any case they are certainly consistent with the condition that $\text{rank}(Dw) = k$. For every $\xi \in \Gamma_v$, and for every $x \in \Gamma_h(\xi)$, basic calculus implies that

$$\text{Im}(Dw(x)) = T_\xi \Gamma_v \tag{1.14}$$

and

$$\ker(Dw(x)) = T_x \Gamma_h(\xi). \tag{1.15}$$

Moreover, the symmetry of $D^2u^i(x)$ implies that $\ker(D^2u^i(x)) = [\text{Im}(D^2u^i(x))]^\perp$, if we identify, in the natural way, the horizontal and vertical spaces to which $T_\xi\Gamma_v$ and $T_x\Gamma_h(\xi)$ belong. Thus,

$$T_x\Gamma_h(\xi) = \ker(Dw(x)) = \bigcap_{i=1}^q \ker(D^2u^i(x)) = \bigcap_{i=1}^q [\text{Im}(D^2u^i(x))]^\perp.$$

The space on the right is completely determined by $T_\xi\Gamma_v$ — in fact it can be written $\cap_{i=1}^q [P_i T_\xi \Gamma_v]^\perp$, where P_i denotes orthonormal projection of $\mathbb{R}^{nq} = (\mathbb{R}^n)^q$ onto the ith copy of $\mathbb{R}^n$. Thus, the tangent space $T_x\Gamma_h(\xi)$ does not depend at all on $x \in \Gamma_h(\xi)$, but only on ξ. Since the tangent space is constant, $\Gamma_h(\xi)$ must be a union of $n-k$-planes in Ω, all orthogonal to $\cap_{i=1}^j [P_i T_\xi \Gamma_v]^\perp$.

The rigorous version of this argument starts in Sect. 3, where we use the machinery of geometric measure theory to establish facts about

- the structure of Γ_v and $\Gamma_h(\xi)$, which in our actual proof will be, not exactly the image and the level sets of w, but closely related sets, and
- the relationship between their tangent spaces and the derivatives of w, along the lines of (1.14) and (1.15) above

that are (barely) strong enough to justify some form of the proof sketched above. These arguments apply to general mappings (without a gradient structure) $w \in W^{1,k+1}(\Omega;\mathbb{R}^\ell)$ such that $\operatorname{rank}(Dw) \le k$ a.e. Under these assumptions, we obtain Γ_v and $\Gamma_h(\xi)$ as, essentially, the vertical projection and horizontal slices, respectively, of a set

$$\Gamma := \{(x, w(x)) \in \Omega \times \mathbb{R}^\ell : x \text{ is a Lebesgue point of both } w \text{ and } Dw\}.$$

(See 3.5, 3.4) for the actual definitions.) Appealing to results of Giaquinta, Modica and Souček [12], we find that Γ is n-rectifiable and that an integral n-current G_w, canonically associated with the graph of w and carried by Γ, has no boundary in $\Omega \times \mathbb{R}^\ell$. Then, the rectifiability of Γ_v and of $\mathcal{H}^k$ almost every $\Gamma_h(\xi)$ follows from classical results and the definitions of these sets, as does a version of (1.14). Additional work is required to establish a version of (1.15) and to show that the slices $\Gamma_h(\xi)$ have enough regularity (in particular, they carry integer $n-k$-currents with no boundary) to conclude from the constancy of the tangent spaces that they are in fact planar.

In Section 4, we use these facts to prove that if $w \in W^{1,k+1}_{\text{loc}}$ satisfies (1.13), then w is densely weakly $(n-k)$ flatly foliated. More precisely, we define

$$\Omega^k := \{x \in \Omega : x \text{ is a Lebesgue point of } w \text{ and } Dw, \text{ and } \operatorname{rank}(Dw) = k\},$$

and we give a rigorous version of the formal argument sketched above to show, roughly speaking, that Ω^k is almost everywhere foliated by level sets of w that are $n-k$-planes in Ω. (We remark that this is the *only* place in the paper where we use the gradient structure of w.) To deduce that w is densely weakly $(n-k)$-flatly foliated, we define $F_k := \bar{\Omega}^k$ and $\Omega_{k-1} := \Omega \setminus F_k$, and we note that rank$(D^2u) \le k-1$ a.e. in Ω_{k-1}. Hence, the above machinery could be reapplied to the new set with the new rank condition. More generally, letting $\Omega_k = \Omega$, and for $j \in \{k, \dots, 0\}$, defining (working downwards)

$$\begin{aligned} \Omega^j &:= \{x \in \Omega_j : x \text{ is a Lebesgue point of } Du \text{ and } D^2u, \text{ and } \operatorname{rank}(D^2u) = j\}, \\ F_j &:= \bar{\Omega}^j \cap \Omega_j, \\ \Omega_{j-1} &:= \Omega_j - F_j = \Omega_j - \bar{\Omega}^j, \end{aligned}$$

we obtain a partition of Ω into disjoint sets F_j, $j = 0, 1, \dots, k$ such that every F_j has a dense subset foliated by $n-j$-planes on which w is $\mathcal{H}^{n-j}$ a.e. constant.

Following this, we prove in Sect. 5 that if $w \in W^{1,k+1}_{\text{loc}}(\Omega;\mathbb{R}^\ell)$ is *densely* weakly $(n-k)$-flatly foliated, then w is *pointwise* weakly $(n-k)$-flatly foliated. (In fact here we only need $W^{1,p}_{\text{loc}}$ for some $p > k$.) The hypothesis already yields a partition of Ω into sets F_j satisfying properties (1.9), (1.10), and so the point is to show that (1.11) together with the assumed

Sobolev regularity implies (1.12). To do this, we obtain a planar level set of w through a given point as a limit of planar level sets through nearby points. We remark that it is possible, as illustrated in Example 3, for F_k to contain a subset of $\Omega \setminus \Omega^k$ of positive measure to be foliated by $n-k$-planes on which w is constant.

The arguments of Sects. 3, 4 and 5require only the weaker regularity assumption (1.6), and this hypothesis is sharp in a sense; this follows from Example 1 below. The stronger assumption (1.5) is needed for Sect. 6, in which prove that if $p = \min\{2k, n\}$ and $w \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^\ell)$ is *pointwise* weakly $(n-k)$-flatly foliated, then w is continuous and hence $(n-k)$-flatly foliated. This will complete the proof of our main results. For the proof, we first show that if a point $x \in F_k$ is contained in two distinct $n-k$-planes in Ω on which w is *a.e.* constant, then the two constants are in fact equal. (Example 5 shows that this situation can in fact arise.) It follows rather easily from this that the restriction of w to F_k is C^0 and indeed that the same holds in F_j for all $j \leq k$. To conclude that w is continuous in Ω, it remains to show that it is continuous at points of $\partial\Omega_j \cap \Omega$. This is a little more subtle and is proved by showing that any such discontinuity is inconsistent with the p-quasicontinuity of w, given facts we have already established about w.

The condition $p \geq \{2k, n\}$ is sharp for the results of Sect. 6, at least for certain values of k, including $k = 2, 4, 8$. This follows from Examples 6–8 in Sect. 6. These results, however, apply to vector-valued maps $w : \Omega \to \mathbb{R}^\ell$ that are pointwise a.e. flatly foliated. As suggested above, we believe that if one considers maps that in addition possess a gradient structure, that is, maps of the form $w = (Du^1, \ldots, Du^q)$ for some q, then it should be possible to weaken the regularity requirements.

2 Degenerate Hessians for Sobolev isometric immersions

In this section we prove a proposition that reduces the case of isometries to that of maps whose Hessian satisfies a degeneracy condition. This is a variant of a classical lemma of Cartan [4], which concerns smooth maps and has a correspondingly stronger conclusion.

Proposition 2.1 *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set and that $U \in W^{2,2}(\Omega, \mathbb{R}^{n+k})$ is an isometric immersion of Ω into $\mathbb{R}^{n+k}$ for some $k \in \{1, \ldots, n-1\}$, i.e. U satisfies*

$$U_{x^i} \cdot U_{x^j} = \delta_{ij}, \quad \forall i, j \in \{1, \ldots, n\}. \tag{2.1}$$

Let $w := DU : \Omega \to \mathbb{R}^n \otimes \mathbb{R}^{n+k} \cong \mathbb{R}^\ell$ for $\ell = n(n+k)$. Then,

$$rank(Dw) \leq k \quad a.e.\ in\ \Omega.$$

In the proof of this result only, to simplify notation we will write $U_{,i}$ to denote partial differentiation with respect to the ith coordinate direction.

Proof We will first establish the following identity:

$$U_{,ij} \cdot U_{,kl} - U_{,il} \cdot U_{,jk} = 0 \quad \forall i, j, k, l \in \{1, \ldots, n\} \quad \text{a.e.} in\ \Omega. \tag{2.2}$$

Let $U_m \in C^\infty(\Omega, \mathbb{R}^{n+k})$ be a sequence of mappings converging to U in the $W^{2,2}$-norm, and let $g^m_{ij} := U_{m,i} \cdot U_{m,j}$. Twice differentiating g^m_{ij}, we obtain for all i, j, k, l:

$$g^m_{ij,kl} = U_{m,ikl} \cdot U_{m,j} + U_{m,ik} \cdot U_{m,jl} + U_{m,il} \cdot U_{m,jk} + U_{m,i} \cdot U_{m,jkl}.$$

Permuting the indices and cancelling the terms in third derivatives yields:

$$g^m_{ij,kl} + g^m_{kl,ij} - g^m_{il,jk} - g^m_{jk,il} = -2(U_{m,ij} \cdot U_{m,kl} - U_{m,il} \cdot U_{m,jk}).$$

Passing to the limit as $m \to \infty$, we observe that the left-hand side converges in the sense of distributions to 0, while the right-hand side converges in L^1 to $-2(U_{,ij} \cdot U_{,kl} - U_{,il} \cdot U_{,jk})$. This establishes (2.2). Our second observation is that

$$U_{,ij} \cdot U_{,k} = 0 \quad \forall i, j, k \in \{1, \ldots, n\} \quad \text{a.e.} in\ \Omega. \tag{2.3}$$

This is straightforward to see, as differentiating the isometry constraint (2.1) we obtain for all i, j, k:

$$0 = U_{,ik} \cdot U_{,j} + U_{,i} \cdot U_{,jk} = U_{,ij} \cdot U_{,k} + U_{,i} \cdot U_{,kj} = U_{,ki} \cdot U_{,j} + U_{,k} \cdot U_{,ji},$$

where the two last identities are obtained by permutations in i, j, k and all three are valid a.e. in Ω. Now, adding the first two identities and subtracting the third implies (2.3), considering that $U_{,ij} = U_{,ji}$ for all choices of i, j a.e. in Ω.

