Annali di Matematica (2016) 195:1833–1863
DOI 10.1007/s10231-015-0531-x

On a concave–convex elliptic problem with a nonlinear boundary condition

Humberto Ramos Quoirin[1] · **Kenichiro Umezu**[2]

Received: 5 May 2015 / Accepted: 28 August 2015 / Published online: 12 September 2015
© Fondazione Annali di Matematica Pura ed Applicata and Springer-Verlag Berlin Heidelberg 2015

Abstract We investigate an indefinite superlinear elliptic equation coupled with a sublinear Neumann boundary condition (depending on a positive parameter λ), which provides a concave–convex nature to the problem. We establish a global multiplicity result for positive solutions in the spirit of Ambrosetti–Brezis–Cerami and obtain their asymptotic profiles as $\lambda \to 0$. Furthermore, we also analyse the case where the nonlinearity is concave. Our arguments are based on a bifurcation analysis, a comparison principle, and variational techniques.

Keywords Semilinear elliptic problem · Concave–convex nonlinearity · Nonlinear boundary condition · Positive solution · Bifurcation · Super and subsolutions · Nehari manifold

Mathematics Subject Classification 35J25 · 35J61 · 35J20 · 35B09 · 35B32

1 Introduction and statements of main results

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. We consider in this article the nonlinear elliptic problem

$$\begin{cases} -\Delta u = a(x)|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda |u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

The first author was supported by the FONDECYT Grant 11121567.

✉ Kenichiro Umezu
kenichiro.umezu.math@vc.ibaraki.ac.jp

Humberto Ramos Quoirin
humberto.ramos@usach.cl

1 Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile

2 Department of Mathematics, Faculty of Education, Ibaraki University, Mito 310-8512, Japan

where

- $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ is the usual Laplacian in $\mathbb{R}^N$,
- $\lambda > 0$,
- $1 < q < 2 < p < \infty$,
- $a \in \mathcal{C}^{\alpha}(\overline{\Omega})$ with $\alpha \in (0, 1)$,
- $\mathbf{n}$ is the unit outer normal to the boundary $\partial\Omega$.

A function $u \in X := H^1(\Omega)$ is said to be a *weak solution* of (P_λ) if it satisfies

$$\int_{\Omega} \nabla u \nabla w - \int_{\Omega} a|u|^{p-2}uw - \lambda \int_{\partial\Omega} |u|^{q-2}uw = 0, \quad \forall w \in X.$$

A weak solution u of (P_λ) is said to be *nontrivial and non-negative* if it satisfies $u \geq 0$ and $u \not\equiv 0$. Under the condition

$$p \leq 2^* = \frac{2N}{N-2} \quad \text{if } N > 2, \tag{1.1}$$

we shall prove that such solutions are strictly positive on $\overline{\Omega}$ (Proposition 2.1) and belong to $\mathcal{C}^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0, 1)$ (Remark 2.2). To this end, we use the weak maximum principle [15] to deduce that any nontrivial non-negative weak solution u of (P_λ) is strictly positive in Ω. In addition, by making good use of a comparison principle [19, Proposition A.1], we shall prove that u is positive on the whole of $\overline{\Omega}$. Finally, a bootstrap argument will provide $u \in \mathcal{C}^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0, 1)$, so that u is a (*classical*) *positive solution*. Note that the standard boundary point lemma (as in [17]) cannot be applied directly to nontrivial non-negative weak solutions of (P_λ).

The purpose of this paper is to study existence, non-existence, and multiplicity of positive solutions of (P_λ), as well as their asymptotic properties as the parameter λ approaches 0. It is promptly seen that (P_λ) has no positive solution if $a \geq 0$. More precisely, we shall see that (P_λ) has a positive solution only if $\int_\Omega a < 0$ (cf. Proposition 2.3). This condition is known to be necessary for the existence of positive solutions of problems with Neumann boundary conditions at least since the work of Bandle–Pozio–Tesei [4]. Therefore, we shall assume that either a changes sign or $a \leq 0$.

In view of the condition $1 < q < 2 < p$, we note that if a changes sign, then (P_λ) belongs to the class of concave–convex type problems with nonlinear boundary conditions. The main reference on concave–convex type problems is the work of Ambrosetti–Brezis–Cerami [3], which deals with

$$\begin{cases} -\Delta u = \lambda|u|^{q-2}u + |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < q < 2 < p$. Under the condition (1.1), the authors proved a *global multiplicity result*, namely the existence of some $\Lambda > 0$ such that (1.2) has at least two positive solutions for $\lambda \in (0, \Lambda)$, at least one positive solution for $\lambda = \Lambda$, and no positive solution for $\lambda > \Lambda$. In addition, they analysed the asymptotic behaviour of the solutions as $\lambda \to 0^+$. Tarfulea [22] considered a similar problem with an indefinite weight and a Neumann boundary condition, namely

$$\begin{cases} -\Delta u = \lambda|u|^{q-2}u + a(x)|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $a \in \mathcal{C}(\overline{\Omega})$. He proved that $\int_\Omega a < 0$ is a necessary and sufficient condition for the existence of a positive solution of (1.3). Making use of the sub-supersolutions technique, he has also shown the existence of $\Lambda > 0$ such that problem (1.3) has at least one positive solution for $\lambda < \Lambda$ which converges to 0 in $L^\infty(\Omega)$ as $\lambda \to 0^+$, and no positive solution for $\lambda > \Lambda$. Garcia-Azorero et al. [11] have considered the problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

By means of a variational approach, they proved that if $1 < q < 2 < p$ and $p < 2^*$ when $N > 2$, then there exists $\Lambda_0 > 0$ such that (1.4) has infinitely many nontrivial weak solutions for $0 < \lambda < \Lambda$. Moreover, they have also proved that if $1 < q < 2$ and $p = 2^*$ when $N > 2$, then there exists $\Lambda_1 > 0$ such that (1.4) has at least two positive solutions for $\lambda < \Lambda_1$, at least one positive solution for $\lambda = \Lambda_1$, and no positive solution for $\lambda > \Lambda_1$.

When a changes sign, we shall prove a global multiplicity result in the style of Ambrosetti–Brezis–Cerami result. However, in doing so we shall encounter some particular difficulties. First of all, the obtention of a first solution by the sub-supersolution method seems difficult since the existence of a strict supersolution of (P_λ) for $\lambda > 0$ small is not evident at all. As a matter of fact, in [22] the author shows that this is a rather delicate issue. Another difficulty in this case is related to the variational structure: note that unlike in problems with Dirichlet boundary conditions, the left-hand side of (P_λ) lacks coercivity, since the term $\int_\Omega |\nabla u|^2$ does not correspond to $\|u\|^2$ in X. This sort of problems has been considered in [18,19] for other kinds of nonlinearities and we shall use a similar approach here to prove existence results for (P_λ). This approach is based on the Nehari manifold method, which is known to be useful when dealing with elliptic problems with powerlike nonlinearities and sign-changing weights. Brown and Wu [6] used this method to deal with the problem

$$\begin{cases} -\Delta u = \lambda m(x)|u|^{q-2}u + a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where m, a are smooth functions which are positive somewhere in Ω. We refer also to Brown [5] for a combination of sublinear and linear terms and to Wu [24] for a problem with a nonlinear boundary condition.

On the other hand, if $a \le 0$ then $a(x)|u|^{p-2}u$ and $\lambda|u|^{q-2}u$ are both concave and (P_λ) shares then some features with the logistic equation. The structure of the positive solution set of (P_λ) with $a \le 0$ and $q = 2$ has been considered by Garcia-Melián et al. [12]. They proved that there exists $0 < \sigma_1 \le \infty$ such that (P_λ) has a positive solution if and only if $0 < \lambda < \sigma_1$. Moreover, this positive solution is unique. We shall prove a similar result for (P_λ) with $\sigma_1 = \infty$.

Whenever $\int_\Omega a < 0$, we set

$$c^* = \left(\frac{|\partial\Omega|}{-\int_\Omega a} \right)^{\frac{1}{p-q}}. \tag{1.6}$$

We also set

$$\overline{\lambda} = \sup\{\lambda > 0 : (P_\lambda) \text{has a positive solution}\}.$$

Let us recall that a positive solution u of (P_λ) is said to be *asymptotically stable* (respect. *unstable*) if $\gamma_1(\lambda, u) > 0$ (respect. < 0), where $\gamma_1(\lambda, u)$ is the smallest eigenvalue of the linearized eigenvalue problem at u, namely

$$\begin{cases} -\Delta\phi = (p-1)a(x)u^{p-2}\phi + \gamma\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = \lambda(q-1)u^{q-2}\phi + \gamma\phi & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

In addition, u is said *weakly stable* if $\gamma_1(\lambda, u) \geq 0$.

We state now our main result:

Theorem 1.1 (1) *(P_λ) has a positive solution for $\lambda > 0$ sufficiently small if*

$$\int_\Omega a < 0. \tag{1.8}$$

Conversely, if (P_λ) has a positive solution for some $\lambda > 0$, then (1.8) is satisfied.

(2) *Assume (1.8). Then the following assertions hold:*

(a) *$0 < \overline{\lambda} \leq \infty$ and (P_λ) has a minimal positive solution $\underline{u}_\lambda$ for $\lambda \in (0, \overline{\lambda})$, i.e. any positive solution u of (P_λ) satisfies $\underline{u}_\lambda \leq u$ in $\overline{\Omega}$. Furthermore, $\underline{u}_\lambda$ has the following properties:*

(i) *$\lambda \mapsto \underline{u}_\lambda(x)$ is strictly increasing in $(0, \overline{\lambda})$.*

(ii) *$\underline{u}_\lambda$ is asymptotically stable for every $\lambda \in (0, \overline{\lambda})$.*

(iii) *$\lambda \mapsto \underline{u}_\lambda$ is $\mathcal{C}^\infty$ from $(0, \overline{\lambda})$ to $\mathcal{C}^{2+\alpha}(\overline{\Omega})$.*

(iv) *$\underline{u}_\lambda \to 0$ and $\lambda^{-\frac{1}{p-q}}\underline{u}_\lambda \to c^*$ in $\mathcal{C}^{2+\alpha}(\overline{\Omega})$ as $\lambda \to 0^+$.*

(b) *Assume (1.1). If $\overline{\lambda} < \infty$, then (P_λ) has a minimal positive solution $\underline{u}_{\overline{\lambda}}$ for $\lambda = \overline{\lambda}$. Moreover, the solution set around $(\overline{\lambda}, \underline{u}_{\overline{\lambda}})$ consists of a $\mathcal{C}^\infty$-curve $(\lambda(s), u(s)) \in \mathbb{R} \times \mathcal{C}^{2+\alpha}(\overline{\Omega})$ of positive solutions, which is parametrized by $s \in (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$, and satisfies $(\lambda(0), u(0)) = (\overline{\lambda}, \underline{u}_{\overline{\lambda}})$, $\lambda'(0) = 0$, $\lambda''(0) < 0$, and $u(s) = \underline{u}_{\overline{\lambda}} + s\phi_1 + z(s)$, where ϕ_1 is a positive eigenfunction associated with the smallest eigenvalue $\gamma_1(\overline{\lambda}, \underline{u}_{\overline{\lambda}})$ of (1.7), and $z(0) = z'(0) = 0$. Finally, the lower branch $(\lambda(s), u(s))$, $s \in (-\varepsilon, 0)$, is asymptotically stable, whereas the upper branch $(\lambda(s), u(s))$, $s \in (0, \varepsilon)$, is unstable.*

(c) *Assume $p < 2^*$ if $N > 2$. Then the set of positive solutions of (P_λ) for $\lambda > 0$ around $(\lambda, u) = (0, 0)$ in $\mathbb{R} \times X$ consists of $\{(\lambda, \underline{u}_\lambda)\}$.*

(d) *Bifurcation from zero of (P_λ) never occurs at any $\lambda > 0$, i.e. there is no sequence (λ_n, u_n) of positive solutions of (P_λ) such that $u_n \to 0$ in $\mathcal{C}(\overline{\Omega})$ and $\lambda_n \to \lambda^* > 0$.*

(e) *(P_λ) has at most one weakly stable positive solution.*

Remark 1.2 (1) Under conditions (1.8) and (1.1), by the left continuity of $\underline{u}_\lambda$ [1, Theorem 20.3], we infer that $(\lambda(s), u(s))$, $s \in (-\varepsilon, 0)$, in Theorem 1.1(2)(b) represents minimal positive solutions. In particular, the mapping $\lambda \mapsto \underline{u}_\lambda$ is continuous from $(0, \overline{\lambda}]$ into $\mathcal{C}(\overline{\Omega})$.

