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Quaternionic contact hypersurfaces in hyper-Kähler manifolds

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Abstract We describe explicitly all quaternionic contact hypersurfaces (qc-hypersurfaces) in the flat quaternion space $\mathbb{H}^{n+1}$ and the quaternion projective space. We show that up to a quaternionic affine transformation a qc-hypersurface in $\mathbb{H}^{n+1}$ is contained in one of the three qc-hyperquadrics in $\mathbb{H}^{n+1}$. Moreover, we show that an embedded qc-hypersurface in a hyper-Kähler manifold is qc-conformal to a qc-Einstein space and the Riemannian curvature tensor of the ambient hyper-Kähler metric is degenerate along the hypersurface.

Keywords Quaternionic contact · Hypersurfaces · Hyper-Kähler · Quaternionic projective space · 3-Sasaki

Mathematics Subject Classification 53C40 · 53C26

1 Introduction

It is well known that the sphere at infinity of a non-compact symmetric space M of rank one carries a natural Carnot–Carathéodory structure, see [20,22]. Quaternionic contact (abbr. qc) structures were introduced by Biquard [3] modeling the conformal boundary at infinity of the quaternionic hyperbolic space. Biquard showed that the infinite dimensional family of complete quaternionic-Kähler deformations of the quaternion hyperbolic metric [18] have conformal infinities which provide an infinite dimensional family of examples of qc-

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structures. Conversely, according to [3,6] every real analytic qc-structure is the conformal infinity of a unique quaternionic-Kähler metric defined in a neighborhood of M.

The basic concrete examples of qc-manifolds are provided by the extensively studied 3-Sasakian spaces and the quaternionic version of the Heisenberg group. As well known [5], see also [4] for a recent complete account, 3-Sasakian manifolds are characterized as Riemannian manifolds whose cone is a hyper-Kähler manifold. In terms of the Riemannian structure, [5] and [8] show that 3-Sasakian manifolds are extrinsic spheres (totally umbilic hypersurfaces with non-vanishing parallel mean curvature vector) in a hyper-Kähler manifold and this is the only way a 3-Sasakian manifold embeds "naturally" in a hyper-Kähler manifold. The considered embedding is "natural" in the sense that the 3-contact structure induced on the hypersurface coincides with the one inducing the 3-Sasakian structure. Clearly, such an embedding imposes rather stringent Riemannian conditions. Hypersurfaces with induced geometric structures in complex and quaternion space forms have been studied imposing usually assumptions such as: (i) the maximal invariant subspace of the hypersurface invariant under the complex or quaternion structure (called horizontal space in this paper) is invariant space for the shape operator; (ii) the normal Jacobi operator commutes with the shape operator; or (iii) the shape operator is parallel, see for example [1,2,15,16,21,23,24] among many others.

The results in this paper are of different nature since the embeddings considered here are the quaternion analog of those studied in the CR case where the horizontal (holomorphic) geometry plays a fundamental role, replaced here by the quaternion structure of the qc-manifold. In other words the qc geometry imposes no other restrictions on the maximal quaternion invariant distribution besides some positivity which is the quaternion counterpart of a strictly pseudo-convex CR structure. The "sub-Riemannian" nature of our problem requires a rather intricate analysis.

A quaternionic contact hypersurface of a quaternionic manifold $(N, \mathcal{Q})$ was defined by Duchemin [7] as a hypersurface M endowed with a qc-structure compatible with the induced quaternion structure on the maximal quaternion invariant subspace H of the tangent space of M. It was shown in [7, Theorem 1.1] that a qc-manifold can be realized as a qc-hypersurface of an abstract quaternionic manifold. In this paper we investigate qc-hypersurfaces embedded in a hyper-Kähler manifold and, in particular, qc-hypersurfaces of the flat quaternion space $\mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$.

A hypersurface of a hyper-Kähler manifold inherits a quaternionic contact structure from the ambient hyper-Kähler structure if the second fundamental form restricted to H is Sp(1)-invariant and definite quadratic tensor, [7,14]. Considering $\mathbb{H}^{n+1}$ as a flat hyper-Kähler manifold, a natural question is the embedding problem for an abstract qc-manifold.

Our first main result describes the embedded in $\mathbb{H}^{n+1}$ qc-hypersurfaces.

Theorem 1.1 *If M is a connected qc-hypersurface of $\mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$ then, up to a quaternionic affine transformation of $\mathbb{H}^{n+1}$, M is contained in one of the following three hyperquadrics:*

$$\text{(i)}\ \ |q_1|^2 + \cdots + |q_n|^2 + |p|^2 = 1, \qquad \text{(ii)}\ \ |q_1|^2 + \cdots + |q_n|^2 - |p|^2 = -1,$$
$$\text{(iii)}\ \ |q_1|^2 + \cdots + |q_n|^2 + \mathbb{R}e(p) = 0.$$

Here $(q_1, q_2, \ldots q_n, p)$ denote the standard quaternionic coordinates of $\mathbb{H}^{n+1}$.

In particular, if M is a compact qc-hypersurface of $\mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$ then, up to a quaternionic affine transformation of $\mathbb{H}^{n+1}$, M is the standard 3-Sasakian sphere.

The second main result of the paper concerns qc embeddings in a hyper-Kähler manifold, which also imposes a restriction on the qc-structure. Recall that a conformal change of the *horizontal (sub-Riemannian)* metric is called a qc-conformal transformations. We show

Theorem 1.2 *If M is a qc-manifold embedded as a hypersurface in a hyper-Kähler manifold, then M is qc-conformal to a qc-Einstein structure.*

In other words, the qc-conformal class of M contains a qc-Einstein structure, i.e., a qc-structure for which the horizontal Ricci tensor of the associated Biquard connection is proportional to the metric on the horizontal distribution. Another geometric way of understanding qc-Einstein structures was provided in [10,11,13,14] where it was shown that a qc-Einstein manifold M is of constant qc-scalar curvature and in the non-vanishing case M is locally qc-homothetic to a 3-Sasakian or negative 3-Sasakian space, i.e., the Riemannian cone over M is hyper-Kähler of signature $(4n+4, 0)$ or $(4n, 4)$, depending on the sign of the qc-scalar curvature.

We obtain our second main result in the course of the proof of a stronger result, cf. Theorem 3.1 and Lemma 3.7

We also find necessary conditions for the existence of a qc-hypersurface in a hyper-Kähler manifold, namely, the Riemannian curvature R of the ambient space has to be degenerate along the normal to the qc-hypersurface vector field, see Theorem 3.10. From this point of view the "richest" ambient space is the flat space $\mathbb{H}^{n+1} \cong \mathbb{R}^{4n+1}$ in which case Theorem 1.1 provides a complete description.

Our approach to the considered problems is partially motivated by [19, Corollary B] who showed that a non-degenerate CR manifold embedded as a hypersurface in $\mathbb{C}^{n+1}$, $n \geq 2$, admits a pseudo-Einstein structure, i.e., there is a contact form for which the pseudo-hermitian Ricci tensor of the Tanaka–Webster connection is proportional to the Levi form. A key insight of [19, Theorem 4.2] is that a contact form θ defines a pseudo-Hermitian structure which is pseudo-Einstein iff locally there exists a closed section of the canonical bundle with respect to which θ is volume-normalized. In the considered here quaternionic setting, we show the existence of a "calibrated" qc-structure which is volume normalizing in a certain sense, see Lemma 3.3 and (3.4).

Convention 1.3 *Throughout the paper, unless explicitly stated otherwise, we will use the following notation.*

a. *The triple* (i, j, k) *denotes any cyclic permutation of* $(1, 2, 3)$.
b. s, t *are any numbers from the set* $\{1, 2, 3\}$, $s, t \in \{1, 2, 3\}$.
c. *For a given decomposition* $TM = V \oplus H$ *we denote by* $[\cdot]_V$ *and* $[\cdot]_H$ *the corresponding projections to* V *and* H.
d. A, B, C, *etc. will denote sections of the tangent bundle of* M, $A, B, C \in TM$.
e. X, Y, Z, U *will denote horizontal vector fields,* $X, Y, Z, U \in H$.

2 Preliminaries

2.1 QC-manifolds

We refer to [3,11,14] for a more detailed exposition of the definitions and properties of qc-structures and the associated Biquard connection. Here, we recall briefly the relevant facts needed for this paper. A quaternionic contact (qc)-manifold is a $4n+3$-dimensional manifold M with a codimension three distribution H equipped with an $Sp(n)Sp(1)$ structure locally defined by an $\mathbb{R}^3$-valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$. Thus, $H = \cap_{s=1}^3 \mathrm{Ker}\ \eta_s$ carries a positive definite symmetric tensor g, called the horizontal metric, and a compatible rank-three bundle $\mathbb{Q}^M$ consisting of endomorphisms of H locally generated by three orthogonal almost complex structures I_s, satisfying the unit quaternion relations: (i) $I_1 I_2 = -I_2 I_1 = I_3$, $I_1 I_2 I_3 = -\mathrm{id}_{|H}$; (ii) $g(I_s\cdot, I_s\cdot) = g(\cdot, \cdot)$; and (iii) the compatibility conditions $2g(I_s X, Y) = \mathrm{d}\eta_s(X, Y)$, $X, Y \in H$ hold true. In particular, a quaternionic contact manifold is orientable.

The transformations preserving a given quaternionic contact structure η, i.e., $\bar{\eta} = \mu\Psi\eta$ for a positive smooth function μ and an $SO(3)$ matrix Ψ with smooth functions as entries are called *quaternionic contact conformal (qc-conformal) transformations*. The qc-conformal curvature tensor W^{qc}, introduced in [9], is the obstruction for a qc-structure to be locally qc-conformal to the standard 3-Sasakian structure on the $(4n+3)$-dimensional sphere [9,11].

It is a noteworthy and well known fact that, unlike the CR geometry, in the qc case the horizontal space determines uniquely the qc-conformal class, see Lemma 5.1. Accordingly, we will denote by $(M, H, \mathbb{Q})$ a qc-conformal structure on the $4n+3$ dimensional manifold M with a fixed horizontal space H equipped with the quaternionic structure $\mathbb{Q} = \mathbb{Q}^M$; this data determines (local) one-forms η_s, $s = 1, 2, 3$, annihilating H up to a local qc-conformal transformation. On the other hand, (M, η) will denote a qc-manifold with a fixed $\mathbb{R}^3$-valued one form, which determines the horizontal space H and the quaternion structure $\mathbb{Q}$ on H uniquely.

As shown in [3] there is a "canonical" connection associated to every qc-manifold of dimension at least eleven. In the seven dimensional case the existence of such a connection requires the qc-structure to be integrable [6]. The integrability condition is equivalent to the existence of Reeb vector fields [6], which (locally) generate the supplementary to H distribution V. The Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ are determined by [3]

$$\eta_s(\xi_t) = \delta_{st}, \qquad (\xi_s \lrcorner \mathrm{d}\eta_s)_{|H} = 0, \quad (\xi_s \lrcorner \mathrm{d}\eta_t)_{|H} = -(\xi_t \lrcorner \mathrm{d}\eta_s)_{|H}, \tag{2.1}$$

where $\lrcorner$ denotes the interior multiplication. Henceforth, by a qc-structure in dimension 7, we shall mean a qc-structure satisfying (2.1) and refer to the "canonical" connection as *the Biquard connection*. The Biquard connection is the unique linear connection preserving the decomposition $TM = H \oplus V$ and the $Sp(n)Sp(1)$ structure on H with torsion T determined by $T(X, Y) = -[X, Y]_{|V}$ while the endomorphisms $T(\xi_s, \cdot) : H \to H$ belong to the orthogonal complement $(sp(n) + sp(1))^{\perp} \subset GL(4n, R)$.