In order to proceed, for any $x \in \Omega$ for which the identities (2.1), (2.2) and (2.3) are valid, hence for a.e. $x \in \Omega$, we define the orthogonal space to the image $U(\Omega)$ at the point $U(x)$ to be:

$$O(x) := \text{span} < U_{,1}(x), \ldots, U_{,n}(x) >^{\perp},$$

and the symmetric bilinear form $\mathcal{B}(x) : \mathbb{R}^n \times \mathbb{R}^n \to O(x)$ by

$$\mathcal{B}(x)(V, W) = W \cdot D^2U(x)V := \sum_{m=1}^{n+k} (W \cdot D^2U^m(x)V)\mathbf{e}_m,$$

where $U = (U^1, \ldots, U^{n+k})$. Evidently, (2.3) implies that $\mathcal{B}(x)$ takes values in $O(x)$. On the other hand, (2.2) implies that for all $X, W, Y, Z \in \mathbb{R}^n$ we have

$$\mathcal{B}(x)(X, W) \cdot \mathcal{B}(x)(Y, Z) - \mathcal{B}(x)(X, Z) \cdot \mathcal{B}(x)(Y, W) = 0,$$

i.e. the symmetric bilinear form $\mathcal{B}(x)$ is flat with respect to the Euclidean scalar product on $O(x)$. Hence, we can apply a result due to E. Cartan [4] (see also [35, Lemma 1] for a proof), to obtain that

$$\dim(\ker \mathcal{B}(x)) \geq \dim(\mathbb{R}^n) - \dim(O(x)) = n - k,$$

where

$$\ker(\mathcal{B}(x)) := \{V \in \mathbb{R}^n; \mathcal{B}(x)(V, W) = 0\ \forall W \in \mathbb{R}^n\} = \ker(Dw(x)).$$

This completes the proof of the proposition. $\square$

3 Degenerate Cartesian maps

In this section, Ω is as usual a bounded, open subset of $\mathbb{R}^n$, and w is a map satisfying

$$w \in W^{1,k+1}_{\text{loc}}(\Omega, \mathbb{R}^\ell), \qquad \text{rank}\,(Dw) \leq k \ \text{ a.e.} \tag{3.1}$$

for some $k \in \{1, \dots, n-1\}$ and some $\ell \geq 1$. We will use the notation

$$\Lambda_w := \{x \in \Omega : x \text{ is a Lebesgue point of both } w \text{ and } Dw\} \tag{3.2}$$

$$\Gamma := \{(x, w(x)) : x \in \Lambda_w\} \subset \Omega \times \mathbb{R}^\ell \tag{3.3}$$

$$\Gamma_h(\xi) := \{x \in \Lambda_w : w(x) = \xi\} \tag{3.4}$$

$$\Gamma_v := \{\xi \in \mathbb{R}^\ell \ : \ \mathcal{H}^{n-k}(\Gamma_h(\xi)) > 0\} \tag{3.5}$$

$$\Omega^k = \{x \in \Lambda_w : \mathrm{rank}(Dw(x)) = k\}. \tag{3.6}$$

The main result of this section, stated below, will be used to make precise the formal arguments discussed in Sect. 1.4. Terminology appearing in the proposition will be recalled after its statement.

Proposition 3.1 *Assume that w satisfies (3.1). Then, Γ_v is k-rectifiable, and for $\mathcal{H}^k$* a.e. *$\xi \in \Gamma_v$, the following hold:*

$$\Gamma_h(\xi) \textit{ is } \mathcal{H}^{n-k}\textit{-measurable and } n-k\textit{-rectifiable} \tag{3.7}$$

$$T_\xi \Gamma_v = Im(Dw(x)) \ \ and \ \ \ker(Dw(x)) = T_x \Gamma_h(\xi), \quad \mathcal{H}^{n-k} a.e. in \Gamma_h(\xi). \tag{3.8}$$

In addition, for $\mathcal{H}^k$ a.e. *$\xi \in \Gamma_v$, there exists an integral current H_ξ in $\Omega \times \mathbb{R}^\ell$, defined explicitly in (3.24) below, represented by integration over $\Gamma_h(\xi) \times \{\xi\}$ such that $\partial H_\xi = 0$. Finally,*

$$\mathcal{L}^n \left(\Omega^k \setminus \cup_{\xi \in \Gamma_v^*} \Gamma_h(\xi) \right) = 0, \tag{3.9}$$

where

$$\Gamma_v^* := \{\xi \in \Gamma_v : \partial H_\xi = 0, \ and \ (3.7) and (3.8) \ hold\}. \tag{3.10}$$

This is related to results in [18], proved in the more abstract setting of Monge–Ampère functions. Here, we are able to exploit the Sobolev regularity and results of Giaquinta *et al* [12] to extract more information than in [18], such as conclusions (3.8), which are new. We also believe that the arguments given here are more transparent than those of [18].

Remark 3.2 We emphasize that Γ and Γ_v may differ from the graph $\{(x, w(x)) : x \in \Omega\}$ and the image $w(\Omega)$ by sets of positive $\mathcal{H}^n$ measure. Indeed, [26] establishes the existence of a continuous mapping $w \in W^{1,n}(\Omega; \mathbb{R}^n)$ with vanishing Jacobian (i.e. $k = n-1$), for which $w(\Omega)$ has positive measure. In this construction, the bulk of the image is obtained by applying w to the null set $\Omega \setminus \Lambda_w$, and in fact Proposition 3.1 shows that Γ_v is an $n-1$-rectifiable set.

We start by recalling some definitions. For more background, one can consult, for example, [12] for a general introduction to geometric measure theory in product spaces and whose notation we have tried to follow.

If $U \subset \mathbb{R}^L$ for some L, then we say that $\Gamma \subset U$ is j-rectifiable if

$$\Gamma \subset M_0 \cup \bigcup_{q=1}^{\infty} f_q(\mathbb{R}^j), \qquad \text{where } \mathcal{H}^j(M_0) = 0 \text{ and } f_q : \mathbb{R}^j \to U \ \text{ is Lipschitz.}$$

It is a standard fact that a j-rectifiable set Γ has a j-dimensional approximate tangent plane, denoted $T_y \Gamma$, at $\mathcal{H}^j$ almost every $y \in \Gamma$.

If $\mathbb{P}$ is a j-dimensional plane in some $\mathbb{R}^L$, then a *unit j-vector orienting* $\mathbb{P}$ is a j-vector (that is, an element of the space $\Lambda_j\mathbb{R}^L$) of the form $\tau = \tau_1 \wedge \cdots \wedge \tau_j$, where $\{\tau_i\}_{i-1}^j$ form an orthonormal basis for the tangent space to $\mathbb{P}$.

Let $\mathcal{D}^j(U)$ denote the space of smooth, compactly supported j-forms on U.

Heuristically, j-currents supported in U are "generalized submanifolds" of dimension j, defined by duality to $\mathcal{D}^j(U)$. Integer multiplicity (henceforth abbreviated as *i.m.*) rectifiable currents are those which are represented by a superposition of rectifiable sets. More precisely, an i.m. rectifiable j-current T in U is a bounded linear functional on $\mathcal{D}^j(U)$ that may be represented in the form

$$T(\phi) = \int_\Gamma \langle\phi, \tau\rangle\ \theta\, d\mathcal{H}^n \tag{3.11}$$

where

- Γ is a j-rectifiable set,
- $\theta : \Gamma \to \mathbb{N}$ is a $\mathcal{H}^j$-measurable function, locally integrable with respect to $\mathcal{H}^j \llcorner \Gamma$; and
- τ is a $\mathcal{H}^j$-measurable function from Γ into the space $\Lambda_j\mathbb{R}^L$ of j-vectors on $\mathbb{R}^L$, such that $\tau(y)$ is a unit j-vector that orients the approximate tangent space $T_y\Gamma$, for a.e. $y \in \Gamma$.

In (3.11), we write $\langle\phi(y), \tau(y)\rangle$ to denote the dual pairing between a j-covector $\phi(y) \in \Lambda^j\mathbb{R}^L$ and a j-vector $\tau(y) \in \Lambda_j\mathbb{R}^L$; see (3.15) below for a concrete definition in the product space setting.

When (3.11) holds, we say that T is represented by integration over Γ.

We next introduce notation needed to write these objects more explicitly and in particular to write currents and differential forms in the product space $U = \Omega \times \mathbb{R}^\ell$. For $1 \le j \le m$, we define

$$I(j, m) := \{\alpha = (\alpha_1, \ldots, \alpha_j) : 1 \le \alpha_1 < \ldots < \alpha_j \le m\}. \tag{3.12}$$

If $\alpha \in I(j, m)$, then $|\alpha| := j$. We will think of $I(0, m)$ as consisting of a single element, "the empty multiindex", which we will denote 0.

If $S = (S^i_j)$ is an $\ell \times n$ matrix (with i running from 1 to ℓ and j from 1 to n) and $\beta \in I(j, \ell)$, $\gamma \in I(j, n)$ for some j, then

$$S^\beta_\gamma = (S^{\beta_i}_{\gamma_{i'}})^j_{i,i'=1}\,, \qquad M^\beta_\gamma(S) := \mathrm{det}S^\beta_\gamma. \tag{3.13}$$

We refer to $M^\beta_\gamma(S)$ as a *minor* of S of order j.

We will write points in $\Omega \times \mathbb{R}^\ell$ in the form (x, ξ), and we will write $\{e_i\}_{i=1}^n$ and $\{\varepsilon_j\}_{j=1}^\ell$ to denote the standard bases for the spaces

$$\mathbb{R}^n_h := \mathbb{R}^n \times \{0\} \qquad \text{and} \qquad \mathbb{R}^\ell_v := \{0\} \times \mathbb{R}^\ell$$

of "horizontal" and "vertical" vectors. For $\alpha \in I(j, n)$, we set

$$\mathrm{d}x^\alpha := \mathrm{d}x^{\alpha_1} \wedge \ldots \wedge \mathrm{d}x^{\alpha_j}, \qquad e_\alpha := e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_j}$$

and similarly $\mathrm{d}\xi^\beta$ and e_β, for $\beta \in I(j, \ell)$. Thus, for example, every n-form in $\Omega \times \mathbb{R}^\ell$ may be written

$$\phi = \sum_{|\alpha|+|\beta|=n} \phi_{\alpha\beta}(x, \xi)\mathrm{d}x^\alpha \wedge \mathrm{d}\xi^\beta, \tag{3.14}$$

where it is understood that $\alpha \in I(*, n)$ and $\beta \in I(*, \ell)$. The dual pairing appearing in (3.11) is defined by

$$\left\langle \sum_{|\alpha|+|\beta|=n} \phi_{\alpha\beta} \mathrm{d}x^{\alpha} \wedge \mathrm{d}\xi^{\beta}, \sum_{|\delta|+|\gamma|=n} \tau^{\delta\gamma} e_{\delta} \wedge \varepsilon_{\gamma} \right\rangle = \sum_{|\alpha|+|\beta|=n} \phi_{\alpha\beta} \tau^{\alpha\beta}. \tag{3.15}$$

Given $\alpha \in I(j, n)$, we will write $\bar{\alpha}$ to denote the complementary multiindex, such that $(\alpha, \bar{\alpha})$ is a permutation of $(1, \ldots, n)$, and we write $\sigma(\alpha, \bar{\alpha})$ to denote the sign of this permutation. Hence, $\bar{\alpha}$ and $\sigma(\alpha, \bar{\alpha})$ are characterized by the conditions

$$|\alpha| + |\bar{\alpha}| = n \qquad \text{and} \qquad \mathrm{d}x^{\alpha} \wedge \mathrm{d}x^{\bar{\alpha}} = \sigma(\alpha, \bar{\alpha}) \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n.$$

We then define the n-current G_w by

$$G_w(\phi \, \mathrm{d}x^{\alpha} \wedge \mathrm{d}\xi^{\beta}) = \sigma(\alpha, \bar{\alpha}) \int_{\Omega} \phi(x, w(x)) M_{\bar{\alpha}}^{\beta}(Dw) \, \mathrm{d}x, \tag{3.16}$$

for $\phi \in C_c^{\infty}(\Omega \times \mathbb{R}^n)$ and $|\alpha| + |\beta| = n$. (We use the convention that $M_0^0(Dw) = 1$.)

We will repeatedly use the fact that

$$G_w(\phi \, \mathrm{d}x^{\alpha} \wedge \mathrm{d}\xi^{\beta}) = 0 \qquad \text{if } |\beta| \geq k+1, \tag{3.17}$$

which is a direct consequence of (3.1). A computation (see [12], section 3.2.1) shows that

$$G_w(\phi) = \int_{\Lambda_w} W^* \phi, \qquad \text{for every } n\text{-form } \phi \text{ in } \Omega \times \mathbb{R}^{\ell}, \text{ where } W(x) := (x, w(x))$$

and the pullback $W^*\phi$ is defined pointwise in Λ_w. Thus, G_w formally looks like integration over the (oriented) graph of w; this is the motivation for the definition of G_w. The next lemma collects some useful observations of Giaquinta, Modica and Souček [12] which clarify the sense in which this is, and is not, the case.

