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On iterations of Steiner symmetrizations

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Abstract We prove in this paper that, given a countable and dense set of directions $U \subset \mathbb{R}^N$, it can be ordered in such a way that if we iterate the Steiner symmetrization of any compact set K in that order, the sequence of Steiner symmetrizations of K converges in the Hausdorff distance to the ball K^* centered at the origin and having the same volume as K. This result provides a generalization of a theorem from Bianchi et al. (Adv Appl Math 47:869–873, 2011) in two directions. On the one hand, the seed of the iteration is allowed to be compact rather than just convex, and on the other hand, the ordering of U is universal and does not depend on the seed.

Keywords Steiner symmetrization · Isoperimetric problem · Caccioppoli sets

Mathematics Subject Classification 60D05 · 52A40 · 28A20

1 Introduction

Recently, several papers appeared concerning convergence of iterations of Steiner symmetrizations stressing the geometric point of view, like [2,4,5,7,11,16]. Other papers appeared in the same period more oriented to applications in calculus of variations and partial differential equations, like [1,3,6,8,15]. This paper can be placed in the first stream.

The initial geometric motivation was Mani's problem [12] concerning almost sure convergence of iterations of random Steiner symmetrizations of compact sets. The problem has been solved by Van Schaftingen in [15] in 2006, and an independent proof has been obtained in [16].

Two of these papers, [4] and [16], investigate the relations between convergence of Steiner symmetrizations of measurable sets with respect to L_1 distance and convergence of Steiner symmetrizations of compact sets with respect to Hausdorff distance. Convex bodies, while

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present in Mani's paper, as well as in [2] and in [11], turn out to be the natural background where to pose problems, but the most general and interesting setting is represented by compact sets. One of the aims of this paper is to support the validity of this claim.

The main result of this paper is the following result. It will be proved in Sect. 3.

Theorem 1.1 *If U is countable and dense in the unit N-dimensional sphere S^{N-1}, then there exists an appropriate ordering $\{u_n\}$ of U such that the Steiner symmetrizations of any compact set in $K \subset \mathbb{R}^N$, taken successively in the directions u_n, $n \in \mathbb{N}$, converge in the Hausdorff distance to the ball K^* centered at the origin having the same volume as K.*

Theorem 1.1 extends the analogous result proved in [2] for convex bodies. The other interesting improvement is that the ordering of U is universal and that it does not depend on the seed K.

It is interesting to note that there exist dense sequences of directions $\{u_n\}$ and convex bodies K such that the Steiner symmetrizations of K, taken successively in the directions u_n, do not converge in the Hausdorff distance (see [4] and [2]).

In Sect. 4, we will prove for measurable sets a proposition which parallels the main result from [11], proved there for convex sets. In [4], Klain's result has been extended to the case of compact sets. Here, we show how from the latter result the measurable case follows easily by inner approximation with compact sets.

2 Preliminaries

We will be using two distances. In the class of all compact sets (or convex bodies), it is natural to use the Hausdorff distance d_H (see, for instance, [13]). On the other hand, in the class of measurable sets having finite measure, the natural distance is the L_1 distance, denoted by d_1.

We will denote by λ_N the N-dimensional Lebesgue outer measure in $\mathbb{R}^N$.

Let us now define the Steiner symmetrization. While for convex bodies and compact sets all authors adopt the same definition, there are some variants in the literature when it comes to measurable sets. We shall follow [6].

Definition 2.1 Let E be a measurable set in $\mathbb{R}^N$, u a unit vector identifying a direction and l_u the line through the origin parallel to u. We denote by $u^\perp$ the $(N-1)$-dimensional subspace orthogonal to u. For each $x \in u^\perp$, let $c(x)$ be defined as follows. If $E \cap (l_u + x)$ is empty, let $c(x) = \emptyset$. Otherwise, let $c(x)$ be the possibly degenerate (and possibly infinite) closed segment on $l_u + x$ centered at $x \in u^\perp$ whose length is equal to $\lambda_1(E \cap (l_u + x))$ (outer measure).

The union of all the line segments $c(x)$, for $x \in u^\perp$, is called the *Steiner symmetral* of E and will be denoted by $S_u E$. The mapping S_u from the family of measurable sets into itself is called *Steiner symmetrization*.

