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Semilinear delay evolution equations with measures subjected to nonlocal initial conditions

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Abstract We prove a global existence result for bounded solutions to a class of abstract semilinear delay evolution equations with measures subjected to nonlocal initial data of the form

$$\begin{cases} \mathrm{d}u(t) = \{Au(t) + f(t, u_t)\}\mathrm{d}t + \mathrm{d}g(t), & t \in \mathbf{R}_+, \\ u(t) = h(u)(t), & t \in [\,-\tau, 0\,], \end{cases}$$

where $\tau \geq 0$, $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup, $f : \mathbf{R}_+ \times \mathcal{R}([\,-\tau, 0\,]; X) \to X$ is continuous, $g \in BV_{\mathrm{loc}}(\mathbf{R}_+; X)$, and $h : \mathcal{R}_b(\mathbf{R}_+; X) \to \mathcal{R}([\,-\tau, 0\,]; X)$ is nonexpansive.

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1 Introduction

The goal of this paper is to prove some sufficient conditions for the global existence and boundedness of $\mathcal{L}^\infty$-solutions for the nonlocal initial problem with measures:

$$\begin{cases} \mathrm{d}u(t) = \{Au(t) + f(t, u_t)\}\mathrm{d}t + \mathrm{d}g(t), & t \in \mathbf{R}_+, \\ u(t) = h(u)(t), & t \in [-\tau, 0]. \end{cases} \tag{1}$$

Here $\tau \geq 0$, A generates a C_0-semigroup of contractions, $\{S(t) : X \to X; t \geq 0\}$, in a real Banach space X, $f : \mathbf{R}_+ \times \mathcal{R}([-\tau, 0]; X) \to X$, $g \in BV_{\mathrm{loc}}(\mathbf{R}_+; X)$, and $h : \mathcal{R}_b(\mathbf{R}_+; X) \to \mathcal{R}([-\tau, 0]; X)$ is nonexpansive.

If I is an interval, we denote by $\mathcal{R}(I; X)$ the space of all piecewise continuous functions from I to X, i.e., the space of all functions having only discontinuities of the first kind. Further, we denote by $\mathcal{R}_b(I; X)$ the space of all bounded functions belonging to $\mathcal{R}(I; X)$.

We notice that, equipped with the uniform convergence on compact intervals topology, $\mathcal{R}(I; X)$ is a separated, locally convex space. Moreover, endowed with the sup-norm, $\mathcal{R}_b(I; X)$ is a real Banach space. If I is a compact interval, each function $u \in \mathcal{R}_b(I; X)$ is automatically bounded, and thus, the space $\mathcal{R}_b(I; X)$ coincides with $\mathcal{R}(I; X)$. If $u \in \mathcal{R}_b([-\tau, +\infty); X)$ and $t \in \mathbf{R}_+$, we denote by u_t the function from $[-\tau, 0]$ to X, defined by

$$u_t(s) := u(t + s)$$

for each $s \in [-\tau, 0]$. Clearly $u_t \in \mathcal{R}([-\tau, 0]; X)$ and the mapping $t \mapsto u_t$ belongs to $\mathcal{R}(\mathbf{R}_+; \mathcal{R}([-\tau, 0]; X))$. For fundamental results on delay equations, see Hale [19].

For delay problems subjected to usual local initial conditions, see the book of Hale [19]. See also Mitidieri and Vrabie [20]. In the specific case where $\tau = 0$ and $h(u) := \xi$, the problem above was studied by Ahmed [1], Amann [2], Grosu [18] and Vrabie [28,29], Benedetti and Rubbioni [7]. We notice that such kind of equations arise in the study of optimal control problems with state constraints. See Barbu and Precupanu [4]. Problem (1) without delay and subjected to a usual initial condition was studied by Vrabie [28, Theorem 8.1] and [29, Theorem 12.2.2, p. 275]. Some extensions to the case in which g may depend on u as well were obtained later by Grosu [18].

As far as nondelay evolution equations subjected to nonlocal initial conditions are concerned, we mention the pioneering works of Byszewski [13,14] and Byszewski and Lakshmikantham [15]. More recent results in this topic are due to Benedetti, Loi and Malaguti [5], Benedetti, Malaguti and Taddei [6], Benedetti, Taddei and Väth [8], García-Falset and Reich [17] and Paicu and Vrabie [23]. For delay evolution equations and inclusions with nonlocal conditions, see Burlică and Roşu [10,11], Necula, Popescu and Vrabie [21], Necula and Vrabie [22], Vrabie [30–33].

The main difficulty occurring in the study of nonlocal initial problems consists in the lack of the semigroup property. More precisely, in this general case, we cannot proceed as in the case of initial local conditions when we prove first a local existence result, and then, under some appropriate hypotheses, by using Zorn's Lemma or one of its weaker variants as, for instance, the Brezis–Browder Ordering Principle [9], we can obtain noncontinuable or even global solutions. By contrary, in this frame, we are forced to solve the problem of global existence directly. So, one may easily realize why this requires some apparently stronger hypotheses than in the classical case of local initial conditions. We mean here the assumption that A generates a semigroup, $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \geq 0\}$, decaying exponentially to 0 when t tends to $+\infty$ and dominates f, i.e., $\ell < \omega$, where $\ell > 0$ is the Lipschitz constant of f with respect to its second argument, while $\omega > 0$ satisfies $\|S(t)\xi\| \leq e^{-\omega t}\|\xi\|$ for each $t > 0$ and $\xi \in \overline{D(A)}$. It is well known that, in the case of initial-value problems, this condition usually ensures the global asymptotic stability of solutions and it is not needed for local or even global existence.

The paper is divided into seven sections. For the sake of clarity, in Sect. 2 we have included some background material on the existence and regularity of $\mathcal{L}^\infty$-solutions and we recall the main compactness arguments we are going to use later on. In Sect. 3, we state the main result and a consequence referring to the nondelay case which also is new. In Sect. 4, we prove some particular cases referring either to local initial-value problems or even to nonlocal initial-value problems for which the history function is a strict contraction. Section 5 is devoted to an auxiliary global existence, uniqueness and boundedness result. In Sect. 6, we prove our main result, while Sect. 7 presents an application to a delay parabolic problem subjected to nonlocal initial conditions.

2 Preliminaries

For easy reference, we begin by recalling some results established in Vrabie [28] and [29].

Let $\mathcal{D}[\,0, T\,]$ be the set of all partitions of the interval $[\,0, T\,]$. If $g : [\,0, T\,] \to X$, then for each $\Delta \in \mathcal{D}[\,0, T\,]$, $\Delta : 0 = t_0 < t_1 < \cdots < t_k = T$, and the number

$$Var_\Delta(g, [\,0, T\,]) = \sum_{i=0}^{k-1} \|g(t_{i+1}) - g(t_i)\|$$

is called the *variation of the function* g *relative to the partition* Δ. If

$$\sup_{\Delta\in\mathcal{D}[\,0,T\,]} Var_\Delta(g, [\,0, T\,]) < \infty,$$

then g is said to be *of bounded variation*, and the number

$$Var(g, [\,0, T\,]) = \sup_{\Delta\in\mathcal{D}[\,0,T\,]} Var_\Delta(g, [\,0, T\,])$$

is called *the variation* of the function g on the interval $[\,0, T\,]$. We denote by $BV([\,0, T\,]; X)$ the space of all functions of bounded variation from $[\,0, T\,]$ to X. Endowed with the norm

$$\|g\| := \|g(0)\| + Var(g, [\,0, T\,])$$

for each $g \in BV([\,0, T\,]; X)$, this is a real Banach space. We also denote by $BV_{\rm loc}(\mathbf{R}_+; X)$ the space of all functions $u : \mathbf{R}_+ \to X$ with $u \in BV([0, T]; X)$ for each $T > 0$.

Let $\{S(t) : X \to X; t \geq 0\}$ be a C_0-semigroup. We say that it is compact if $S(t)$ is a compact operator for each $t > 0$. Basic facts on C_0-semigroups we need in this paper can be found in Pazy [24] and Vrabie [29].