Lemma 3.3 *Assume that w satisfies (3.1). Then:*

(1) The restriction of $W(x) = (x, w(x))$ to Λ_w maps $\mathcal{L}^n$ null sets to $\mathcal{H}^n$ null sets.
(2) Γ is n-rectifiable.
(3) For $\mathcal{H}^n$ a.e. *point $W(x) \in \Gamma$, with $x \in \Lambda_w$,*

$$T_{W(x)}\Gamma = Im(DW(x)) \tag{3.18}$$

(4) G_w is an i.m. rectifiable n-current represented by integration over Γ. Indeed, for every compactly supported n-form ϕ in $\Omega \times \mathbb{R}^{\ell}$,

$$G_w(\phi) = \int_{\Gamma} \langle \phi, \tau \rangle d\mathcal{H}^n, \qquad \textit{where} \quad \tau(x, \xi) = \frac{W_{x^1}(x) \wedge \ldots \wedge W_{x^n}(x)}{|W_{x^1}(x) \wedge \ldots \wedge W_{x^n}(x)|}. \tag{3.19}$$

(5) If K is a compact subset of Ω, then $\|G_w\|(K \times \mathbb{R}^{\ell}) = \mathcal{H}^n(\Gamma \cap (K \times \mathbb{R}^{\ell})) < \infty$, where $\|G_w\|$ denotes the total variation measure associated with G_w.

Proof It follows from assumption (3.1) that w is a.e. approximately differentiable, and all minors of Dw are locally integrable. These are exactly the hypotheses of results in Giaquinta et al. [12], see in particular sections 3.1.5 and 3.2.1 which establish all the conclusions of the lemma. □

Under the conditions of Lemma 3.3, the set Γ which carries G_w can differ from the actual graph $\{(x, w(x)) : x \in \Omega\}$ by a set of positive $\mathcal{H}^n$ measure; see, for example, [26]. As we show below, it is nonetheless true that the current G_w associated with Γ has no boundary in $\Omega \times \mathbb{R}^\ell$. For this, we need the full strength of assumption (3.1); for Lemma 3.3 above, it in fact suffices to assume that $w \in W^{1,k}_{\mathrm{loc}}$.

Lemma 3.4 *If w satisfies* (3.1) *and G_w is the n-current defined in* (3.16)*, then*

$$\partial G_w = 0 \qquad \text{in } \Omega \times \mathbb{R}^\ell. \tag{3.20}$$

Remark 3.5 The Lemma implies that if u is a scalar function and $w = Du$ satisfies (3.1), then u is a Monge–Ampère function, see [11,18], and moreover that $\mathrm{Det}\, \mathrm{D}^2\mathrm{u} = 0$ in the sense of [17, Equation (1.14)]. We mention that, while the functions constructed in Examples 1 and 2 are also Monge–Ampere functions, they do not satisfy $\mathrm{Det}\, \mathrm{D}^2\mathrm{u} = 0$ in the above sense. It would also be possible to construct a function $u \in W^{2,k}$ on a bounded, open subset of $\mathbb{R}^n$ (say the unit ball) such that rank $D^2u \le k$ a.e. but u is not Monge–Ampère, due to an accumulation of conical singularities.

Proof We must check that

$$0 = G_w\left(d(\phi\, \mathrm{d}x^\alpha \wedge \mathrm{d}\xi^\beta)\right) = G_w\left(\phi_{x^i} \mathrm{d}x^i \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}\xi^\beta\right) + G_w\left(\phi_{\xi^j} d\xi^j \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}\xi^\beta\right) \tag{3.21}$$

for all $\phi \in C_c^\infty(\Omega \times \mathbb{R}^\ell)$ and α, β such that $|\alpha| + |\beta| = n - 1$. The terms on the right-hand side have the form

$$\int_\Omega \phi_{x^i}(x, w) \cdot (\text{minor of order } |\beta|) + \int_\Omega \phi_{\xi^i}(x, w) \cdot (\text{minor of order } |\beta| + 1). \tag{3.22}$$

If $|\beta| \ge k + 1$, then the assumption that $\mathrm{rank}(Dw) \le k$ a.e. implies that all such terms vanish and hence that (3.21) holds. If $|\beta| \le k$, then let w_q be a sequence of smooth functions converging to w in $W^{1,k+1}_{\mathrm{loc}}(\Omega, \mathbb{R}^\ell)$. For each w_q, (3.21) holds (with w replaced by w_q). Also, all minors of Dw_q appearing in (3.22) have order at most $k + 1$ and hence converge in L^1_{loc} to the corresponding minors of Dw. And we can arrange after passing to a subsequence that

$$\left.\begin{array}{l} \phi_{x^i}(x, w_q(x)) \to \phi_{x^i}(x, w(x)) \\ \phi_{\xi^j}(x, w_q(x)) \to \phi_{\xi^j}(x, w(x)) \end{array}\right\} \qquad \mathcal{L}^n \text{a.e.} x, \text{ as } q \to \infty$$

for all i and j. These terms are also pointwise bounded uniformly in q (by $\|\nabla\phi\|_\infty$). We can thus send $q \to \infty$ to conclude that (3.21) holds for w. □

Below, we write $J_k p_v$ for the k-dimensional Jacobian (in the sense of [8] 3.2.22) of $p_v : \Gamma \to \mathbb{R}^\ell_v$, the point being that we implicitly restrict the domain of p_v to Γ. Similarly, for $A \subset \mathbb{R}^\ell_v$, we understand $p_v^{-1}(A)$ to mean $\{(x, \xi) \in \Gamma : \xi \in A\}$.

We can now prove Proposition 3.1. In doing so, we establish a number of additional facts that we record here:

Lemma 3.6 *Assume that w satisfies* (3.1) *and let G_w, Γ_v and Γ_h be defined, respectively, as in* (3.16), (3.5) *and* (3.4). *Then, there exist measurable mappings $\tau_v : \Gamma_v \to \Lambda_k \mathbb{R}^\ell_v$ and $\tau_h : p_v^{-1}(\Gamma_v) \to \Lambda_{n-k}(\mathbb{R}^n_h)$ such that τ_v and τ_h are* a.e. *unit simple multivectors orienting $T_\xi \Gamma_v$ and $T_{(x,\xi)}(\Gamma_h(\xi) \times \{\xi\})$, and*

$$G_w(\chi\, \mathrm{d}\xi^\beta \wedge \psi) = \int_{\Gamma_v} H_\xi(\psi) \langle \mathrm{d}\xi^\beta, \tau_v\rangle\, \chi\, d\mathcal{H}^k \tag{3.23}$$

for $\beta \in I(k,\ell)$, $\psi \in \mathcal{D}^{n-k}(\Omega \times \mathbb{R}^\ell_v)$ *and* $\chi \in C^\infty(\mathbb{R}^\ell)$, *where*

$$H_\xi(\psi) := \int_{\Gamma_h(\xi)\times\{\xi\}} \langle \psi, \tau_h \rangle \, d\mathcal{H}^{n-k} \qquad \textit{for } \psi \in \mathcal{D}^{n-k}(\Omega \times \mathbb{R}^\ell). \tag{3.24}$$

Proof of Proposition 3.1 and Lemma 3.6 **1**. Given that Γ is rectifiable, see Lemma 3.3, the measurability and rectifiability of Γ_v are immediate consequences of [8] 3.2.31, and then, the *a.e.* measurability and rectifiability of $\Gamma_h(\xi)$ follow directly from [8] 3.2.22(2).

Next, the coarea formula [8] 3.2.22(3) states that for any $\mathcal{H}^n \llcorner \Gamma$-integrable function g,

$$\int_\Gamma g \; J_k p_v \; d\mathcal{H}^n = \int_{\Gamma_v} \left(\int_{p_v^{-1}\{\xi\}} g \; d\mathcal{H}^{n-k} \right) d\mathcal{H}^k.$$

It follows that

$$J_k p_v(x,\xi) > 0 \quad \mathcal{H}^{n-k} \text{ a.e.}\, in\, \Gamma_h(\xi), \text{ for } \mathcal{H}^k \text{ a.e. } \xi \quad \in \Gamma_v. \tag{3.25}$$

Moreover,

$$T_\xi \Gamma_v = p_v(T_{(x,\xi)}\Gamma) = \mathrm{Im}(Dw(x)), \quad \mathcal{H}^{n-k}\text{a.e.} \quad in \Gamma_h(\xi), \text{ for } \mathcal{H}^k \text{a.e.} \xi \in \Gamma_v, \tag{3.26}$$

using [8] 3.2.22(1) for the first equality and (3.18) for the second.

2. Let $\tau_v : \Gamma_v \to \Lambda_k \mathbb{R}^\ell_v$ be any fixed measurable unit simple k-vectorfield that orients $T_\xi \Gamma_v$ a.e.. We will construct $\mathcal{H}^n$-measurable $\tau_h : p_v^{-1}(\Gamma_v) \to \Lambda_{n-k}(\mathbb{R}^n_h)$ characterized (up to null sets) by the identity

$$\langle \mathrm{d}\xi^\beta \wedge \mathrm{d}x^\alpha, \tau(x,\xi) \rangle = J_k p_v(x,\xi) \; \langle \mathrm{d}\xi^\beta, \tau_v(\xi) \rangle \; \langle \mathrm{d}x^\alpha, \tau_h(x,\xi) \rangle \tag{3.27}$$

for all multiindices such that $|\beta| = n - |\alpha| = k$, where τ was defined in (3.19). In fact, since τ_v and τ are measurable, this identity is automatically the measurability of τ_h.

To prove (3.27), we fix some point $(x,\xi) \in p_v^{-1}\Gamma_v$ such that $\mathrm{rank}(Dw(x)) = k$ and (3.18) holds. These conditions hold $\mathcal{H}^n$ a.e. by (3.25) and Lemma 3.3. We will find τ_h by first selecting a basis $\{b_i\}_{i=1}^n$ for $\mathbb{R}^n_h$ with a number of good properties and then defining

$$\tau_i := DW(x) b_i, \quad i = 1, \ldots, n, \qquad \tau_h := \tau_{k+1} \wedge \ldots \wedge \tau_n. \tag{3.28}$$

In view of (3.18), any such $\{\tau_i\}_{i=1}^n$ is a basis for $T_{(x,\xi)}\Gamma$. We choose $\{b_i\}$ to satisfy the following:

- $\{b_i\}_{i=k+1}^n$ are an orthonormal basis for $\ker(Dw(x))$.
- $\{b_i\}_{i=1}^k$ are orthogonal to $\ker(Dw(x))$ and are chosen so that $\{\tau_i\}_{i=1}^k$ are orthonormal.
- $b_1, \ldots, b_k$ are ordered so that $Dw(x)b_1 \wedge \ldots \wedge Dw(x)b_k$ is a positive multiple of $\tau_v(\xi)$.
- $\{b_1, \ldots, b_n\}$ is positively oriented with respect to the standard basis $\{e_1, \ldots, e_n\}$.

The first two conditions can be satisfied since $\mathrm{rank}(Dw(x)) = k$. The third condition can be achieved due to (3.8), by changing the sign of b_1 if necessary. Having fixed $\{b_1, \ldots, b_k\}$, we can adjust the sign of b_{k+1} to arrange the final condition.

We now verify (3.27). Note that $\tau_i = DW(x)b_i = (b_i, Dw(x)b_i) \in \mathbb{R}^n_h \times \mathbb{R}^\ell_v$. It follows that $\tau_i = (b_i, 0)$ for $i > k$ and hence that $\{\tau_i\}_{i=1}^n$ are orthonormal. This and the ordering of $\{b_1, \ldots, b_n\}$ imply that $\tau_1 \wedge \ldots \wedge \tau_n = \tau(x,\xi)$.