Definition 2.2 A measurable subset C of $\mathbb{R}^N$ is called a *Caccioppoli set* if

$$\lambda_N((C + x) \triangle C) \leq p \cdot \|x\|$$

for some constant p and for every $x \in \mathbb{R}^N$.

This concept is due to Caccioppoli [C] and has been extensively studied and used by De Giorgi (in [9] and subsequent papers, and in [10], which is devoted to the isoperimetric problem).

For a Caccioppoli set C let us denote by $p(C)$ the De Giorgi–Caccioppoli perimeter of C (see [14]) which can be defined for instance by

$$p(C) = \inf \left\{ \liminf_{n\to\infty} p(E_n) \right\},$$

where $\{E_n\}$ is a sequence of smooth subsets of $\mathbb{R}^N$ such that $d_1(C, E_n) \to 0$, the infimum is taken over all such sequences, and the perimeter for smooth sets is understood in the ordinary sense.

We will need several properties of Caccioppoli sets and its perimeter. It is well known that the Steiner symmetral of a Caccioppoli set is a Caccioppoli set and that Steiner symmetrization does not increase the perimeter of a Caccioppoli set.

We will also need the following compactness criterion for Caccioppoli sets: If $\mathcal{F}$ is a collection of Caccioppoli sets which are contained in a bounded set and have uniformly bounded perimeters, then $\mathcal{F}$ is relatively compact with respect to the L_1 distance.

Finally, since finite unions of rectangles are Caccioppoli sets, the latter are dense in the L_1 distance among measurable sets having finite measure.

Since the perimeter is lower semicontinuous, we need a more adequate functional on the family of all (bounded, say) measurable sets.

Definition 2.3 Given a measurable set E, its *(central) moment of inertia* is defined by

$$\mu(E) = \int_E \|z\|^2 \, d\lambda_N(z) \,.$$

By Lemma 2.6 of [16], we know that the moment of inertia is uniformly continuous on any family of measurable sets contained in a bounded set.

The next two results are Lemmas 2.11 and 3.2 of [16] and will be employed in the sequel. Let us recall that the distance between two directions u and v is defined as $d(u, v) = \min\{\|u - v\|, \|u + v\|\}$, since the unit vectors u and $-u$ define the same direction.

Lemma 2.4 *If E has finite moment of inertia and is essentially different from a ball centered at the origin, then there exist a direction v and a positive δ such that*

$$\mu(S_u E) < \mu(E)$$

for all u such that $d(u, v) < \delta$.

Let us denote by κ_N the volume of the N-dimensional unit ball B and put $\mu_N = \mu(B)$.

The following lemma is the key to the main result. Its proof depends on the compactness criterion for Caccioppoli sets.

Lemma 2.5 *Fix $\rho_0 > 1$, $p_0 > 0$ and $\varepsilon_0 > 0$ and consider the family $\mathcal{F} = \mathcal{F}(\rho_0, p_0, \varepsilon_0)$ of all Caccioppoli sets contained in the ball $B(o, \rho_0)$ centered in 0 and having radius ρ_0, whose perimeters are bounded by p_0, such that $\lambda_N(E) = \kappa_N$ and*

$$\mu(E) \geq \mu_N + \varepsilon_0 \,.$$

Then, there exist a $\delta_0 > 0$, and for each $E \in \mathcal{F}$ a direction v_E, such that

$$\mu(S_v E) < \mu(E) - \delta_0$$

for every v such that $d(v, v_E) < \delta_0$.

3 Dense set of directions

Given a measurable set E having finite measure, we will denote by E^* the closed ball centered at the origin having the same measure.

If $V = \{v_1, \ldots, v_n\}$ is a finite ordered set of directions, we will denote with $S_V(E)$, or also with $S_{v_1,\ldots,v_n}(E) = S_{v_n}(S_{v_{n-1}} \ldots (S_{v_1}(E)) \ldots)$ the set obtained iterating the Steiner symmetrization with respect to v_1 first, then with respect to v_2 and so on, to finish with v_n.

Lemma 3.1 *If $U \subset S^{N-1}$ is countable and dense and if E is a measurable set having finite measure, then*

$$\inf\{\mu(S_V E) : V \subset U, \text{ with } V \text{ finite}\} = \mu(E^*) .$$

We may assume that $\lambda(E) = \kappa_N$ applying, if necessary, a homothety.

Let us first assume that E is a bounded Caccioppoli set. The proof is by contradiction. Assume the infimum above equals $\mu_0 > \mu_N$.