The covariant derivatives with respect to the Biquard connection of the endomorphisms I_s and the Reeb vector fields are given by

$$\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \qquad \nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j.$$

The $\mathfrak{sp}(1)$-connection 1-forms $\alpha_1, \alpha_2, \alpha_3$, defined by the above equations satisfy [3]

$$\alpha_i(X) = \mathrm{d}\eta_k(\xi_j, X) = -\mathrm{d}\eta_j(\xi_k, X), \qquad X \in H.$$

Let $R = [\nabla, \nabla] - \nabla_{[\cdot,\cdot]}$ be the curvature tensor of ∇ and $R(A, B, C, D) = g(R_{A,B}C, D)$ be the corresponding curvature tensor of type (0,4). The qc-Ricci tensor Ric and the normalized qc-scalar curvature S are defined by

$$Ric(A, B) = \sum_{a=1}^{4n} R(e_a, A, B, e_a) \qquad 8n(n+2)S = Scal = \sum_{a=1}^{4n} Ric(e_a, e_a),$$

where $e_1, \ldots, \mathbf{e}_{4n}$ is a g-orthonormal frame of H.

We say that (M, η) is a qc-Einstein manifold if the restriction of the qc-Ricci tensor to the horizontal space H is trace-free, i.e.,

$$Ric(X, Y) = \frac{Scal}{4n} g(X, Y) = 2(n+2)Sg(X, Y), \quad X, Y \in H.$$

The qc-Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection, $T(\xi_s, X) = 0$ [14]. It is also known [13,14] that the qc-scalar curvature of a qc-Einstein manifold is constant.

The structure equations of a qc-manifold [10, Theorem 1.1] are given by

$$d\eta_i = 2\omega_i - \eta_j \wedge \alpha_k + \eta_k \wedge \alpha_j - S\eta_j \wedge \eta_k, \tag{2.2}$$

where ω_s are the fundamental 2-forms defined by the equations

$$2\omega_{s|H} \;=\; d\eta_{s|H}, \qquad \xi_t \lrcorner \omega_s = 0.$$

By [13, Theorem 5.1], see also [10] and [11, Theorem 4.4.4] for alternative proofs in the case $Scal \neq 0$, a qc-Einstein structure is characterised by either of the following equivalent conditions:

i) locally, the given qc-structure is defined by 1-form (η_1, η_2, η_3) such that for some constant S we have

$$d\eta_i = 2\omega_i + S\eta_j \wedge \eta_k; \tag{2.3}$$

ii) locally, the given qc-structure is defined by a 1-form (η_1, η_2, η_3) such that the corresponding connection 1-forms vanish on H and (cf. the proof of Lemma 4.18 of [14])

$$\alpha_s = -S\eta_s. \tag{2.4}$$

2.2 QC-hypersurfaces

Let $(K, \mathcal{Q})$ be a quaternionic manifold with quaternionic bundle $\mathcal{Q}$. Thus, $\mathcal{Q}$ is a 3-dimensional subbundle of the endomorphism bundle $End(TK)$ that is locally generated by a pointwise quaternionic structure J_1, J_2, J_3, such that there exists a torsion free connection $\nabla^{\mathcal{Q}}$ on TK with $\nabla^{\mathcal{Q}}_A \mathcal{Q} \subset \mathcal{Q}$ for all tangent vectors $A \in TK$.

Let M be a hypersurface of K and H be the maximal $\mathcal{Q}$-invariant subspace of TM. M is a qc-hypersurface if it is a qc manifold with respect to the induced quaternionic structure on the *horizontal space* H. Formally, we rely on the following definition [7, Proposition 2.1] which uses the notation introduced at the beginning of Sect. 2.1.

Definition 2.1 Let $(M, H, \mathcal{Q}^M)$ be a qc-manifold, and $\iota : M \to K$ an embedding. We say that M is a qc-embedded hypersurface of K if $\iota_*(H)$ is a codimension four subbundle of TK and the map ι_* intertwines $\mathcal{Q}^M$ and $\mathcal{Q}$.

In order to simplify the notation, we will frequently identify the corresponding points and tensor fields on M with their images through the map ι in K. In particular, in the embedded case, we will use $\mathbb{Q}^M = \mathcal{Q}$ for the quaternion structure on H. We note that the above definition determines the conformal class of the given qc-structure rather than a particular qc-structure inside this conformal class, cf. Lemma 5.1. An equivalent characterization of a qc-hypersurface M is that the restriction of the second fundamental form of M to the horizontal space is a definite symmetric form, which is invariant with respect to the quaternion structure, see [7, Proposition 2.1]. After choosing the unit normal vector N to M appropriately we can *and will* assume that the second fundamental form of M is negative definite on the horizontal space.

Remark 2.2 For practical purposes, it is useful to keep in mind the description through a locally defining function ρ with a non-vanishing differential $\mathrm{d}\rho$ for which $M = \rho^{-1}(0)$. By [7, Proposition 2.1], M is a qc-hypersurface iff pointwise $\nabla^{\mathbb{Q}}\mathrm{d}\rho(X, Y)$ is a $\mathbb{Q}$-invariant positive or negative definite quadratic form on the maximal $\mathbb{Q}$-invariant subspace H of TM.

For the rest of this section we shall assume K is a hyper-Kähler manifold with hyper-complex structure (J_1, J_2, J_3), quaternionic bundle $\mathbb{Q}$, and hyper-Kähler metric G. In particular, the Levi–Civita connection D will be used as the torsion free connection on K preserving the quaternion bundle of $\mathbb{Q}$. We note that the qc-structure on the hypersurface M is generated by globally defined 1-forms $\hat{\eta}_s$ determined by the unit normal N to M as follows. With $|.|$ denoting the length of a tensor determined by the metric G, consider

$$\hat{\eta}_s(A) = G(J_s N, A) = \frac{1}{|\mathrm{d}\rho|} J_s \mathrm{d}\rho(A), \ A \in TM, \tag{2.5}$$

so that $H = \cap_{s=1}^{3}\mathrm{Ker}\ \hat{\eta}_s$. Let $II(A, B)$ be the second fundamental form of M, $II(A, B) = -G(D_A N, B)$. Since the complex structures J_s are parallel with respect to the Levi–Civita connection D, it follows

$$\begin{aligned} \mathrm{d}\hat{\eta}_s(A, B) &= (D_A\hat{\eta}_s)(B) - (D_B\hat{\eta}_s)(A) = G(J_s(D_A N), B) - G(J_s(D_B N), A) \\ &= II(A, [J_s B]_{TM}) - II(B, [J_s A]_{TM}), \qquad A, B \in TM. \end{aligned} \tag{2.6}$$

Defining $\hat{g}(X, Y) = -II(X, Y),\ X, Y \in H$, (2.6) yields $\mathrm{d}\hat{\eta}_s(X, Y) = 2g(I_s X, Y)$, which defines a qc-structure $(M, \hat{\eta}_s, I_s, \hat{g})$ in the qc-conformal class determined by the qc-embedding.

The associated Reeb vector fields $\hat{\xi}_s$, fundamental 2-forms $\hat{\omega}_s$, and $\mathfrak{sp}(1)$-connection 1-forms $\hat{\alpha}_s$ are determined easily as follows. For $\hat{r}_s = \hat{\xi}_s - J_s N$, since $\hat{\eta}_t(\hat{r}_s) = 0$ we have $\hat{r}_s \in H$. Using the equation $\mathrm{d}\hat{\eta}_s(\hat{\xi}_s, X) = 0,\ X \in H$ and (2.6) we obtain

$$2II(\hat{r}_i, X) = -II(J_i N, X).$$

In addition, we have

$$\begin{aligned} \hat{\alpha}_i(X) &= \mathrm{d}\hat{\eta}_k(\hat{r}_j, X) + \mathrm{d}\hat{\eta}_k(J_j N, X) = 2II(\hat{r}_j, I_k X) + \mathrm{d}\hat{\eta}_k(J_j N, X) \\ &= 2II(\hat{r}_j, I_k X) + II(J_j N, I_k X) + II(X, J_i N) \\ &= -II(J_j N, I_k X) + II(J_j N, I_k X) + II(X, J_i N) = II(J_i N, X). \end{aligned}$$

Notice that, unless the three 1-forms $II(J_s N, \cdot)$ vanish on H, the qc-structure $(\hat{\eta}_s, I_s, \hat{g})$ does not satisfy the structure equations $\mathrm{d}\hat{\eta}_i = 2\hat{\omega}_i + \hat{S}\hat{\eta}_j \wedge \hat{\eta}_k$, (cf. formula 2.2), and the vector fields $J_s N$ differ from the Reeb vector fields $\hat{\xi}_s$.

3 QC-hypersurfaces of hyper-Kähler manifolds

Let M be a qc-hypersurface of the hyper-Kähler manifold K as in Sect. 2.2. Summarizing the notation from Sect. 2.2 we have that the defining tensors of the embedded qc-structure on M are given by

$$\begin{aligned} &\hat{\eta}_s(A) = G(J_s N, A), \quad \hat{\xi}_s = J_s N + \hat{r}_s, \quad \hat{\omega}_s(X, Y) = -II(I_s X, Y), \\ &\hat{g}(X, Y) = -\hat{\omega}_s(I_s X, Y). \end{aligned} \tag{3.1}$$

Notice that Theorem 1.2 claims that the qc-conformal class of any embedded qc-hypersurface in a hyper-Kähler manifold contains a qc-Einstein structure. In turn, this follows from the following stronger result.

Theorem 3.1 *Let $\iota : M \to K$ be an oriented qc-hypersurface of a hyper-Kähler manifold K with parallel quaternion structures J_s, $s \in \{1, 2, 3\}$, and hyper-Kähler metric G. There exists a unique up to a multiplicative constant symmetric J_s-invariant bilinear form $\mathfrak{W}$ on the pull-back bundle $TK|_M \overset{def}{=} \iota^*(TK) \to M$ such that $\mathfrak{W}$ is parallel with respect to the pull-back of the Levi–Civita connection and whose restriction to TM is proportional to the second fundamental form of M. Furthermore, the restriction of $\mathfrak{W}$ to H is the horizontal metric of a qc-Einstein structure in the qc-conformal class defined by the (second fundamental form of the) qc-embedding.*

We note that the existence is the main difficulty in the above result, since the uniqueness up to a multiplicative constant is trivial. Indeed, if $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are two such forms, then from $\mathfrak{W}_1|_{TM} = e^{2\phi}\mathfrak{W}_2|_{TM}$ for some function ϕ on M, the J_s-invariance implies the same relation on $TK|_M$. Therefore, $\mathrm{d}\phi(A) = 0$ for any $A \in TM$ since the bilinear forms are parallel.

Before we turn to the proof of Theorem 3.1, we give an example of the above construction and Theorem 3.1 by considering the standard embedding of the quaternionic Heisenberg group in the $n + 1$-dimensional quaternion space.

Example 3.2 An embedding of the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$, see [14, Sect. 5.2].