Also, it is a fact that $J_k p_v = |p_v \tau_1 \wedge \ldots \wedge p_v \tau_k|$; this is a straightforward consequence of the definition of the Jacobian. Since $|\tau_v(\xi)| = 1$ and $p_v \tau_i = Dw(x) b_i$, the ordering of $b_1, \ldots, b_k$ implies that

$$\tau_v(\xi) = \frac{p_v \tau_1 \wedge \ldots \wedge p_v \tau_k}{|p_v \tau_1 \wedge \ldots \wedge p_v \tau_k|} = \frac{p_v \tau_1 \wedge \ldots \wedge p_v \tau_k}{J_k p_v(x,\xi)}.$$

Since $p_v\tau_i = 0$ for $i > k$, it follows that

$$\begin{aligned}\tau(x,\xi) &= \tau_1 \wedge \ldots \wedge \tau_n\\ &= (p_h\tau_1 + p_v\tau_1) \wedge \ldots \wedge (p_h\tau_k + p_v\tau_k) \wedge \tau_h\\ &= J_k p_v(x,\xi)\, \tau_v \wedge \tau_h + (\text{terms involving at most } k-1 \text{ vertical vectors}).\end{aligned}$$

Then, the claim (3.27) follows by letting $\mathrm{d}\xi^\beta \wedge \mathrm{d}x^\alpha$ act by duality on both sides of the above expression, since

$$\left\langle \mathrm{d}\xi^\beta \wedge \mathrm{d}x^\alpha, \text{terms involving at most } k-1 \text{ vertical vectors}\right\rangle = 0.$$

3. We will now show that if $|\beta| = n - |\alpha| \geq k$, then

$$\int_\Gamma \left\langle \phi\, \mathrm{d}\xi^\beta \wedge \mathrm{d}x^\alpha, \tau\right\rangle d\mathcal{H}^n \;=\int_{p_v^{-1}\Gamma_v} \left\langle \phi\, \mathrm{d}\xi^\beta \wedge \mathrm{d}x^\alpha, \tau\right\rangle d\mathcal{H}^n \qquad \text{for } \phi \in C_c^\infty(\Omega\times\mathbb{R}^n). \tag{3.29}$$

This is clear whether $|\beta| = n - |\alpha| > k$, in which case both sides vanish. For $|\beta| = k$, this follows from a classical argument, dating back at least to Fu [11], which we recall for the convenience of the reader. First, we rewrite the left-hand side in terms of slices $\langle G_w, q_\beta, \cdot\rangle$ of G_w by level sets of q_β, where $q_\beta(x,\xi) = (\xi^{\beta_1}, \ldots, \xi^{\beta_k}) \in \mathbb{R}^k$. This leads to

$$\int_\Gamma \langle \phi\, \mathrm{d}\xi^\beta \wedge \mathrm{d}x^\alpha, \tau\rangle d\mathcal{H}^n \;=\; G_w(\mathrm{d}\xi^\beta \wedge \phi\, \mathrm{d}x^\alpha) = \int_{\mathbb{R}^k} \langle G_w, q_\beta, y\rangle(\phi\, \mathrm{d}x^\alpha)\, \mathrm{d}y. \tag{3.30}$$

Fix some $i \in \{1, \ldots, \ell\}$. We will write $q_i(x,\xi) = \xi^i$ and $q_{\beta,i}(x,\xi) = (q_\beta(\xi), \xi^i) \in \mathbb{R}^{k+1}$. We claim that

$$\Big\langle \langle G_w, q_\beta, y\rangle, q_i, s\Big\rangle = 0 \qquad \text{for a.e. } (y,s) \in \mathbb{R}^k \times \mathbb{R}. \tag{3.31}$$

To see this, note that for $\mathcal{L}^{k+1}$ a.e. $(y,s) \in \mathbb{R}^k \times \mathbb{R}$,

$$\Big\langle \langle G_w, q_\beta, y\rangle, q_i, s\Big\rangle = \langle G_w, q_{\beta,i}, (y,s)\rangle$$

(see [8] 4.3.5). Then, basic properties of slicing imply that for any $\psi \in \mathcal{D}^{n-k-1}(\Omega \times \mathbb{R}^\ell_v)$ and $\chi \in C_c^\infty(\mathbb{R}^k \times \mathbb{R})$,

$$\int_{\mathbb{R}^k\times\mathbb{R}} \langle G_w, q_{\beta,i}, (y,s)\rangle(\psi)\, \chi(y,s)\, \mathrm{d}\, ds = G_w(\chi \circ q_{\beta,i}\, \mathrm{d}\xi^\beta \wedge d\xi^i \wedge \psi) \overset{(3.17)}{=} 0.$$

It follows that for every ψ as above,

$$\Big\langle \langle G_w, q_\beta, y\rangle, q_i, s\Big\rangle(\psi) = 0 \qquad \text{for a.e. } (y,s) \in \mathbb{R}^k \times \mathbb{R}.$$

Then, (3.31) follows by considering a countable dense subset of $\mathcal{D}^{n-k-1}(\Omega \times \mathbb{R}^\ell_v)$.

Now according to Solomon's Separation Lemma (Lemma 3.3 of [33]), it is a consequence of (3.31) that for $\mathcal{L}^k$ a.e. y, every indecomposable component of $\langle G_w, q_\beta, y\rangle$ is carried by a level set of q_i. Since this holds for all i, we infer that for a.e y, every indecomposable component of $\langle G_w, q_\beta, y\rangle$ is carried by $p_v^{-1}\{\xi\}$ for some $\xi \in \mathbb{R}^\ell$. From general properties of slicing, each such indecomposable component can be represented by integration with respect to $\mathcal{H}^{n-k}$ over $p_v^{-1}\{\xi\}$. In particular, for each such indecomposable component, $\mathcal{H}^{n-k}(p_v^{-1}\{\xi\}) > 0$, so $\xi \in \Gamma_v$. Hence, $\langle G_w, q_\beta, y\rangle$ is carried by $p_v^{-1}\Gamma_v$. We combine this fact with (3.30) to deduce (3.29).

4. We now prove (3.23). Thus, for $\beta \in I(k,\ell)$, $\psi \in \mathcal{D}^{n-k}(\Omega \times \mathbb{R}^\ell_v)$ and $\chi \in C^\infty(\mathbb{R}^\ell_v)$, we find from (3.19), (3.27), (3.29) and the coarea formula [8] 3.2.22 that

$$
\begin{aligned}
G_w(\chi\, \mathrm{d}\xi^\beta \wedge \psi) &= \int_{p_v^{-1}\Gamma_v} \langle \mathrm{d}\xi^\beta \wedge \psi, \tau\rangle \chi\, d\mathcal{H}^n \\
&= \int_{p_v^{-1}\Gamma_v} \langle \psi, \tau_h(x,\xi)\rangle \langle \mathrm{d}\xi^\beta, \tau_v(\xi)\rangle J_k p_v(x,\xi)\chi(\xi)\, d\mathcal{H}^n \\
&= \int_{\Gamma_v} \left(\int_{p_v^{-1}\{\xi\}} \langle \psi, \tau_h\rangle d\mathcal{H}^{n-k} \right) \langle \mathrm{d}\xi^\beta, \tau_v\rangle\ \chi\ d\mathcal{H}^k
\end{aligned}
$$

This is (3.23).

5. Since $\partial G_w = 0$ in $\Omega \times \mathbb{R}^\ell$, it follows from (3.23) that

$$
\int_{\Gamma_v} \partial H_\xi(\psi)\ \langle \mathrm{d}\xi^\beta, \tau_v\rangle \chi(\xi)\, d\mathcal{H}^k = 0
$$

for all $\psi \in \mathcal{D}^{n-k-1}(\Omega \times \mathbb{R}^\ell)$, $\chi \in C^\infty(\mathbb{R}^\ell)$, and $\beta \in I(k,\ell)$. For every such ψ, it follows that $\partial H_\xi(\psi) = 0$ for $\mathcal{H}^k$ a.e. $\xi \in \Gamma_v$. By considering a countable dense subset of $\mathcal{D}^{n-k-1}(\Omega \times \mathbb{R}^\ell)$, we conclude that

$$
\partial H_\xi = 0 \quad \text{in } \Omega \times \mathbb{R}^\ell, \qquad \text{for } \mathcal{H}^k \text{ a.e. } \xi \in \Gamma_v. \tag{3.32}
$$

Then, a standard blow-up argument shows that at any point (x,ξ) of $\Gamma_h(\xi) \times \{\xi\}$ which is a Lebesgue point of τ_h and at which $\Gamma_h(\xi) \times \{\xi\}$ has an $n-k$-dimensional approximate tangent space P, suitable rescalings of H_ξ converge to the current

$$
T(\psi) = \int_P \langle \psi(y), \tau_h(x,\xi)\rangle d\mathcal{H}^{n-k}(y),
$$

and moreover that $\partial T = 0$. It follows that at such points, which comprise $\mathcal{H}^{n-k}$ almost all of $\Gamma_h(\xi) \times \{\xi\}$, the approximate tangent space P is oriented by $\tau_h(x,\xi)$. Projecting this statement onto the horizontal component, and recalling the choice of $\{\tau_i\}$ in Step 1 above, we deduce that

$$
T_x\Gamma_h(\xi) = \mathrm{span}\{p_h\tau_i\}_{i=k+1}^n = \mathrm{span}\{b_i\}_{i=k+1}^n = \ker(Dw(x)).
$$

This completes the proof of (3.8), recalling that we have already verified (3.26).

6. Finally, comparing (3.16) and (3.23),

$$
\int_{\Lambda_w} \phi(x, w(x)) M^\beta_{\bar\alpha}(Dw)\, \mathrm{d}x = \pm \int_{\Gamma_v} \left(\int_{\{\xi\}\times\Gamma_h(\xi)} \phi(x,\xi)\langle \mathrm{d}x^\alpha, \tau_h\rangle d\mathcal{H}^{n-k} \right) \langle \mathrm{d}\xi^\beta, \tau_v\rangle\ d\mathcal{H}^k
$$

if $|\beta| = n - |\alpha| = k$, for $\phi \in C^\infty_c(\Omega \times \mathbb{R}^\ell)$. By an approximation argument, this also holds for $\phi \in L^\infty(\Omega \times \mathbb{R}^\ell)$ with compact support. Also, we may replace Γ_v by Γ_v^*, defined in (3.10), since it follows from what we have already proved that the latter has full $\mathcal{H}^k$ measure in Γ_v. We deduce that for any compact set $K \subset \Omega \times \mathbb{R}^\ell$, if we define

$$
\Omega^k_{\alpha,\beta,K} := \{x \in \Lambda_w : (x, w(x)) \in K,\ M^\beta_{\bar\alpha}(Dw(x)) \neq 0\}
$$

then

$$
\mathcal{L}^n \left(\Omega^k_{\alpha,\beta,K} \setminus \cup_{\xi\in\Gamma_v^*}\Gamma_h(\xi) \right) = 0.
$$

Since

$$\Omega^k = \bigcup_{|\beta| = n - |\alpha| = k} \; \bigcup_{K \text{compact}} \Omega^k_{\alpha,\beta,K},$$

and indeed this can be written as a countable union via a suitable sequence of compact sets $\{K_j\}_{j=1}^{\infty}$, this implies (3.9). □

4 Dense weak flat foliation

The main result of this section is the following.

Proposition 4.1 *Assume that* Ω *is a bounded, open subset of* $\mathbb{R}^n$ *and that*

$$w \in W^{1,k+1}_{\text{loc}}(\Omega), \qquad \textit{rank}\,(Dw) \le k \;\; \textit{a.e.} \tag{4.1}$$

for some $k \in \{1, \dots, n-1\}$*, and*

$$w = (Du^1, \dots, Du^q) \; \textit{for some} \; q \ge 1. \tag{4.2}$$

Then, w *is densely weakly* $(n-k)$*-flatly foliated.*

This will be a straightforward consequence of the following lemma, which gives a more detailed description of w in the set Ω^k in which Dw has maximal rank k, see (3.6).