(2) Under (1.1), the minimal positive solution $\underline{u}_{\overline{\lambda}}$ obtained for $\lambda = \overline{\lambda}$ satisfies in addition $\gamma_1(\overline{\lambda}, \underline{u}_{\overline{\lambda}}) = 0$.

Theorem 1.3 *Assume $a \leq 0$, $a \not\equiv 0$. Then the following assertions hold:*

(1) *If (P_λ) has a positive solution for some $\lambda > 0$, then it is unique and asymptotically stable.*

(2) *If, in addition, (1.1) is satisfied, then $\overline{\lambda} = \infty$. Moreover, denoting by u_λ the unique positive solution of (P_λ), the mapping $\lambda \mapsto u_\lambda$ is $\mathcal{C}^\infty$ in $(0, \infty)$.*

Theorem 1.4 *Assume that a changes sign and (1.8) is satisfied. Then the following assertions hold:*

(1) *If $a > 0$ on $\partial\Omega$, then $\overline{\lambda} < \infty$.*

(2) *Assume in addition* $p < \frac{2N}{N-2}$ *if* $N > 2$. *Then* (P_λ) *has a second positive solution* $u_{2,\lambda}$ *satisfying* $\underline{u}_\lambda < u_{2,\lambda}$ *in* $\overline{\Omega}$ *for every* $\lambda \in (0, \overline{\lambda})$. *Moreover,* $u_{2,\lambda}$ *is unstable for every* $\lambda \in (0, \overline{\lambda})$ *and there exists* $\lambda_n \to 0^+$ *such that* $u_{2,\lambda_n} \to u_{2,0}$ *in* $C^{2+\theta}(\overline{\Omega})$ *for any* $\theta \in (0, \alpha)$ *as* $n \to \infty$, *where* $u_{2,0}$ *is a positive solution of*

$$\begin{cases} -\Delta u = a(x)u^{p-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Remark 1.5 (1) In the case $a \le 0$, $a \not\equiv 0$, the following remarks are in order:

- (a) The condition (1.1) can be removed when dealing with weak solutions. In other words, if $a \le 0$, $a \not\equiv 0$ and $p > 1$, then (P_λ) has a unique nontrivial non-negative weak solution u_λ for every $\lambda > 0$, see Proposition 4.3. This has been observed in [12, Theorem 2] in the case $q = 2$.
- (b) In [12], it has been proved that if $q = 2$, then (P_λ) has a positive solution if and only if $0 < \lambda < \sigma_1$, where σ_1 is the first eigenvalue of the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_0, \\ \frac{\partial u}{\partial \mathbf{n}} = \sigma u & \text{on } \Sigma_1, \\ u = 0 & \text{on } \Sigma_2. \end{cases} \tag{1.10}$$

Here Ω_0 is the interior of $\{a = 0\}$ and it is assumed that $\partial\Omega_0 = \Sigma_1 \cup \Sigma_2$ with $\Sigma_1 = \partial\Omega \cap \partial\Omega_0$ and $\Sigma_2 = \Omega \cap \partial\Omega_0$ such that $\overline{\Sigma_2} \subset \Omega$. Moreover, if $\Sigma_1 = \emptyset$, then $\sigma_1 = \infty$. According to Theorem 1.3, in the case $1 < q < 2$ we have $\sigma_1 = \infty$ regardless of $\{a = 0\}$. Biologically, this result would be interpreted in the following way: an incoming flux on $\partial\Omega$ occurs in both cases $q = 2$ and $1 < q < 2$, but a grow-up phenomenon occurs in the refuge $\{a = 0\}$ in the case $q = 2$, whereas no such phenomenon occurs in the case $1 < q < 2$. The difference between them might be caused by the fact that the incoming flux u^{q-1} on $\partial\Omega$ in the case $1 < q < 2$ is much smaller than in the case $q = 2$ when u is large. Here our situation is that the intrinsic growth rate of population with density u is 0, a reaction on $\partial\Omega$, which is given by λu^{q-1}, is assumed with its amplitude λ, and we consider a decay of the population following self-limitation $a(x)u^{p-1}$ with spatially inhomogeneous rate $a(x)$ inside Ω.

(2) In accordance with Theorems 1.1, 1.3 and 1.4, some possible positive solutions sets of (P_λ) are depicted in Fig. 1.

The outline of this article is the following: in Sect. 2, we show that nontrivial non-negative solutions of (P_λ) are positive on $\overline{\Omega}$ and that (1.8) is a necessary condition for the existence of positive solutions of (P_λ). In Sect. 3, we carry out a bifurcation analysis and consider the existence of a minimal positive solution of (P_λ). In Sect. 4, we use variational techniques to prove Theorems 1.3 and 1.4. Finally, in Sect. 5 we establish the existence of a smooth curve of positive solutions.

2 Positivity and a necessary condition

We begin this section showing the positivity on $\partial\Omega$ of nontrivial non-negative weak solutions of (P_λ). As mentioned in the Introduction, the boundary point lemma is difficult to apply directly to (P_λ) since $0 < q-1 < 1$. However, by making good use of a comparison principle

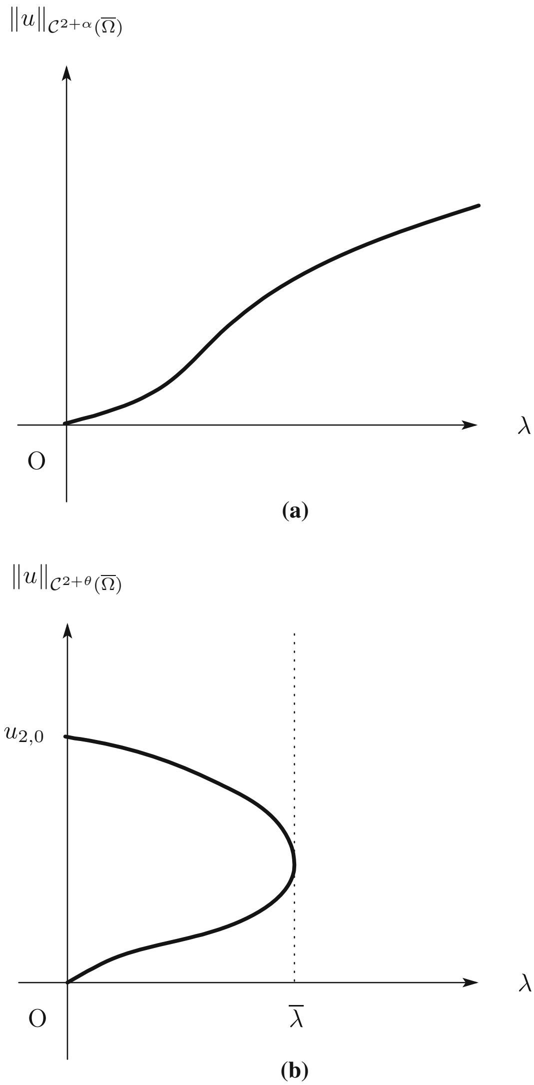


Fig. 1 Possible bifurcation diagrams for (P_λ) when $\int_\Omega a < 0$. **a** Bifurcation diagram in the case $a \le 0$ and $a \not\equiv 0$. **b** Bifurcation diagram in the case a changes sign

for a class of nonlinear boundary value problems of concave type, we are able to show that nontrivial non-negative weak solutions of (P_λ) with $\lambda > 0$ are positive on the whole of $\overline{\Omega}$:

Proposition 2.1 *Assume* (1.1)*. Then any nontrivial non-negative weak solution of* (P_λ) *is strictly positive on* $\overline{\Omega}$*.*

Proof First of all, we note that under (1.1) any nontrivial non-negative weak solution belongs to $X \cap \mathcal{C}^\theta(\overline{\Omega})$ for some $\theta \in (0, 1)$, cf. Rossi [21, Theorem 2.2]. We consider the following boundary value problem of concave type

$$\begin{cases} -\Delta u = -a_0 u^{p-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda u^{q-1} & \text{on } \partial\Omega, \end{cases}$$

where $a^- = a^+ - a$, and $a_0 = \sup_\Omega a^-$. A nontrivial non-negative weak solution u_λ of (P_λ) for $\lambda > 0$ satisfies

$$\int_\Omega \nabla u_\lambda \nabla w + a_0 \int_\Omega u_\lambda^{p-1} w - \lambda \int_{\partial\Omega} u_\lambda^{q-1} w \geq 0,$$

for every $w \in X$ such that $w \geq 0$. On the other hand, we consider the following eigenvalue problem:

$$\begin{cases} -\Delta\phi = \sigma\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = \lambda\phi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is easy to see that for any $\lambda > 0$, this problem has a smallest eigenvalue σ_1, which is negative. So, using a positive eigenfunction ϕ_1 associated with σ_1, we deduce that if ε is sufficiently small, then $\varepsilon\phi_1$ satisfies

$$\int_\Omega \nabla(\varepsilon\phi_1)\nabla w + a_0 \int_\Omega (\varepsilon\phi_1)^{p-1} w - \lambda \int_{\partial\Omega} (\varepsilon\phi_1)^{q-1} w \leq 0,$$

for every $w \in X$ such that $w \geq 0$. By the comparison principle [19, Proposition A.1], we infer that $\varepsilon\phi_1 \leq u_\lambda$ on $\overline{\Omega}$. In particular, we have $0 < \varepsilon\phi_1 \leq u_\lambda$ on $\partial\Omega$. □

Remark 2.2 Thanks to the positivity property, the assumption $a \in \mathcal{C}^\alpha(\overline{\Omega})$, $0 < \alpha < 1$, allows us to prove that under (1.1), any nontrivial non-negative weak solution u of (P_λ) belongs to $\mathcal{C}^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0, 1)$, by elliptic regularity. Proposition 2.1 will be needed in a combination argument of bifurcation and variational techniques, since our purpose in this paper is to discuss the existence of a classical solution of (P_λ) which is positive on $\overline{\Omega}$.

We prove now that (1.8) is a necessary condition for (P_λ) to have a positive solution for some $\lambda > 0$.