Let us identify $\boldsymbol{G}\,(\mathbb{H})$ with the boundary Σ of a Siegel domain in $\mathbb{H}^n \times \mathbb{H}$, $\Sigma = \{(q', p') \in \mathbb{H}^n \times \mathbb{H} : \Re\, p' = -|q'|^2\}$, by using the map $\iota\left((q', \omega')\right) = (q', -|q'|^2 + \omega') = (q, p) \in \mathbb{H}^n \times \mathbb{H}$, where $p = t + \omega = t + ix + jy + kz \in \mathbb{H}$, $q = (q_1, \ldots, q_n) \in \mathbb{H}^n$, and $q_\alpha = t_\alpha + ix_\alpha + jy_\alpha + kz_\alpha \in \mathbb{H}$, $\alpha = 1, \ldots, n$. The "standard" contact form on $\boldsymbol{G}\,(\mathbb{H})$, written as a purely imaginary quaternion valued form, is given by

$$\begin{aligned}\tilde{\Theta} &= \frac{1}{2}\left(-\mathrm{d}\omega + \mathrm{d}\bar{q}\cdot q - q\cdot \mathrm{d}\bar{q}\right) = i\left(-\frac{1}{2}\mathrm{d}x - t_\alpha \mathrm{d}x_\alpha + x_\alpha \mathrm{d}t_\alpha + y_\alpha \mathrm{d}z_\alpha - z_\alpha \mathrm{d}y_\alpha\right)\\ &\quad + j\left(-\frac{1}{2}\mathrm{d}y - t_\alpha \mathrm{d}y_\alpha - x_\alpha \mathrm{d}z_\alpha + y_\alpha \mathrm{d}t_\alpha + z_\alpha \mathrm{d}x_\alpha\right)\\ &\quad + k\left(-\frac{1}{2}\mathrm{d}z - t_\alpha \mathrm{d}z_\alpha + x_\alpha \mathrm{d}y_\alpha - y_\alpha \mathrm{d}x_\alpha + z_\alpha \mathrm{d}t_\alpha\right),\end{aligned} \tag{3.2}$$

where $\cdot$ denotes the quaternion multiplication. We note that the complex structures $J_1,\ J_2,\ J_3$ on $\mathbb{R}^{4n+4}$ are, respectively, the multiplication on the *right* by $-i,\ -j,\ -k$ in $\mathbb{H}^{n+1}$, hence

$$\begin{aligned} &J_1\mathrm{d}t_\alpha = -\mathrm{d}x_\alpha, \quad J_1\mathrm{d}y_\alpha = \mathrm{d}z_\alpha, \quad J_1\mathrm{d}t = -\mathrm{d}x, \quad J_1\mathrm{d}y = \mathrm{d}z,\\ &J_2\mathrm{d}t_\alpha = -\mathrm{d}y_\alpha, \quad J_2\mathrm{d}z_\alpha = \mathrm{d}x_\alpha, \quad J_2\mathrm{d}t = -\mathrm{d}y, \quad J_2\mathrm{d}z = \mathrm{d}x.\end{aligned}$$

Clearly, Σ is the 0-level set of $\rho = |q|^2 + t$ and we have

$$\begin{aligned} &J_s\mathrm{d}\rho = \sqrt{1+4|q|^2}\,\hat{\eta}_s, \quad N = \frac{2}{\sqrt{1+4|q|^2}}\left(\frac{1}{2}\partial_t + t_\alpha\partial_{t_\alpha} + x_\alpha\partial_{x_\alpha} + y_\alpha\partial_{y_\alpha} + z_\alpha\partial_{z_\alpha}\right),\\ &\hat{\eta} = i\hat{\eta}_1 + j\hat{\eta}_2 + k\hat{\eta}_3 = \frac{1}{\sqrt{1+4|q|^2}}\left(-\mathrm{d}\omega + \mathrm{d}\bar{q}\cdot q - \bar{q}\cdot \mathrm{d}q\right),\end{aligned}$$

$$II(A,B) = -\frac{1}{|\mathrm{d}\rho|}\, D\mathrm{d}\rho\,(A,B) = -\frac{2}{\sqrt{1+4|q|^2}}\,\langle A_H, B_H\rangle$$
$$= -\frac{2}{\sqrt{1+4|q|^2}}\,(\mathrm{d}t_\alpha \odot \mathrm{d}t_\alpha + \mathrm{d}x_\alpha \odot \mathrm{d}x_\alpha + \mathrm{d}y_\alpha \odot \mathrm{d}y_\alpha + \mathrm{d}z_\alpha \odot \mathrm{d}z_\alpha)\,(A,B),$$

where for a tangent vector A we use $A_H = A - \mathrm{d}t(A)\partial_t - \mathrm{d}x(A)\partial_x - \mathrm{d}y(A)\partial_y - \mathrm{d}z(A)\partial_z$ for the orthogonal projection from $\mathbb{H}^{n+1}$ to the horizontal space, which is given by $H = \operatorname{Ker}\,\mathrm{d}\rho \cap \{\cap_{s=1}^3 \operatorname{Ker}\,\hat\eta_s\}$. From the above formulas we see that $\Theta \overset{def}{=} \iota^*\hat\eta$ is conformal to $\tilde\Theta$. Therefore, the qc-structure $\eta_s = \frac{\sqrt{1+4|q|^2}}{2}\hat\eta_s$, i.e., the standard qc-structure (3.2), has horizontal metric given by the restriction of the bilinear form $\mathfrak{W} = \mathrm{const}\,\Re(dq_\alpha \cdot \mathrm{d}\bar q_\alpha)|_M$, which is parallel along M. This is the symmetric form whose existence is claimed by Theorem 3.1, while the calibrating function is a certain multiple of $\sqrt{1+4|q|^2}$, cf. (3.4).

It is worth noting that the qc-Einstein structures in the qc-conformal class of the standard qc-structure were essentially classified in [14, Theorem 1.1] where it was shown that all qc-Einstein structures of positive qc-scalar curvature globally conformal to the standard qc-structure are obtained from the standard qc-structure on the quaternionic Heisenberg group with a qc-automorphism, see also [12, Theorem 6.2] for the general case.

3.1 Proof of Theorem 3.1

A key point of our analysis is a volume normalization condition, which is based on Lemma 3.3. To this effect we consider a qc-conformal transformation $\eta_s = f\,\hat\eta_s$ where f is a positive smooth function on M. Let ξ_s, ω_s, ∇ and α_s be the Reeb vector fields, the fundamental 2-forms, the Biquard connection and the $\mathfrak{sp}(1)$-connection 1-forms of the qc-structure defined by θ_s. The orthogonal complement $V = \operatorname{span}\{\xi_1,\xi_2,\xi_3\}$ of H and the endomorphism I_1, defined on the horizontal space H, induce a decomposition of the complexified tangent bundle of M (we use the same notation TM for both the tangent bundle and its complexification), $TM = V \oplus H_{I_1}^{1,0} \oplus H_{I_1}^{0,1}$, and consequently of the whole complexified tensor bundle of M. We shall need the type decomposition of the 1- and 2-forms on M,

$$T^*M = H^*_{1,0} \oplus H^*_{0,1} \oplus L^*, \qquad L^* = \operatorname{span}\{\eta_1,\eta_2,\eta_3\},$$
$$\Lambda^2(T^*M) = \Lambda^2(H^*_{1,0}) \oplus \Lambda^2(H^*_{0,1}) \oplus (H^*_{1,0}\otimes H^*_{0,1}) \oplus \Lambda^2(L^*) \oplus (L^*\otimes H^*).$$

In particular, $H^*_{1,0}$ is the $2n$-dimensional space of all complex one-forms which vanish on ξ_1,ξ_2,ξ_3 and are of type $(1,0)$ with respect to I_1 when restricted to H. Similarly, using the endomorphism I_2 or I_3 we obtain corresponding decompositions. We shall write explicitly the analysis with respect to I_1, but keep in mind that the arguments remain true if we cyclicly permute the indices 1, 2 and 3.

Consider the following complex 2-forms on M,

$$\hat\gamma_i = \hat\omega_j + \sqrt{-1}\,\hat\omega_k, \qquad \gamma_i = f\hat\gamma_i = \omega_j + \sqrt{-1}\,\omega_k,$$
$$\Gamma_i(A,B) = G(J_jA,B) + \sqrt{-1}\,G(J_kA,B).$$

We have $\xi_t \lrcorner \gamma_s = 0$ and $\gamma_1, \hat\gamma_1|_H, \Gamma_1|_H \in \Lambda^2(H^*_{1,0})$. Moreover, since K is a hyper-Kähler manifold, the three 2-forms Γ_s are closed, $\mathrm{d}\Gamma_s = 0$. The volume normalization relies on the following algebraic lemma.

Lemma 3.3 *Let $\mathcal{H}^{4n}$ be a real vector space with hyper-complex structure (I_1,I_2,I_3), i.e., $I_1^2 = I_2^2 = I_3^2 = -\mathrm{Id}$, $I_1I_2 = -I_2I_1 = I_3$ and $\hat g$ and g be two positive definite inner* *products on $\mathcal{H}^{4n}$ satisfying $\hat{g}(I_sX, I_sY) = \hat{g}(X, Y)$, and $g(I_sX, I_sY) = g(X, Y)$ for all $X, Y \in \mathcal{H}^{4n}$, $s = 1, 2, 3$. If*

$$\hat{\gamma}_i(X, Y) = \hat{g}(I_jX, Y) + \sqrt{-1}\,\hat{g}(I_kX, Y), \quad \gamma_i(X, Y) = g(I_jX, Y) + \sqrt{-1}\,g(I_kX, Y),$$

then there exists a positive real number μ such that $\underbrace{\hat{\gamma}_s \wedge \cdots \wedge \hat{\gamma}_s}_{n\ times} = \mu\,\underbrace{(\gamma_s \wedge \cdots \wedge \gamma_s)}_{n\ times}$, $s = 1, 2, 3$.

Proof A small calculation shows that both γ_1 and $\hat{\gamma}_1$ are of type (2, 0) with respect to I_1. The complex vector space $\Lambda^{2n}(\mathcal{H}^*_{1,0})$ is one dimensional, and γ_1^n and $\hat{\gamma}_1^n$ are non zero elements of it, hence there exists a non zero complex number μ such that $\gamma_1^n = \mu\,\hat{\gamma}_1^n$. Note that $I_2\gamma_1 = \overline{\gamma_1}$ and the same holds true for $\hat{\gamma}_1$. It follows that

$$(I_2\gamma_1)^n = \overline{\gamma_1^n} \ \text{ i.e., } \ \mu\overline{\hat{\gamma}_1^n} = \bar{\mu}\,\overline{\hat{\gamma}_1^n},$$

thus $\mu = \bar{\mu} \neq 0$. The group $GL(n, \mathbb{H})$ acts transitively on the set of all positive definite inner products g of $\mathcal{H}$, compatible with the hyper-complex structure, and hence also on the set of all corresponding 2-forms γ_1. The group $GL(n, \mathbb{H})$ is connected, therefore each orbit is connected as well, which implies $\mu > 0$. It remains to show that the constant μ in the equation $\hat{\gamma}_s^n = \mu\,\gamma_s^n$ is independent of s. For this we use that the $4n$-form $\gamma_1^n \wedge \overline{\gamma_1^n}$ equals the volume form of the metric g and hence it is independent of s. This implies that μ^2 does not depend on s, and therefore the same is true for μ. □

From Lemma 3.3 applied to the metrics $\hat{g}$ and $G|_H$ on H it follows that there exists a positive function μ on M such that $\Gamma_s^n|_H = \mu\hat{\gamma}_s^n|_H$, $s = 1, 2, 3$ i.e.,

$$\Gamma_s^n \equiv \mu\hat{\gamma}_s^n \mod \{\eta_1, \eta_2, \eta_3\}. \tag{3.3}$$

At this point we define the "*calibrated*" qc-structure using the function f defined by

$$f = \mu^{\frac{1}{n+2}}. \tag{3.4}$$

The reminder of this section is devoted to showing that with this choice of f the qc-structure determined by η_s satisfies all the requirements of the theorem.