Lemma 4.2 *Assume that* w *satisfies the hypotheses of Proposition 4.1.*

Then, for $\mathcal{L}^n$ a.e. $x \in \Omega^k$, $w^{-1}\{w(x)\}$ *coincides, up to a* $\mathcal{H}^{n-k}$ *null set, with a countable union of* $(n-k)$*-planes in* Ω*, all of them parallel to* $\ker(Dw(x))$.

In particular, for $\mathcal{L}^n$ a.e. $x \in \Omega^k$, w *is* $\mathcal{H}^{n-k}$ a.e. *constant on the* $n-k$*-plane in* Ω *that passes through* x *and whose tangent space is* $\ker(Dw(x))$.

This is essentially proved in [18] in the case $k = 1, n = 2$.

Note that for $w \in W^{1,k+1}_{\text{loc}}$, the set of points that fail to be Lebesgue points of w has dimension less than $n-k$, as discussed in Remark 1.11, so the conclusions of the proposition make sense.

The proof of Lemma uses the geometric measure theory results of the previous section to give a rigorous version of the formal argument sketched in the introduction. It is the only point in this paper at which we use the gradient structure (4.2) of w.

In the proof, we will identify $\mathbb{R}^n_h$ and $\mathbb{R}^n_v$ via the natural isomorphism $e_j \leftrightarrow \varepsilon_j$. In particular, for each $i \in \{1, \dots, q\}$, we use the identity $\ker(D^2u^i(x)) = [\operatorname{Im}(D^2u^i(x))]^{\perp}$.

Proof of Lemma 4.2 **1**. We fix $\xi \in \Gamma^*_v$, defined in (3.10), and we first claim that

$$T_x\Gamma_h(\xi) \text{ is } \mathcal{H}^{n-k} \text{a.e.} constant for x \in \Gamma_h(\xi). \tag{4.3}$$

Indeed, since $D^2u^i(x)$ is symmetric for every i, at $\mathcal{H}^{n-k}$ a.e. $x \in \Gamma_h(\xi)$ we have

$$T_x\Gamma_h(\xi) \overset{(3.8)}{=} \ker(Dw(x)) \overset{(4.2)}{=} \cap_{i=1}^q \ker(D^2u^i(x)) = \cap_{i=1}^q [\operatorname{Im}(D^2u^i(x))]^{\perp}.$$

Moreover, if we write $P^i : (\mathbb{R}^n)^q \to \mathbb{R}^n$ to denote orthonormal projection of $\mathbb{R}^{nq} = (\mathbb{R}^n)^q$ onto the ith copy of $\mathbb{R}^n$, then $D^2u^i(x) = P^i \circ Dw(x)$. Thus,

$$\operatorname{Im}(D^2u^i(x)) = \operatorname{Im}(P^i \circ Dw(x)) = P^i(Im(Dw_i)) \overset{(3.8)}{=} P^i(T_\xi\Gamma_v).$$

The term on the right depends only on ξ, so (4.3) follows from the previous two identities.

2. For $\xi \in \Gamma_v^*$, we will write $T(\xi) := \cap_{i=1}^j [P^i(T_\xi \Gamma_v)]^\perp = T_x\Gamma_h(\xi)$ for a.e. $x \in \Gamma_h(\xi)$. We next claim that

$$\text{if } \xi \in \Gamma_v^*, \text{ then } \Gamma_h(\xi) \text{ is a union of } (n-k)\text{-planes in } \Omega, \text{ all parallel to } T(\xi). \tag{4.4}$$

Since the current H_ξ from Proposition 3.1 is represented by integration over $\Gamma_h(\xi) \times \{\xi\}$, it suffices to show that every indecomposable component of H_ξ is supported on exactly a set of the form $P \times \{\xi\}$, where P is an $(n-k)$-plane in Ω with tangent space $T(\xi)$.

This follows from (4.3) and the fact that $\partial H_\xi = 0$ in $\Omega \times \mathbb{R}^n$, by classical arguments that we have already seen in the proof of Proposition 3.6. In detail, by changing coordinates we may arrange that $T_x\Gamma_h(\xi) = \text{span}\{e_1, \ldots, e_{n-k}\}$ for a.e. $x \in \Gamma_h(\xi)$. Since H_ξ is carried by $\Gamma_h(\xi) \times \{\xi\}$, it follows that for $H_\xi(\phi \wedge df) = 0$ for every $n-k-1$-form ϕ with compact support in Ω, whenever f has the form $f(x) = x^j$ for some $j \in \{n-k+1, \ldots, n\}$. In this situation, Solomon's Separation Lemma (Lemma 3.3 of [33]) states that every indecomposable component of H_ξ is carried by a level set of f. It follows that every indecomposable piece of H_ξ is contained in an $n-k$ plane in which x^j is constant for all $j = n-k+1, \ldots, n$ (in the coordinates we have chosen, which depended on ξ.) described above. This completes the proof of (4.4).

3. Now the conclusions of the lemma follow directly from (4.4), the definition (3.4) of $\Gamma_h(\xi)$, which implies in particular that w is a.e. constant in each of these sets, and (3.9), which asserts that $\cup_{\xi\in\Gamma_v^*}\Gamma_h(\xi)$ contains almost every point of Ω^k. □

Having Lemma 4.2 at hand, the proof that w is densely weakly flatly foliated is straightforward.

Proof of Proposition 4.1 **1**. We recall from Definition 1.9 that the definition of densely weakly flatly foliated involves a partition of Ω into sets F_j such that $\Omega_j := \cup_{m=0}^j F_m$ is open for every j, and satisfying a property recalled in (4.8) below. We define these sets as follows. As before,

$$\Omega^k := \{x \in \Omega : x \text{ is a Lebesgue point of } w \text{ and } Dw, \text{ and rank}\,(D^2u(x)) = k\}.$$

We also let $\Omega_k = \Omega$, and for $j \in \{k-1, \ldots, 0\}$, we recursively define (working downwards)

$$\Omega_j = \Omega_{j+1} - \bar{\Omega}^{j+1} \tag{4.5}$$

$$\Omega^j = \{x \in \Omega_j : x \text{ is a Lebesgue point of } Du \text{ and } D^2u, \text{ and rank}\,(D^2u) = j\}, \tag{4.6}$$

Finally, we set

$$F_j := \bar{\Omega}^j \cap \Omega_j = \Omega_j \setminus \Omega_{j-1}. \tag{4.7}$$

This indeed defines a partition of Ω such that every Ω_j is open, as required.

Note that by our convention $F_k = \bar{\Omega}^k$.

We must show that for every $j \in \{0, \ldots, k\}$,

$$\begin{aligned} &\text{for every } x \text{ in a dense subset of } F_j, \text{ there exists at least one } n-j\text{-plane } P \text{ in } \Omega_j \\ &\quad \text{such that } x \in P \text{ and } w \text{ is } \mathcal{H}^{n-j} \text{a.e. constant on} P. \end{aligned} \tag{4.8}$$

Observe for every $j \le k$, Ω_j is open, and $w \in W^{1,j+1}_{\text{loc}}(\Omega_j; \mathbb{R}^\ell) \subset W^{1,k+1}_{\text{loc}}(\Omega; \mathbb{R}^\ell)$, with $\text{rank}(Dw) \le j$ a.e. in Ω_j. In other words, $w|_{\Omega_j}$ satisfies (4.1) with k replaced by j, and hence, Lemma 4.2 holds, with k replaced by j in $\Omega^j \subset \Omega_j$. It follows that

$$\text{for every } x \text{ in a full measure subset of } \Omega^j, \text{ there exists at least one } n-j\text{-plane } P \text{ in } \Omega_j \text{ such that } x \in P \text{ and } w \text{ is } \mathcal{H}^{n-j} \text{ a.e. constant on } P. \tag{4.9}$$

Since Ω^j is manifestly dense in F_j, to deduce (4.8) from (4.9) it suffices to prove that every full measure subset of Ω^j is in fact dense in Ω^j.

To see this, consider some $x_0 \in \Omega^j$, and fix $\delta > 0$ such that $\operatorname{rank}(A) \geq j$ for all matrices with $|A - Dw(x_0)| < \delta_0$. Then, for every $r > 0$ such that $B_r(x_0) \subset \Omega_j$, since x_0 is a Lebesgue point of w and Dw, the set

$$\{x \in B_r(x_0) : x \text{ is a Lebesgue point of } w \text{ and } Dw, \text{ and } |Dw(x) - Dw(x_0)| < \delta_0\}$$

has positive measure. Since $\operatorname{rank}(Dw) \leq j$ a.e in $B_r(x_0) \subset \Omega_j$, the above set intersects Ω^j in a set of positive measure. Since x_0 and r were arbitrary, this completes the proof of (4.8). □

5 Pointwise weak developability

In this section, we will prove the following statement, which is an important step in establishing Theorem 4.

Proposition 5.1 *Assume that*

$$w \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^\ell), \qquad \mathit{rank}\,(Dw) \leq k \ \text{ a.e.} \tag{5.1}$$

for some $k \in \{1, \ldots, n-1\}$ and some $p > k$. If w is densely weakly $(n-k)$-flatly foliated, then w is pointwise weakly $(n-k)$-flatly foliated.

Remark 5.2 In view of Definition 1.6, we could say that Propositions 2.1, 4.1 and 5.1 together imply a pointwise weak developability result for $W^{2,k+1}(\Omega; \mathbb{R}^{n+k})$ isometric immersions and also for such $u \in W^{2,k+1}$ such that $\operatorname{rank}(D^2u) \leq k$ a.e.

The proposition will follow from a couple of lemmas.

Lemma 5.3 *Assume that k, n are integers such that $1 \leq k < n$. Let U be an open subset of $\mathbb{R}^{n-k}$, and for $r > 0$ let $S := U \times B^k_r$ for some $r > 0$.*

Assume that $w \in W^{1,p}(S; \mathbb{R}^\ell)$ for some $p > k$, and for $i = 1, 2$ let $\zeta_i : U \to B^k_s$ be continuous functions. Then, (writing points in S in the form $x = (y, z)$ with $y \in U$, $z \in B^k_s$)

$$\left(\int_U |w(y, \zeta_1(y)) - w(y, \zeta_2(y))|^p \mathrm{d}\right)^{1/p} \leq C\|w\|_{W^{1,p}(S)} \|\zeta_1 - \zeta_2\|^\alpha_{L^\infty(U)}$$

for $\alpha = 1 - \frac{k}{p}$, for a constant C depending only on k and p.

Proof We compute

$$\begin{aligned}
\|w\|^p_{W^{1,p}(S)} &\geq \int_U \|w(y, \cdot)\|^p_{W^{1,p}(B^k_r)} \,\mathrm{d} \\
&\geq C^{-1} \int_U \frac{|w(y, \zeta_1(y)) - w(y, \zeta_2(y))|^p}{|\zeta_1(y) - \zeta_2(y)|^{\alpha p}} \,\mathrm{d}
\end{aligned}$$

by the (k-dimensional) Sobolev Embedding, from which we also know that the constant C depends only on p and k and in particular is independent of r. □

Our next lemma will be used again in Sect. 6.

Lemma 5.4 *Assume that Ω is a bounded, open subset of $\mathbb{R}^n$ and that $w \in W^{1,p}(\Omega, \mathbb{R}^\ell)$ for some $p > j \in \{1, \dots, n-1\}$ and some ℓ.*

Assume also that $x_0 \in \Omega$ and that there exists a sequence of points $(x_m) \subset \Omega$ and values $(\xi_m) \in \mathbb{R}^\ell$ such that $x_m \to x_0$ as $m \to \infty$, and $w = \xi_m$ at $\mathcal{H}^{n-j}$ a.e. *point on an $(n-j)$-plane P_m in Ω that contains x_m.*

Then, $w = \lim_{m\to\infty} \xi_m$ at $\mathcal{H}^{n-j}$ a.e. *point on some $n-j$ plane P in Ω that contains x_0. (In particular, $\lim_{m\to\infty} \xi_m$ exists).*

As before, note that in view of Remark 1.11 the assumptions and the conclusion of the lemma make sense for the considered class of mappings.