Let $t \in (0, T]$, $\Delta : 0 = t_0 < t_1 < \cdots < t_k = t$ and let $\tau_i \in [t_i, t_{i+1}]$, for $i = 0, 1, \cdots, k-1$. Let us consider *the Riemann–Stieltjes sum* of $\tau \to S(t-\tau)$ over $[0, t]$ with respect to g, i.e.,

$$\sigma_{[0,t]}(\Delta, S, g, \tau_i) = \sum_{i=0}^{k-1} S(t-\tau_i)(g(t_{i+1}) - g(t_i)).$$

If $\Delta \in \mathcal{D}[0, t]$, we denote by $\mu(\Delta) = \max_{i=0,1,\ldots k-1}(t_{i+1} - t_i)$.

Theorem 1 *If* $\{S(t) : X \to X; t \geq 0\}$ *is continuous from* $(0, +\infty)$ *to* $\mathcal{L}(X)$ *in the uniform operator topology, then, for each* $g \in BV([0, T]; X)$ *and* $t \in (0, T]$*, the limit*

$$\int_0^t S(t-s)\mathrm{d}g(s) = \lim_{\mu(\Delta)\downarrow 0} \sum_{i=0}^{k-1} S(t-\tau_i)(g(t_{i+1}) - g(t_i)) \tag{2}$$

exists in the norm topology of X.

See Vrabie [28, Theorem 2.1] or Vrabie [29, Theorem 9.1.1, p. 208].

For more details on the integral in (2)—called the Riemann–Stieltjes integral of $s \mapsto S(t-s)$ with respect to g —see Vrabie [29, Chapter 9, pp. 205–222]. A different approach to defining $\int_0^t S(t-s)\mathrm{d}g(s)$ is due to Amann [2] who considered only the case of analytic C_0-semigroups.

Next, let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of contractions $\{S(t) : X \to X; t \geq 0\}$, $\xi \in X$ and $g \in BV([0, T]; X)$. Let us consider the nonhomogeneous Cauchy problem with measures:

$$\begin{cases} \mathrm{d}u(t) = [Au(t)]\mathrm{d}t + \mathrm{d}g(t), & t \in [0, T] \\ u(0) \;\; = \xi. \end{cases} \tag{3}$$

Definition 1 A function $u : [0, T] \to X$ is called an $\mathcal{L}^\infty$*-solution on* $[0, T]$ of the Problem (3) if

$$u(t) = S(t)\xi + \int_0^t S(t-s)\mathrm{d}g(s)$$

for each $t \in [0, T]$, where the integral on the left-hand side is defined by (2).

If $g \in BV_{\mathrm{loc}}(\mathbf{R}_+; X)$, the function $u : \mathbf{R}_+ \to X$ is called an $\mathcal{L}^\infty$*-solution on* $\mathbf{R}_+$ if for each $T > 0$ u is an $\mathcal{L}^\infty$-solution on $[0, T]$ of the Problem (3) in the sense specified above.

See Vrabie [28, Definition 2.1].

Theorem 2 (Regularity of $\mathcal{L}^\infty$-solutions) *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* C_0*-semigroup of contractions* $\{S(t) : X \to X; t \geq 0\}$ *which is continuous from* $(0, +\infty)$ *to* $\mathcal{L}(X)$ *in the uniform operator topology. Let* $\xi \in X$*,* $g \in BV_{\mathrm{loc}}(\mathbf{R}_+; X)$ *and let u be the* $\mathcal{L}^\infty$*-solution of (3) corresponding to* ξ *and g. Then, for each* $t \in \mathbf{R}_+$ *and each* $s \in (0, +\infty)$*, we have:*

$$\begin{cases} u(t+0) - u(t) = g(t+0) - g(t), \\ u(s) - u(s-0) = g(s) - g(s-0). \end{cases}$$

So, in this case, u is continuous from the right (*left*) *at* $t \in \mathbf{R}_+$ *if and only if g is continuous from the right* (*left*) *at t. In particular, u is continuous at any point at which g is continuous, and thus, u is piecewise continuous on* $\mathbf{R}_+$.

See Vrabie [28, Theorem 3.1] or Vrabie [29, Theorem 9.2.1, p. 210] .

Remark 1 We notice that each $\mathcal{L}^\infty$-solution u satisfies

$$\|u(t)\| \le \|S(t)\|\|\xi\| + \int_0^t \|S(t-s)\| \mathrm{d}Var(g, [\,0, s\,]),$$

for each $T > 0$ and $t \in [\,0, T\,]$.

Theorem 3 (Tychonoff) *Let $\mathcal{X}$ be a separated locally convex topological vector space and let K be a nonempty, convex and closed subset in $\mathcal{X}$. If $Q : K \to K$ is continuous and $Q(K)$ is relatively compact, then it has at least one fixed point, i.e., there exists $\xi \in K$ such that $Q(\xi) = \xi$.*

See Tychonoff [25] or Edwards [16, Theorem 3.6.1, 161].

Throughout, for each $\xi \in X$ and $f \in L^1(0, T; X)$, we will denote $u := \mathcal{M}(\xi, f)$, the mild, or $\mathcal{L}^\infty$-solution of the problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [\,0, T\,], \\ u(0) = \xi. \end{cases}$$

i.e., the function $u : [\,0, T\,] \to X$, defined by

$$u(t) = S(t)\xi + \int_0^t S(t-s) f(s)\, \mathrm{d}s$$

for each $t \in [\,0, T\,]$.

A set $\mathcal{F} \subseteq L^1(0, T; X)$ is called *uniformly integrable* if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, for each $E \subseteq [\,0, T\,]$ whose Lebesgue measure satisfies $\mu(E) \le \delta(\varepsilon)$, we have

$$\int_E \|f(t)\|\, \mathrm{d}t \le \varepsilon,$$

uniformly for $f \in \mathcal{F}$.

Theorem 4 (Baras–Hassan–Veron) *Let $A : D(A) \subseteq X \to X$ be the generator of a compact C_0-semigroup, let $D \subset X$ be a bounded subset, and let $\mathcal{F}$ be a uniformly integrable subset in $L^1(0, T; X)$. Then $\mathcal{M}(\mathcal{D}, \mathcal{F})$ is relatively compact in $C([\,\delta, T\,]; X)$ for each $\delta \in (0, T)$. If, in addition, D is relatively compact, then $\mathcal{M}(\mathcal{D}, \mathcal{F})$ is relatively compact even in $C([\,0, T\,]; X)$.*

See Baras et al. [3] or Vrabie [29, Theorem 8.4.2, p. 196]. Some extensions of this result to the case in which $A : D(A) \subseteq X \to X$ is a nonlinear m-dissipative operator can be found in Vrabie [27, Theorem 2.3.3, p. 47] and Vrabie [26].

3 A global existence theorem

Let X be a real Banach space, $A : D(A) \subseteq X \to X$ the infinitesimal generator of a C_0-semigroup $\{S(t) : X \to X; t \ge 0\}$, let $\xi \in X$, let $g \in BV_{\mathrm{loc}}(\mathbf{R}_+; X)$ be a given function, and let $h : \mathcal{R}_b(\mathbf{R}_+; X) \to \mathcal{R}([\,-\tau, 0\,]; X)$ be a nonexpansive function. Let us consider the Cauchy problem:

$$\begin{cases} \mathrm{d}u(t) = [Au(t) + f(t, u_t)]\, \mathrm{d}t + \mathrm{d}g(t), & t \in \mathbf{R}_+, \\ u(t) \;\; = h(u)(t), & t \in [\,-\tau, 0\,]. \end{cases} \tag{4}$$

By an L^∞-solution of the Problem (4), we mean a piecewise continuous function $u : [-\tau, +\infty) \to X$ satisfying

$$u(t) = \begin{cases} h(u)(t), & t \in [-\tau, 0], \\ S(t)h(u)(0) + \int_0^t S(t-s) f(s, u_s)\, ds + \int_0^t S(t-s) dg(s), & t \in \mathbf{R}_+. \end{cases}$$

Remark 2 One may easily see that an $\mathcal{L}^\infty$-solution of the Problem (4) is a function $\widetilde{u} : [-\tau, +\infty) \to X$,

$$\widetilde{u}(t) := \begin{cases} h(u)(t), & t \in [-\tau, 0], \\ u(t), & t \in \mathbf{R}_+, \end{cases}$$

where, for each $T > 0$, u is an $\mathcal{L}^\infty$-solution in the sense of Definition 1 of the problem below

$$\begin{cases} du(t) = [Au(t)]dt + d\widetilde{g}(t), & t \in [0, T], \\ u(0) \ = h(u)(0), \end{cases}$$

where

$$\widetilde{g}(t) := g(t) + \int_0^t f(s, u_s)\, ds$$

for each $t \in [0, T]$.