Proposition 2.3 *If (P_λ) has a positive solution for some $\lambda > 0$, then (1.8) is satisfied.*

Proof Let u be a positive solution of (P_λ). Then we have

$$\int_\Omega \nabla u \nabla w - \int_\Omega a u^{p-1} w - \lambda \int_{\partial\Omega} u^{q-1} w = 0, \quad \forall w \in X.$$

Since $u^{1-p} \in X$, we deduce that

$$\int_\Omega a = \int_\Omega \nabla u \nabla \left(u^{1-p}\right) - \lambda \int_{\partial\Omega} u^{q-1} \frac{1}{u^{p-1}} = (1-p) \int_\Omega u^{-p} |\nabla u|^2 - \lambda \int_{\partial\Omega} u^{-(p-q)} < 0,$$

as desired. □

Remark 2.4 By virtue of Proposition 2.1, under (1.1) we can prove that Proposition 2.3 holds for nontrivial non-negative weak solutions of (P_λ).

3 Bifurcation and minimal positive solutions

Throughout this section, we assume (1.8). As we shall discuss bifurcation from the zero solution, the following result will be useful (see [20] for a similar proof):

Lemma 3.1 *Assume* (1.1). *If* (λ_n, u_n) *are weak solutions of* (P_λ) *with* (λ_n) *bounded, then* $\|u_n\|_X \to 0$ *if and only if* $\|u_n\|_{\mathcal{C}(\overline{\Omega})} \to 0$.

We use now a bifurcation technique to show the existence of at least one positive solution of (P_λ) for $\lambda > 0$ close to 0. To this end, we consider positive solutions of the following problem, which corresponds to (P_λ) after the change of variable $w = \lambda^{-\frac{1}{p-q}} u$:

$$\begin{cases} -\Delta w = \lambda^{\frac{p-2}{p-q}} a w^{p-1} & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = \lambda^{\frac{p-2}{p-q}} w^{q-1} & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Proposition 3.2 (1) *If* (3.1) *has a sequence of positive solutions* (λ_n, w_n) *such that* $\lambda_n \to 0^+$, $w_n \to c$ *in* $\mathcal{C}(\overline{\Omega})$ *and* c *is a positive constant, then* $c = c^*$, *where* c^* *is given by* (1.6).

(2) *Conversely,* (3.1) *has for* $|\lambda|$ *sufficiently small a secondary bifurcation branch* $(\lambda, w(\lambda))$ *of positive solutions (parametrized by* λ*) emanating from the trivial line* $\{(0, c) : c$ *is a positive constant*$\}$ *at* $(0, c^*)$ *and such that, for* $0 < \theta \le \alpha$, *the mapping* $\lambda \mapsto w(\lambda) \in \mathcal{C}^{2+\theta}(\overline{\Omega})$ *is continuous. Moreover, the set* $\{(\lambda, w)\}$ *of positive solutions of* (3.1) *around* $(\lambda, w) = (0, c^*)$ *consists of the union*

$$\left\{(0, c) : c \text{ is a positive constant},\ |c - c^*| \le \delta_1\right\} \cup \{(\lambda, w(\lambda)) : |\lambda| \le \delta_1\}$$

for some $\delta_1 > 0$.

Proof The proof is similar to the one of [19, Proposition 5.3]:

(1) Let w_n be positive solutions of (3.1) with $\lambda = \lambda_n$, where $\lambda_n \to 0^+$. By the Green formula, we have

$$\int_\Omega a w_n^{p-1} + \int_{\partial\Omega} w_n^{q-1} = 0.$$

Passing to the limit as $n \to \infty$, we deduce the desired conclusion.

(2) We reduce (3.1) to a bifurcation equation in $\mathbb{R}^2$ by the Lyapunov–Schmidt procedure: we use the usual orthogonal decomposition

$$L^2(\Omega) = \mathbb{R} \oplus V,$$

where $V = \{v \in L^2(\Omega) : \int_\Omega v = 0\}$ and the projection $Q : L^2(\Omega) \to V$, given by

$$v = Qu = u - \frac{1}{|\Omega|} \int_\Omega u.$$

The problem of finding a positive solution of (3.1) reduces then to the following problems:

$$\begin{cases} -\Delta v + \frac{\mu}{|\Omega|} \int_{\partial\Omega} (t+v)^{q-1} = \mu Q\left[a(t+v)^{p-1}\right] & \text{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}} = \mu (t+v)^{q-1} & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

$$\mu \left(\int_\Omega a(t+v)^{p-1} + \int_{\partial\Omega} (t+v)^{q-1} \right) = 0, \tag{3.3}$$

where $\mu = \lambda^{\frac{p-2}{p-q}}$, $t = \frac{1}{|\Omega|} \int_\Omega w$, and $v = w - t$. To solve (3.2) in the framework of Hölder spaces, we set

$$Y = \left\{ v \in \mathcal{C}^{2+\theta}(\overline{\Omega}) : \int_\Omega v = 0 \right\},$$

$$Z = \left\{ (\phi, \psi) \in \mathcal{C}^{\theta}(\overline{\Omega}) \times \mathcal{C}^{1+\theta}(\partial\Omega) : \int_{\Omega} \phi + \int_{\partial\Omega} \psi = 0 \right\}.$$

Let $c > 0$ be a constant and $U \subset \mathbb{R} \times \mathbb{R} \times Y$ be a small neighbourhood of $(0, c, 0)$. We consider the nonlinear mapping $F : U \to Z$ given by

$$F(\mu, t, v) = \left(-\Delta v - \mu Q\left[a(t+v)^{p-1}\right] + \frac{\mu}{|\Omega|} \int_{\partial\Omega} (t+v)^{q-1},\ \frac{\partial v}{\partial \mathbf{n}} - \mu(t+v)^{q-1} \right).$$

The Fréchet derivative F_v of F with respect to v at $(0, c, 0)$ is given by the formula

$$F_v(0, c, 0)v = \left(-\Delta v,\ \frac{\partial v}{\partial \mathbf{n}} \right).$$

Since $F_v(0, c, 0)$ is a homeomorphism, the implicit function theorem implies that the set $F(\mu, t, v) = 0$ around $(0, c, 0)$ consists of a unique $\mathcal{C}^{\infty}$ function $v = v(\mu, t)$ in a neighbourhood of $(\mu, t) = (0, c)$ and satisfying $v(0, c) = 0$. Now, plugging $v(\mu, t)$ in (3.3), we obtain the bifurcation equation

$$\Phi(\mu, t) = \int_{\Omega} a(t + v(\mu, t))^{p-1} + \int_{\partial\Omega} (t + v(\mu, t))^{q-1} = 0, \quad \text{for } (\mu, t) \simeq (0, c).$$

It is clear that $\Phi(0, c^*) = 0$. Differentiating Φ with respect to t at $(0, c^*)$, we get

$$\begin{aligned}
\Phi_t\left(0, c^*\right) &= \int_{\Omega} a(p-1)\left(c^* + v(0, c^*)\right)^{p-2}\left(1 + v_t(0, c^*)\right) \\
&\quad + \int_{\partial\Omega} (q-1)\left(c^* + v(0, c^*)\right)^{q-2}\left(1 + v_t(0, c^*)\right) \\
&= (p-1)(c^*)^{p-2} \int_{\Omega} a\left(1 + v_t(0, c^*)\right) + (q-1)(c^*)^{q-2} \int_{\partial\Omega} \left(1 + v_t(0, c^*)\right).
\end{aligned}$$

Differentiating now (3.2) with respect to t, and plugging $(\mu, t) = (0, c^*)$ therein, we have $v_t(0, c^*) = 0$. Hence,

$$\Phi_t\left(0, c^*\right) = (p-1)(c^*)^{p-2}\left(\int_{\Omega} a\right) + (q-1)(c^*)^{q-2}|\partial\Omega| = (c^*)^{q-2}(q-p) < 0$$

By the implicit function theorem, the function $w(\lambda) = t(\mu) + v(\mu, t(\mu))$ with $\mu = \lambda^{\frac{p-2}{p-q}}$ satisfies the desired assertion.

□

By considering the transform $u(\lambda) = \lambda^{\frac{1}{p-q}} w(\lambda)$, we get the following result:

Proposition 3.3 *Let $0 < \theta \leq \alpha$ and $w(\lambda)$ be given by Proposition 3.2. If $\lambda > 0$ is sufficiently small, then $u(\lambda) = \lambda^{\frac{1}{p-q}} w(\lambda)$ is a positive solution of (P_λ) which satisfies $\lambda^{-\frac{1}{p-q}} u(\lambda) \to c^*$ in $\mathcal{C}^{2+\theta}(\overline{\Omega})$ as $\lambda \to 0^+$. In particular, $u(\lambda) \to 0$ in $\mathcal{C}^{2+\theta}(\overline{\Omega})$ as $\lambda \to 0^+$.*

Now, in association with the first positive solution, we discuss the existence of a minimal positive solution of (P_λ). For this purpose, we reduce (P_λ) to an operator equation in $\mathcal{C}(\overline{\Omega})$. As in [23], a positive solution u of (P_λ) can be characterized as a positive solution of the following operator equation

$$u = \mathcal{F}_\lambda(u) := \mathcal{K}\left(Mu + au^{p-1}\right) + \lambda \mathcal{R}(u^{q-1}) \quad \text{in } \mathcal{C}(\overline{\Omega}), \tag{3.4}$$

where $M > 0$ is a constant and $\mathcal{K}, \mathcal{R}$ are the resolvents of the following linear boundary value problems, respectively.

$$\begin{cases} (-\Delta + M)v = f(x) & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} (-\Delta + M)w = 0 & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \mathbf{n}} = \xi(x)|_{\partial\Omega} & \text{on } \partial\Omega. \end{cases}$$

We recall that $\mathcal{K}, \mathcal{R}$ are both compact and positive in $\mathcal{C}(\overline{\Omega})$, see Amann [2]. In particular, $\mathcal{K}$ is *strongly positive*, in the sense that for any $u \in \mathcal{C}(\overline{\Omega})$ which is nontrivial and non-negative, $\mathcal{K}u$ is strictly positive on $\overline{\Omega}$, i.e. $\mathcal{K}u$ is an interior point of the positive cone $P = \{u \in \mathcal{C}(\overline{\Omega}) : u \geq 0\}$. We denote this property by $\mathcal{K}u \gg 0$. Functions $v, w \in \mathcal{C}(\overline{\Omega})$ which are positive on $\overline{\Omega}$ are called a supersolution and a subsolution of (3.4) if $v \geq \mathcal{F}_\lambda(v)$ and $w \leq \mathcal{F}_\lambda(w)$, respectively.

Let us prove now the existence of positive subsolutions of (3.4). We recall that σ_λ and ϕ_λ are the smallest eigenvalue and the corresponding positive eigenfunction of (2.1) with $\lambda > 0$. Note that $\sigma_\lambda < 0$.