Let ρ_0 be such that $B(0, \rho_0) \supset E$, let p_0 be the perimeter of E, let $\varepsilon_0 = \mu_N - \mu_0$ and denote by $\mathcal{F}$ the set $\mathcal{F}(\rho_0, p_0, \varepsilon_0)$ defined in Lemma 2.5. It contains E and all its symmetrizations whose moment of inertia is at least μ_0. Then, there exist a $\delta_0 > 0$, and a direction v_E, such that

$$\mu(S_v E) < \mu(F) - \delta_0$$

for every v such that $d(v, v_F) < \delta_0$.

Since U is dense, we may select $v = v_1$ in U.

If $\mu(S_{v_1}(E)) < \mu_0$ we are done, otherwise we apply the same argument to $S_{v_1}E$ to conclude that there exists $v_2 \in U$, different from v_1, such that

$$\mu(S_{v_1 v_2}(E)) < \mu(S_{v_1}(E)) - \delta_0 < \mu(E) - 2\delta_0.$$

We can repeat the same construction diminishing the moment of inertia by at least δ_0 as long as the set (symmetrized with respect to distinct and appropriate $v_3, \ldots, v_n$ from U), belongs to $\mathcal{F}$.

There exists n such that

$$\mu(E) - (n+1)\delta_o < \mu_0 \leq \mu(S_{v_1,\ldots,v_n}(E)) \leq \mu(E) - n\delta_0.$$

Symmetrizing $S_{v_1,\ldots,v_n}(E)$ with respect to v_{n+1} (selected as before), we contradict the assumption on μ_0.

It is easy now to extend the result to general measurable sets E having finite measure. We may suppose again $\lambda_N(E) = \kappa_N$.

For any $\varepsilon > 0$, there exists a finite union of rectangles G (which is a Caccioppoli set) having measure κ_N and such that $d_1(G, E) < \varepsilon$. Since Steiner symmetrization is Lipschitz with constant 1, $d_1(S_{v_1,\ldots,v_n}(G), S_{v_1,\ldots,v_n}(E)) \leq d_1(G, E)$ and since $\{\mu(S_{v_1,\ldots,v_n})(G)\}$ tends to μ_N, the conclusion follows.

Let us now prove preliminarily a theorem which parallels the result from [2] in the context of measurable sets.

Theorem 3.2 *If U is countable and dense in the unit N-dimensional sphere S^{N-1} and if $E \subset \mathbb{R}^N$ has finite measure, then there exists an appropriate ordering $\{u_n\}$ of U such that the Steiner symmetrizations of E, taken successively in the directions u_n, converge in the L_1 distance to the ball E^*.*

Let $\{w_n\}$ be an ordering of U. From the previous lemma, we know that we can select distinct vectors $u_1, \ldots, u_{n_1}$ from U such that $\mu(S_{u_1,\ldots,u_{n_1}}(E)) < \mu(E^*) + 1$. We may assume that w_1 is among the selected vectors because if it were not so, we could add it, decreasing further the moment of inertia.

Since $U \backslash \{u_1, \ldots, u_{n_1}\}$ is still dense, we can select in the reduced set another group of vectors $\{u_{n_1+1}, \ldots, u_{n_2}\}$ such that w_2 is among them and

$$S_{V_2}(E) < \mu(E^*) + \frac{1}{2},$$

where V_2 denotes the ordered set of directions $v_1, \ldots, v_{n_1}, v_{n_1+1}, \ldots, v_{n_2}$ and the symmetrizations are taken in the natural order.

We iterate this construction and obtain an increasing family of ordered sets V_n (with the order on V_{i+1} extending the order on V_i), such that $w_n \in V_n$ (implying that sooner or later we symmetrize with respect to each $w_n \in U$) and

$$S_{V_n}(E) < \mu(E^*) + \frac{1}{2^{n-1}} .$$

Therefore $\mu(S_{V_n}(E))$ converges, when n tends to infinity, to $\mu(E^*)$ and since Steiner symmetrization is continuous with respect to the L_1 distance, by Lemma 2.4 the sequence $\{S_{V_n}(E)\}$ tends to E^* in the L_1 distance.

We will now extend the previous result, showing that there exist *universal* rearrangements of U which make the successive Steiner symmetrizations converge for every seed E which is measurable and has finite measure. The result is of independent interest, but it is also a tool for proving our main result.