Proof Let $\xi_m \in \mathbb{R}^\ell$ denote the value of w on $\mathcal{H}^{n-j}$ a.e. point of P_m, and let $\mathbb{P}_m$ denote the $(n-j)$-plane such that P_m is a connected component of $\mathbb{P}_m \cap \Omega$.

Since the Grassmannian of unoriented $(n-j)$-dimensional subspaces in $\mathbb{R}^n$ is compact, we may assume, after passing to subsequences (still labelled (P_m), (ξ_m)), that there is a $(n-j)$-plane $\mathbb{P}$ passing through x_0 such that $\mathbb{P}_m \to \mathbb{P}$ in the Hausdorff distance on $B_R(0) \subset \mathbb{R}^n$ as $m \to \infty$, for every $R > 0$. Now let P be the $(n-j)$-plane in Ω consisting of the connected component of $\mathbb{P} \cap \Omega$ that contains x_0.

We may arrange, after a translation and a rotation, that $x_0 = 0$ and $\mathbb{P} = \mathbb{R}^{n-j} \times \{0\}$, and we write $\mathbb{R}^n = \mathbb{R}^{n-j}_y \times \mathbb{R}^j_z$ as in Lemma 5.3. Fix a connected, relatively open set $U \subset P$, containing x_0 and having compact closure in Ω. Then, there exists an open ball B^j_r such that $S := U \times B^j_r \Subset \Omega$. The convergence $\mathbb{P}_m \to \mathbb{P}$ implies that for every sufficiently large m, there is an affine function $\zeta_m : U \to B^j_r$ such that $\mathbb{P}_m \cap S = \{(y, \zeta_m(y)) : y \in U\}$ and moreover that $\|\zeta_m\|_{L^\infty(U)} \to 0$ as $m \to \infty$.

Also, for m large enough that $x_m \in S$, we have that $P_m \cap S$ is nonempty and hence (since $S \subset \Omega$ is convex and P_m is a connected component of $\mathbb{P}_m \cap \Omega$) that $\mathbb{P}_m \cap S = P_m \cap S \subset P_m$. So $w = \xi_m\ \mathcal{H}^{n-j}$ a.e. in $\mathbb{P}_m \cap S$, and by applying Lemma 5.3 to $\zeta = 0$ and ζ_m, we find that

$$\int_U |w(y,0) - \xi_m|^p \mathrm{d} = \int_U |w(y,0) - w(y, \zeta_m(y))|^p \mathrm{d}$$
$$\leq C\|w\|^p_{W^{1,p}(S)} \|\zeta_m\|^{\alpha p}_{L^\infty(U)} \to 0 \quad \text{as } m \to \infty,$$

where $\alpha = 1 - \frac{j}{p}$. It follows that there exists some $\xi \in \mathbb{R}^\ell$ such that $\xi_m \to \xi$ and moreover that $w(\cdot, 0) = \xi$ a.e. on U. Since U was arbitrary, it follows that $w = \xi$ at $\mathcal{H}^{n-j}$ a.e. point of P. □

Now we complete the

Proof of Proposition 5.1 By assumption, Ω is partitioned into sets F_j, $j = 0, \dots, n-k$ such that $\Omega_j := \cup_{m=0}^j F_m$ is open for every j, and in addition, there is a dense subset of F_j in which every point is contained in a $n-j$-plane in Ω_j on which w is $\mathcal{H}^{n-j}$ a.e. constant.

To prove the proposition (with the same partition (F_j) of Ω), it suffices to show that *every* point in F_j is contained in a $n-j$-plane in Ω_j on which w is $\mathcal{H}^{n-j}$ a.e. constant. This follows directly from Lemma 5.4, since every point in F_j satisfies the hypotheses of the lemma, with Ω replaced by Ω_j. □

Remark 5.5 We note in passing that a slightly more careful version of the above argument would prove the following statement: For *every* $x \in \Omega^j$ as defined in (4.6), w is $\mathcal{H}^{n-j}$ a.e. constant on the $n-j$-plane in Ω_j that passes through x and whose tangent space is $\ker(Dw(x))$, and the constant value is equal to $w(x)$.

6 Strong developability

In this section, we prove the following

Proposition 6.1 *Assume that Ω is an open subset of $\mathbb{R}^n$ and that $w \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^\ell)$ for some $p \geq \min\{2k, n\}$. If w is pointwise weakly $(n-k)$-flatly foliated, then w is continuous. As a result, if P is any $n-j$-plane in Ω_j (as in Definition 1.10) on which w is $\mathcal{H}^{n-j}$* a.e. *constant, then in fact w is constant on P. In particular, w is $(n-k)$-flatly foliated.*

For the convenience of the reader, the proof will be split in a series of Lemmas which will follow and will be completed in Lemma 6.7. This will complete the proof of Theorems 3 and 4, which follow immediately from combining Propositions 4.1, 5.1 and 6.1 and, for Theorem 3 only, Proposition 2.1 as well.

The following examples show that the condition $p \geq \min\{2k, n\}$ cannot be weakened, at least for certain values of n and k.

Example 6 Consider the map $w : \mathbb{R}^4 \to S^2 \subset \mathbb{R}^3$ defined by

$$w(x) = H\left(\frac{x}{|x|}\right) \quad \text{if } x \neq 0, \quad w(0) = 0,$$

where $H : S^3 \to S^2$ is the Hopf fibration. Recall that every level set of H has the form $\{(z, \zeta) \in \mathbb{C}^2 \cong \mathbb{R}^4 : |z|^2 + |\zeta|^2 = 1,\ \alpha z = \beta\zeta\}$ for some fixed $\alpha, \beta \in \mathbb{C}$ (one of which can always be taken to equal 1). From this, one easily checks that w is a 2-plane passing through the origin and that the intersection of any two level sets is $\{0\}$. Thus, w is pointwise weakly $(n-k)$-flatly foliated (see Definition 1.10) with $n = 4$, $k = 2$ and $F_2 = \mathbb{R}^4$, $F_0 = F_1 = \emptyset$, and $w \in W^{1,p}$ for all $p < 4 = \min\{2k, n\}$. But clearly w is not continuous.

This example shows the hypothesis $p \geq \min\{2k, n\}$ of Proposition 6.1 cannot be weakened when $n = 2k = 4$.

Example 7 Next, for $n \geq 5$ define $w_1 : \mathbb{R}^n \to \mathbb{R}^3$ by $w_1(x^1, \ldots, x^n) = w(x^1, \ldots, x^4)$ where w is the function from the above example. Then, w_1 is pointwise weakly $(n-k)$-flatly foliated with $k = 2$ and $F_2 = \mathbb{R}^n$, $F_0 = F_1 = \emptyset$. Also, $w_1 \in W^{1,p}_{\text{loc}}$ for all $p < 4 = \min\{2k, n\}$. But again w_1 is not continuous.

So the condition $p \geq \min\{2k, n\}$ cannot be weakened whenever $k = 2$ and $n > 4$.

Example 8 One can construct a function similar to that of Example 6 when $n = 2k = 8$ or 16 by using Hopf fibrations $S^7 \to S^4$ and $S^{15} \to S^8$, and similarly a function similar to the one in Example 7 when $n > 2k = 8$ or 16. It follows that the condition $p \geq \min\{2k, n\}$ cannot be weakened whenever $k = 4$ or 8 and $n \geq 2k$.

Remark 6.2 One can check that the $w : \mathbb{R}^n \to \mathbb{R}^\ell$ constructed in the above examples are not gradients of scalar functions. In fact, we conjecture that if we add to Proposition 6.1 the assumption that $w = Du$ for some scalar function u, then the conclusions of the proposition should still be true if we merely assume $p \geq k + 1$.

Before proceeding, we remind the reader once more that the assumptions of the next couple of lemmas regarding the $\mathcal{H}^{n-k}$ a.e. value of w on $n-k$-planes are justified (see Remark 1.11). The next lemma, whose proof is very similar to that of Lemma 5.3, still only needs the minimal regularity assumptions $p > k$.

Lemma 6.3 *Assume that k, n are integers such that $1 \le k < n$. Let Ω be an open subset of $\mathbb{R}^n$, and assume that $w \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^\ell)$ for some $p > k$. Finally, assume that P is an $n-k$-plane in Ω such that $w = \xi$* a.e. *on P for some $\xi \in \mathbb{R}^\ell$.*

If $x \in P$ is a Lebesgue point of $|Dw|^p$, then x is a Lebesgue point of w, and $w(x) = \xi$.

Proof We may assume after a translation and a rotation that P is a connected component of $\Omega \cap (\mathbb{R}^{n-k} \times \{0\})$ and that $x = 0$. Fix $R > 0$ such that $B^{n-k}_R \times B^k_R \subset \Omega$ and let $\alpha = 1 - \frac{k}{p}$. Then, for any positive $r < R$, writing $[f]_\alpha$ to denote the α-Hölder seminorm,

$$\fint_{B^{n-k}_r \times B^k_r} |w - \xi|^p \,\mathrm{d}z = \fint_{B^{n-k}_r} \left(\fint_{B^k_r} |w(y,z) - w(y,0)|^p \mathrm{d}z \right) \mathrm{d}y$$

$$\le \fint_{B^{n-k}_r} \left(\fint_{B^k_r} |z|^{p\alpha} [w(y,\cdot)]^p_\alpha \mathrm{d}z \right) \mathrm{d}y.$$

Also, by the k-dimensional Sobolev embedding,

$$\fint_{B^k_r} |z|^{p\alpha} [w(y,\cdot)]^p_\alpha \mathrm{d}z \le C r^{\alpha p - k} \int_{B^k_r} |Dw(y,z)|^p \,\mathrm{d}z = C r^{\alpha p} \fint_{B^k_r} |Dw(y,z)|^p \,\mathrm{d}z$$

with a constant C independent of r. Thus,

$$\fint_{B^{n-k}_r \times B^k_r} |w - \xi|^p \,\mathrm{d}z \ \le r^{\alpha p} \fint_{B^{n-k}_r \times B^k_r} |Dw|^p \,\mathrm{d}z.$$

Since x is a Lebesgue point of $|Dw|^p$, the right-hand side is bounded by $Cr^{p\alpha}$ for all small r, proving the lemma.

□

The restriction $p \ge \min\{2k, n\}$ in Proposition 6.1 arises from the following lemma.

Lemma 6.4 *Assume that Ω is an open subset of $\mathbb{R}^n$ and that $w \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^\ell)$ for some ℓ and some $p \ge 1$. Suppose that for $i = 1, 2$, there exist values $\xi^i \in \mathbb{R}^n$, planes P_i in Ω of dimension $n - k$ such that*

$$P_1 \cap P_2 \ne \emptyset, \qquad \text{and} \quad w = \xi^i, \quad \mathcal{H}^{n-k} a.e. in P_i$$

for $i = 1, 2$. If $p \ge \min\{n, 2k\}$, then $\xi_1 = \xi_2$.

Proof **1**. We first consider the case $2k < n$.