We start by proving in Lemma 3.5 a few important preliminary technical facts. Let us define the following three vector fields r_s

$$r_s = \xi_s - \frac{1}{f}\,J_sN. \tag{3.5}$$

Since $\eta_t(r_s) = \delta_{ts} - \hat{\eta}_t(J_sN) = 0$, it follows that r_s are horizontal vector field, $r_s \in H$. We will denote by r_s also the corresponding 1-forms, defined by $r_s(A) = G(r_s, A)$, $A \in TM$.

Remark 3.4 Note that in general expressions of the type $\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \delta$, with δ being differential form on M, depend only on the restriction of δ to H. This fact will be used repeatedly hereafter.

Lemma 3.5 *We have*

$$\eta_2 \wedge \Gamma_1^{n+1} = (n+1)\,\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \gamma_1^n \tag{3.6}$$

$$\Gamma_1^{n+1} = \sqrt{-1}(n+1)\eta_1 \wedge \left(\eta_2 + \sqrt{-1}\eta_3\right) \wedge \gamma_1^n$$
$$+n(n+1)f^{-2}\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \left(-J_3r_3 + \sqrt{-1}J_2r_3 + J_2r_2 + \sqrt{-1}J_3r_2\right) \wedge \Gamma_1^{n-1}. \tag{3.7}$$

Furthermore, the above equations hold after any cyclic permutation of the indices 1, 2 and 3.

Proof Let us define $\Gamma_1^{'}$ and $\Gamma_1^{''}$ to be 2-forms on M which coincide with the 2-form Γ_1 when restricted to the distribution H and satisfy the additional conditions $\xi_s \lrcorner \Gamma_1^{'} = 0$, $(J_s N) \lrcorner \Gamma_1^{''} = 0$. In order to find the relation between Γ_1 and $\Gamma_1^{'}$, we compute

$$\begin{aligned}\Gamma_1'(A, B) &= \Gamma_1(A - \eta_s(A)\xi_s, B - \eta_t(B)\xi_t)\\ &= \Gamma_1(A, B) - \eta_s(B)\Gamma_1(A, \xi_s) - \eta_s(A)\Gamma_1(\xi_s, B) + \Gamma_1(\xi_s, \xi_t)\eta_s(A)\eta_t(B)\\ &= \Gamma_1(A, B) - \eta_t \wedge (\xi_t \lrcorner \Gamma_1)(A, B) + \frac{1}{2}\Gamma_1(\xi_s, \xi_t)\eta_s \wedge \eta_t(A, B).\end{aligned}$$

A short calculation gives

$$\begin{aligned}(\xi_t \lrcorner \Gamma_1)(A) &= G(J_2\xi_t, A) + \sqrt{-1}G(J_3\xi_t, A)\\ &= G\left(J_2\left(r_t + \frac{1}{f}J_t N\right), A\right) + \sqrt{-1} + G\left(J_3\left(r_t + \frac{1}{f}J_t N\right), A\right)\\ &= (J_2 r_t + J_3 r_t)(A) \quad \text{mod } \{\eta_1, \eta_2, \eta_3\},\end{aligned}$$

which shows that for some functions $\Gamma_1^{s,t}$ on M we have

$$\Gamma^{'}{}_1 = \Gamma_1 - \sum_{t=1}^{3} \eta_t \wedge \left(J_2 r_t + \sqrt{-1}\, J_3 r_t\right) + \sum_{s,t=1}^{3} \Gamma_1^{s,t} \eta_s \wedge \eta_t. \tag{3.8}$$

Similarly to the derivation of (3.8) we can find the relation between $\Gamma_1^{''}$ and Γ_1,

$$\Gamma_1^{''} = \Gamma_1 \; - \; f^{-2}\left(\eta_3 \wedge \eta_1 + \sqrt{-1}\,\eta_1 \wedge \eta_2\right),$$

which gives

$$\begin{aligned}\Gamma_1^{n+1} &= \sqrt{-1}\,(n+1) f^{-2}\, \eta_1 \wedge \left(\eta_2 + \sqrt{-1}\,\eta_3\right) \wedge \left(\Gamma_1^{''}\right)^n\\ &= \sqrt{-1}\,(n+1) f^{-2}\, \eta_1 \wedge (\eta_2 + \sqrt{-1}\,\eta_3) \wedge \Gamma_1^n.\end{aligned} \tag{3.9}$$

Clearly, $\Gamma_s^{'} \in \Lambda^2\left(H_{1,0}^*\right)$ and $\left(\Gamma_s^{'}\right)^{n+1} = \left(\Gamma_s^{''}\right)^{n+1} = 0$. Noting that (3.3) are equivalent to the equations

$$\left(\Gamma_s^{'}\right)^n = f^2 \gamma_s^n$$

we obtain from (3.8) the identity

$$\begin{aligned}\Gamma_1^n &= \left(\Gamma_1^{'}\right)^n \; + \; n\sum_{s=1}^{3} \eta_s \wedge \left(J_2 r_s + \sqrt{-1}\, J_3 r_s\right) \wedge \left(\Gamma_1^{'}\right)^{n-1} \text{mod}\langle \eta_s \wedge \eta_t\rangle\\ &= \; f^2 \gamma_1^n \; + n\sum_{s=1}^{3} \eta_s \wedge \left(J_2 r_s + \sqrt{-1}\, J_3 r_s\right) \wedge \left(\Gamma_1^{'}\right)^{n-1} \text{mod } \langle \eta_s \wedge \eta_t\rangle.\end{aligned} \tag{3.10}$$

Finally, a substitution of (3.10) in (3.9) gives

$$\begin{aligned}\Gamma_1^{n+1} &= \sqrt{-1}(n+1)\eta_1 \wedge \left(\eta_2 + \sqrt{-1}\eta_3\right) \wedge \gamma_1^n\\ &\quad + n(n+1) f^{-2} \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \left(-J_3 r_3 + \sqrt{-1} J_2 r_3 + J_2 r_2 + \sqrt{-1} J_3 r_2\right) \wedge \left(\Gamma_1^{'}\right)^{n-1},\end{aligned}$$

which, in view of the relation $\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \left(\Gamma_1'\right)^{n-1} = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \Gamma_1^{n-1}$, yields (3.7). The Eq. (3.6) follows now by taking the wedge products of both sides of (3.7) with the 1-form η_2. □

Following is a technical lemma which will be used in the proof of Lemma 3.7 below.

Lemma 3.6 *For any $\lambda \in H^*_{1,0}$ (considered with respect to I_1) we have*

$$\lambda \wedge \omega_1 \wedge \gamma_1^{n-1} = \frac{\sqrt{-1}}{2n}(I_2\lambda) \wedge \gamma_1^n.$$

Proof We can take a basis of the cotangent space of M in the form

$$\eta_1, \eta_2, \eta_3, \epsilon_1, \dots, \epsilon_n, I_1\epsilon_1, \dots, I_1\epsilon_n, I_2\epsilon_1, \dots, I_2\epsilon_n, I_3\epsilon_1, \dots, I_3\epsilon_n,$$

where $\xi_s \lrcorner \epsilon_t = 0$, $s = 1, 2, 3$, $t = 1, 2, \dots, n$, which is orthonormal in the sense that the following equations hold

$$\omega_1 = \sum_{s=1}^{n} (\epsilon_s \wedge I_1\epsilon_s + I_2\epsilon_s \wedge I_3\epsilon_s), \quad \omega_2 = \sum_{s=1}^{n} (\epsilon_s \wedge I_2\epsilon_s + I_3\epsilon_s \wedge I_1\epsilon_s),$$
$$\omega_3 = \sum_{s=1}^{n} (\epsilon_s \wedge I_3\epsilon_s + I_1\epsilon_s \wedge I_2\epsilon_s).$$

For $\phi_t = \epsilon_t + \sqrt{-1}I_1\epsilon_t$ and $\psi_t = I_2\epsilon_t + \sqrt{-1}I_3\epsilon_t$ the forms $\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_n$ form a basis of $H^*_{1,0}$. Moreover, we have

$$I_2\phi_s = \bar{\psi}_s, \quad I_2\psi_s = -\bar{\phi}_s, \quad s = 1, \dots, n,$$
$$\omega_1 = \frac{\sqrt{-1}}{2}\sum_{s=1}^{n}\left(\phi_s \wedge \bar{\phi}_s + \psi_s \wedge \bar{\psi}_s\right), \quad \gamma_1 = \sum_{s=1}^{n}\phi_s \wedge \psi_s,$$
$$\gamma_1^n = n!\,\phi_1 \wedge \psi_1 \wedge \dots \wedge \phi_n \wedge \psi_n,$$
$$\gamma_1^{n-1} = (n-1)!\sum_{s=1}^{n}\phi_1 \wedge \psi_1 \wedge \dots \wedge \widehat{\phi_s \wedge \psi_s} \wedge \dots \wedge \phi_n \wedge \psi_n,$$
$$\omega_1 \wedge \gamma_1^{n-1} = \frac{\sqrt{-1}(n-1)!}{2}\sum_{s=1}^{n}\phi_1 \wedge \psi_1 \wedge \dots \wedge (\phi_s \wedge \bar{\phi}_s + \psi_s \wedge \bar{\psi}_s) \wedge \dots \wedge \phi_n \wedge \psi_n.$$

Since $\lambda \in H^*_{1,0}$ there exist constants $a_s, b_s, s = 1, \dots, n$ such that $\lambda = \sum_{s=1}^n (a_s\phi_s + b_s\psi_s)$. It follows that $I_2\lambda = \sum_{s=1}^n (a_s\bar{\psi}_s - b_s\bar{\phi}_s)$. Finally we compute (omitting the sum symbols)

$$\begin{aligned}
&\lambda \wedge \omega_1 \wedge \gamma_1^n \\
&= \frac{\sqrt{-1}(n-1)!}{2}(a_t\phi_t + b_t\psi_t) \wedge \left(\phi_1 \wedge \psi_1 \wedge \dots \wedge \left(\phi_s \wedge \bar{\phi}_s + \psi_s \wedge \bar{\psi}_s\right) \wedge \dots \wedge \phi_n \wedge \psi_n\right) \\
&= \frac{\sqrt{-1}(n-1)!}{2}\left(a_s\bar{\psi}_s - b_s\bar{\phi}_s\right) \wedge \phi_1 \wedge \psi_1 \wedge \dots \wedge \phi_n \wedge \psi_n = \frac{\sqrt{-1}}{2n}(I_2\lambda) \wedge \gamma_1^n.
\end{aligned}$$

□

Lemma 3.7 *The calibrated qc-structure $\eta_s = f\hat{\eta}_s$, where f is given by (3.4), satisfies the structure equations (2.3). In particular, (M, H, η_s) is a qc-Einstein structure. Furthermore, we have*

$$I_1 r_1 = I_2 r_2 = I_3 r_3.$$

Proof Taking the exterior derivative of (3.6) and recalling that Γ_1 is a closed form, we obtain

$$\begin{aligned} 0 = \ & n(n+1)\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \mathrm{d}\gamma_1 \wedge \gamma_1^{n-1} \\ & + \mathrm{d}\eta_2 \wedge \left(\Gamma_1^{n+1} + (n+1)\eta_1 \wedge (\eta_3 - \sqrt{-1}\eta_2) \wedge \gamma_1^n\right) \\ & - (n+1)\left(\mathrm{d}\eta_1 \wedge \eta_2 \wedge (\eta_3 - \sqrt{-1}\eta_2) \ + \ \eta_1 \wedge \eta_2 \wedge d(\eta_3 - \sqrt{-1}\eta_2)\right) \wedge \gamma_1^n. \end{aligned} \tag{3.11}$$