Theorem 3.3 *If U is countable and dense in the unit N-dimensional sphere S^{N-1}, then there exists an appropriate ordering $\{u_n\}$ of U such that the Steiner symmetrizations of any measurable set having finite measure $E \subset \mathbb{R}^N$ taken successively in the directions u_n, converge in the L_1 distance to the ball E^*.*

Let, as in the previous theorem, $\{w_n\}$ be an ordering of U. Let $\{E_m\}$, for $m \in \mathbb{N}$, be a countable dense family of measurable sets having finite measure.

Choose $V_1 = (v_1, \ldots, v_{n_1})$ such that w_1 is among the selected vectors and

$$\mu(S_{V_1} E_1) < \mu(E_1^*) + 1 .$$

We can now choose as before $(v_{n_1+1}, \ldots, v_{n_2})$ in $U \setminus \{v_1, v_2, \ldots, v_{n_1}\}$ such that w_2 is among the selected vectors and letting $V_2 = (v_1, \ldots, v_{n_1}, v_{n_1+1}, \ldots, v_{n_2})$,

$$\mu(S_{V_2} E_m) < \mu(E_m^*) + \frac{1}{2},$$

for $m = 1, 2$.

Iterating this construction, we get at the ith step a finite set of directions $V_i = (v_1, \ldots, v_{n_1}, v_{n_1+1}, \ldots, v_{n_i})$ such that all the w_j's, for $1 \leq j \leq i$, belong to V_i and

$$\mu(S_{V_i} E_m) < \mu(E_m^*) + \frac{1}{2^{i-1}},$$

for $m \leq i$.

It follows that for any $m \in \mathbb{N}$

$$\lim_{n\to\infty} \mu(S_{V_n}(E_m)) = \mu(E_m^*) .$$

Let now E be any measurable set having finite measure. There exists a sequence $\{E_{m_j}\}$ such that $\lim_{j\to\infty} d_1(E_{m_j}, E) = 0$. Then, we have that

$$d_1(S_{V_n}(E), E^*) \leq d_1(S_{V_n}(E), S_{V_n}(E_{m_j})) + d_1(S_{V_n}(E_{m_j}), E^*_{m_j}) + d_1(E^*_{m_j}, E^*) \leq$$
$$d_1(E, E_{m_j}) + d_1(S_{V_n}(E_{m_j}), E^*_{m_j}) + d_1(E^*_{m_j}, E^*)\,.$$

Given $\varepsilon > 0$, we may fix a j such that the first and the last terms are both smaller than that $\frac{\varepsilon}{3}$. Given j, the intermediate term tends to zero because of the previous theorem, and the proof is complete.

To prove Theorem 1.1, we will use the previous results and follow the route we took in Section 4 of [16].

Given a nonnegative integrable function f on $\mathbb{R}^N$, its superlevel set, for $y > 0$, is defined as

$$E_f(y) = \{x : x \in \mathbb{R}^N, f(x) \geq y\}\,.$$

The *Steiner symmetrization* in direction u of $f \in L_1^+$ is the function $S_u(f(x)) = \sup\{y : x \in S_u(E_f(y))\}$.

The function $f^* = sup\{y : x \in E_f(y)^*\}$ is called the *spherical symmetrization* of f.

Following now verbatim the proofs of Theorem 4.3 and 4.4 of [16], we obtain the following results.

Proposition 3.4 *If $f \in L_1^+$ and $\{u_n\}$ is the sequence of directions constructed in Theorem 3.3, then the sequence $\{S_{u_1,\ldots,u_n}(f)\}$ converges in L_1 to f^*.*

Proposition 3.5 *If f is continuous with compact support and $\{u_n\}$ is the sequence of directions constructed in Theorem 3.3, the sequence $\{S_{u_1,\ldots,u_n}(f)\}$ converges uniformly to f^*.*

We can now prove our main result.

Proof of Theorem 1.1 If U is countable and dense in the unit N-dimensional sphere S^{N-1}, let us order it as in Theorem 3.3.

If K is a compact set in $\mathbb{R}^N$, consider the continuous function with compact support

$$f(x) = \max\{0, 1 - d(x, K)\}\,.$$

Observe (see Theorem 5.1 of [16]) that uniform convergence of the sequence $\{S_{u_1,\ldots,u_n}(f)\}$ to f^* implies convergence in the Hausdorff distance of the superlevel sets to the superlevel sets of f^*, which are appropriate balls. In particular, this holds for the superlevel set corresponding to $y = 1$, and the conclusion follows.