For the sake of simplicity, we confine ourselves to the case of C_0-semigroups of contractions although our results can be extended to general C_0-semigroups. More precisely, we will use the following general hypotheses:

(H_A) $A : D(A) \subseteq X \to X$ generates a C_0-semigroup on X satisfying:

(A_1) $\|S(t)\| \le e^{-\omega t}$ for each $t \in \mathbf{R}_+$;
(A_2) the semigroup $\{S(t) : X \to X; t \ge 0\}$ is compact.

(H_f) $f : \mathbf{R}_+ \times \mathcal{R}([-\tau, 0]; X) \to X$ is continuous and satisfies:

(f_1) there exist $\ell > 0$ and $m \ge 0$ such that $\|f(t, u)\| \le \ell \|u\|_{\mathcal{R}([-\tau,0];X)} + m$ for each $t \in \mathbf{R}_+$ and $u \in \mathcal{R}([-\tau, 0]; X)$;
(f_2) with $\ell > 0$ given by (f_1), we have

$$\|f(t, u) - f(t, v)\| \le \ell \|u - v\|_{\mathcal{R}([-\tau,0];X)}$$

for each $t \in \mathbf{R}_+$ and each $u, v \in \mathcal{R}([-\tau, 0]; X)$.

(H_g) The function $g \in BV_{\text{loc}}(\mathbf{R}_+; X)$, and there exists $m_1 > 0$ such that

$$\int_0^t e^{\omega s} d\text{Var}\,(g, [0, s]) \le m_1$$

for each $t \in \mathbf{R}_+$.
(H_h) The restriction of the function $h : \mathcal{R}_b(\mathbf{R}_+; X) \to \mathcal{R}([-\tau, 0]; X)$ on every bounded subset in $\mathcal{R}_b(\mathbf{R}_+; X)$ is continuous with respect to the induced locally convex topology of $\mathcal{R}(\mathbf{R}_+; X)$ to $\mathcal{R}([-\tau, 0]; X)$ and there exists $a > 0$ such that

$$\|h(u) - h(v)\|_{\mathcal{R}([-\tau,0];X)} \le \|u - v\|_{\mathcal{R}_b([a,+\infty);X)}$$

for each $u, v \in \mathcal{R}_b(\mathbf{R}_+; X)$.
(H_c) The constants ℓ and ω satisfy $\ell < \omega$.

Remark 3 Let $\{S(t) : X \to X; t \geq 0\}$ be a C_0-semigroup continuous on $(0, +\infty)$ in the uniform operator topology, satisfying (A_1) in (H_A). Let $g \in BV_{\mathrm{loc}}(\mathbf{R}_+; X)$ be a function satisfying (H_g). Then

$$\left\| \int_0^t S(t-s)\mathrm{d}g(s) \right\| \leq e^{-\omega t} m_1$$

for each $t \in \mathbf{R}_+$. Indeed,

$$\left\| \int_0^t S(t-s)\mathrm{d}g(s) \right\| \leq e^{-\omega t} \int_0^t e^{\omega s} \mathrm{d}\mathrm{Var}\,(g, [\,0, s\,]) \leq e^{-\omega t} m_1,$$

as claimed.

Our main result is:

Theorem 5 *Let us assume that (H_A), (H_f), (H_g) (H_h) and (H_c) are satisfied. Then, the Problem (1) has at least one $\mathcal{L}^\infty$-solution $u \in \mathcal{R}_b([\,-\tau, +\infty); X)$.*

As far as the nondelay case is concerned, let us observe first that, for $\tau = 0$, $\mathcal{R}([\,-\tau, 0\,]; X)$ reduces to X and, for each $u \in \mathcal{R}_b(\mathbf{R}_+; X)$ and $t \in \mathbf{R}_+$, $u_t = u(t)$. So the Problem (4) rewrites as:

$$\begin{cases} \mathrm{d}u(t) = [Au(t) + f(t, u(t))]\,\mathrm{d}t + \mathrm{d}g(t),\ t \in \mathbf{R}_+, \\ u(0) = h(u). \end{cases} \tag{5}$$

So, in this specific case, the hypotheses (H_f) and (H_h) take the form:

$(H_f^{[\tau=0]})$ $f : \mathbf{R}_+ \times X \to X$ is continuous and satisfies:

$(f_1^{[\tau=0]})$ there exist $\ell > 0$ and $m \geq 0$ such that $\|f(t,u)\| \leq \ell\|u\| + m$ for each $t \in \mathbf{R}_+$ and $u \in X$;

$(f_2^{[\tau=0]})$ with $\ell > 0$ given by (f_1), we have

$$\|f(t,u) - f(t,v)\| \leq \ell\|u - v\|$$

for each $t \in \mathbf{R}_+$ and each $u, v \in X$.

$(H_h^{[\tau=0]})$ The restriction of the function $h : \mathcal{R}_b(\mathbf{R}_+; X) \to X$ is continuous on every bounded subset in $\mathcal{R}_b(\mathbf{R}_+; X)$ with respect to the induced locally convex topology of $\mathcal{R}(\mathbf{R}_+; X)$ to X and there exists $a > 0$ such that

$$\|h(u) - h(v)\| \leq \|u - v\|_{\mathcal{R}_b([\,a,+\infty);X)}$$

for each $u, v \in \mathcal{R}_b(\mathbf{R}_+; X)$.

and from Theorem 5, we deduce

Theorem 6 *Let us assume that (H_A), $(H_f^{[\tau=0]})$, (H_g) $(H_h^{[\tau=0]})$ and (H_c) are satisfied. Then, the Problem (5) has at least one $\mathcal{L}^\infty$-solution $u \in \mathcal{R}_b(\mathbf{R}_+; X)$.*

Remark 4 We notice that the nonlocal initial condition contains as particular cases:

(i) The periodic condition $u(0) = u(T)$ which corresponds to the choice of h as $h(u) := u(T)$;
(ii) The anti-periodic condition $u(0) = -u(T)$ which corresponds to the choice of h as $h(u) := -u(T)$;

(iii) The mean condition

$$u(0) = \sum_{i=1}^{\infty} \alpha_i u(t_i),$$

where $0 < t_1 < t_2 < \cdots < t_n < \cdots$ and $\alpha \in \mathbf{R}$ satisfy

$$\sum_{i=1}^{\infty} \alpha_i \leq 1,$$

which corresponds to the choice of h as

$$h(u) := \sum_{i=1}^{\infty} \alpha_i u(t_i).$$

A similar remark applies to the general delay case.

4 Preliminary results

For the sake of simplicity, we divide the proof of Theorem 5 into several steps, the first one being the following lemma which is generalization, in the semilinear case, to delay evolutions with measures, of a result in Vrabie [31, Lemma 4.3].

Lemma 1 *Let us assume that* (H_A), (H_f), (H_g) *and* (H_c) *are satisfied. Then, for each* $\varphi \in \mathcal{R}([-\tau, 0]; X)$*, the problem*

$$\begin{cases} \mathrm{d}u(t) = [Au(t) + f(t, u_t)]\,\mathrm{d}t + \mathrm{d}g(t), & t \in \mathbf{R}_+, \\ u(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{6}$$

has a unique $\mathcal{L}^\infty$*-solution* $u \in \mathcal{R}_b([-\tau, +\infty); X)$.

Proof Let $v \in \mathcal{R}_b([-\tau, +\infty); X)$. Since, by (A_2) of (H_A), the C_0-semigroup generated by A is compact, it is continuous on $(0, +\infty)$ in the uniform operator topology. So, Theorem 1 applies and we can define the function $u : \mathbf{R}_+ \to X$ by

$$u(t) := S(t)\varphi(0) + \int_0^t S(t-s) f(s, v_s)\,\mathrm{d}s + \int_0^t S(t-s)\mathrm{d}g(s),$$

for each $t \in \mathbf{R}_+$. By (f_1) in (H_f), (H_g) and Theorem 2, it follows that the function u belongs to $\mathcal{R}_b([-\tau, +\infty); X)$.