Lemma 3.4 *Let $\mu > 0$ be fixed. Then there exists $\varepsilon_\mu > 0$ such that $\varepsilon\phi_\mu$ is a subsolution of (3.4) if $0 < \varepsilon \leq \varepsilon_\mu$ and $\lambda \geq \mu$.*

Proof Note that

$$\varepsilon\phi_\mu = \mathcal{K}(\varepsilon\phi_\mu + \sigma_\mu\varepsilon\phi_\mu) + \mathcal{R}(\mu\varepsilon\phi_\mu).$$

By direct computations, there exists $\varepsilon_\mu > 0$ such that if $\lambda \geq \mu$ and $0 < \varepsilon \leq \varepsilon_\mu$, then we have

$$\sigma_\mu\varepsilon\phi_\mu - a(x)(\varepsilon\phi_\mu)^{p-1} = \varepsilon\phi_\mu\left(\sigma_\mu - a(x)(\varepsilon\phi_\mu)^{p-2}\right) \leq 0 \quad \text{in } \Omega,$$
$$\mu\varepsilon\phi_\mu - \lambda(\varepsilon\phi_\mu)^{q-1} \leq \mu\left(\varepsilon\phi_\mu - (\varepsilon\phi_\mu)^{q-1}\right) \leq 0 \quad \text{on } \partial\Omega.$$

Hence, for $\lambda \geq \mu$ and $0 < \varepsilon \leq \varepsilon_\mu$, we deduce that

$$\varepsilon\phi_\mu \leq \mathcal{K}\left(\varepsilon\phi_\mu + a(\varepsilon\phi_\mu)^{p-1}\right) + \mathcal{R}\left(\lambda(\varepsilon\phi_\mu)^{q-1}\right) = \mathcal{F}_\lambda(\varepsilon\phi_\mu),$$

as desired. □

From Lemma 3.4, we can deduce the following *a priori* lower bound for positive solutions of (P_λ):

Proposition 3.5 *Let $\mu > 0$ be fixed. Given any positive solution u of (P_λ) with $\lambda \geq \mu$, we have $u \geq \varepsilon_\mu\phi_\mu$ on $\overline{\Omega}$, where ε_μ is given by Lemma 3.4.*

Proof Let u be a positive solution of (P_λ) for $\lambda \geq \mu$. We pick M such that $Mt + a(x)t^{p-1}$ is strictly increasing in $t \in [0, \sup_\Omega u]$ for every $x \in \Omega$. Assume by contradiction that $u \not\geq \varepsilon_\mu\phi_\mu$. Then, since $u > 0$ on $\overline{\Omega}$, there exists $s \in (0, 1)$ such that $u \geq s\varepsilon_\mu\phi_\mu$ and $u - s\varepsilon_\mu\phi_\mu$ is on the boundary of the positive cone P. Lemma 3.4 tells us that $0 \leq \mathcal{F}_\lambda(s\varepsilon_\mu\phi_\mu) - s\varepsilon_\mu\phi_\mu$. On the other hand, since $\mathcal{K}$ is strongly positive, we have $0 \ll \mathcal{F}_\lambda(u) - \mathcal{F}_\lambda(s\varepsilon_\mu\phi_\mu)$. Hence, from $u = \mathcal{F}_\lambda(u)$, we deduce $0 \ll u - s\varepsilon_\mu\phi_\mu$, which is a contradiction. □

Now, using Proposition 3.5, we establish the existence of a minimal positive solution of (P_λ):

Proposition 3.6 *Let $\lambda > 0$ be such that (P_λ) has a positive solution. Then (P_λ) has a minimal positive solution $\underline{u}_\lambda$.*

Proof Let u_λ be a positive solution of (P_λ). Consider the interval in $\mathcal{C}(\overline{\Omega})$

$$[\varepsilon_\lambda \phi_\lambda, u_\lambda] := \left\{u \in \mathcal{C}(\overline{\Omega}) : \varepsilon_\lambda \phi_\lambda \le u \le u_\lambda\right\},$$

and recall that $\varepsilon_\lambda \phi_\lambda$ is a subsolution of (3.4) from Lemma 3.4 with $\mu = \lambda$. Since u_λ is a supersolution of (3.4), by the super and subsolution technique of [2], there exist a minimal solution $\underline{u}_\lambda$ and a maximal solution $\overline{u}_\lambda$ of (3.4) which are in $[\varepsilon_\lambda \phi_\lambda, u_\lambda]$, in the sense that any solution $u \in [\varepsilon_\lambda \phi_\lambda, u_\lambda]$ of (3.4) satisfies $\underline{u}_\lambda \le u \le \overline{u}_\lambda$.

We show now that $\underline{u}_\lambda$ is minimal among the positive solutions of (P_λ). Let u be an arbitrary positive solution of (P_λ). We choose $M > 0$ such that $Mt + a(x)t^{p-1}$ is increasing in $[0, \sup_\Omega u + \sup_\Omega u_\lambda]$, implying that if $v, w \in [0, \sup_\Omega u + \sup_\Omega u_\lambda]$ satisfy that $v - w \in P$, then we have $0 \le \mathcal{F}_\lambda(v) - \mathcal{F}_\lambda(w)$. Put $u_\lambda \wedge u = \min(u_\lambda, u)$. Since $u - (u_\lambda \wedge u) \in P$ and $u_\lambda - (u_\lambda \wedge u) \in P$, we see that

$$0 \le \mathcal{F}_\lambda(u) - \mathcal{F}_\lambda(u_\lambda \wedge u) \quad \text{and} \quad 0 \le \mathcal{F}_\lambda(u_\lambda) - \mathcal{F}_\lambda(u_\lambda \wedge u).$$

It follows that

$$\mathcal{F}_\lambda(u_\lambda \wedge u) \le \mathcal{F}_\lambda(u_\lambda) \wedge \mathcal{F}_\lambda(u) = u_\lambda \wedge u.$$

This means that $u_\lambda \wedge u$ is a supersolution of (3.4). Now, from Proposition 3.5, we obtain $\varepsilon_\lambda \phi_\lambda \le u_\lambda \wedge u$. Applying the sub- and supersolution method in the interval $[\varepsilon_\lambda \phi_\lambda, u_\lambda \wedge u]$, we get a solution u' of (3.4) such that $\varepsilon_\lambda \phi_\lambda \le u' \le u_\lambda \wedge u$. Since u' is a solution in $[\varepsilon_\lambda \phi_\lambda, u_\lambda]$, we get $\underline{u}_\lambda \le u'$. However, it is clear that $u' \le u$. Therefore, we have $\underline{u}_\lambda \le u$, as desired. □

As a consequence of Proposition 3.5, we also have:

Proposition 3.7 *Bifurcation from zero never occurs for (P_λ) at any $\lambda > 0$. More precisely, it never occurs that there exist $\lambda_n, \lambda^* > 0$, and positive solutions u_{λ_n} of (P_{λ_n}) such that $\lambda_n \to \lambda^*$ and $\|u_n\|_{\mathcal{C}(\overline{\Omega})} \to 0$.*

Now, by Proposition 3.3, we deduce that

$$\overline{\lambda} = \sup\{\lambda > 0 : (P_\lambda) \text{ has a positive solution}\} > 0.$$

Proposition 3.8 *Assume $a > 0$ on $\partial\Omega$. Then $\overline{\lambda} < \infty$.*

Proof First of all, since $a > 0$ on $\partial\Omega$, we can choose a constant $\varepsilon_0 > 0$ such that

$$\{x \in \Omega : d(x, \partial\Omega) < \varepsilon_0\} \subset \{x \in \Omega : a(x) > 0\}, \tag{3.5}$$

where $d(x, A) = \inf\{|x - y| : y \in A\}$ for a set $A \subset \mathbb{R}^N$. Consider a positive eigenfunction Φ_1 associated with the positive principal eigenvalue Λ_1 of the problem

$$\begin{cases} -\Delta\varphi = \lambda a(x)\varphi & \text{in } D, \\ \frac{\partial\varphi}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_1, \\ \varphi = 0 & \text{on } \Gamma_0, \end{cases}$$

where

$$D = \{x \in \Omega : d(x, \partial\Omega) < \varepsilon_0\}, \quad \Gamma_1 = \partial\Omega, \quad \text{and} \quad \Gamma_0 = \{x \in \Omega : d(x, \partial\Omega) = \varepsilon_0\}.$$

By (3.5), we have $a > 0$ in D. Let u be a positive solution of (P_λ). It follows that

$$\int_\Omega \nabla u \nabla \Phi_1 - \int_\Omega a u^{p-1}\Phi_1 - \lambda \int_{\Gamma_1} u^{q-1}\Phi_1 = 0,$$

where Φ_1 is extended by zero in $\Omega \setminus D$. On the other hand, the divergence theorem shows that

$$\int_D \operatorname{div}(u\nabla\Phi_1) = \int_{\Gamma_0} u \frac{\partial \Phi_1}{\partial \nu} < 0,$$

where ν denotes the unit outer normal to Γ_0. It follows that

$$\lambda \int_{\Gamma_1} u^{q-1}\Phi_1 < \int_D a\Phi_1(\Lambda_1 u - u^{p-1}).$$

Lemma 3.4 and Proposition 3.5 allow us to deduce that given $\mu > 0$ there exists $\varepsilon_\mu > 0$ such that

$$\lambda \varepsilon_\mu^{q-1} \int_{\Gamma_1} \phi_\mu^{q-1}\Phi_1 < \sup_{t\geq 0}(\Lambda_1 t - t^{p-1}) \int_D a\Phi_1 \quad \text{if } \lambda \geq \mu.$$

Therefore, we must have $\overline{\lambda} < \infty$. □

4 Variational approach

We associate to (P_λ) the $\mathcal{C}^1$ functional

$$I_\lambda(u) := \frac{1}{2}E(u) - \frac{1}{p}A(u) - \frac{\lambda}{q}B(u), \quad u \in X,$$

where

$$E(u) = \int_\Omega |\nabla u|^2, \quad A(u) = \int_\Omega a(x)|u|^p, \quad \text{and} \quad B(u) = \int_{\partial\Omega} |u|^q.$$

Let us recall that $X = H^1(\Omega)$ is equipped with the usual norm $\|u\| = \left[\int_\Omega \left(|\nabla u|^2 + u^2\right)\right]^{\frac{1}{2}}$. We denote by $\rightharpoonup$ the weak convergence in X.

The following result will be used repeatedly in this section.

Lemma 4.1 (1) *If (u_n) is a sequence such that $u_n \rightharpoonup u_0$ in X and* $\liminf E(u_n) \leq 0$, *then u_0 is a constant and $u_n \longrightarrow u_0$ in X.*
(2) *Assume (1.8). If $v \neq 0$ and $A(v) \geq 0$, then v is not a constant.*

Proof (1) Since $u_n \rightharpoonup u_0$ in X and E is weakly lower semicontinuous, we have $E(u_0) \leq \liminf E(u_n)$, so that

$$0 \leq E(u_0) \leq \liminf E(u_n) \leq 0.$$

Hence, $E(u_0) = 0$, which implies that u_0 is a constant. Assume $u_n \not\to u_0$ in X. Then $E(u_0) < \liminf E(u_n) \leq 0$, which is a contradiction. Therefore, $u_n \to u_0$ in X.
(2) If $v_0 \neq 0$ is a constant, then $0 \leq A(v_0) = |v_0|^p \int_\Omega a < 0$, a contradiction.

□

4.1 The case $a \leq 0$

In this subsection, we assume $a \leq 0$, $a \not\equiv 0$, and (1.1) is satisfied.