Remark 3.6 From Theorem 5.1 of [16] or also from Corollary 1 of [5], we deduce the following result.

Suppose P is any probability on U such that $P(\{u\}) > 0$ for each $u \in U$. Then with probability one for a random and independently selected sequence $\{u_n\}$, given any compact set K, $\{S_{u_1,\ldots,u_n}(K)\}$ converges in the Hausforff distance to K^*.

By the Borel–Cantelli lemma, a random and independently selected sequence contains with probability one every $u \in U$, but repetitions are inevitable.

This statement is powerful since it shows the myriad of sequences which make the iterated Steiner symmetrals converge, but it is not an alternative way for proving Theorem 1.1, because with probability one each $u \in U$ is selected infinitely often.

4 Klain's theorem for measurable sets

Our last result concerns iterations of Steiner symmetrization with respect to directions taken from a finite set F. Klain proved that when the seed is a convex body then such sequences always converge in the Hausdorff distance to a convex body which is symmetric under reflection in each of the directions in F which are taken infinitely often.

This result has been extended in [4] to the case of a compact seed.

We will show here that an analogous result holds if the seed in measurable and convergence is taken with respect to d_1.

Theorem 4.1 *Let $\{u_n\}$ be a sequence of vectors chosen from a finite set $F = \{v_1, v_2, \ldots, v_k\}$. Then, for every measurable set $E \subset \mathbb{R}^N$, the symmetrals*

$$S_{u_1,\ldots,u_n} E$$

converge in L_1 distance to a measurable set G. Furthermore, G is symmetric under reflection in each of the directions $v \in F$ that appear in the sequence infinitely often.

For any $m \in \mathbb{N}$, let K^m be a compact set contained in E such that $\lambda_N(K^m) > \lambda_N(E) - \frac{1}{m}$. From the analogous theorem for compact sets proved in [4], we have that for any $m \in \mathbb{N}$, the sequence $\{S_{u_1,\ldots,u_n}(K^m)\}$ tends, in the Hausdorff distance, to a compact set L^m. Put $L = \cup_{m=1}^{\infty} L^m$. Each L^m is symmetric under reflection in each of the directions $v \in F$ that appear in the sequence infinitely often. Therefore, L has the same property.

We shall show that $\{S_{u_1,\ldots,u_n}(E)\}$ converges to L with respect to the d_1 distance.

$$\begin{aligned} &d_1(S_{u_1,\ldots,u_n}(E), L) \\ &\quad \le d_1(S_{u_1,\ldots,u_n}(E), S_{u_i,\ldots,u_n}(K^m)) + d_1(S_{u_1,\ldots,u_n}(K^m), L^m) + d_1(L^m, L)\,. \end{aligned}$$

Fix $\varepsilon > 0$. The first term in the last line is bounded by $\frac{\varepsilon}{3}$ for m large enough. So is the third. Now, for a fixed m large enough, the second term tends to zero by (ii) of Theorem 3.1 of [4], and therefore, for n large enough, it is smaller than $\frac{\varepsilon}{3}$ and the conclusion follows.

The previous theorem shows how simple is to prove a result concerning Steiner symmetrizations of measurable sets when an analogous result for compact sets is available. On the other hand, compact sets include convex bodies; therefore, it is clear that results for compact sets play a central role and that Mani, with his question, drew the attention to an important issue.

Example 4.2 Let K be the triangle with vertices in $(-1, 0)$, $(0, 1)$ and $(1, -1)$ and let $u = (1, 0)$ and $v = (0, 1)$. It is easy to check that $L = S_{u,v}(K) \neq S_{v,u}(K) = H$. The limit depends on the order in which the vectors from $F = \{u, v\}$ are taken. In fact, it depends on the first choice.

Open problems 4.3 1) Under which conditions the limit does not depend on the order in which the directions of $F = \{v_1, v_2, \ldots, v_k\}$ are taken (if we assume that each direction from F is taken infinitely many often)?

2) Given F with at least three elements, there are continuously many different sequences using infinitely many times each $v \in F$. How many different limits may exist?
3) If we take in Example 4.2 the directions in F randomly, then L and H have the same probability to be the limits of $\{S_{u_1,\ldots,v_n}(E)\}$. Can we say something similar in the general case?

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