We will show first that u is bounded. Indeed, if $t \geq 0$, thanks to (H_A), (H_f), (H_g) and Remark 3, we deduce

$$\|u(t)\| \leq e^{-\omega t}\|\varphi(0)\| + \ell\left[\int_0^t e^{-\omega(t-s)}\left(\|v_s\|_{\mathcal{R}([-\tau,0];X)} + \frac{m}{\ell}\right)\right]\mathrm{d}s + e^{-\omega t} m_1$$

and thus

$$\|u(t)\| \leq e^{-\omega t}\|\varphi\|_{\mathcal{R}([-\tau,0];X)} + (1 - e^{-\omega t})\frac{\ell}{\omega}\left[\|v\|_{\mathcal{R}_b([-\tau,+\infty);X)} + \frac{m}{\ell}\right] + e^{-\omega t} m_1,$$

for each $t \in \mathbf{R}_+$. Therefore, u belongs to $\mathcal{R}_b([-\tau, +\infty); X)$.

So, the operator

$$Q : \mathcal{R}_b([-\tau, +\infty); X) \to \mathcal{R}_b([-\tau, +\infty); X),$$
$$Q(v)(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0], \\ S(t)\varphi(0) + \int_0^t S(t-s) f(s, v_s)\,ds + \int_0^t S(t-s)dg(s), & t \in \mathbf{R}_+ \end{cases} \tag{7}$$

is well defined. Clearly (6) has a unique $\mathcal{L}^\infty$-solution if and only if the operator Q has a unique fixed point.

Let $v, w \in \mathcal{R}_b([-\tau, +\infty); X)$ and let us observe that

$$\begin{aligned} \|Q(v)(t) - Q(w)(t)\| &\le \ell \int_0^t e^{-\omega(t-s)} \|v_s - w_s\|_{\mathcal{R}([-\tau,0];X)}\,ds \\ &\le \frac{\ell}{\omega}\left(1 - e^{-\omega t}\right)\|v - w\|_{\mathcal{R}_b([-\tau,+\infty);X)} \\ &\le \frac{\ell}{\omega}\|v - w\|_{\mathcal{R}_b([-\tau,+\infty);X)} \end{aligned}$$

for each $t \in \mathbf{R}_+$.

Since

$$\|Q(v) - Q(w)\|_{\mathcal{R}([-\tau,0];X)} = \|\varphi - \varphi\|_{\mathcal{R}([-\tau,0];X)} = 0,$$

it follows that

$$\|Q(v) - Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)} = \|Q(v) - Q(w)\|_{\mathcal{R}_b(\mathbf{R}_+;X)}.$$

So, we get

$$\|Q(v) - Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)} \le \frac{\ell}{\omega}\|v - w\|_{\mathcal{R}_b([-\tau,+\infty);X)}.$$

Consequently, by (H_c), Q is strict contraction on $\mathcal{R}_b([-\tau, +\infty); X)$, and by Banach's fixed point theorem, it has a unique fixed point u. The proof is complete. □

Theorem 7 *Let us assume that (H_A), (f_1) in (H_f), (H_g) and (H_c) are satisfied. Then, for each $\varphi \in \mathcal{R}([-\tau, 0]; X)$, the problem*

$$\begin{cases} du(t) = [Au(t) + f(t, u_t)]\,dt + dg(t), & t \in \mathbf{R}_+, \\ u(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{8}$$

has at least one $\mathcal{L}^\infty$-solution $u \in \mathcal{R}_b([-\tau, +\infty); X)$.

Proof We define the operator Q by (7), and we observe that (8) has at least one $\mathcal{L}^\infty$-solution if and only if Q has at least one fixed point.

We will show that, for a suitably chosen $r > 0$, Q maps the closed ball with center 0 and radius r in $\mathcal{R}_b(\mathbf{R}_+; X)$, denoted by $B(0, r)$, into itself, is continuous and compact in the locally convex topology of $\mathcal{R}(\mathbf{R}_+; X)$ on both domain and range.

Let $v \in \mathcal{R}_b([-\tau, +\infty); X)$. By Remark 1, Remark 3, (A_1) in (H_A), (f_1) in (H_f) and (H_g), we conclude that $Q(v)$, defined by (7), satisfies

$$\begin{aligned} \|Q(v)(t)\| &\le e^{-\omega t}\|\varphi(0)\| + \ell \int_0^t e^{-\omega(t-s)}\left(\|v_s\|_{\mathcal{R}([-\tau,0];X)} + \frac{m}{\ell}\right) ds + e^{-\omega t} m_1 \\ &\le e^{-\omega t}\|\varphi\|_{\mathcal{R}([-\tau,0];X)} + \ell \int_0^t e^{-\omega(t-s)}\left(\|v\|_{\mathcal{R}_b([-\tau,+\infty);X)} + \frac{m}{\ell}\right) ds + e^{-\omega t} m_1 \end{aligned}$$

for each $t \in \mathbf{R}_+$. Hence, if $r > 0$ and

$$\|v\|_{\mathcal{R}_b([-\tau,+\infty);X)} \leq r,$$

we get

$$\|Q(v)(t)\| \leq e^{-\omega t}\|\varphi\|_{\mathcal{R}([-\tau,0];X)} + \frac{\ell}{\omega}(1-e^{-\omega t})\left(r+\frac{m}{\ell}\right) + e^{-\omega t}m_1$$

for each $t \in \mathbf{R}_+$. Thus

$$\|Q(v)(t)\| \leq \|\varphi\|_{\mathcal{R}([-\tau,0];X)} + \frac{\ell}{\omega}\left(r+\frac{m}{\ell}\right) + m_1$$

for each $t \in \mathbf{R}_+$. Next, let us fix $r > 0$ satisfying

$$\|\varphi\|_{\mathcal{R}([-\tau,0];X)} + \frac{\ell}{\omega}\left(r+\frac{m}{\ell}\right) + m_1 \leq r \tag{9}$$

which is always possible because, by (H_c), $\ell < \omega$.

So, if $r > 0$ is as above and $v \in B(0,r)$, by (9), we get

$$\begin{aligned}\|Q(v)\|_{\mathcal{R}_b([-\tau,+\infty);X)} &= \max\left\{\|Q(v)\|_{\mathcal{R}([-\tau,0];X)}, \|Q(v)\|_{\mathcal{R}_b(\mathbf{R}_+;X)}\right\}\\ &= \max\left\{\|\varphi\|_{\mathcal{R}([-\tau,0];X)}, r\right\} = r\end{aligned}$$

which shows that Q maps $B(0,r)$ into itself. It is easy to observe that $B(0,r)$ is closed in $\mathcal{R}([-\tau,+\infty);X)$ being closed in $\mathcal{R}_b([-\tau,+\infty);X)$.

We will prove next that Q is continuous from $B(0,r)$ into itself with respect to the topology of $\mathcal{R}([-\tau,+\infty);X)$ on both domain and range. To this aim, let $(v_n)_n$ be a sequence in $B(0,r)$ and $v \in B(0,r)$ with

$$\lim_n v_n = v \tag{10}$$

in $\mathcal{R}([-\tau,+\infty);X)$, i.e., uniformly on compact subsets in $[-\tau,+\infty)$. We have

$$\|Q(v_n)(t) - Q(v)(t)\| \leq \int_0^t e^{-\omega(t-s)}\|f(s,v_{ns}) - f(s,v_s)\|\,\mathrm{d}s$$

for each $n \in \mathbf{N}$, each $p \in \mathbf{N}$ and each $t \in [0,p]$. From (10), we conclude that

$$\lim_n v_{ns} = v_s$$

for each $s \in \mathbf{R}_+$. Since f is continuous and satisfies (f_1) in (H_f), an appeal to Lebesgue's dominated convergence theorem shows that

$$\lim_n Q(v_n) = Q(v)$$

uniformly on $[0,p]$, for each $p \in \mathbf{N}$. So, Q is continuous from $B(0,r)$ into itself with respect to the topology of $\mathcal{R}([-\tau,+\infty);X)$ on both domain and range.