The structure equations (2.2) and the identities $\omega_2 = \frac{1}{2}(\gamma_1 + \bar{\gamma}_1)$, $\omega_3 = \frac{\sqrt{-1}}{2}(\bar{\gamma}_1 - \gamma_1)$ and $\omega_1 \wedge \gamma_1^n = 0$ imply

$$\begin{aligned} & \mathrm{d}\eta_1 \equiv 0 \quad \mathrm{mod}\ \{\eta_2, \eta_3, H^*_{1,0}\}, \\ & \mathrm{d}\eta_2 \equiv \bar{\gamma}_1 \quad \mathrm{mod}\ \{\eta_1, \eta_3, H^*_{1,0}\}, \quad \mathrm{d}\eta_3 \equiv \sqrt{-1}\bar{\gamma}_1 \quad \mathrm{mod}\ \{\eta_1, \eta_2, H^*_{1,0}\}, \\ & d(\eta_3 - \sqrt{-1}\eta_2) \equiv -2\sqrt{-1}\gamma_1 + \sqrt{-1}\eta_3 \wedge \alpha_1 \quad \mathrm{mod}\ \{\eta_1, \eta_2\}, \\ & \mathrm{d}\gamma_1 \equiv -\sqrt{-1}\alpha_1 \wedge \gamma_1 + (-\alpha_3 + \sqrt{-1}\alpha_2) \wedge \omega_1 \quad \mathrm{mod}\ \{\eta_1, \eta_2, \eta_3\}. \end{aligned}$$

From (3.7) and the above identities applied to (3.11) we find

$$\begin{aligned} 0 = \ & \mathrm{d}\eta_2 \wedge \left(n(n+1)f^{-2}\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge (-J_3r_3 + \sqrt{-1}J_2r_3 + J_2r_2 + \sqrt{-1}J_3r_2) \wedge \Gamma_1^{n-1}\right) \\ & - (n+1)\eta_1 \wedge \eta_2 \wedge \sqrt{-1}\eta_3 \wedge \alpha_1 \wedge \gamma_1^n \ - n(n+1)\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \sqrt{-1}\alpha_1\gamma_1^n \\ & + \ n(n+1)\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge (-\alpha_3 + \sqrt{-1}\alpha_2) \wedge \omega_1 \wedge \gamma_1^{n-1} \\ = \ & n(n+1)f^{-2}\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \bar{\gamma}_1 \wedge \Gamma_1^{n-1} \wedge (-J_3r_3 + \sqrt{-1}J_2r_3 + J_2r_2 + \sqrt{-1}J_3r_2) \\ & + \ n(n+1)\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge (-\alpha_3 + \sqrt{-1}\alpha_2) \wedge \omega_1 \wedge \gamma_1^{n-1} \\ & - \sqrt{-1}(n+1)^2\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \gamma_1^n \wedge \alpha_1. \end{aligned}$$

The last expression is a $(2n+4)$-form which belongs to the space (decomposition with respect to I_1)

$$\Lambda^3(L^*) \otimes \Lambda^2(H^*_{0,1}) \otimes \Lambda^{2n-1}(H^*_{1,0}) \ \oplus \ \Lambda^3(L^*) \otimes \Lambda^1(H^*_{0,1}) \otimes \Lambda^{2n}(H^*_{1,0}).$$

Hence, we obtain the next two identities

$$\begin{aligned} & \sqrt{-1}(n+1)^2\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \gamma_1^n \wedge \alpha_1 \\ & \quad = n(n+1)\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \frac{1}{2}\left(-\alpha_3 - \sqrt{-1}I_1\alpha_3 + \sqrt{-1}\alpha_2 - I_1\alpha_2\right) \wedge \omega_1 \wedge \gamma_1^{n-1}. \end{aligned} \tag{3.12}$$

and also

$$\begin{aligned} & -n(n+1)f^{-2}\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \bar{\gamma}_1 \wedge \Gamma_1^{n-1} \wedge (-J_3r_3 + \sqrt{-1}J_2r_3 + J_2r_2 + \sqrt{-1}J_3r_2) \\ & \quad = n(n+1)\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \frac{1}{2}(-\alpha_3 + \sqrt{-1}I_1\alpha_3 + \sqrt{-1}\alpha_2 + I_1\alpha_2) \wedge \omega_1 \wedge \gamma_1^{n-1}. \end{aligned} \tag{3.13}$$

Equation (3.12) yields

$$\begin{aligned} & n(-\alpha_3 - \sqrt{-1}I_1\alpha_3 + \sqrt{-1}\alpha_2 - I_1\alpha_2) \wedge \omega_1 \wedge \gamma_1^{n-1} \\ & \quad \equiv \sqrt{-1}(n+1)\gamma_1^n \wedge (\alpha_1 - \sqrt{-1}I_1\alpha_1) \quad \mathrm{mod}\ \{\eta_1, \eta_2, \eta_3\}. \end{aligned} \tag{3.14}$$

With the help of Lemma 3.6 we can write (3.14) in the form

$$\frac{\sqrt{-1}}{2} I_2 \left(-\alpha_3 - \sqrt{-1} I_1 \alpha_3 + \sqrt{-1}\alpha_2 - I_1\alpha_2\right) \\ \equiv \sqrt{-1}(n+1)\left(\alpha_1 - \sqrt{-1} I_1 \alpha_1\right) \qquad \text{mod } \{\eta_1, \eta_2, \eta_3\}.$$

Taking the real part of the last identity we come to $2(n+1)I_1\alpha_1 + I_2\alpha_2 + I_3\alpha_3 \equiv 0$ mod $\{\eta_1, \eta_2, \eta_3\}$.

A cyclic rotation of the indices 1, 2, 3 in the above arguments gives the following system $mod\{\eta_1, \eta_2, \eta_3\}$

$$2(n+1)I_1\alpha_1 + I_2\alpha_2 + I_3\alpha_3 \equiv 0 \\ I_1\alpha_1 + 2(n+1)I_2\alpha_2 + I_3\alpha_3 \equiv 0 \\ I_1\alpha_1 + I_2\alpha_2 + 2(n+1)I_3\alpha_3 \equiv 0,$$

which has the unique solution $I_1\alpha_1 \equiv I_2\alpha_2 \equiv I_3\alpha_3 \equiv 0 \mod \{\eta_1, \eta_2, \eta_3\}$. Therefore, the calibrated qc-structure has vanishing $\mathfrak{sp}(1)$-connection 1-forms

$$(\alpha_1)|_H = (\alpha_2)|_H = (\alpha_3)|_H = 0, \tag{3.15}$$

hence by (2.4) it is a qc-Einstein structure. From (3.13) (and a cyclic rotation of the indeces) we also conclude that $I_1r_1 = I_2r_2 = I_3r_3$. □

We shall denote by r the common vector defined above by $I_s r_s$ in Lemma 3.7, see also (3.5),

$$r = -I_s r_s \in H, \qquad \text{hence } r_s = I_s r.$$

The calibrated qc-structure constructed in Lemma 3.7 enjoys further useful technical properties recorded below.

Lemma 3.8 *The second fundamental form II of the qc-embedding $M \subset K$ and the calibrating function f defined by (3.4) satisfy the identities:*

i. $II(X, Y) = -f^{-1}g(X, Y)$;
ii. $II(J_sN, J_sX) = -f^{-1}\mathrm{d}f(X) = g(r, X),\ X \in H$;
iii. $II(J_sN, J_tN) = -\delta_{st} f(S/2 + g(r, r))$;
iv. $\mathrm{d}f(J_sN) = \mathrm{d}f(\xi_s) = 0$.

Proof (i) The identity $II(X, Y) = -f^{-1}g(X, Y)$ holds by the definition of g, also recall (3.1).

(ii) Using the fact that the complex structures J_s are D-parallel, the relation $\eta_s = fG(J_sN, .)$ and the formula $\mathrm{d}\eta_s(A, B) = (D_A\eta_s)(B) - (D_B\eta_s)(A)$ we find

$$\mathrm{d}\eta_s(A, B) = f^{-1}\mathrm{d}f \wedge \eta_s(A, B) + f\,II(A, [J_sB]_{TM}) - f\,II(B, [J_sA]_{TM}). \tag{3.16}$$

The above formula implies

$$\mathrm{d}\eta_i(J_jN, J_kX) = -f\,II(J_jN, J_jX) - f\,II(J_kN, J_kX), \\ \mathrm{d}\eta_i(J_iN, X) = -\mathrm{d}f(X) + f\,II(J_iN, J_iX). \tag{3.17}$$

On the other hand, since $\xi_s = \frac{1}{f}J_sN + J_sr$ and $\alpha_{i\,|H} = (\xi_j \lrcorner \mathrm{d}\eta_k)_{|H} = 0$, we have

$$0 = \mathrm{d}\eta_i(\xi_j, J_kX) = f^{-1}\mathrm{d}\eta_i(J_jN, J_kX) + 2g(r, X), \\ 0 = \mathrm{d}\eta_i(\xi_i, X) = f^{-1}\mathrm{d}\eta_i(J_iN, X) - 2g(r, X). \tag{3.18}$$

The first of the above identities together with the first identity in (3.17) imply the equation $II(J_iN, J_iX) = g(r, X)$, which together with the second identity in (3.17) and (3.18) give the identities in (ii).

(iii) and (iv). From (3.16)

we have

$$\begin{aligned} d\eta_i(J_iN, J_jN) &= -\, df(J_jN) + f\,II(J_iN, J_kN),\\ d\eta_i(J_iN, J_kN) &= -\, df(J_kN) - f\,II(J_iN, J_jN),\\ d\eta_i(J_jN, J_kN) &= -\, f\,II(J_jN, J_jN) - f\,II(J_kN, J_kN), \end{aligned} \tag{3.19}$$

which give the wanted identities. From (3.15) and (2.2)–(2.4) we have $d\eta_s(\xi_j, \xi_k) = 2\delta_{si}S$. Therefore, we obtain

$$\begin{aligned} &0 = d\eta_i(\xi_i, \xi_j) = d\eta_i(f^{-1}J_iN + J_ir, f^{-1}J_jN + J_jr) = f^{-2}d\eta_i(J_iN, J_jN)\\ &0 = d\eta_i(\xi_i, \xi_k) = d\eta_i(f^{-1}J_iN + J_ir, f^{-1}J_kN + J_kr) = f^{-2}d\eta_i(J_iN, J_kN)\\ &S = d\eta_i(\xi_j, \xi_k) = f^{-2}d\eta_i(J_jN, J_kN) - 2g(r, r). \end{aligned} \tag{3.20}$$

The first two identities of (3.19) and the first two equations in (3.20) give

$$II(J_iN, J_jN) = -df(J_kN), \qquad II(J_jN, J_iN) = df(J_kN),$$

hence $df(J_kN) = 0$. Finally, recalling (3.5), we compute

$$\begin{aligned} df(\xi_s) = df(r_s + f^{-1}J_sN) = df(I_sr) &= \sum_{a=1}^{4n} df(I_se_a)g(r, e_a)\\ &= -f^{-1}\sum_{a=1}^{4n} df(I_se_a)df(e_a) = 0. \end{aligned}$$

The third identity of (3.19) and the third line of (3.20) imply

$$II(J_iN, J_iN) = -f\,(S/2 + g(r, r))\,,$$

which completes the proof of parts (iii) and (iv) of Lemma 3.8. □

The next lemma gives an explicit formula for the horizontal metric of the calibrated qc-Einstein structure.