Let $x_0 \in \Omega \cap P_1 \cap P_2$. Any two planes of dimension $n - k$ that intersect at a point must intersect along a plane of dimension $n - 2k$. We may assume after a translation that x_0 is the origin, and after a rotation that $P_1 \cap P_2 = \mathbb{R}^{n-2k} \times \{0\}$. We write y and z, respectively, to denote points in $\mathbb{R}^{n-2k}$ and in $\mathbb{R}^{2k}$, and we fix r and s such that $B^{n-2k}_r \times B^{2k}_s \subset \Omega$. Then, for $\mathcal{H}^{n-2k+1}$ a.e. $(y, \sigma) \in B^{n-2k}_r \times (0, s)$,

$$\operatorname{ess\,osc}_{\{y\} \times \partial B^{2k}_\sigma} |w| \ge |\xi_1 - \xi_2|,$$

so that by the Sobolev embedding theorem,

$$|\xi_1 - \xi_2|^{2k} \le C\sigma \int_{\{y\} \times \partial B^{2k}_\sigma} |Dw|^{2k} \, d\mathcal{H}^{2k-1}.$$

Thus,

$$\int_{B_r^{n-2k}\times B_s^{2k}} |Dw|^{2k} = \int_{B_r^{n-2k}} \int_0^s \int_{\{y\}\times\partial B_\sigma^{2k}} |Dw|^{2k}\, d\mathcal{H}^{2k-1}\, d\sigma\, \mathrm{d}y$$
$$\geq c|\xi_1-\xi_2|^{2k} \int_{B_r^{n-2k}} \int_0^s \frac{1}{\sigma} d\sigma\, \mathrm{d}y.$$

The left-hand side is finite, so it follows that $|\xi_1-\xi_2| = 0$.

2. The case $2k \geq n$ is similar but easier. Here, all we can say about any two $n-k$-planes with nonempty intersection is that their intersection must contain a point x_0. Hence, the essential oscillation of w on a.e. small sphere centred at x_0 is bounded below by $|\xi_1-\xi_2|$, and as a result

$$\int_{B_s^n(x_0)} |Dw|^n = \int_0^s \int_{\partial B_\sigma^n(x_0)} |Dw|^n \geq c|\xi_1-\xi_2|^n \int_0^s \frac{1}{\sigma}\, d\sigma. \tag{6.1}$$

We conclude as before that $|\xi_1-\xi_2| = 0$. □

Remark 6.5 If $2k \geq n$, then a small modification of the above proof shows that the conclusion remains true if we assume $w = \xi_1$ a.e. in P_1 and that $w = \xi_2$ at $\mathcal{H}^1$ a.e. point of a connected, relatively open subset $U \subset P_2$, with $P_1 \cap \bar{U} \neq \emptyset$. Indeed, these hypotheses imply the existence of an open line segment containing x_0 on which $w = \xi_1$ a.e., and a second open line segment with an endpoint at x_0 on which $w = \xi_2$ a.e., and these conditions imply that the essential oscillation of w on a.e. small sphere centred at x_0 is bounded below by $|\xi_1-\xi_2|$, allowing us to conclude as in (6.1).

Our next result follows rather easily from the above two lemmas.

Lemma 6.6 *Assume that* $w \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^\ell)$ *for some* $p \geq \min\{2k, n\}$. *If* w *is pointwise weakly* $(n-k)$*-flatly foliated, then there exists a function* $\overline{w} : \Omega \to \mathbb{R}^n$ *such that*

$$\overline{w}|_{F_j} \text{ is continuous for every } j \in \{0,\ldots,k\} \tag{6.2}$$

and

$$\overline{w} = w \text{ a.e. in } \Omega. \tag{6.3}$$

In particular, for every $x \in F_j$, *there is an* $n-j$*-plane in* Ω_j *containing* x *on which* $\overline{w} = \overline{w}(x)$ *everywhere, where* F_j *and* Ω_j *are given as in Definition* 1.10.

Proof **1**. We define $\overline{w}$ by requiring that

$$\overline{w}(x) = \xi \text{ if } x \in F_j \text{ and } w = \xi \text{a.e.}\mathit{onsome} n-j\text{-plane } P \text{ in } \Omega_j \text{ passing through } x.$$

We claim that $\overline{w}$ is well defined. Towards this end, note that every x belongs to a unique F_j by (1.9) and hence by (1.11) belongs to at least one $n-j$-plane in F_j on which w is a.e. constant. Then, by Lemma 6.4, the values of w on any two such planes must agree a.e., so the claim follows.

2. It follows from the definition of $\overline{w}$ and Lemma 6.3 that $w = \overline{w}$ at every Lebesgue point of $|Dw|^p$, which implies (6.3).

3. To verify that (6.2) holds, assume towards a contradiction that $\overline{w}|_{F_j}$ is not continuous at some point $x_0 \in F_j$. Then, there exists a sequence (x_m) in F_j such that

$$|x_m - x_0| < \frac{1}{m}, \qquad |\overline{w}(x_m) - \overline{w}(x_0)| \geq c_0$$

for some $c_0 > 0$. Let $\xi_m := \overline{w}(x_m)$, and let P_m be a $n-j$-plane in Ω_j such that $\overline{w} = \xi_m$ on P_m. Then,

$$P_m \cap B_{1/m}(x_0) \neq \emptyset \qquad w = \xi_m \text{ a.e.} \quad on \quad P_m. \tag{6.4}$$

Then, Lemma 5.4 implies that there exists some exactly $(n-j)$-plane P' in Ω_j and some $\xi' \in \mathbb{R}^n$ such that

$$x_0 \in P', \qquad \xi_m \to \xi', \qquad \text{and } w = \xi' \ \mathcal{H}^{n-j} \text{ a.e.} \ \ on \ P'.$$

The definition of $\overline{w}$ implies that $\overline{w}(x_0) = \xi'$. This, however, is impossible, since $\xi_m \to \xi'$ and $|\xi_m - \overline{w}(x_0)| \geq c_0$ for all m. This contradiction shows that $\overline{w}|_{F_j}$ is continuous on F_j. □

Our next goal is to show that the function $\overline{w}$ found above is continuous in all of Ω. This will directly imply the continuity of w and hence will conclude the proof of our main results.

Lemma 6.7 *Assume that $w \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^\ell)$ for some $p \geq \min\{2k, n\}$ and that w is pointwise weakly $(n-k)$-flatly foliated. Let $\overline{w}$ be the function found in Lemma 6.6. Then, $\overline{w}$ is continuous in Ω, and as a result, w is continuous in Ω.*

Before giving the proof, we recall that every $f \in W^{1,p}(\Omega, \mathbb{R}^\ell)$ is *p-quasicontinuous*, which means that for every $\varepsilon > 0$, there exists an open set $O \subset \Omega$ such that $\text{Cap}_p(O) < \varepsilon$ and $f|_{\Omega \setminus O}$ is continuous. For the definition and the few properties of capacity that are needed for our argument (e.g. the above statement) refer to [7], unless another reference is provided.

The idea of the proof below is to show that, given what we already know about w, if it is discontinuous anywhere, then it must fail to be p-quasicontinuous, for $p = \min\{2k, n\}$, which is impossible. That is, we will argue (in the more difficult case $2k < n$) that, in view of (6.2), any discontinuity of $\overline{w}$ would involve the intersection of (the closure of) portions of planes on which $\overline{w}$ is constant, one having dimension at least $n-k$ and the other dimension at least $n-k+1$. This would lead to a discontinuity set for w of dimension at least $n-2k+1$, along which the discontinuity cannot be eliminated by cutting out an open set of small enough p-capacity, the point being that a set of p-capacity zero has dimension strictly less than $n-2k+1$.

Proof of Lemma 6.7 First, since $\overline{w} = w$ a.e., if $\overline{w}$ is continuous, then every $x \in \Omega$ is a Lebesgue point of w, and the Lebesgue value at x equals $\overline{w}(x)$. So $w = \overline{w}$ pointwise in Ω, and the continuity of w follows. Thus, we only need to show that $\overline{w}$ is continuous.

It is convenient to write $F_{\geq j} := \bigcup_{m \geq j} F_m$, and similarly $F_{>j} := \bigcup_{\ell > j} F_m = F_{\geq j+1}$. With this notation, we will prove that by (downward) induction on j that

$$\overline{w}|_{F_{\geq j}} \text{ is continuous for every } j \in \{k, \ldots, 0\} \tag{6.5}$$

which in particular will imply that $\overline{w}$ is continuous on $F_{\geq 0} = \Omega$.

From Lemma 6.6 we already know that (6.5) holds for $j = k$. Now we assume by induction that $\overline{w}|_{F_{>j}}$ is continuous for some nonnegative $j < k$, and we prove that $\overline{w}|_{F \geq j}$ is continuous.

Step 1 We first show that

$$\text{if } P \text{ is an } n-j\text{-plane in } \Omega_j \text{ for which } \overline{w} = \xi \text{ on } P, \text{ then } \overline{w} = \xi \text{ on } \bar{P} \cap F_{>j}. \tag{6.6}$$

This is a key point of the proof. In the case $2k \geq n$, this follows in a straightforward way from Remark 6.5, so we focus on the case $2k < n$.

Step 1a. Assume towards a contradiction that (6.6) fails, so that for some $n-j$-plane P in Ω_j and $x_0 \in \bar{P} \cap F_{>j}$ such that

$$\overline{w} = \xi \text{ on } P, \text{ and } \overline{w}(x_0) = \xi_0, \qquad \text{for some } \xi \neq \xi_0 \in \mathbb{R}^\ell. \tag{6.7}$$

Then, $x_0 \in F_i$ for some $i > j$, so there exists an $n-i$-plane P_0 in Ω_i such that $x_0 \in P_0$ and $\overline{w} = \xi_0$ in P_0.

We may assume that

$$P \cap P_0 = \emptyset \tag{6.8}$$

because if there exists some $y_0 \in P \cap P_0$, then since both P and P_0 are relatively open, we could apply Lemma 6.4 on a small ball containing y_0 to conclude that $\xi = \xi_0$.

We may also assume (after a translation) that $x_0 = 0$. We write $\mathbb{P}$ and $\mathbb{P}_0$ to denote the planes (of dimension $n-j$ and $n-i$, respectively) that contain P and P_0, and we let d denote the dimension of $\mathbb{P} \cap \mathbb{P}_0$, so that $d \geq n-i-j \geq n-2k+1$, recalling that $j < i \leq k$. Also, $d < n-i = \dim(\mathbb{P}_0) < n-j$.

We can arrange by a suitable rotation that

$$\mathbb{P} = \mathbb{R}^{n-j} \times \{0\} \subset \mathbb{R}^n, \qquad \mathbb{P} \cap \mathbb{P}_0 = \mathbb{R}^d \times \{0\} \subset \mathbb{R}^n.$$

We will write points in $\mathbb{R}^n$ in the form $x = (y, z)$ with $y \in \mathbb{R}^d$, $z \in \mathbb{R}^{n-d}$.

By the induction hypothesis, we may fix $r > 0$ so small that $B_r^d \times B_r^{n-d} \subset \Omega_i$ and

$$|\overline{w}(x) - \xi| > \delta := \frac{1}{2}|\xi_0 - \xi| \qquad \text{for all} \quad x \in (B_r^d \times B_r^{n-d}) \cap F_{>j}. \tag{6.9}$$

Let B be a relatively open ball in $P \cap (B_r^d \times B_r^{n-d})$, and let B_0 denote the orthogonal projection of B onto $\mathbb{R}^d \times \{0\}$, so that B_0 is a relatively open subset of $B_r^d \times \{0\}$.

Step 1b. We claim that for every $y \in B_0$, the restriction of w to $\{y\} \times B_r^{n-d}$ is discontinuous.

This is a consequence of the following two facts, which we will prove below. First,

$$\forall y \in B_0, \quad (\{y\} \times B_r^{n-d}) \cap \partial_{\mathbb{P}} P \text{ is nonempty}, \tag{6.10}$$

where $\partial_{\mathbb{P}} P$ denotes the boundary of P in $\mathbb{P}$. Second,

$$w \text{ is discontinuous at every point of } \partial_{\mathbb{P}} P \cap (B_r^d \times B_r^{n-d}). \tag{6.11}$$

(Recall that w is identified with its precise representative and that the complement of the set of Lebesgue points has dimension less than $n-p-\varepsilon$ for every $\varepsilon > 0$ and in particular is a $\mathcal{H}^{n-p+1}$ null set).