Finally, we will show that $Q(B(0,r))$ is relatively compact in $\mathcal{R}([-\tau,+\infty);X)$. Let $p \in \mathbf{N}$ be arbitrary and let us observe that, by (f_1) in (H_f), the set

$$\{s \mapsto f(s,v_s);\ v \in B(0,r)\}$$

is uniformly bounded and thus uniformly integrable on $[0,p]$. As by (A_2) in (H_A), the C_0-semigroup generated by A is compact, we are in the hypotheses of Baras, Hassan and Veron Theorem 4, wherefrom it follows that, for each $p \in \mathbf{N}$, the family of functions

$$\mathcal{F} := \left\{t \mapsto S(t)\varphi(0) + \int_0^t S(t-s)f(s,v_s)\,\mathrm{d}s;\ v \in B(0,r)\right\}$$

is relatively compact in $C([0, p]; X)$. Consequently, for each $p \in \mathbf{N}$,

$$Q(B(0,r)) = \begin{cases} \mathcal{F} + \left\{ t \mapsto \int_0^t S(t-s)\mathrm{d}g(s) \right\}, & t \in \mathbf{R}_+ \\ \{t \mapsto \varphi(t)\}, & t \in [-\tau, 0] \end{cases}$$

is relatively compact in $\mathcal{R}([-\tau, p]; X)$ or, equivalently, in $\mathcal{R}([-\tau, +\infty); X)$. By Tychonoff's fixed point Theorem 3, it follows that Q has at least one fixed point $u \in B(0, r)$, which clearly is an $\mathcal{L}^\infty$-solution of the Problem (8) as claimed. □

Theorem 8 *Let us assume that* (H_A), (H_f), (H_g), (H_h) *and* (H_c) *are satisfied and, in addition, h is a strict contraction. Then the problem*

$$\begin{cases} \mathrm{d}u(t) = [Au(t) + f(t, u_t)]\,\mathrm{d}t + \mathrm{d}g(t), & t \in \mathbf{R}_+, \\ u(t) \quad = h(u)(t), & t \in [-\tau, 0], \end{cases} \tag{11}$$

has a unique $\mathcal{L}^\infty$*-solution* $u \in \mathcal{R}_b([-\tau, +\infty); X)$.

Proof Let $v \in \mathcal{R}_b([-\tau, +\infty); X)$ and let us consider the problem

$$\begin{cases} \mathrm{d}u(t) = [Au(t) + f(t, u_t)]\,\mathrm{d}t + \mathrm{d}g(t), & t \in \mathbf{R}_+, \\ u(t) \quad = h(v)(t), & t \in [-\tau, 0]. \end{cases} \tag{12}$$

By Lemma 1, the Problem (12) has a unique $\mathcal{L}^\infty$-solution $u \in \mathcal{R}_b([-\tau, +\infty); X)$. So, we can define the operator

$$Q : \mathcal{R}_b([-\tau, +\infty); X) \to \mathcal{R}_b([-\tau, +\infty); X),$$

defined by

$$Q(v) := u,$$

where u is the unique $\mathcal{L}^\infty$-solution of (12), i.e.,

$$u(t) = \begin{cases} h(v)(t), & t \in [-\tau, 0], \\ S(t)h(v)(0) + \int_0^t S(t-s) f(s, u_s)\,\mathrm{d}s + \int_0^t S(t-s)\mathrm{d}g(s), & t \in \mathbf{R}_+. \end{cases}$$

At this point, let us observe that $Q(u) = u$ if and only if u is an $\mathcal{L}^\infty$-solution of the Problem (11). Thus, to complete the proof, it suffices to show that Q has a unique fixed point in $\mathcal{R}_b([-\tau, +\infty); X)$. To this aim, we will show that Q is a strict contraction of constant $k \in (0, 1)$, where k is the Lipschitz constant of h. Let $v, w \in \mathcal{R}_b([-\tau, +\infty); X)$ and let us observe that, by (f_1) in (H_f), we have

$$\begin{aligned} \|Q(v)(t) - Q(w)(t)\| &\le e^{-\omega t}\|Q(v)(0) - Q(w)(0)\| \\ &\quad + \ell \int_0^t e^{-\omega(t-s)} \|[Q(v)]_s - [Q(w)]_s\|_{\mathcal{R}([-\tau,0];X)}\,\mathrm{d}s \\ &\le e^{-\omega t}\|Q(v) - Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)} \\ &\quad + \frac{\ell}{\omega}\left(1 - e^{-\omega t}\right)\|Q(v) - Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)} \end{aligned}$$

for each $t \in \mathbf{R}_+$. We distinguish between (noncomplementary) cases.
Case 1. If there exists $t_0 > 0$ such that

$$\|Q(v) - Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)} = \limsup_{t \to t_0} \|Q(v)(t) - Q(w)(t)\|,$$

then passing to the lim sup in the last inequality, we obtain

$$\begin{aligned}&\left(1-e^{-\omega t_0}\right)\|Q(v)-Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)}\\&\leq\frac{\ell}{\omega}\left(1-e^{-\omega t_0}\right)\|Q(v)-Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)}\end{aligned}$$

which shows that

$$\|Q(v)-Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)}=0.$$

Case 2. If there exists $t_0\in[-\tau,0]$ such that

$$\|Q(v)-Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)}=\limsup_{t\to t_0}\|Q(v)(t)-Q(w)(t)\|,$$

then

$$\|Q(v)-Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)}=\|Q(v)-Q(w)\|_{\mathcal{R}([-\tau,0];X)},$$

and so, by (H_h), we obtain

$$\|Q(v)-Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)}\leq k\|v-w\|_{\mathcal{R}_b([a,+\infty);X)}\leq k\|v-w\|_{\mathcal{R}_b([-\tau,+\infty);X)}.$$

We notice that if $t_0=0$ and

$$\|Q(v)-Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)}=\limsup_{t\downarrow 0}\|Q(v)(t)-Q(w)(t)\|,$$

then we can pass to the limit directly in the inequality preceding **Case 1** to get the same conclusion as above.
Case 3. If

$$\|Q(v)-Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)}=\limsup_{t\to+\infty}\|Q(v)(t)-Q(w)(t)\|,$$

then reasoning as in **Case 1**, we deduce that

$$\|Q(v)-Q(w)\|_{\mathcal{R}_b([-\tau,+\infty);X)}=0.$$

So, Q is a strict contraction and this completes the proof. □

5 An auxiliary lemma

We need the following extension of a result in Burlică and Roşu [12, Lemma 4.1, formula (4.3)] valid for continuous functions.

Lemma 2 *Let us assume that (H_A), (H_h), (H_g) and (H_c) are satisfied. Then, for each $f\in L^\infty(\mathbf{R}_+;X)$, the problem*

$$\begin{cases}\mathrm{d}u(t)=[Au(t)+f(t)]\mathrm{d}t+\mathrm{d}g(t), & t\in\mathbf{R}_+,\\ u(t)\;=h(u)(t), & t\in[-\tau,0]\end{cases}\tag{13}$$

has a unique $\mathcal{L}^\infty$-solution $u\in\mathcal{R}_b([-\tau,+\infty);X)$ satisfying

$$\|u\|_{\mathcal{R}_b(\mathbf{R}_+;X)}\leq\frac{e^{\omega a}}{e^{\omega a}-1}m_2+\frac{1}{\omega}\|f\|_{L^\infty(\mathbf{R}_+;X)},\tag{14}$$

where $m_2:=m_0+m_1$, $m_0=\|h(0)\|_{\mathcal{R}([-\tau,0];X)}$ and m_1 is given by (H_g).