Lemma 3.9 *The horizontal metric g of the calibrated by* (3.4) *qc-structure is related to the second fundamental form of the qc-embedding by the formula*

$$g(A_H, B_H) = -f\,II(A, B) - \frac{S}{2}\sum_{s=1}^{3}\eta_s(A)\eta_s(B), \quad A, B \in TM, \tag{3.21}$$

where for $A \in TM$ *we let* $A_H = A - \sum_{s=1}^{3}\eta_s(A)\xi_s$ *be the horizontal part of* A.

Proof A few calculations give the next three identities

$$\begin{aligned}
II(\xi_s, X) &= II(I_s r + f^{-1} J_s N, X) \\
&= II(I_s r, X) - f^{-1} II(J_s N, J_s(J_s X)) \\
&= -f^{-1} g(I_s r, X) - f^{-1} g(r, I_s X) = 0, \\
II(\xi_s, \xi_s) &= II(I_s r + f^{-1} J_s N, I_s r + f^{-1} J_s N) \\
&= II(I_s r, I_s r) + 2f^{-1} II(J_s N, J_s r) + f^{-2} II(J_s N, J_s N) \\
&= -f^{-1} g(r, r) + 2f^{-1} g(r, r) - f^{-1}(S/2 + g(r, r)) \\
&= -f^{-1} S/2, \\
II(\xi_i, \xi_j) &= II(I_i r + f^{-1} J_i N, I_j r + f^{-1} J_j N) \\
&= II(I_i r, I_j r) + f^{-1} II(J_i N, J_j r) \\
&\quad + f^{-1} II(J_i r, J_j N) + f^{-2} II(J_i N, J_j N) = 0.
\end{aligned}$$

The above identities together with $II(X, Y) = -f^{-1} g(X, Y)$ yield (3.21), which completes the proof. □

At this point we are ready to complete the proof of Theorem 3.1. We proceed by showing that there exists a unique section $\mathfrak{W}$ of the pullback bundle $(T^*K \otimes T^*K)|_M \to M$, which is J_s-invariant, and whose restriction to TM coincides with the tensor $-fII$. It will be convenient to consider the *calibrated transversal* to M vector field

$$\xi(p) = f^{-1}(p)N(p) + r(p), \ \ p \in M, \tag{3.22}$$

which is a section of the vector bundle $TK|_M \to M$. Clearly, $J_s \xi = \xi_s$ by (3.5), which together with the J_s invariance of II on the horizontal space H gives the existence of J_s-invariant bilinear form on $TK|_M \to M$ by adding a bilinear form on the complement $V \oplus \mathbb{R}\,\xi$. In fact, with the obvious identifications, since the fiber of $TK|_M$ over any $p \in M \subset K$ decomposes as a direct sum of subspaces as $H_p \oplus V_p \oplus \mathbb{R}\,\xi(p)$, for a $v \in T_p K$ we define

$$v' = v - \lambda(v)\xi(p) \in T_p M = H_p \oplus V_p, \qquad v'' = v' - \sum_{s=1}^{3} \eta_s \left(v'\right) \xi_s \in H_p,$$

where λ is a 1-form, $\lambda = fG(N, .)$, so that v' is the projection of v on $T_p M = H_p \oplus V_p$ parallel to the calibrated transversal field ξ. We can rewrite formula (3.21) in terms of the introduced decomposition as follows

$$-fII(A, B) = g\left(A'', B''\right) + \frac{S}{2} \sum_{s=1}^{3} \eta_s(A)\eta_s(B), \qquad A, B \in T_p M,$$

which leads to the following definition of the symmetric bilinear form $\mathfrak{W}$,

$$\begin{aligned}
\mathfrak{W}(v, w) &\overset{def}{=} -fII\left(v', w'\right) + \frac{S}{2}\lambda(v)\lambda(w) \\
&= g\left(v'', w''\right) + \frac{S}{2} \sum_{s=1}^{3} \eta_s\left(v'\right)\eta_s\left(w'\right) + \frac{S}{2}\lambda(v)\lambda(w), \ \ v, w \in T_p K.
\end{aligned} \tag{3.23}$$

We shall prove that this symmetric form is parallel as required, i.e, for any $A \in TM$ and $v, w \in TK$ we have $(D_A \mathfrak{W})(v, w) = 0$. From the symmetry and $Sp(1)$ invariance of $\mathfrak{W}$ we have trivially for $v, w \in TK$ the identities

$$(D_A\mathfrak{W})(v,w)=(D_A\mathfrak{W})(w,v),\qquad (D_A\mathfrak{W})(J_sv,J_sw)=(D_A\mathfrak{W})(v,w). \tag{3.24}$$

Furthermore, the restrictions of $\mathfrak{W}(J_s\cdot,\cdot)$ to TM are closed 2-forms on M. Indeed, let $\mathfrak{W}_s$ be the 2-form on M defined by

$$\mathfrak{W}_s(A,B)=\mathfrak{W}(J_sA,B).$$

Using the identity $(J_iA)'=(J_iA)''+\eta_j(A)\xi_k-\eta_k(A)\xi_j$ in (3.23) we see that

$$\mathfrak{W}_i(A,B)=\omega_i(A,B)+\frac{S}{2}\sum_{s=1}^{3}\eta_s\left((J_iA)'\right)\eta_s(B)=\left(\omega_i+\frac{S}{2}\eta_j\wedge\eta_k\right)(A,B)$$
$$=\frac{1}{2}\mathrm{d}\eta_i(A,B),$$

which implies $\mathrm{d}\mathfrak{W}_i(A,B,C)=0$. On the other hand, the exterior derivative $\mathrm{d}\mathfrak{W}_i$ can be expressed in terms of the covariant derivative $D\mathfrak{W}_i$ through the well know formula

$$\mathrm{d}\mathfrak{W}_i(A,B,C)=(D_A\mathfrak{W}_i)(B,C)+(D_B\mathfrak{W}_i)(C,A)+(D_C\mathfrak{W}_i)(A,B). \tag{3.25}$$

Since by assumption $DJ_s=0$ we have $(D_A\mathfrak{W}_s)(B,C)=(D_A\mathfrak{W})(J_sB,C)$, Eq. (3.25) gives

$$(D_A\mathfrak{W})(J_sB,C)+(D_B\mathfrak{W})(J_sC,A)+(D_C\mathfrak{W})(J_sA,B)=0,\qquad A,B,C\in TM. \tag{3.26}$$

We will show that the identities (3.24) and (3.26) yield $(D_A\mathfrak{W})(v,w)=0$. An application of (3.26) gives

$$-(D_X\mathfrak{W})(Y,Z)+(D_{J_iY}\mathfrak{W})(J_iZ,X)+(D_Z\mathfrak{W})(X,Y)=0,$$
$$-(D_{J_kX}\mathfrak{W})(J_kY,Z)+(D_{J_iY}\mathfrak{W})(J_iZ,X)+(D_Z\mathfrak{W})(X,Y)=0.$$

Therefore, $(D_{J_sX}\mathfrak{W})(J_sY,Z)=(D_X\mathfrak{W})(Y,Z)=(D_X\mathfrak{W})(J_sY,J_sZ)$, which by (3.24), implies $(D_{J_sX}\mathfrak{W})(Y,J_sZ)=(D_X\mathfrak{W})(Y,Z)$. It follows

$$(D_{J_sX}\mathfrak{W})(Y,Z)=-(D_X\mathfrak{W})(Y,J_sZ)=(D_X\mathfrak{W})(J_sY,Z)=-(D_{J_sX}\mathfrak{W})(Y,Z),$$

thus $(D_X\mathfrak{W})(Y,Z)=0$.

Another use of (3.26) gives

$$(D_{\xi_i}\mathfrak{W})(J_iY,Z)+(D_Y\mathfrak{W})(Z,\xi)-(D_Z\mathfrak{W})(\tilde{N},Y)=0, \tag{3.27}$$

which implies

$$(D_{\xi_1}\mathfrak{W})(J_1Y,Z)=(D_{\xi_2}\mathfrak{W})(J_2Y,Z)=(D_{\xi_3}\mathfrak{W})(J_3Y,Z),$$
$$(D_{\xi_1}\mathfrak{W})(Y,J_1Z)=(D_{\xi_2}\mathfrak{W})(Y,J_2Z)=(D_{\xi_3}\mathfrak{W})(Y,J_3Z).$$

Therefore, we have

$$(D_{\xi_i}\mathfrak{W})(Y,Z)=(D_{\xi_i}\mathfrak{W})(J_iY,J_iZ)$$
$$=(D_{\xi_j}\mathfrak{W})(J_jY,J_iZ)=(D_{\xi_j}\mathfrak{W})(J_jY,J_jJ_kZ)=(D_{\xi_i}\mathfrak{W})(J_jY,J_iJ_kZ)$$
$$=-(D_{\xi_i}\mathfrak{W})(J_jY,J_jZ)=-(D_{\xi_i}\mathfrak{W})(Y,Z),$$

thus

$$(D_{\xi_s}\mathfrak{W})(Y,Z)=0. \tag{3.28}$$

Now, a substitution in (3.27) gives

$$(D_Y\mathfrak{W})(Z,\xi)=(D_Z\mathfrak{W})(Y,\xi). \tag{3.29}$$

Invoking again (3.26) we find

$$(D_{\xi_j}\mathfrak{W})(J_iY, Z) + (D_Y\mathfrak{W})(J_kZ, \xi) - (D_Z\mathfrak{W})(\tilde{N}, J_kY) = 0,$$

which together with (3.28) and (3.29) give $(D_{J_sX}\mathfrak{W})(Y, \xi) = (D_X\mathfrak{W})(J_sY, \xi)$. In addition, it also follows

$$D_{J_kX}\mathfrak{W})(Y, \xi) = (D_X\mathfrak{W})(J_iJ_jY, \tilde{N}) = (D_{J_jJ_iX}\mathfrak{W})(Y, \xi) = -(D_{J_kX}\mathfrak{W})(Y, \xi),$$

thus $(D_X\mathfrak{W})(Y, \xi) = 0$ as well.

Next, we apply (3.26) as follows

$$\begin{aligned} &-(D_{\xi_j}\mathfrak{W})(\xi_j, Z) - (D_{\xi_k}\mathfrak{W})(\xi_k, Z) + (D_Z\mathfrak{W})(\tilde{N}, \tilde{N}) = 0,\\ &-(D_{\xi_i}\mathfrak{W})(\xi_i, Z) - (D_{\xi_j}\mathfrak{W})(\xi_j, Z) - (D_{J_jZ})\mathfrak{W}(\tilde{N}, \xi_j) = 0. \end{aligned} \tag{3.30}$$

Since, $(D_{J_jZ}\mathfrak{W})(\xi, \xi_j) = (D_{J_jZ}\mathfrak{W})(\xi, J_j\xi) = 0$, the second equation in (3.30) implies $(D_{\xi_s}\mathfrak{W})(\xi_s, X) = 0$, which together with the first equation in (3.30) give $(D_{\xi_s}\mathfrak{W})(\xi, X) = (D_X\mathfrak{W})(\xi, \xi) = 0$.