Proposition 4.2 *I_λ is coercive for any $\lambda > 0$.*

Proof Let $(u_n) \subset X$ be such that $\|u_n\| \to \infty$ and assume by contradiction that $I_\lambda(u_n)$ is bounded from above. Then

$$C \geq I_\lambda(u_n) = \frac{1}{2}E(u_n) - \frac{1}{p}A(u_n) - \frac{\lambda}{q}B(u_n) \geq \frac{1}{2}E(u_n) - \frac{\lambda}{q}B(u_n).$$

Let $v_n := \frac{u_n}{\|u_n\|}$. We may assume that $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^q(\partial\Omega)$. Hence, since $q < 2 < p$, from the above inequalities we have $\limsup E(v_n) \leq 0$. By Lemma 4.1 (1), we infer that $v_n \to v_0$ in X and v_0 is a constant. On the other hand, from

$$C \geq I_\lambda(u_n) = \|u_n\|^p \left(-\frac{1}{p}A(v_n) + o(1)\right),$$

we get $A(v_0) \geq 0$, so that $A(v_0) = 0$. By Lemma 4.1 (2), we must have $v_0 \equiv 0$, which contradicts $\|v_n\| = 1$. Therefore, we reach a contradiction, which shows that I_λ is coercive for any $\lambda > 0$. □

Proposition 4.3 *(P_λ) has a unique positive solution u_λ for any $\lambda > 0$.*

Proof Let $\lambda > 0$. From Proposition 4.2, we know that I_λ is coercive. Thus, it achieves a global minimum at some $u_\lambda \in X$, which can be taken non-negative since I_λ is even. Moreover, it is clear that this global minimum is negative, and consequently $u_\lambda \not\equiv 0$. Finally, let $f(x,s) = a(x)s^{p-1}$ and $h(s) = \lambda s^{q-1}$. Since $\frac{f(x,s)}{s}$ and $\frac{h(s)}{s}$ are non-increasing in $(0,\infty)$ and $\frac{h(s)}{s}$ is decreasing, by [16, Theorem 1.2], (P_λ) has at most one positive solution. Therefore, u_λ is the unique positive solution of (P_λ). □

Remark 4.4 Proposition 4.2 holds for any $p > 1$ if we allow I_λ to take infinite values. In this case, it can be shown that the global minimum of I_λ is achieved at some u_λ such that $A(u_\lambda) > -\infty$. It follows that (P_λ) has a weak solution for any $\lambda > 0$ and $p > 1$. We refer to the proof of [12, Theorem 2] for similar arguments.

Proposition 4.5 *For any $\mu > 0$, there exists a constant $K_\mu > 0$ such that $\|u\|_\infty \leq K_\mu$ for any positive solution of (P_λ) with $\lambda \in (0,\mu)$. In particular, bifurcation from infinity cannot occur for (P_λ) at any $\lambda \geq 0$.*

Proof Fix $\mu > 0$ and assume by contradiction that $(\lambda_n) \subset (0,\mu)$, and $\|u_n\| \to \infty$ for some positive solutions u_n of (P_{λ_n}). Set $v_n = \frac{u_n}{\|u_n\|}$. We can assume that $v_n \rightharpoonup v_0$ in X. From

$$E(u_n) = A(u_n) + \lambda_n B(u_n) \leq \mu B(u_n)$$

we get $E(v_n) \to 0$, so $v_n \to v_0$ in X and v_0 is a constant. Moreover, we have $A(v_n) \to 0$, so $A(v_0) = 0$, which is impossible since $\int_\Omega a < 0$. Therefore, there exists $K_\mu > 0$ such that $\|u\| \leq K_\mu$ for any positive solution u of (P_λ) with $\lambda \in (0,\mu)$. By elliptic regularity, we get the conclusion. □

Proposition 4.6 *Let u_λ be the unique positive solution of (P_λ) for $\lambda > 0$, given by Proposition 4.3. Then u_λ satisfies the following two assertions:*

(1) $\lambda^{-\frac{1}{p-q}} u_\lambda \to c^*$ *in* $\mathcal{C}^{2+\alpha}(\overline{\Omega})$ *as* $\lambda \to 0^+$.
(2) *The mapping* $\lambda \mapsto u_\lambda$, *from* $(0, \infty)$ *to* $\mathcal{C}^{2+\alpha}(\overline{\Omega})$, *is* $\mathcal{C}^\infty$.

Proof (1) Since u_λ is the unique positive solution of (P_λ), the assertion is a direct consequence of Proposition 3.3.
(2) In view of the uniqueness of u_λ and the concavity of $u \mapsto au^{p-1}$ and $u \mapsto \lambda u^{q-1}$ for $u > 0$, by the implicit function theorem we deduce that $\lambda \mapsto u_\lambda$ is a smooth curve. Moreover, as $u_\lambda > 0$ in $\overline{\Omega}$, this mapping is $\mathcal{C}^\infty$.

□

4.2 The indefinite case

Throughout this subsection, in addition to (1.1) and (1.8), we assume that a changes sign. Moreover, we assume $p < \frac{2N}{N-2}$ if $N > 2$ (except in Proposition 4.22). We shall prove the existence of two positive solutions of (P_λ) for $0 < \lambda < \overline{\lambda}$ and characterize their asymptotic profiles as $\lambda \to 0^+$. To this end, we use the Nehari manifold and the fibering maps associated with I_λ. Let us introduce some useful subsets of X:

$$E^+ = \{u \in X : E(u) > 0\},$$
$$A^\pm = \{u \in X : A(u) \gtrless 0\}, \quad A_0 = \{u \in X : A(u) = 0\}, \quad A_0^\pm = A^\pm \cup A_0,$$
$$B^+ = \{u \in X : B(u) > 0\}.$$

The Nehari manifold associated with I_λ is given by

$$N_\lambda := \{u \in X \setminus \{0\} : \langle I_\lambda'(u), u\rangle = 0\} = \{u \in X \setminus \{0\} : E(u) = A(u) + \lambda B(u)\}.$$

We shall use the splitting

$$N_\lambda = N_\lambda^+ \cup N_\lambda^- \cup N_\lambda^0,$$

where

$$N_\lambda^\pm := \{u \in N_\lambda : \langle J_\lambda'(u), u\rangle \gtrless 0\} = \left\{u \in N_\lambda : E(u) \lessgtr \lambda \frac{p-q}{p-2} B(u)\right\}$$
$$= \left\{u \in N_\lambda : E(u) \gtrless \frac{p-q}{2-q} A(u)\right\},$$

and

$$N_\lambda^0 = \{u \in N_\lambda : \langle J_\lambda'(u), u\rangle = 0\}.$$

Note that any nontrivial weak solution of (P_λ) belongs to N_λ. Furthermore, it follows from the implicit function theorem that $N_\lambda \setminus N_\lambda^0$ is a $\mathcal{C}^1$ manifold and every critical point of the restriction of I_λ to this manifold is a critical point of I_λ (see for instance [7, Theorem 2.3]).

To analyse the structure of $N_\lambda^\pm$, we consider the fibering maps corresponding to I_λ for $u \neq 0$ in the following way:

$$j_u(t) := I_\lambda(tu) = \frac{t^2}{2} E(u) - \frac{t^p}{p} A(u) - \lambda \frac{t^q}{q} B(u), \quad t > 0.$$

It is easy to see that

$$j_u'(1) = 0 \lessgtr j_u''(1) \Longleftrightarrow u \in N_\lambda^\pm,$$

and more generally,

$$j_u'(t) = 0 \lessgtr j_u''(t) \iff tu \in N_\lambda^{\pm}.$$

Having this characterization in mind, we look for conditions under which j_u has a critical point. Set

$$i_u(t) := t^{-q} j_u(t) = \frac{t^{2-q}}{2} E(u) - \frac{t^{p-q}}{p} A(u) - \lambda B(u), \quad t > 0.$$

Let $u \in E^+ \cap A^+ \cap B^+$. Then i_u has a global maximum $i_u(t^*)$ at some $t^* > 0$, and moreover, t^* is unique. If $i_u(t^*) > 0$, then j_u has a global maximum which is positive and a local minimum which is negative. Moreover, these are the only critical points of j_u. We shall require a condition on λ that provides $i_u(t^*) > 0$. Note that

$$i_u'(t) = \frac{2-q}{2} t^{1-q} E(u) - \frac{p-q}{p} t^{p-q-1} A(u) = 0$$

if and only if

$$t = t^* := \left(\frac{p(2-q)E(u)}{2(p-q)A(u)} \right)^{\frac{1}{p-2}}.$$

Moreover,

$$i_u(t^*) = \frac{p-2}{2(p-q)} \left(\frac{p(2-q)}{2(p-q)} \right)^{\frac{2-q}{p-2}} \frac{E(u)^{\frac{p-q}{p-2}}}{A(u)^{\frac{2-q}{p-2}}} - \frac{\lambda}{q} B(u) > 0$$

if and only if

$$0 < \lambda^{\frac{p-2}{p-q}} < C_{pq} \frac{E(u)}{B(u)^{\frac{p-2}{p-q}} A(u)^{\frac{2-q}{p-q}}}, \tag{4.1}$$

where $C_{pq} = \left(\frac{q(p-2)}{2(p-q)}\right)^{\frac{p-2}{p-q}} \left(\frac{p(2-q)}{2(p-q)}\right)^{\frac{2-q}{p-q}}$. Note that $F(u) = \frac{E(u)}{B(u)^{\frac{p-2}{p-q}} A(u)^{\frac{2-q}{p-q}}}$ satisfies $F(tu) = F(u)$ for $t > 0$, i.e. F is homogeneous of order 0 (Fig. 2).

We deduce then the following result, which provides sufficient conditions for the existence of critical points of j_u:

Proposition 4.7 *The following assertions hold:*

Fig. 2 The case $i_u(t^*) > 0$

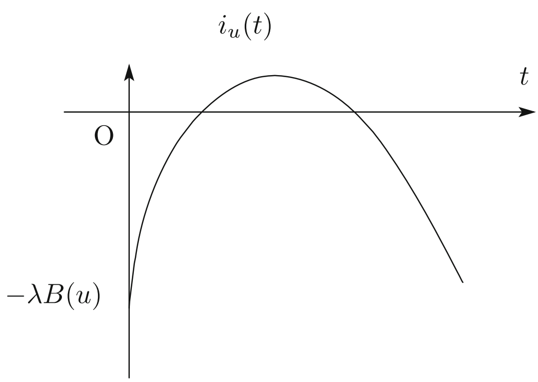
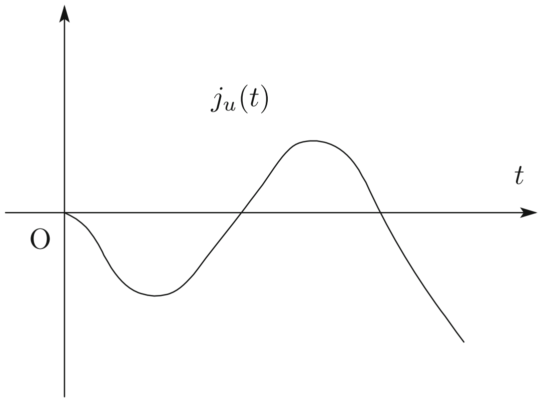


Fig. 3 A case of j_u having a global maximum and a local minimum



(1) *If either* $u \in E^+ \cap A_0^- \cap B^+$ *or* $u \in A^- \cap B^+$, *then* $j_u(t)$ *has a negative global minimum at some* $t_1 > 0$, *i.e.* $j_u'(t_1) = 0 < j_u''(t_1)$, *and* $j_u(t) > j_u(t_1)$ *for* $t \neq t_1$. *Moreover,* t_1 *is the unique critical point of* j_u *and* $j_u(t) \to \infty$ *as* $t \to \infty$.