Proof Let $f \in L^\infty(\mathbf{R}_+; X)$ be arbitrary but fixed. Let $w \in \mathcal{R}_b([-\tau, +\infty); X)$ and let us consider the auxiliary problem

$$\begin{cases} du(t) = [Au(t) + f(t)]dt + dg(t), & t \in \mathbf{R}_+, \\ u(t) = h(w)(t), & t \in [-\tau, 0]. \end{cases} \tag{15}$$

Reasoning as in Lemma 5 with $f(t, u_t)$ replaced by $f(t)$, it follows that (15) has a unique $\mathcal{L}^\infty$-solution $u \in \mathcal{R}_b([-\tau, +\infty); X)$ satisfying

$$\|u(t)\| \le e^{-\omega t}\left[\|w\|_{\mathcal{R}_b([a,+\infty);X)} + m_2\right] + \left(1 - e^{-\omega t}\right)\frac{1}{\omega}\|f\|_{L^\infty(\mathbf{R}_+;X)}. \tag{16}$$

Thus, we can define the operator

$$T : \mathcal{R}_b([-\tau, +\infty); X) \to \mathcal{R}_b([-\tau, +\infty); X)$$

by

$$T(w) = u,$$

where $u \in \mathcal{R}_b([-\tau, +\infty); X)$ is the unique $\mathcal{L}^\infty$-solution of the Problem (15). Obviously, (15) has a unique $\mathcal{L}^\infty$-solution if and only if T has a unique fixed point.

Then, to complete the proof, it suffices to show that T^2 is a strict contraction. The idea below goes back to Vrabie [30]. Here, we follow the arguments in Burlică and Roşu [10] slightly modified to handle the case of $\mathcal{L}^\infty$-solutions which may fail to be continuous. So, let $w, z \in \mathcal{R}_b([-\tau, +\infty); X)$. We have

$$\|Tw - Tz\|_{\mathcal{R}_b([a,+\infty);X)} \le e^{-\omega a}\|w - z\|_{\mathcal{R}_b([a,+\infty);X)}. \tag{17}$$

Indeed, if $t \ge a$ using (H_h), we obtain

$$\begin{aligned} \|(Tw)(t) - (Tz)(t)\| &\le e^{-\omega t}\|h(w)(0) - h(z)(0)\| \\ &\le e^{-\omega t}\|h(w) - h(z)\|_{\mathcal{R}([-\tau,0];X)} \\ &\le e^{-\omega a}\|w - z\|_{\mathcal{R}_b([a,+\infty);X)}. \end{aligned}$$

Now we prove that

$$\|T^2 w - T^2 z\|_{\mathcal{R}_b([-\tau,+\infty);X)} \le e^{-\omega a}\|w - z\|_{\mathcal{R}_b([-\tau,+\infty);X)}. \tag{18}$$

First, let $t \ge 0$. By (H_h) and (17), we have

$$\begin{aligned} \|(T^2 w)(t) - (T^2 z)(t)\| &\le e^{-\omega t}\|(h(Tw))(0) - (h(Tz))(0)\| \\ &\le e^{-\omega t}\|h(Tw) - h(Tz)\|_{\mathcal{R}([-\tau,0];X)} \\ &\le \|Tw - Tz\|_{\mathcal{R}_b([a,+\infty);X)} \\ &\le e^{-\omega a}\|w - z\|_{\mathcal{R}_b([a,+\infty);X)} \\ &\le e^{-\omega a}\|w - z\|_{\mathcal{R}_b([-\tau,+\infty);X)}. \end{aligned}$$

Second, if $t \in [-\tau, 0)$, using the same inequalities, i.e., (H_h) and (17), we get

$$\begin{aligned} \|(T^2 w)(t) - (T^2 z)(t)\| &= \|(h(Tw))(t) - (h(Tz))(t)\| \\ &\le \|h(Tw) - h(Tz)\|_{\mathcal{R}([-\tau,0];X)} \\ &\le \|Tw - Tz\|_{\mathcal{R}_b([a,+\infty);X)} \\ &\le e^{-\omega a}\|w - z\|_{\mathcal{R}_b([-\tau,+\infty);X)}. \end{aligned}$$

Thus, for each $t \in [-\tau, +\infty)$, we have

$$\|(T^2 w)(t) - (T^2 z)(t)\| \leq e^{-\omega a} \|w - z\|_{\mathcal{R}_b([-\tau,+\infty);X)}$$

which implies (18). But (18) shows that T^2 is a contraction of constant $e^{-\omega a}$. So T^2 has a unique fixed point u which is also a fixed point of T. Indeed, we have

$$\|Tu - u\|_{\mathcal{R}_b([a,+\infty);X)} = \|T^3 u - T^2 u\|_{\mathcal{R}_b([a,+\infty);X)} \leq e^{-\omega a} \|Tu - u\|_{\mathcal{R}_b([a,+\infty);X)}$$

for each $t \in [-\tau, +\infty)$. Since $e^{-\omega a} < 1$, this inequality shows that $Tu = u$ and so u is a fixed point of T which clearly is unique and is an $\mathcal{L}^\infty$-solution of the Problem (13).

This completes the proof of the existence and uniqueness part.

Finally, we will estimate $\|u(t)\|$ for $t \in [-\tau, 0]$ and for $t \in (0, +\infty)$ separately. For $t \in [-\tau, 0]$, using (H_h), we get

$$\|u(t)\| = \|h(u)(t)\| \leq \|u\|_{\mathcal{R}_b([a,+\infty);X)} + m_0.$$

On the other hand, from (16), setting $w = u$, we have

$$\|u(t)\| \leq e^{-\omega t} \|u\|_{\mathcal{R}_b([a,+\infty);X)} + e^{-\omega t} m_2 + (1 - e^{-\omega t}) \frac{1}{\omega} \|f\|_{L^\infty(\mathbf{R}_+;X)} \tag{19}$$

for each $t > 0$. So, if for some $t \geq a$, we have

$$\limsup_{s \to t} \|u(s)\| = \|u\|_{\mathcal{R}_b([a,+\infty);X)},$$

then, from the last inequality, by observing that the function $x \mapsto \dfrac{1}{e^{\omega x} - 1}$ is strictly decreasing on $[a, +\infty)$, we deduce

$$\|u\|_{\mathcal{R}_b([a,+\infty);X)} \leq \frac{m_2}{e^{\omega a} - 1} + \frac{1}{\omega} \|f\|_{L^\infty(\mathbf{R}_+;X)}. \tag{20}$$

If for each $t \geq a$ we have $\limsup_{s \to t} \|u(s)\| < \|u\|_{\mathcal{R}_b([a,+\infty);X)}$, then there exists $t_n \to \infty$ such that

$$\lim_{n \to \infty} \|u(t_n)\| = \|u\|_{\mathcal{R}_b([a,+\infty);X)}.$$

Setting $t = t_n$ in (19) and passing to the limit for $n \to +\infty$, we get

$$\|u\|_{\mathcal{R}_b([a,+\infty);X)} \leq \frac{1}{\omega} \|f\|_{L^\infty(\mathbf{R}_+;X)}.$$

So, in any case, (20) holds true.

Taking $w = u$ in (16) and using (20), we obtain

$$\|u(t)\| \leq e^{-\omega t} \left(\frac{m_2}{e^{\omega a} - 1} + \frac{1}{\omega} \|f\|_{L^\infty(\mathbf{R}_+;X)} + m_2 \right) + (1 - e^{-\omega t}) \frac{1}{\omega} \|f\|_{L^\infty(\mathbf{R}_+;X)}$$

for each $t \in \mathbf{R}_+$. After some obvious rearrangements, we get (14) and this completes the proof. □

6 Proof of the main result

Proof Let $\varepsilon \in (0, 1)$ and let us consider the approximate problem

$$\begin{cases} \mathrm{d}u(t) = [Au(t) + f(t, u_t)]\,\mathrm{d}t + \mathrm{d}g(t), & t \in \mathbf{R}_+, \\ u(t) = (1-\varepsilon)h(u)(t), & t \in [\,-\tau, 0\,]. \end{cases} \tag{21}$$

By Theorem 8, (21) has a unique $\mathcal{L}^\infty$-solution $u_\varepsilon \in \mathcal{R}_b([\,-\tau, +\infty); X)$. We will show first that the $\mathcal{L}^\infty$-solution set $\{u_\varepsilon;\ \varepsilon \in (0, 1)\}$ is uniformly bounded on $[\,-\tau, +\infty)$. To this aim, let us observe that, by (14) in Lemma 2 and (f_1) in (H_f), we have

$$\|u_\varepsilon\|_{\mathcal{R}_b(\mathbf{R}_+;X)} \le \frac{e^{\omega a}}{e^{\omega a}-1}m_2 + \frac{\ell}{\omega}\|u_\varepsilon\|_{\mathcal{R}_b([\,-\tau,+\infty);X)} + \frac{m}{\omega}. \tag{22}$$

Let $E \subseteq (0, 1)$ and $F \subseteq (0, 1)$ be defined by

$$E := \left\{\varepsilon \in (0,1);\ \|u_\varepsilon\|_{\mathcal{R}_b(\mathbf{R}_+;X)} = \|u_\varepsilon\|_{\mathcal{R}_b([\,-\tau,+\infty);X)}\right\}$$

and by

$$F := \left\{\varepsilon \in (0,1);\ \|u_\varepsilon\|_{\mathcal{R}([\,-\tau,0\,];X)} = \|u_\varepsilon\|_{\mathcal{R}_b([\,-\tau,+\infty);X)}\right\},$$

respectively.