Finally, from (3.26) we have $(D_{\xi_i}\mathfrak{W})(\xi, \xi) + (D_{\xi_j}\mathfrak{W})(J_k\xi, \tilde{N}) - (D_{\xi_k}\mathfrak{W})(\xi, J_j\xi) = 0$, which implies $(D_{\xi_s}\mathfrak{W})(\xi, \xi) = 0$. This completes the proof of Theorem 3.1.

We record an important relation between the calibrating function and the parallel bilinear form,

$$\mathfrak{W}(N, A) = -f II\left(N', A\right) = f^2 II(r, A) = -f g\left(r, A''\right) = \mathrm{d}f\left(A''\right) = \mathrm{d}f(A), \tag{3.31}$$

which follows from Lemma 3.8 and the definition of $\mathfrak{W}$, (3.23).

As an application of Theorem 3.1 we have the following result.

Theorem 3.10 *Let (K, G) be a hyper-Kähler manifold with Riemannian curvature tensor $\hat{R}$. If M is a qc-hypersurface of K with normal vector field N then we have that $\hat{R}_{vw}N = 0$ for all $p \in M$ and $v, w \in T_pK$. In particular, the Riemannian curvature tensor $\hat{R}$ is degenerate at each point p of the hypersurface M.*

Proof Let M be a qc-hypersurface of the hyper-Kähler manifold (K, G, J_1, J_2, J_3). Let f and η_s be the calibrating function and calibrated qc-structure determined in Theorem 3.1, see also (3.4). Let us extend the second fundamental form II of the embedding to a section of the bundle $TK|_M \otimes TK|_M \to M$ by setting $II(N, A) = II(N, N) = II(A, N) = 0,\ A \in TM \subset TK$. For any $v, w \in TK$ we have

$$\begin{aligned} II(v, w) &= -\frac{1}{f}\mathfrak{W}\left(v - G(v, N)N, w - G(w, N)\right)\\ &= -\frac{1}{f}\left\{\mathfrak{W}(v, w) - G(v, N)\mathfrak{W}(N, w) - G(w, N)\mathfrak{W}(N, v)\right.\\ &\quad \left.+ G(v, N)G(w, N)\mathfrak{W}(N, N)\right\}. \end{aligned}$$

Using the Levi–Civita connection D of the hyper-Kähler manifold K we differentiate the above equation to obtain

$$\begin{aligned} (D_AII)(B, C) &= \frac{\mathrm{d}f(A)}{f^2}\mathfrak{W}(B, C) + \frac{1}{f}\left\{G(B, D_AN)\mathrm{d}f(C) + G(C, D_AN)\mathrm{d}f(B)\right\}\\ &= \frac{1}{f^2}\left\{\mathrm{d}f(A)\mathfrak{W}(B, C) + \mathrm{d}f(B)\mathfrak{W}(C, A) + \mathrm{d}f(C)\mathfrak{W}(B, A)\right\}, \end{aligned}$$

which, in particular, implies

$(D_A II)(B,C) - (D_B II)(A,C) = 0.$
On the other hand we compute

$$
\begin{aligned}
(D_A II)(B,C) =& A(II(B,C)) - II(D_A B, C) - II(B, D_A C)\\
=& - AG(D_B N, C) + G(D_{D_A B} N, C) + G(D_B N, D_A C)\\
=& - G(D_A D_B N, C) + G(D_{D_A B} N, C).
\end{aligned}
$$

For the curvature tensor $\hat{R}$ of D we obtain

$$
\begin{aligned}
0 &= (D_A II)(B,C) - (D_B II)(A,C) = -G(D_A D_B N, C) + G(D_{D_A B} N, C)\\
&\quad + G(D_B D_A N, C) - G(D_{D_B A} N, C)\\
&= G(\hat{R}_{AB} N, C),
\end{aligned}
$$

thus $\hat{R}_{AB} N = 0,\ A, B \in TM$. Furthermore, since $\hat{R}$ is the curvature of a hyper-Kähler manifold, it has the property $\hat{R}(J_s v, J_s w) = \hat{R}(v,w),\ v, w \in TK$. Hence, $\hat{R}_{XN} N = \hat{R}_{J_s X, J_s N} N = 0$ and $\hat{R}_{J_i N, N} N = \hat{R}_{J_k N, J_j N} N = 0$, which completes the proof of the theorem. □

4 QC-hypersurfaces in the flat hyper-Kähler manifold $\mathbb{H}^{n+1}$

As usual, we consider the flat hyper-Kähler quaternion space $\mathbb{H}^{n+1}$ with its standard quaternionic structure $\mathbb{Q} = \mathrm{span}\{J_1, J_2, J_3\}$, determined by the multiplication on the right by $-i$, $-j$ and $-k$, respectively. Let

$$
\langle q, q' \rangle = Re \left(\sum_{a=1}^{n+1} q_a \overline{q'_a} \right), \qquad q_a = t_a + i x_a + j y_a + k z_a,
$$

be the flat hyper-Kähler metric of $\mathbb{H}^{n+1}$. If M is a qc-hypersurface of $\mathbb{H}^{n+1}$ and $(A, \omega, q_0) \in GL(n+1, \mathbb{H}) \times Sp(1) \times \mathbb{H}^{n+1}$, then the quaternionic affine map $F : \mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$, defined by $F(q) = Aq\bar{\omega} + q_0$, transforms M into another qc-hypersurface $F(M)$ of $\mathbb{H}^{n+1}$ since F preserves the quaternion structure of $\mathbb{H}^{n+1}$. In this section we will prove, as another application of Theorem 3.1, that in fact any qc-hypersurface of $\mathbb{H}^{n+1}$ is congruent by the action of the quaternion affine group $GL(n+1, \mathbb{H}) \times Sp(1) \rtimes \mathbb{H}^{n+1}$ to one of the standard examples: the quaternionic Heisenberg group, the round sphere or the qc-hyperboloid, see Example 3.2, (4.4) and (4.5), respectively.

4.1 Proof of Theorem 1.1

Let $\iota : M \to \mathbb{H}^{n+1}$ be a qc-embedding, with N and II the unit normal and the second fundamental form of M. Recall, we assume II to be negative definite on the maximal J_s-invariant distribution H of M. From Theorem 3.1, we obtain a calibrating function f on M and a parallel, J_s-invariant section $\mathfrak{W}$ of the bundle $(T^*K \otimes T^*K)|_M$. Clearly, since $\mathfrak{W}$ is parallel, we can find an endomorphism of the real vector space $\mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$, which we denote again by $\mathfrak{W}$, such that

$$
\mathfrak{W}(v, w) = \langle \mathfrak{W}(v), w \rangle, \qquad v, w \in \mathbb{R}^{4n+4}.
$$

By (3.23) in Theorem 3.1 and (3.31) we have the identities

$$\mathfrak{W} \circ J_s = J_s \circ \mathfrak{W}, \quad \mathrm{d}f(A) = \langle \mathfrak{W}N, \iota_* A\rangle, \quad -f\,II(A,B) = \langle \mathfrak{W}\,\iota_* A, \iota_* B\rangle, \quad A, B \in TM. \tag{4.1}$$

With the help of the matrix $\mathfrak{W}$ and the above identities we can express the derivative of the unit normal to M vector N along tangent fields as follows

$$D_A N = \frac{1}{f}\Big(\mathfrak{W}\iota_* A - \mathrm{d}f(A)N\Big). \tag{4.2}$$

Indeed, an orthogonal decomposition and the last equation of (4.1) give

$$\begin{aligned}\langle D_A N, v\rangle =&\langle D_A N, [v]_{TM}\rangle + \langle D_A N, N\rangle\langle v, N\rangle = -II(A, [v]_{TM}) = \frac{1}{f}\langle \mathfrak{W}\iota_* A, [v]_{TM}\rangle\\ =&\frac{1}{f}\Big(\langle \mathfrak{W}\iota_* A, v\rangle - \langle \mathfrak{W}\iota_* A, N\rangle\langle v, N\rangle\Big)\end{aligned}$$

using the second formula in (4.1). Moreover, formula (3.23) from the proof of Theorem 3.1 shows that, depending on the constant S, we have exactly one of the following three cases: (i) $\mathfrak{W}$ is positive definite; (ii) $\mathfrak{W}$ is of signature $(4n, 4)$; (iii) $\mathfrak{W}$ is degenerate of signature $(4n, 0)$.

Let us consider the most interesting case (iii). Assume $\mathfrak{W}$ is degenerate of signature $(4n, 0)$ and $\ker\ \mathfrak{W} = \{v_0, J_1 v_0, J_2 v_0, J_3 v_0\}$ for some unit $v_0 \in \mathbb{R}^{4n+4}$, so that $\mathbb{R}^{4n+4} = \mathrm{im}\mathfrak{W} \oplus \ker\ \mathfrak{W}$. We define the symmetric endomorphism $\mathfrak{W}'$ of $\mathbb{R}^{4n+4}$ which is inverse to $\mathfrak{W}$ on $\mathrm{im}\,\mathfrak{W}$ and satisfies $\ker\ \mathfrak{W}' = \ker\ \mathfrak{W}$. Thus, we have

$$\mathfrak{W} \circ \mathfrak{W}'(v) = \mathfrak{W}' \circ \mathfrak{W}(v) = v - \langle v, v_0\rangle v_0 - \sum_{s=1}^{3}\langle v, J_s v_0\rangle J_s v_0, \quad v \in \mathbb{R}^{4n+4}.$$

Consider the functions $h, t_m, l_m : M \to \mathbb{R}$, $m = 0, 1, 2, 3$, defined by

$$h(p) = \langle \mathfrak{W}'N, N\rangle, \quad t_0(p) = \langle v_0, \iota(p)\rangle, \quad t_s(p) = \langle J_s v_0, \iota(p)\rangle, \quad l_0(p) = \langle v_0, N\rangle, \quad l_s(p) = \langle J_s v_0, N\rangle.$$

Invoking (4.2) we compute

$$\begin{aligned}\mathrm{d}l_0(A) =&\langle v_0, D_A N\rangle = \frac{1}{f}\langle v_0, \mathfrak{W}\iota_*(A) - \mathrm{d}f(A)N\rangle\\ =&\frac{1}{f}\langle \mathfrak{W}v_0, \iota_*(A)\rangle \ - \ \frac{\mathrm{d}f(A)}{f}l_0 = -\frac{\mathrm{d}f(A)}{f}l_0,\end{aligned}$$

which implies that the product $f\,l_0$ is constant on M, $f\,l_0 = C_0$, $C_0 \in \mathbb{R}$. Similarly we have $\mathrm{d}l_s = -l_s\frac{\mathrm{d}f}{f}$ and therefore $f\,l_s = C_s$, $s = 1, 2, 3$, where C_s are constants. Furthermore,

$$\begin{aligned}\mathrm{d}h(A) &= 2\langle \mathfrak{W}'N, D_A N\rangle = \frac{2}{f}\langle \mathfrak{W}'N, \mathfrak{W}\iota_*(A) - \mathrm{d}f(A)N\rangle\\ &= \frac{2}{f}\langle \mathfrak{W}\mathfrak{W}'N, \iota_*(A)\rangle - \frac{2h\mathrm{d}f(A)}{f}\\ &= -\frac{2h\mathrm{d}f(A)}{f} - \frac{2}{f}\sum_{m=0}^{3} l_m \mathrm{d}t_m(A) = -\frac{1}{f^2}\left\{2\sum_{m=0}^{3} l_m \mathrm{d}t_m(A) + h\,\mathrm{d}(f^2)(A)\right\}.\end{aligned}$$