(2) *If* $u \in E^+ \cap A^+ \cap B_0$, *then* $j_u(t)$ *has a positive global maximum at some* $t_2 > 0$, *i.e.* $j_u'(t_2) = 0 > j_u''(t_2)$ *and* $j_u(t) < j_u(t_2)$ *for* $t \neq t_1$. *Moreover,* t_2 *is the unique critical point of* j_u *and* $j_u(t) \to -\infty$ *as* $t \to \infty$.

(3) *Assume* (1.8). *If we set*

$$\lambda_0^{\frac{p-2}{p-q}} = \inf \left\{ E(u) : u \in E^+ \cap A^+ \cap B^+,\ C_{pq}^{-1} B(u)^{\frac{p-2}{p-q}} A(u)^{\frac{2-q}{p-q}} = 1 \right\}, \tag{4.2}$$

then $\lambda_0 > 0$. *Moreover, for any* $0 < \lambda < \lambda_0$ *and* $u \in E^+ \cap A^+ \cap B^+$, *the map* j_u *has a negative local minimum at* $t_1 > 0$ *and a positive global maximum at* $t_2 > t_1$. *Furthermore,* t_1, t_2 *are the only critical points of* j_u *and* $j_u(t) \to -\infty$ *as* $t \to \infty$ *(see Fig. 3).*

Proof Assertions (1) and (2) are straightforward from the definition of j_u. We prove now assertion (3). First, we show that $\lambda_0 > 0$. Assume $\lambda_0 = 0$, so that we can choose $u_n \in E^+ \cap A^+ \cap B^+$ satisfying

$$E(u_n) \longrightarrow 0, \quad \text{and} \quad C_{pq}^{-1} B(u_n)^{\frac{p-2}{p-q}} A(u_n)^{\frac{2-q}{p-q}} = 1.$$

If (u_n) is bounded in X, then we may assume that $u_n \rightharpoonup u_0$ for some $u_0 \in X$ and $u_n \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows from Lemma 4.1(1) that u_0 is a constant and $u_n \to u_0$ in X. From $u_n \in A^+$, we deduce that $u_0 \in A_0^+$. In addition, we have

$$C_{pq}^{-1} B(u_0)^{\frac{p-2}{p-q}} A(u_0)^{\frac{2-q}{p-q}} = 1,$$

so that $u_0 \not\equiv 0$. From Lemma 4.1(2), we get a contradiction.

Let us assume now that $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$, so that $\|v_n\| = 1$. We may assume that $v_n \rightharpoonup v_0$ and $v_n \to v_0$ in $L^p(\Omega)$. Since $E(v_n) \to 0$ and $v_n \in A^+$, we have $v_n \to v_0$ in X, v_0 is a constant, and $v_0 \in A_0^+$. In particular, $\|v_0\| = 1$, i.e. $v_0 \not\equiv 0$. Lemma 4.1 provides again a contradiction.

Finally, for any $u \in E^+ \cap A^+ \cap B^+$, we have

$$\lambda_0^{\frac{p-2}{p-q}} \leq C_{pq} \frac{E(u)}{B(u)^{\frac{p-2}{p-q}} A(u)^{\frac{2-q}{p-q}}}.$$

Thus, if $0 < \lambda < \lambda_0$ then $i_u(t^*) > 0$ from (4.1). This completes the proof of assertion (3). □

Proposition 4.8 *We have, for* $0 < \lambda < \lambda_0$*:*

(1) N_λ^0 *is empty.*
(2) $N_\lambda^\pm$ *are non-empty.*

Proof (1) From Proposition 4.7, it follows that there is no $t > 0$ such that $j_u'(t) = j_u''(t) = 0$, i.e. N_λ^0 is empty.

(2) Consider the following eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda a(x)\varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Under (1.8), it is known that this problem has a unique positive principal eigenvalue λ_N with a positive principal eigenfunction φ_N. From $\varphi_N > 0$ on $\partial\Omega$ and the fact that φ_N is not a constant, we deduce that $\varphi_N \in E^+ \cap A^+ \cap B^+$. Since $0 < \lambda < \lambda_0$, Proposition 4.7(3) provides the desired conclusion. □

The following result provides some properties of N_λ^+:

Lemma 4.9 *Let* $0 < \lambda < \lambda_0$. *Then, we have the following two assertions:*

(1) N_λ^+ *is bounded in* X.
(2) $I_\lambda(u) < 0$ *for any* $u \in N_\lambda^+$ *and moreover* $t > 1$ *if* $j_u'(t) > 0$.

Proof (1) Assume $(u_n) \subset N_\lambda^+$ and $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$. It follows that $\|v_n\| = 1$, so we may assume that $v_n \rightharpoonup v_0$, $B(v_n)$ is bounded, and $v_n \to v_0$ in $L^p(\Omega)$ (implying $A(v) \to A(v_0)$). Since $u_n \in N_\lambda^+$, we see that

$$E(v_n) < \lambda\frac{p-q}{p-2}B(v_n)\|u_n\|^{q-2},$$

and thus $\limsup_n E(v_n) \le 0$. Lemma 4.1(1) yields that v_0 is a constant and $v_n \to v_0$ in X. Consequently, $\|v_0\| = 1$, and v_0 is a nonzero constant. However, since $u_n \in N_\lambda$, we see that

$$0 \le E(u_n) = A(u_n) + \lambda B(u_n),$$

and it follows that

$$0 \le A(v_n) + \lambda B(v_n)\|u_n\|^{q-p}.$$

Passing to the limit as $n \to \infty$, we deduce $0 \le A(v_0)$. Lemma 4.1(2) leads us to a contradiction. Therefore, N_λ^+ is bounded in X.

(2) Let $u \in N_\lambda^+$. Then

$$0 \le E(u) < \lambda\frac{p-q}{p-2}B(u),$$

so that $B(u) > 0$. First we assume that u is not a constant. In this case, $E(u) > 0$. If $A(u) > 0$, then Proposition 4.7(3) tells us that $I_\lambda(u) < 0$ and $t > 1$ if $j_u'(t) > 0$. On the other hand, if $A(u) \le 0$, then $u \in E^+ \cap A_0^- \cap B^+$. So Proposition 4.7(1) gives the same conclusion. Assume now that u is a constant. In this case, $A(u) = |u|^p \int_\Omega a < 0$, so that $u \in A^- \cap B^+$. Proposition 4.7(1) again yields the desired conclusion. □

Next we prove that $\inf_{N_\lambda^+} I_\lambda$ is achieved by some $u_{1,\lambda} > 0$ for $\lambda \in (0, \lambda_0)$, which implies the estimate $\overline{\lambda} \geq \lambda_0$. Furthermore, we will show that $u_{1,\lambda}$ is in fact the minimal positive solution of (P_λ) for $\lambda > 0$ sufficiently small (see Corollary 4.21).

Proposition 4.10 *For any* $0 < \lambda < \lambda_0$, *there exists* $u_{1,\lambda}$ *such that* $I_\lambda(u_{1,\lambda}) = \min_{N_\lambda^+} I_\lambda$. *In particular,* $u_{1,\lambda}$ *is a positive solution of* (P_λ).

Proof Let $0 < \lambda < \lambda_0$. We consider a minimizing sequence $(u_n) \subset N_\lambda^+$, i.e.

$$I_\lambda(u_n) \longrightarrow \inf_{N_\lambda^+} I_\lambda < 0.$$

Since (u_n) is bounded in X, we may assume that $u_n \rightharpoonup u_0$, $u_n \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows that

$$I_\lambda(u_0) \leq \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda(u) < 0,$$

so that $u_0 \not\equiv 0$. We claim that $u_n \to u_0$ in X. We have two possibilities:

- If u_0 is a constant, then $0 = E(u_0) \leq \lambda \frac{p-q}{p-2} B(u_0)$. If $B(u_0) = 0$, then $u_0 = 0$ on $\partial\Omega$, so that $u_0 = 0$ in Ω, which yields a contradiction. Hence, $B(u_0) > 0$. In this case, we have $A(u_0) = |u_0|^p \int_\Omega a < 0$, so that $u_0 \in A^- \cap B^+$. Proposition 4.7(1) implies that $t_1 u_0 \in N_\lambda^+$ and j_{u_0} has a global minimum at t_1. If $u_n \not\to u_0$, then
$$I_\lambda(t_1 u_0) = j_{u_0}(t_1) \leq j_{u_0}(1) < \liminf_n j_{u_n}(1) = \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda, \tag{4.3}$$
which is a contradiction since $t_1 u_0 \in N_\lambda^+$. Therefore, $u_n \to u_0$.
- If u_0 is not a constant, then $E(u_0) > 0$ and $B(u_0) > 0$. So either $u_0 \in E^+ \cap A_0^- \cap B^+$ or $u_0 \in E^+ \cap A^+ \cap B^+$. In the first case, j_{u_0} has a global minimum point t_1 and we can argue as in the previous case. In the second case, since $0 < \lambda < \lambda_0$, Proposition 4.7 yields that $t_1 u_0 \in N_\lambda^+$ for some $t_1 > 0$. Assume $u_n \not\to u_0$. If $1 < t_1$, then we have again
$$I_\lambda(t_1 u_0) = j_{u_0}(t_1) \leq j_{u_0}(1) < \liminf_n j_{u_n}(1) = \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda, \tag{4.4}$$
If $t_1 < 1$, then $j'_{u_n}(t_1) < 0$ for every n, so that $j'_{u_0}(t_1) < \liminf j'_{u_n}(t_1) \leq 0$, which is a contradiction. Therefore, $u_n \to u_0$.

Now, since $u_n \to u_0$ we have $j'_{u_0}(1) = 0 \leq j''_{u_0}(1)$. But $j''_{u_0}(1) = 0$ is impossible by Proposition 4.8(1). Thus, $u_0 \in N_\lambda^+$ and $I_\lambda(u_0) = \inf_{N_\lambda^+} I_\lambda$. □

Remark 4.11 From Proposition 4.10, we derive $\overline{\lambda} \geq \lambda_0$.

Next we obtain a second nontrivial non-negative weak solution of (P_λ), which achieves $\inf_{N_\lambda^-} I_\lambda$ for $\lambda \in (0, \lambda_0)$. The following result provides some properties of N_λ^-:

Lemma 4.12 *Let* $0 < \lambda < \lambda_0$. *Then we have* $I_\lambda(u) > 0$ *for any* $u \in N_\lambda^-$. *Moreover,* $t < 1$ *if* $j'_u(t) > 0$.