Clearly the set

$$\left\{\|u_\varepsilon\|_{\mathcal{R}_b([\,-\tau,+\infty);X)};\ \varepsilon \in E\right\}$$

is bounded. Indeed, if we assume the contrary, from (22), we get $\omega \le \ell$ which is a contradiction.

Also the set

$$\left\{\|u_\varepsilon\|_{\mathcal{R}_b([\,-\tau,+\infty);X)};\ \varepsilon \in F\right\}$$

is bounded. Indeed, if $\varepsilon \in F$, from (22) and (H_h), we get

$$\|u_\varepsilon(t)\| \le \|u_\varepsilon\|_{\mathcal{R}_b([\,a,+\infty);X)} + m_0 \le \frac{e^{\omega a}}{e^{\omega a}-1}m_2 + \frac{\ell}{\omega}\|u_\varepsilon\|_{\mathcal{R}([\,-\tau,0\,];X)} + \frac{m}{\omega} + m_0$$

for each $t \in [\,-\tau, 0\,]$. So,

$$\|u_\varepsilon\|_{\mathcal{R}([\,-\tau,0\,];X)} \le \frac{e^{\omega a}}{e^{\omega a}-1}m_2 + \frac{\ell}{\omega}\|u_\varepsilon\|_{\mathcal{R}([\,-\tau,0\,];X)} + \frac{m}{\omega} + m_0$$

which shows that the set

$$\left\{\|u_\varepsilon\|_{\mathcal{R}([\,-\tau,0\,];X)};\ \varepsilon \in F\right\} = \left\{\|u_\varepsilon\|_{\mathcal{R}_b([\,-\tau,+\infty);X)};\ \varepsilon \in F\right\}$$

is bounded.

From the arguments above, it readily follows that, for each sequence $\varepsilon_n \downarrow 0$, the set $\{u_n;\ n \in \mathbf{N}\}$, where, for each $n \in \mathbf{N}$, $u_n := u_{\varepsilon n}$, is bounded in $\mathcal{R}_b([\,-\tau, +\infty); X)$. From (f_1) in (H_f), we conclude that $\{t \mapsto f(t, u_{nt});\ n \in \mathbf{N}\}$ is uniformly bounded and thus uniformly integrable in $L^1(0, T; X)$ for each $T > 0$. From Theorem 4, we deduce that the family of functions

$$\left\{t \mapsto \left[u_n(t) - \int_0^t S(t-s)\mathrm{d}g(s)\right];\ n \in \mathbf{N}\right\}$$

is relatively compact in $C([\,0, T\,]; X)$ for each $T > 0$. Indeed, taking

$$\mathcal{D} = \{h(u_n)(0);\ n \in \mathbf{N}\}\,,$$

which is bounded and

$$\mathcal{F} = \{t \mapsto f(t, u_{nt});\ n \in \mathbf{N}\}\,,$$

we get that

$$\mathcal{M}(\mathcal{D}, \mathcal{F}) = \left\{t \mapsto \left[u_n(t) - \int_0^t S(t-s)\mathrm{d}g(s)\right];\ n \in \mathbf{N}\right\}$$

is relatively compact in $C([\,\delta, T\,]; X)$ for each $T > 0$ and $\delta \in (0, T)$. In particular, $\mathcal{M}(\mathcal{D}, \mathcal{F})$ is relatively compact in $C([\,a, +\infty); X)$ endowed with the uniform convergence on compacta topology. So, $\{u_n\}$ is relatively compact in $\mathcal{R}([\,a, +\infty); X)$. So, on some subsequence, denoted for simplicity again by $(u_n)_n$, we have $\lim_n u_n = u$ in $\mathcal{R}([\,a, +\infty); X)$. By (H_h), it follows that $\lim_n h(u_n) = h(u)$ in $\mathcal{R}[\,-\tau, 0\,]; X)$ and so $\lim_n h(u_n)(0) = h(u)(0)$, which means that $\mathcal{D}$ is relatively compact in X. Hence, we are in the hypotheses of Theorem 4 which implies that $\mathcal{M}(\mathcal{D}, \mathcal{F})$ is relatively compact in $C(\mathbf{R}_+; X)$ endowed with the convergence on compacta topology. This clearly shows that $\{u_n;\ n \in \mathbf{N}\}$ is relatively compact in $\mathcal{R}(\mathbf{R}_+; X)$.

Accordingly, on a subsequence at least, we can pass to the limit in the variation of constants formula

$$u_n(t) = S(t)u_n(0) + \int_0^t S(t-s)f(s, u_{ns})\,\mathrm{d}s + \int_0^t S(t-s)\mathrm{d}g(s)$$

uniformly for t in compact subsets in $\mathbf{R}_+$. We get

$$u(t) = S(t)u(0) + \int_0^t S(t-s)f(s, u_s)\,\mathrm{d}s + \int_0^t S(t-s)\mathrm{d}g(s)$$

for each $t \in \mathbf{R}_+$. On the other hand, since h is continuous from $\mathcal{R}(\mathbf{R}_+; X)$ to $\mathcal{R}([\,-\tau, 0\,]; X)$, it follows that

$$\lim_n u_n(t) = \lim_n (1 - \varepsilon_n)h(u_n)(t) = h(u)(t)$$

uniformly for $t \in [\,-\tau, 0\,]$. Thus u is an $\mathcal{L}^\infty$-solution of the Problem (1) and this completes the proof. □

Remark 5 A simple analysis of the arguments above shows that we may relax the hypotheses (A_1) in (H_A) and (H_c), imposed in Theorem 5, to

$(\widetilde{A}_1)$ there exist $M \geq 1$ and $\omega > 0$ such that $\|S(t)\| \leq Me^{-\omega t}$ for each $t \in \mathbf{R}_+$;
$(\widetilde{H}_c)$ The constants M, ℓ and ω satisfy $M\ell < \omega$.

7 An example

Example 7.1 Let Ω be a bounded domain in $\mathbf{R}^d$, $d \in \mathbf{N}^*$, with sufficiently smooth boundary Γ and let $\tau \geq 0$. Let $F : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ be a continuous function, let $H : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and let $\psi \in \mathcal{R}([\,-\tau, 0\,]; L^1(\Omega))$.

Let $(t_i)_{i\in\mathbf{N}}$ be an increasing sequence, let $(g_i)_{i\in\mathbf{N}}$ be a sequence in $L^1(\Omega)$, and let $g : \mathbf{R}_+ \to L^1(\Omega)$ be defined by

$$g(t)(x) := \sum_{\{i;\ t_i \le t\}} g_i(x) \tag{23}$$

Clearly, g is a sum of Heaviside-like $L^1(\Omega)$-valued functions and belongs to $BV_{\mathrm{loc}}(\mathbf{R}_+; L^1(\Omega))$. So,

$$\mathrm{d}g(t)(x) = \sum_{i=1}^{\infty} g_i(x)\delta(t - t_i)$$

for $(t, x) \in \mathbf{R}_+ \times \Omega$, where $\delta(t - t_i)$ is the Dirac delta concentrated at the point t_i, $i = 0, 1, 2, \ldots$.