It follows that $f^2 \mathrm{d}h + h\mathrm{d}(f^2) = -2\sum_{m=0}^{3} C_m \mathrm{d}t_m$, which implies that on the manifold M we have

$$f^2 h = c + \sum_{m=0}^{3} c_m t_m \tag{4.3}$$

for some constants c, $c_m \in \mathbb{R}$, $m = 0, \ldots, 3$. Now, consider the vector valued function $V \circ \iota : M \to \mathbb{R}^{4n+4}$,

$$V(p) = f\, \mathfrak{W}'\, N(p) + t_0(p)\, v_0 + \sum_{s=1}^{3} t_s(p)\, J_s v_0, \quad p \in M.$$

Formula (4.3) implies $\langle \mathfrak{W} V, V \rangle = f^2 h = c + \sum_{m=0}^{3} c_m t_m$. On the other hand, using (4.2), we have

$$\begin{aligned}(\iota - V)_* A &= \iota_* A - \mathrm{d}f(A)\, \mathfrak{W}' N - f \mathfrak{W}' \left(\frac{1}{f} \mathfrak{W}(\iota_* A) - \frac{\mathrm{d}f(A)}{f} N \right) \\ &\quad - \mathrm{d}t_0(A)\, v_0 - \sum_{s=1}^{3} \mathrm{d}t_s(A)\, J_s v_0 \\ &= \iota_* A - \mathfrak{W} \circ \mathfrak{W}' \iota_* A - \mathrm{d}t_0(A)\, v_0 - \sum_{s=1}^{3} dt_s(A)\, J_s v_0 = 0, \qquad A \in TM.\end{aligned}$$

Thus, there exists a point $p_0 \in \mathbb{H}^{n+1}$ such that for all $p \in M$ we have

$$\langle \mathfrak{W} \left(\iota(p) - p_0 \right), \iota(p) - p_0 \rangle = c + \sum_{m=0}^{3} c_m t_m(p).$$

A translation $p = \tilde{p} + p_0$ brings us to the case (identifying points on M with their images by ι)

$$\langle \mathfrak{W} \tilde{p}, \tilde{p} \rangle = \tilde{c} + \sum_{m=0}^{3} c_m t_m(\tilde{p}), \quad \tilde{p} \in M.$$

Let $\tilde{q}_0 \in \ker\ \mathfrak{W}$ be such that $\sum_{m=0}^{3} c_m t_m(\tilde{q}_0) = \tilde{c}$, which is possible since $\sum_{m=0}^{3} c_m t_m \not\equiv 0$. Indeed, otherwise $c_m = 0$, $m = 0, \ldots, 3$ implies that the $\ker\ \mathfrak{W} \subset H$, which is a contradiction with the non-integrability of H. We consider the translation $\tilde{p} = \tilde{q} - \tilde{q}_0$ which brings us to

$$\langle \mathfrak{W} \tilde{q}, \tilde{q} \rangle = \sum_{m=0}^{3} c_m t_m(\tilde{q}), \quad \tilde{q} \in M.$$

Let ϵ_{n+1} be a unit vector in the direction of the vector $c_0 v_0 + \sum_{s=1}^{3} c_s\ J_s v_0 \in \ker\ \mathfrak{W}$ and consider $\mathcal{E} = \{\epsilon_1, J_1\epsilon_1, J_2\epsilon_1 J_3\epsilon_1, \ldots, \epsilon_n, J_1\epsilon_n, , J_2\epsilon_n, \ldots, J_3\epsilon_n, \epsilon_{n+1}, J_1\epsilon_{n+1}, J_2\epsilon_{n+1}, J_3\epsilon_{n+1}\}$ where the first $4n$ vectors are an orthonormal basis of eigenvectors of the symmetric J_s-invariant operator $\mathfrak{W}$ on $\operatorname{im} \mathfrak{W}$. For this we note that by the J_s-invariance of $\mathfrak{W}$ it follows that if v is a (real) eigenvector of $\mathfrak{W}$ so are the vectors $J_s v$. In the quaternion coordinate coordinates q_a determined by $\{\epsilon_a, J_1\epsilon_a, J_2\epsilon_a, J_3\epsilon_a\}$, $a = 1, \ldots, n+1$ we come to the desired form. Thus, there is a quaternionic affine transformation of $\mathbb{H}^{n+1}$ which maps $\iota(M)$ into the hypersurface $|q|^2 + t = 0$ described in Example 3.2.

Proceeding similarly in the cases where $\mathfrak{W}$ is positive definite or of signature $(4n, 4)$ we will obtain, respectively,

$$\sum_{a=1}^{n} |q_a|^2 + |p|^2 = 1, \tag{4.4}$$

i.e., the $4n+3$ dimensional round sphere in $\mathbb{R}^{4n+4} = \mathbb{H}^{n+1}$ and the hyperboloid

$$\sum_{a=1}^{n} |q_a|^2 - |p|^2 = -1. \tag{4.5}$$

In these two cases, however, a simpler prove is possible, by first applying an appropriate transformation from the linear group $GL(n+1, \mathbb{H})$, which transforms $\mathfrak{W}$ into a diagonal matrix with entries $+1$ or -1. Then, the transformed hypersurface will be totally umbilical, and one can use the corresponding classification theorem of totally umbilical hypersurfaces in $\mathbb{R}^{4n+4}$ to complete the proof.

4.2 QC-hypersurfaces in the quaternionic projective space $\mathbb{H}P^{n+1}$

Note that, as a quaternionic manifold, $\mathbb{H}^{n+1}$ is equivalent to an open dense subset of the quaternionic projective space $\mathbb{H}P^{n+1}$. Thus, all qc-hypersurfaces of $\mathbb{H}^{n+1}$ are also qc-hypersurfaces of $\mathbb{H}P^{n+1}$. Also, it is well known that $PGL(n+2, \mathbb{H})$ is the group of quaternionic affine transformations of [17] $\mathbb{H}P^{n+1}$. As a direct consequence of Theorem 1.1 we obtain

Corollary 4.1 *If M is a connected qc-hypersurface of the quaternionic projective space $\mathbb{H}P^{n+1}$, then there exists a transformation $\phi \in GL(n+2, \mathbb{H})$ of $\mathbb{H}P^{n+1}$ which transforms M into an open set $\phi(M)$ of the qc-hypersurface M_o, defined by*

$$M_o \;=\; \{[q_1, \ldots, q_{n+2}] \in \mathbb{H}P^{n+1} : |q_1|^2 + \cdots + |q_{n+1}|^2 = |q_{n+2}|^2\},$$

where $[q_1, \ldots, q_{n+2}]$ denote the quaternionic homogeneous coordinates of $\mathbb{H}P^{n+1}$.

In particular, as an abstract qc-manifold, every qc-hypersurface of $\mathbb{H}P^{n+1}$ is qc-conformally equivalent to an open set of the quaternionic contact (3-Sasakian) sphere S^{4n+3}.

Proof The proof relies on the preceding remarks and the fact that the quaternion structures of the flat hyper-Kähler and the quaternion-Kähler spaces are identical. Thus, every affine part of a qc-hypersurface of $\mathbb{H}P^{n+1}$ is one of the quadrics in Theorem 1.1 up to a quaternionic affine transformation. By analytic continuation the quadric has to be the same.

Finally, the three quadrics in Theorem 1.1 are congruent modulo the $GL(n+2, \mathbb{H})$ action on the projective space $\mathbb{H}P^{n+1}$, which completes the proof. □

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5 Appendix

The following property of the qc geometry observed in [3] clarifies the paragraph after Definition 2.1. As well known, this property is particular for the quaternionic contact case in contrast with the situation in the CR case. For completeness, we include the statement and a complete proof.

Lemma 5.1 *Let (M, H) be a qc-manifold and (η_s, I_s, g), (η'_s, I'_s, g') be two local qc-structures on an open set $U \subset M$ with the same horizontal space H. Then, there exist a positive function $f : U \to R$, $f > 0$ and a matrix-valued function $\mathcal{A} = (a_{ij}) : U \to SO(3)$ such that*

$$\left(I'_1, I'_2, I'_3\right) = (I_1, I_2, I_3)\,\mathcal{A}, \quad \left(\eta'_1, \eta'_2, \eta'_3\right) = f\,(\eta_1, \eta_2, \eta_3)\,\mathcal{A}, \quad g' = f\,g.$$

Proof By assumption, $H = \cap_{i=1}^{3}\eta_i = \cap_{i=1}^{3}\eta'_i$, thus there exists a matrix-valued function $\mathcal{A} = (a_{ij}) : U \to GL(3)$ with $\eta'_s = \sum_{t=1}^{3} a_{st}\eta_t$, $s = 1, 2, 3$. Taking the exterior derivative of the above equations we obtain

$$(\mathrm{d}\eta'_s)|_H = \sum_t a_{st}(\mathrm{d}\eta_t)|_H. \tag{5.1}$$

Let us fix a symmetric and positive definite section h of the bundle $H^* \otimes H^*$ which we will use as a "background" metric on H. With respect to this metric, consider the restrictions of the 2-forms $(\mathrm{d}\eta'_s)|_H$ to H as endomorphisms of H, i.e., sections of the bundle $End(H) = H^* \otimes H$. This identification depends on the choice of h. However, it is easy to see that the composition of two endomorphisms of the form $((\mathrm{d}\eta'_s)|_H)^{-1} \circ (\mathrm{d}\eta'_t)|_H$, is an endomorphism independent of the choice of h. For (i, j, k) a cyclic permutation of $(1, 2, 3)$ and $h = g'$ we have

$$\left((\mathrm{d}\eta'_j)|_H\right)^{-1} \circ \left(\mathrm{d}\eta'_i\right)|_H = I'_k. \tag{5.2}$$

The above equation holds for any choice of the metric h on H, in particular, also for $h = g$. Using 5.1, we conclude that

$$I'_k \;=\; \left((\mathrm{d}\eta'_j)|_H\right)^{-1} \circ \left(\mathrm{d}\eta'_i\right)|_H \;\in\; \mathrm{span}_{\mathbb{R}}\,\{\mathrm{id}_H, I_1, I_2, I_3\}.$$

Note that $\mathrm{span}_{\mathbb{R}}\,\{\mathrm{id}_H, I_1, I_2, I_3\} \subset End(H)$ is an algebra with respect to the usual composition of endomorphisms, which is isomorphic to the algebra of the quaternions $\mathbb{H} = \mathrm{span}_{\mathbb{R}}\,\{1, i, j, k\}$. If an element of $\mathbb{H}$ has square -1 then this element belongs to $Im(\mathbb{H})$. Therefore, $I'_s \in Q = \mathrm{span}\,\{I_1, I_2, I_3\}$, hence

$$\mathrm{span}_{\mathbb{R}}\,\{I_1, I_2, I_3\} \;=\; \mathrm{span}_{\mathbb{R}}\,\{I'_1, I'_2, I'_3\}.$$

Now, still identifying $H^* \otimes H$ with $End(H)$, using $h = g$ and using that the metric g is I_s- and I'_s-compatible, then each of the endomorphisms $(\mathrm{d}\eta'_k)_H \in End(H)$ anti-commutes with both I'_i and I'_j.This implies that, as an endomorphism, $(\mathrm{d}\eta'_k)_H$ is proportional to I'_k, which gives $g' = f\,g$ for some $f > 0$. The fact that $\mathcal{A} = (a_{ij})$ takes values in $SO(3)$ follows from the requirement that both (I_1, I_2, I_3) and (I'_1, I'_2, I'_3) satisfy the quaternionic identities. □

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