Proof If $u \in N_\lambda^-$, then $A(u) > 0$ and u is not a constant from Lemma 4.1(2). It follows immediately that $E(u) > 0$. If $B(u) > 0$, then, by Proposition 4.7(3), we have that $I_\lambda(u) > 0$ and $t < 1$ if $j'_u(t) > 0$. If $B(u) = 0$, then Proposition 4.7(2) provides the same conclusion. □

Proposition 4.13 *For any* $\lambda \in (0, \lambda_0)$*, there exists* $u_{2,\lambda}$ *such that* $I_\lambda(u_{2,\lambda}) = \min_{N_\lambda^-} I_\lambda$*. In particular,* $u_{2,\lambda}$ *is a positive solution of* (P_λ)*.*

Proof Since $I_\lambda(u) > 0$ for $u \in N_\lambda^-$, we can choose $u_n \in N_\lambda^-$ such that

$$I_\lambda(u_n) \longrightarrow \inf_{N_\lambda^-} I_\lambda(u) \geq 0.$$

We claim that (u_n) is bounded in X. Indeed, there exists $C > 0$ such that $I_\lambda(u_n) \leq C$. Since $u_n \in N_\lambda$, we deduce

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(u_n) - \lambda \left(\frac{1}{q} - \frac{1}{p}\right) B(u_n) = I_\lambda(u_n) \leq C.$$

Assume $\|u_n\| \to \infty$ and set $v_n = \frac{u_n}{\|u_n\|}$, so that $\|v_n\| = 1$. We may assume that $v_n \rightharpoonup v_0$, and $v_n \to v_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Then, from

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(v_n) \leq \lambda \left(\frac{1}{q} - \frac{1}{p}\right) B(v_n) \|u_n\|^{q-2} + \frac{C}{\|u_n\|^2},$$

we infer that $\limsup_n E(v_n) \leq 0$. Lemma 4.1(1) yields that v_0 is a constant, and $v_n \to v_0$ in X, which implies $\|v_0\| = 1$. However, since $u_n \in N_\lambda^-$, we observe that

$$E(v_n)\|u_n\|^{2-p} < \frac{p-q}{2-q} A(v_n).$$

Passing to the limit $n \to \infty$, we get $0 \leq A(v_0)$, which is contradictory by Lemma 4.1(2). Hence, (u_n) is bounded. We may then assume that $u_n \rightharpoonup u_0$, and $u_n \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. We claim that $u_n \to u_0$ in X. Assume $u_n \not\to u_0$. Then, since $u_n \in N_\lambda^-$, we deduce

$$0 \leq E(u_0) < \liminf_n E(u_n) \leq \liminf_n \frac{p-q}{2-q} A(u_n) = \frac{p-q}{2-q} A(u_0).$$

This implies that u_0 is not a constant by Lemma 4.1(2), so that $E(u_0) > 0$. Since $u_0 \in E^+ \cap A^+$, Proposition 4.7 tells us that there exists $t_2 > 0$ such that $t_2 u_0 \in N_\lambda^-$. Moreover, $0 = j'_{u_0}(t_2) < \liminf_n j'_{u_n}(t_2)$, since $u_n \not\to u_0$. We deduce that $j'_{u_n}(t_2) > 0$ for n large enough. Since $u_n \in N_\lambda^-$, we have $t_2 < 1$ from Lemma 4.12. Then, we observe that

$$I_\lambda(t_2 u_0) = j_{u_0}(t_2) < \liminf_n j_{u_n}(t_2) \leq \liminf_n j_{u_n}(1) = \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^-} I_\lambda.$$

This is a contradiction, which implies that $u_n \to u_0$ and $I_\lambda(u_n) \to I_\lambda(u_0) = \gamma$.

Now we verify that $u_0 \neq 0$. Assume $u_0 = 0$. Then, since $u_n \in N_\lambda$, we have

$$E(v_n)\|u_n\|^{2-q} = A(v_n)\|u_n\|^{p-q} + \lambda B(v_n),$$

where $v_n = \frac{u_n}{\|u_n\|}$. We may assume again that $v_n \rightharpoonup v_0$ and $v_n \to v_0$ in $L^q(\partial\Omega)$ and $L^p(\Omega)$. Passing to the limit as $n \to \infty$, we obtain $0 = \lambda B(v_0)$, so that $v_0 = 0$ on $\partial\Omega$. On the other hand, we observe that

$$0 < I_\lambda(u_n) = \frac{1}{2} E(u_n) - \frac{1}{p} A(u_n) - \frac{\lambda}{q} B(u_n).$$

Since $u_n \in N_\lambda$, we deduce

$$\left(\frac{1}{q} - \frac{1}{2}\right) E(v_n) \leq \left(\frac{1}{q} - \frac{1}{p}\right) A(v_n)\|u_n\|^{p-2}.$$

From the assumption $u_n \to 0$ in X, it follows that $\limsup E(v_n) \leq 0$. By Lemma 4.1(1), we get that v_0 is a constant, and $v_n \to v_0$ in X, so that $\|v_0\| = 1$. Since v_0 is a constant and $v_0 = 0$ on $\partial\Omega$, we have $v_0 = 0$ in Ω. This is a contradiction, as desired.

Finally, since $u_n \to u_0$ in X, we have $j'_{u_0}(1) = 0 \geq j''_{u_0}(1)$. But $j''_{u_0}(1) = 0$ is impossible by Proposition 4.8(1). Thus, $u_0 \in N_\lambda^-$ and $I_\lambda(u_0) = \inf_{N_\lambda^-} I_\lambda$. □

We discuss now the asymptotic profiles of $u_{1,\lambda}, u_{2,\lambda}$ as $\lambda \to 0^+$. The following lemma is concerned with the behaviour of positive solutions in N_λ^+ as $\lambda \to 0^+$:

Proposition 4.14 *If u_λ is a positive solution of (P_λ) such that $u_\lambda \in N_\lambda^+$ for $\lambda > 0$ sufficiently small, then $u_\lambda \to 0$ in X as $\lambda \to 0^+$. Moreover, there holds $\lambda^{-\frac{1}{p-q}} u_\lambda \to c^*$ in $\mathcal{C}^{2+\theta}(\overline{\Omega})$ for any $\theta \in (0, \alpha)$ as $\lambda \to 0^+$.*

Proof First we show that u_λ remains bounded in X as $\lambda \to 0^+$. Indeed, assume that $\|u_\lambda\| \to \infty$ and set $v_\lambda = \frac{u_\lambda}{\|u_\lambda\|}$. We may then assume that for some $v_0 \in X$, we have $v_\lambda \rightharpoonup v_0$ in X, and $v_\lambda \to v_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Since $u_\lambda \in N_\lambda$, we have

$$E(v_\lambda)\|u_\lambda\|^{2-p} = A(v_\lambda) + \lambda B(v_\lambda)\|u_\lambda\|^{q-p}.$$

Passing to the limit as $\lambda \to 0^+$, we obtain $A(v_0) = 0$. From $u_\lambda \in N_\lambda^+$, we have

$$E(v_\lambda) < \lambda \frac{p-q}{p-2} B(v_\lambda)\|u_\lambda\|^{q-2},$$

so that $\limsup_\lambda E(v_\lambda) \leq 0$. By Lemma 4.1(1), we infer that v_0 is a constant and $v_\lambda \to v_0$ in X, so that $\|v_0\| = 1$, i.e. $v_0 \neq 0$. This is contradictory with Lemma 4.1(2), and therefore, u_λ stays bounded in X as $\lambda \to 0^+$.

Hence, we may assume that $u_\lambda \rightharpoonup u_0$ in X and $u_\lambda \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$ as $\lambda \to 0^+$. Since $u_\lambda \in N_\lambda^+$, we observe that

$$E(u_\lambda) < \lambda \frac{p-q}{p-2} B(u_\lambda).$$

Passing to the limit as $\lambda \to 0^+$, we get $\limsup_\lambda E(u_\lambda) \leq 0$. Lemma 4.1(2) provides that u_0 is a constant and $u_\lambda \to u_0$ in X. Since $u_\lambda \in N_\lambda$, we have

$$E(u_\lambda) = A(u_\lambda) + \lambda B(u_\lambda).$$

which implies $A(u_0) = 0$, so that $u_0 = 0$ from Lemma 4.1(2). Therefore, $u_\lambda \to 0$ in X as $\lambda \to 0^+$.

Now we obtain the asymptotic profile of u_λ as $\lambda \to 0^+$. Let $w_\lambda = \lambda^{-\frac{1}{p-q}} u_\lambda$. We claim that w_λ remains bounded in X as $\lambda \to 0^+$. Indeed, since $u_\lambda \in N_\lambda^+$, we have

$$E(w_\lambda) < \frac{p-q}{p-2} \lambda^{\frac{p-2}{p-q}} B(w_\lambda).$$

Let us assume that $\|w_\lambda\| \to \infty$ and set $\psi_\lambda = \frac{w_\lambda}{\|w_\lambda\|}$. We may assume that $\psi_\lambda \rightharpoonup \psi_0$ and $\psi_\lambda \to \psi_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows that

$$E(\psi_\lambda) < \frac{p-q}{p-2} \lambda^{\frac{p-2}{p-q}} B(\psi_\lambda)\|w_\lambda\|^{q-2},$$

so that $\limsup_\lambda E(\psi_\lambda) \leq 0$. By Lemma 4.1(1), we infer that ψ_0 is a constant and $\psi_\lambda \to \psi_0$ in X. On the other hand, from $u_\lambda \in N_\lambda$ it follows that

$$0 \leq A(u_\lambda) + \lambda B(u_\lambda),$$

so that

$$-B(\psi_\lambda)\|w_\lambda\|^{q-p} \leq A(\psi_\lambda).$$

Taking the limit as $\lambda \to 0^+$, we get $0 \leq A(\psi_0)$, which contradicts Lemma 4.1(2). Hence, w_λ stays bounded in X as $\lambda \to 0^+$ and we may assume that $w_\lambda \rightharpoonup w_0$ in X and $w_\lambda \to w_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows that $\limsup_\lambda E(w_\lambda) \leq 0$, and by Lemma 4.1(1), we get that w_0 is a constant and $w_\lambda \to w_0$ in X.

It remains to show that $w_0 = c^*$. We note that w_λ satisfies

$$\int_\Omega \nabla w_\lambda \nabla w - \lambda^{\frac{p-2}{p-q}} \int_\Omega a w_\lambda^{p-1} w - \lambda^{\frac{p-2}{p-q}} \int_{\partial\Omega} w_\lambda^{q-1} w = 0, \quad \forall w \in X, \tag{4.5}$$

since u_λ is a weak solution of (P_λ). Taking $w = 1$, we see that

$$\int_\Omega a w_\lambda^{p-1} + \int_{\partial\Omega} w_\lambda^{q-1} = 0.$$

Passing to the limit as $\lambda \to 0^+$, we see that either $w_0 = 0$ or $w_0 = c^*$. However, taking $w = \frac{1}{w_\lambda^{q-1}}$ in (4.5), we obtain

$$0 > -(q-1)\int_\Omega w_\lambda^{-q}|\nabla w_\lambda|^2 = \lambda^{\frac{p-2}{p-q}}\left(\int_\Omega a w_\lambda^{p-q} + |\partial\Omega|\right),$$

so that

$$|\partial\Omega| < -\int_\Omega a w_\lambda^{p-q}.$$

It is clear then that $w_0 \neq 0$, i.e. $w_0 = c^*$, and consequently we obtain $\lambda^{-\frac{1}{p-q}} u_\lambda \to c^*$ in X. By a standard bootstrap argument, we get the desired conclusion. $\square$

We turn now to the asymptotic behaviour of $u_{2,\lambda}$ as $\lambda \to 0^+$. We shall prove initially that solutions in N_λ^- are bounded away from zero as $\lambda \to 0^+$:

Lemma 4.15 *If u_λ is a positive solution of (P_λ) such that $u_\lambda \in N_\lambda^-$ for $\lambda > 0$ sufficiently small, then $\|u_\lambda\| \geq C$ for some constant $C > 0$ as $\lambda \to 0^+$.*