To prove (6.10), we first note that the definition of B_0 implies directly that

$$(\{y\} \times B_r^{n-d}) \cap P \text{ is nonempty} \qquad \text{for } y \in B_0. \tag{6.12}$$

Also, the definitions imply that

$$B_r^d \times \{0\} \subset P_0. \tag{6.13}$$

This is verified by noting that $P_0 \cap (B_r^d \times B_r^{n-d})$ is nonempty, since $x_0 = (0, 0) \in P_0$, and that in addition P_0 is a connected, relatively open subset of $\mathbb{P}_0 \cap \Omega_i$. Since $(B_r^d \times \mathbb{R}_r^{n-d}) \subset \Omega_i$, it follows that P_0 contains $\mathbb{P}_0 \cap (B_r^d \times B_r^{n-d})$, which implies (6.13).

From (6.13) and (6.8), we see that $(y, 0) \notin P$ and hence that

$$(\{y\} \times B_r^{n-d}) \cap (\mathbb{P} \setminus P) \text{ is nonempty}. \tag{6.14}$$

Since P is a connected, relatively open subset of $\mathbb{P}$, the claim (6.10) follows from (6.12) and (6.14).

To prove (6.11), fix $z \in \partial_{\mathbb{P}} P \cap (B_r^d \times B_r^{n-d})$, and note that $z \in \Omega_i \setminus \Omega_j$, since P is by definition a connected component of $\mathbb{P} \cap \Omega_j$, and Ω_j is open. Thus, $z \in F_m$ for some $j < m \le i$, and so there exists an $n-m$-plane P_1 in Ω_m containing z, and on which $w = \overline{w}(z)$ $\mathcal{H}^{n-m}$ a.e.. So every ball around z contains points at which $w = \overline{w}(z)$. Similarly, (6.9) implies that every ball around z contains points at which $w = \xi \neq \overline{w}(z)$. Therefore, (6.11) follows, completing Step 1b.

We now establish (6.6). Since w is p-quasicontinuous, for any $\varepsilon > 0$, there exists a set S such that the restriction of w to $\Omega \setminus S$ is continuous, and $\mathrm{Cap}_p(S) < \varepsilon$. By Step 1b, the orthogonal projection of S onto $\mathbb{R}^d \times \{0\}$ must contain the open ball B_0. Note that p-capacity is not increased by orthogonal projection, e.g. by [27, Theorem 3] (see also [1, Chapter 5] for further discussion of this type of results). Therefore, it follows that $\mathrm{Cap}_p(B_0) < \varepsilon$ for every $\varepsilon > 0$ and hence that $\mathrm{Cap}_p(B_0) = 0$. This, however, is false, as a set with zero p-capacity has H^s measure 0 for every $s > n-p$, and the dimension d of B_0 satisfies $d \ge n-2k+1 > n-p$. So we have proved (6.6).

Step 2. We now use (6.6) to prove the continuity of $\overline{w}$ on $F_{\ge j}$.

Clearly, $F_{\ge j}$ is partitioned as $F_{>j} \cup F_j$. Since Ω_j is open and $F_j = \Omega_j \cap F_{\ge j}$, we see that F_j is relatively open and $F_{>j}$ relatively closed in $F_{\ge j}$. Thus, in view of the induction hypothesis and Lemma 6.6, it suffices to check that if $x_0 \in F_{>j}$ and (x_m) is a sequence in F_j converging to x_0, then $\overline{w}(x_m) \to \overline{w}(x_0)$.

Thus, we fix some $x_0 \in F_i$ for some $i > j$, and we assume towards a contradiction that there is a sequence (x_m) in F_j such that

$$x_m \to x_0, \qquad |\overline{w}(x_m) - \overline{w}(x_0)| \ge c_0 > 0 \quad \text{for all } m.$$

The definition of $\overline{w}$ implies that there exists an $n-i$-plane P in Ω_i such that $x_0 \in P \subset \Omega_i$, $\overline{w} = \overline{w}(x_0)$ everywhere on P, and $w = \overline{w}(x_0)$ almost everywhere on P. It further implies that for each x_m, there exists a $n-j$-plane P_m in Ω_j such that $x_m \in P_m$, and on which $\overline{w} = \xi_m := \overline{w}(x_m)$ everywhere, and $w = \xi_m$ almost everywhere.

For each m, we write $\mathbb{P}_m$ to denote the $n-j$-plane such that P_m is a connected component of $\Omega_j \cap \mathbb{P}_m$. We now consider two cases.

Case 1. There exists some $\delta > 0$ and a subsequence (m_q) such that $\mathbb{P}_{m_q} \cap B_\delta(x_0) \subset \Omega_j$ for every q.

If this holds, then it follows from Lemma 5.4, with Ω replaced by $B_\delta(x_0)$, that there exists some $n-j$-plane in $B_\delta(x_0)$ that contains x_0, and on which $w = \lim \xi_{m_q}$ a.e.. This, however, would imply that $\overline{w}(x_0) = \lim \xi_{m_q}$, which is impossible.

Case 2. Next we suppose that Case 1 does not hold.

Then, for every q there is some m_q such that

$$\mathbb{P}_{m_q} \cap B_{1/q}(x_0) \not\subset \Omega_j.$$

For q large enough that $B_{1/q}(x_0) \subset \Omega = \Omega_j \cup F_{>j}$, it must then be the case that $\bar{P}_{m_q} \cap F_{>j} \cap B_{1/q}(x_0) \neq \emptyset$. Let $y_{m_q} \in \bar{P}_{m_q} \cap F_{>j} \cap B_{1/q}(x_0)$.

By Step 1, we know that $\overline{w}(y_{m_q}) = \overline{w}(x_{m_q})$.

Also, by construction, $y_{m_q} \to x_0$ as $q \to \infty$. Then, since $y_{m_q} \in F_{>k}$ for every q, it follows from the induction hypothesis that $\overline{w}(x_0) = \lim_{q\to\infty} \overline{w}(y_{m_q}) = \lim_{q\to\infty} \overline{w}(x_{m_q})$, which is impossible in view of the choice of the sequence (x_m). Hence, $\overline{w}$ is continuous as claimed. $\square$

Acknowledgements The first author was partially supported by the National Science and Engineering Research Council of Canada under operating Grant 261955. The work performed on the project by the second author was partially supported by the NSF Grant DMS-1210258.

References

1. Adams, D.R., Hedberg, L.I.: Function Spaces and Potential Theory, Grundlehren der mathematischen Wissenschaften (Book 314). Springer, Berlin (1999)
2. Borisov, Y.F.: Irregular surfaces of the class $C^{1,\beta}$ with an analytic metric. (Russian) Sibirsk. Mat. Zh. **45**(1), 25–61 (2004); translation in Siberian Math. J. **45**(1), 19–52 (2004)
3. Borisov, Yu F.: The parallel translation on a smooth surface. III. Vestnik Leningrad. Univ. **14**(1), 34–50 (1959)
4. Cartan, E.: Bull. Soc. Math. France. **46**, 125, (1919); **48**, 132, (1920)
5. Chern, S.S., Lashof, R.K.: On the total curvature of immersed manifolds. Am. J. Math. **79**(2), 306–318 (1957)
6. Conti, S., De Lellis, C., Székelyhidi Jr. L.: h-principle and rigidity for $C^{1,\alpha}$ isometric embeddings. To appear in the Proceedings of the Abel Symposium (2010)
7. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics. CRC Press, Boca Raton (1992)
8. Federer, H.: Geometric Measure Theory. Springer, Berlin (1969)
9. Fonseca, I., Malý, J.: From Jacobian to Hessian: distributional form and relaxation. Riv. Mat. Univ. Parma **7**(4*), 45–74 (2005)
10. Friesecke, G., James, R., Müller, S.: A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity. Commun. Pure. Appl. Math. **55**, 1461–1506 (2002)
11. Fu, J.H.G.: Monge-Ampère functions I. Indiana Univ. Math. J. **38**(3), 745–771 (1989)
12. Giaquinta, M., Modica, G., Soucek, J.: Cartesian Currents in the Calculus of Variations I. Springer, New York (1998)
13. Hartman, P., Nirenberg, L.: On spherical image maps whose Jacobians do not change sign. Am. J. Math. **81**, 901–920 (1959)
14. Hornung, P.: Approximating $W^{2,2}$ isometric immersions. C. R. Math. Acad. Sci. Paris **346**(3–4), 189–192 (2008)
15. Hornung, P.: Fine level set structure of flat isometric immersions. Arch. Ration. Mech. Anal. **199**, 943–1014 (2011)
16. Hornung, P.: Approximation of flat $W^{2,2}$ isometric immersions by smooth ones. Arch. Ration.l Mech. Anal. **199**, 1015–1067 (2011)
17. Jerrard, R.L.: Some remarks on Monge-Ampère functions, Singularities in PDE and the calculus of variations. CRM Proceedings and Lecture Notes **44**, 89–112 (2008)
18. Jerrard, R.L.: Some rigidity results related to Monge-Ampère functions. Can. J. Math. **62**(2), 320–354 (2010)
19. Kirchheim, B.: Geometry and Rigidity of Microstructures. Habilitation Thesis, Leipzig, Zbl pre01794210 (2001)
20. Kirchhoff, G.: Über das gleichgewicht und die bewegung einer elastischen scheibe. J. Reine Angew. Math. **40**, 51–88 (1850)
21. Kuiper, N.H.: On C^1-isometric imbeddings. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **58**(545–556), 683–689 (1955)
22. Lewicka, M., Pakzad, M.R.: Convex integration for the Monge-Ampère equation in two dimensions. http://arxiv.org/abs/1508.01362
23. Lewicka, M., Pakzad, M.: Scaling laws for non-Euclidean plates and the $W^{2,2}$isometric immersions of Riemannian metrics. ESAIM Control Optim. Calculus Var. **17**, 1158 (2010)
24. Liu, Z., Malý, J.: A strictly convex Sobolev function with null Hessian minors, Preprint (2015)
25. Liu, Z., Pakzad, M.R.: Rigidity and regularity of co-dimension one Sobolev isometric immersions, To appear in Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5). doi:10.2422/2036-2145.201302_001. http://arxiv.org/pdf/1302.0075v2 (2014)
26. Malý, J., Martio, O.: Lusin's condition (N) and mappings of the class $W^{1,n}$. J. Reine Angew. Math. **458**, 19–36 (1995)
27. Meyers, N.G.: Continuity of Bessel potentials. Israel J. Math. **11**(3), 271–283 (1972)
28. Müller, S., Pakzad, M.R.: Regularity properties of isometric immersions. Math. Z. **251**(2), 313–331 (2005)
29. Nash, J.: C^1 isometric imbeddings. Ann. Math. **60**, 383–396 (1954)

30. Pakzad, M.R.: On the Sobolev space of isometric immersions. J. Differ. Geom. **66**(1), 47–69 (2004)
31. Pogorelov, A.V.: Extrinsic geometry of convex surfaces, Translation of mathematical monographs vol. 35. American Math. Soc., (1973)
32. Pogorelov, A.V.: Surfaces with bounded extrinsic curvature (Russian), Kharhov (1956)
33. Solomon, B.: New proof of the Closure Theorem for integral currents. Indiana Univ. Math. J. **33**(3), 393–418 (1984)
34. Šverák, V.: On regularity for the Monge-Ampère equation without convexity assumptions, preprint, Heriot-Watt University (1991)
35. Venkataramani, S.C., Witten, T.A., Kramer, E.M., Geroch, R.P.: Limitations on the smooth confinement of an unstretchable manifold. J. Math. Phys. **41**(7), 5107–5128 (2000)
36. Ziemer, W.P.: Weakly differentiable functions. Sobolev spaces and functions of bounded variation, Graduate Texts in Mathematics, 120. Springer, New York (1989)