Let $\omega > 0$, let $Q_+ := \mathbf{R}_+ \times \Omega$, $\Sigma_+ := \mathbf{R}_+ \times \Gamma$ and $Q_\tau := [-\tau, 0] \times \Omega$, and let us consider the linear delay parabolic equation:

$$\begin{cases} \mathrm{d}u(t) = \left[\Delta u(t) - \omega u(t) + F\left(t, \int_{-\tau}^{0} u(t+s)\mathrm{d}s\right)\right]\mathrm{d}t + \mathrm{d}g(t) & \text{in } Q_+, \\ u = 0 & \text{on } \Sigma_+, \\ u(t,x) = \int_{\tau}^{+\infty} H(s, u(t+s)(x))\,\mathrm{d}\lambda(s) + \psi(t)(x) & \text{in } Q_\tau. \end{cases} \tag{24}$$

In fact (24) describes a diffusion process in which there is a feedback term depending on the cumulative history of the state, i.e., $F\left(t, \int_{-\tau}^{0} u(t+s)\mathrm{d}s\right)$ and a forcing term, $\mathrm{d}g(t)$, responsible for some instantaneous changes of the speed of propagation.

From Theorem 7, we deduce:

Theorem 9 *Let Ω be a bounded domain in $\mathbf{R}^d$, $d \in \mathbf{N}^*$, with sufficiently smooth boundary Γ, let $\tau \ge 0$, and let $\omega > 0$. Let g be defined by (23), let λ be a positive Radon measure on $\mathbf{R}_+$, and let $\psi \in \mathcal{R}([-\tau, 0]; L^1(\Omega))$.*

We assume that:

(F) *The function $F : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ is continuous and there exists $\ell_0 > 0$ and $\widetilde{m} > 0$ such that*

$$|F(t,u)| \le \ell_0|u| + \widetilde{m},$$

for all $(t,u) \in \mathbf{R}_+ \times \mathbf{R}$. Moreover, F satisfies:

$$|F(t,u) - F(t,v)| \le \ell_0|u - v|,$$

for all $(t,u), (t,v) \in \mathbf{R}_+ \times \mathbf{R}$.

(G) *The functions g_i, $i = 0, 1, 2, \ldots$, in (23) satisfy*

$$\sum_{i=0}^{\infty} e^{\omega t_i}\|g_{i+1} - g_i\|_{L^1(\Omega)} < +\infty.$$

(H) *The function $H : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ is continuous and there exists α in $L^1(\mathbf{R}_+; \lambda, \mathbf{R}_+)$ such that*

$$\int_{b}^{+\infty} \alpha(s)\, d\lambda(s) \le 1$$

and

$$|H(s,v)-H(s,w)|\le \alpha(s)|v-w|$$

for each $s\in[b,+\infty)$ *and* $v,w\in\mathbf{R}$.

(C) *The constants* τ, b, ℓ_0 *and* ω *satisfy* $b-\tau>0$ *and* $\ell_0\tau<\omega$.

Then, the Problem (24) has at least one $\mathcal{L}^\infty$*-solution* $u\in\mathcal{R}_b([-\tau,+\infty);L^1(\Omega))$.

Proof In order to use Theorem 7, we shall rewrite (24) as a delay evolution equation with measures subjected to a nonlocal initial condition in an appropriate real Banach space. First, let us define the operator $A: D(A)\subseteq L^1(\Omega)\to L^1(\Omega)$ by

$$\begin{cases} D(A)=\{u\in W_0^{1,1}(\Omega);\,\Delta u\in L^1(\Omega)\};\\ Au=\Delta u-\omega u,\ \text{for each } u\in D(A).\end{cases}$$

It is well known that A generates a compact C_0-semigroup of contractions on the space $X=L^1(\Omega)$. Moreover, ω is exactly that one in (A_1) in (H_A). See for instance Vrabie [29, Theorem 7.2.7, p. 160].

Now, let us observe that (24) may be rewritten as a Cauchy problem in $L^1(\Omega)$ of the form:

$$\begin{cases} \mathrm{d}u(t)=[Au(t)+f(t,u_t)]\mathrm{d}t+\mathrm{d}g(t), & t\in\mathbf{R}_+,\\ u(t)=h(u)(t), & t\in[-\tau,0],\end{cases}$$

where A and g are as above, while $f:\mathbf{R}_+\times\mathcal{R}([-\tau,0];L^1(\Omega))\to L^1(\Omega)$ and $h:\mathcal{R}_b(\mathbf{R}_+;L^1(\Omega))\to\mathcal{R}([-\tau,0];L^1(\Omega))$ are defined by

$$f(t,u)(x)=F\left(t,\int_{-\tau}^0 u(t+s)(x)\mathrm{d}s\right)$$

for each $u\in\mathcal{R}([-\tau,0];L^1(\Omega))$, $t\in\mathbf{R}_+$ and a.e. for $x\in\Omega$ and, respectively, by

$$[h(u)(t)](x):=\int_b^{+\infty} H(s,u(t+s)(x))\,d\lambda(s)+\psi(t)(x)$$

for each $u\in\mathcal{R}_b(\mathbf{R}_+;L^1(\Omega))$, $t\in[-\tau,0]$ and a.e. for $x\in\Omega$.

By (iii) in Vrabie [29, Lemma A.6.1, p. 313], it readily follows that f is well defined and continuous on $\mathbf{R}_+\times\mathcal{R}([-\tau,0];L^1(\Omega))$. Moreover, by (F) and (C), it follows that it satisfies (f_1) in (H_f) with $\ell=\ell_0\tau$ and $m=\widetilde{m}\mu(\Omega)$. By (G), we conclude that g satisfies (H_g), while from (H), we deduce that h satisfies (H_h) with $a=b-\tau$. Again, by (C), we conclude that ℓ and ω satisfy(H_c).

Then, by Theorem 7, we conclude that the Problem (24) has at least one $\mathcal{L}^\infty$-solution and this completes the proof. □

As far as the case of purely singular measures is concerned, we have:

Theorem 10 *Let* Ω *be a bounded domain in* $\mathbf{R}^d$, $d\in\mathbf{N}^*$, *with sufficiently smooth boundary* Γ, *let* $\tau\ge0$, *and let* $\omega>0$. *Let* g *be a function in* $BV_{\mathrm{loc}}(\mathbf{R}_+;L^1(\Omega))\cap C(\mathbf{R}_+;L^1(\Omega))$, *let* λ *be a positive Radon measure on* $\mathbf{R}_+$, *and let* $\psi\in\mathcal{R}([-\tau,0];L^1(\Omega))$.

We assume that:

(F) *The function* $F:\mathbf{R}_+\times\mathbf{R}\to\mathbf{R}$ *is continuous and there exist* $\ell_0>0$ *and* $\widetilde{m}>0$ *such that*

$$|F(t,u)|\le\ell_0|u|+\widetilde{m},$$

for all $(t,u)\in\mathbf{R}_+\times\mathbf{R}$.

$(\widetilde{G})$ *There exists* $m_1 > 0$ *such that*

$$\int_0^t e^{\omega s}\, d\mathrm{Var}(g; [\,0, s\,]) \le m_1$$

for each $t \in \mathbf{R}_+$.

(H) *The function* $H : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ *is continuous and there exists* α *in* $L^1(\mathbf{R}_+; \lambda, \mathbf{R}_+)$ *such that*

$$\int_b^{+\infty} \alpha(s)\, d\lambda(s) \le 1$$

and

$$|H(s, v) - H(s, w)| \le \alpha(s)|v - w|$$

for each $s \in [\,b, +\infty)$ *and* $v, w \in \mathbf{R}$.

(C) *The constants* τ, b, ℓ_0 *and* ω *satisfy* $b - \tau > 0$ *and* $\ell_0\tau < \omega$.

Then, the Problem (24) has at least one $\mathcal{L}^\infty$*-solution* $u \in \mathcal{R}_b([\,-\tau, +\infty); L^1(\Omega))$.

Remark 6 We emphasize that the really interesting case is that in which g is not in $W^{1,1}_{\mathrm{loc}}(\mathbf{R}_+; L^1(\Omega))$ when the Problem (24) reduces to a classical one, i.e., when $\mathrm{d}g(t)$ can be substituted by $f'(t)\mathrm{d}t$.

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