Proof Assume by contradiction that (u_n) is a sequence of positive solutions of (P_{λ_n}) with $\lambda_n \to 0^+$, $u_n \in N_{\lambda_n}^-$ and $\|u_n\| \to 0$. Then, since $u_n \in N_{\lambda_n}^-$, we deduce

$$E(v_n) < \frac{p-q}{2-q} A(v_n)\|u_n\|^{p-2},$$

where $v_n = \frac{u_n}{\|u_n\|}$. We may assume that $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^p(\Omega)$. It follows that $\limsup E(v_n) \leq 0$. By Lemma 4.1(1), we get that v_0 is a constant and $v_n \to v_0$ in X, so that $\|v_0\| = 1$. On the other hand, we see that $A(v_n) > 0$, since $u_n \in N_{\lambda_n}^-$. We obtain then $0 \leq A(v_0)$, which is a contradiction with Lemma 4.1(2). $\square$

We prove now that $u_{2,\lambda}$ is bounded in X as $\lambda \to 0^+$:

Lemma 4.16 *There exists a constant $C > 0$ such that $C^{-1} \leq \|u_{2,\lambda}\| \leq C$ as $\lambda \to 0^+$.*

Proof By Lemma 4.15, we know that $\|u_{2,\lambda}\| \geq C^{-1}$ for some $C > 0$ as $\lambda \to 0^+$. We show now that $u_{2,\lambda}$ is bounded in X as $\lambda \to 0^+$. First, we show that there exists a constant $C_1 > 0$ such that $I_\lambda(u_{2,\lambda}) \leq C_1$ for every $\lambda \in (0, \lambda_0)$. To this end, we consider the following eigenvalue problem with the Dirichlet boundary condition.

$$\begin{cases} -\Delta\varphi = \lambda a(x)\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

We denote by φ_D a positive eigenfunction associated with the positive principal eigenvalue λ_D. Multiplying (4.6) by φ_D^{p-1}, we see that $\varphi_D \in A^+$. Thus, $\varphi_D \in E^+ \cap A^+ \cap B_0$ and

$$j_{\varphi_D}(t) = \frac{t^2}{2}E(\varphi_D) - \frac{t^p}{p}A(\varphi_D),$$

so that j_{φ_D} has a global maximum at some $t_2 > 0$, which implies $t_2\varphi_D \in N_\lambda^-$. Moreover, neither j_{φ_D} nor $t_2\varphi_D$ depend on $\lambda \in (0, \lambda_0)$. Let $C_1 = j_{\varphi_D}(t_2) = I_\lambda(t_2\varphi_D) > 0$. Since $t_2\varphi_D \in N_\lambda^-$, we deduce that $I_\lambda(u_{2,\lambda}) \leq C_1$.

Assume now that $\|u_{2,\lambda}\| \to \infty$ as $\lambda \to 0^+$ and set $v_\lambda = \frac{u_{2,\lambda}}{\|u_{2,\lambda}\|}$. We may assume that $v_\lambda \rightharpoonup v_0$ and $v_\lambda \to v_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Since

$$0 \leq E(u_{2,\lambda}) < \frac{p-q}{2-q}A(u_{2,\lambda}),$$

it follows that $A(v_\lambda) > 0$. Passing to the limit as $\lambda \to 0^+$, we get $A(v_0) \geq 0$. However, we will see that the condition $I_\lambda(u_{2,\lambda}) \leq C_1$ leads us to a contradiction. Indeed, since $u_{2,\lambda} \in N_\lambda$, we deduce

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(u_{2,\lambda}) - \left(\frac{1}{q} - \frac{1}{p}\right) \lambda B(u_{2,\lambda}) = I_\lambda(u_{2,\lambda}) \leq C_1.$$

Hence,

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(v_\lambda) \leq \left(\frac{1}{q} - \frac{1}{p}\right) \lambda B(v_\lambda)\|u_{2,\lambda}\|^{q-2} + C_1\|u_{2,\lambda}\|^{-2}.$$

Letting $\lambda \to 0^+$, we obtain $\limsup_\lambda E(v_\lambda) \leq 0$, and by Lemma 4.1, we infer that v_0 is a constant and $v_\lambda \to v_0$ in X. In particular, $\|v_0\| = 1$, which contradicts Lemma 4.1(2). The proof is now complete. ◻

We establish now (up to a subsequence) the precise limiting behaviour of $u_{2,\lambda}$:

Proposition 4.17 *There exists a sequence $\lambda_n \to 0^+$ such that $u_{2,\lambda_n} \to u_{2,0}$ in $C^{2+\theta}(\overline{\Omega})$ for any $\theta \in (0, \alpha)$, where $u_{2,0}$ is a positive solution of* (1.9).

Proof Since $u_{2,\lambda}$ stays bounded in X as $\lambda \to 0^+$, up to a subsequence, we have $u_{2,\lambda} \rightharpoonup u_{2,0}$, and $u_{2,\lambda} \to u_{2,0}$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$ as $\lambda \to 0^+$. Since $u_{2,\lambda}$ is a weak solution of (P_λ), we have

$$\int_\Omega \nabla u_{2,\lambda}\nabla w - \int_\Omega a u_{2,\lambda}^{p-1} w - \lambda \int_{\partial\Omega} u_{2,\lambda}^{q-1} w = 0, \quad \forall w \in X.$$

Letting $\lambda \to 0^+$, we get

$$\int_\Omega \nabla u_{2,0}\nabla w - \int_\Omega a u_{2,0}^{p-1} w = 0, \quad \forall w \in X,$$

i.e. $u_{2,0}$ is a non-negative weak solution of (1.9). If $u_{2,0} \equiv 0$, then from

$$E(u_{2,\lambda}) < \frac{p-q}{2-q} A(u_{2,\lambda}) \quad \text{and} \quad A(u_{2,0}) = 0,$$

we deduce that $\limsup_{\lambda} E(u_{2,\lambda}) \leq 0$. By Lemma 4.1(1), we infer that u_0 is a constant and $u_{2,\lambda} \to u_{2,0} = 0$ in X, which contradicts Lemma 4.16.

Finally, since $u_{2,0} \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$, and $u_{2,0} > 0$ in $\overline{\Omega}$ by the weak maximum principle and the boundary point lemma, we infer that $u_{2,0}$ is a positive solution of (1.9). By a standard bootstrap argument, we obtain the desired conclusion. □

We show now the uniqueness of positive solutions of (P_λ) converging to 0 as $\lambda \to 0^+$. This will be done combining Proposition 3.2, Proposition 4.14, and Lemma 4.15.

Lemma 4.18 *Any positive solution of* (P_λ) *converging to* 0 *in* X *as* $\lambda \to 0^+$ *belongs to* N_λ^+.

Proof By Proposition 4.8(1), we know that N_λ^0 is empty for $0 < \lambda < \lambda_0$. Furthermore, by Lemma 4.15, if $u_\lambda \in N_\lambda^-$ is a solution of (P_λ) with $\lambda \to 0^+$, then $\|u_\lambda\| \geq C$, for some constant $C > 0$. Therefore, $u_\lambda \in N_\lambda^+$. □

Proposition 4.19 (P_λ) *has a unique positive solution converging to* 0 *in* X *as* $\lambda \to 0^+$. *More precisely, there exists an open neighbourhood* U *of* $(\lambda, u) = (0, 0)$ *in* X *such that if* u *is a positive solution of* (P_λ) *with* $\lambda > 0$ *and* $(\lambda, u) \in U$*, then* $u = u(\lambda)$*, where* $u(\lambda)$ *is given by Proposition* 3.3.

Proof First of all, from Proposition 3.2 with $\theta = \theta_0 < \alpha$, we know that the set of solutions of (3.1) for $\lambda > 0$ around $(\lambda, w) = (0, c^*)$ in $\mathbb{R} \times C^{2+\theta_0}(\overline{\Omega})$ consists of $\{(\lambda, \lambda^{-\frac{1}{p-q}} u(\lambda))\}$. We assume by contradiction that for a open ball $B_{\rho_n}(0, 0)$ in X with $\rho_n \to 0^+$, we can choose $\lambda_n > 0$ and a positive solution u_{λ_n} of (P_{λ_n}) such that $(\lambda_n, u_{\lambda_n}) \in B_{\rho_n}(0, 0)$ but $u_{\lambda_n} \neq u(\lambda_n)$. Since $\lambda_n \to 0^+$ and $u_{\lambda_n} \to 0$ in X, Lemma 4.18 provides that $u_{\lambda_n} \in N_{\lambda_n}^+$ for any n large enough. So Proposition 4.14 yields $\lambda_n^{-\frac{1}{p-q}} u_{\lambda_n} \to c^*$ in $\mathcal{C}^{2+\theta_1}(\overline{\Omega})$ for $\theta_1 \in (\theta_0, \alpha)$. In particular, we have $\lambda_n^{-\frac{1}{p-q}} u_{\lambda_n} \to c^*$ in $\mathcal{C}^{2+\theta_0}(\overline{\Omega})$. It follows that $u_{\lambda_n} = u(\lambda_n)$ for n sufficiently large, which is a contradiction. □

Remark 4.20 From Lemma 4.15 and Proposition 4.19, it follows that if (u_n) is a sequence of positive solutions of (P_{λ_n}) which are not minimal and $\lambda_n \to 0^+$, then (u_n) is bounded from below by a positive constant.

Corollary 4.21 *Let* $u(\lambda)$ *be the positive solution given by Proposition* 3.3*, and let* $u_{1,\lambda}$ *be the positive solution given by Proposition* 4.10*. Then* $u(\lambda)$ *and* $u_{1,\lambda}$ *are both equal to the minimal positive solution of* (P_λ) *for* $\lambda > 0$ *sufficiently small.*

Let us prove now that if $\overline{\lambda} < \infty$, then $(P_{\overline{\lambda}})$ has a positive solution:

Proposition 4.22 *Assume* (1.8) *and* $0 < \overline{\lambda} < \infty$*. Then* (P_λ) *has a positive solution for* $\lambda = \overline{\lambda}$.

Proof By Proposition 3.6, we know that (P_λ) has a minimal positive solution $\underline{u}_\lambda$ for $0 < \lambda < \overline{\lambda}$. We claim that $\underline{u}_\lambda \in N_\lambda^+ \cup N_\lambda^0$. Indeed, we know that $\underline{u}_\lambda$ is weakly stable, i.e. if $\gamma_1(\lambda, u)$ is the smallest eigenvalue of the linearized eigenvalue problem at a positive solution u of (P_λ), namely

$$\begin{cases} -\Delta\phi = (p-1)a(x)u^{p-2}\phi + \gamma\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = \lambda(q-1)u^{q-2}\phi + \gamma\phi & \text{on } \partial\Omega, \end{cases}$$

then we have $\gamma_1(\lambda) := \gamma_1(\lambda, \underline{u}_\lambda) \geq 0$, see [1, Theorem 20.4]. On the other hand, if $\underline{u}_\lambda \in N_\lambda^-$ then

$$E(\underline{u}_\lambda) - (p-1)A(\underline{u}_\lambda) - \lambda(q-1)B(\underline{u}_\lambda) < 0,$$

which provides $\gamma_1(\lambda) < 0$. Therefore, $\underline{u}_\lambda \in N_\lambda^+ \cup N_\lambda^0$. We claim now that $\underline{u}_\lambda$ is bounded in X for $0 < \lambda < \overline{\lambda}$. Assume by contradiction that $\|\underline{u}_{\lambda_n}\| \to \infty$ with $\lambda_n \nearrow \overline{\lambda}$. Set $v_n = \frac{\underline{u}_{\lambda_n}}{\|\underline{u}_{\lambda_n}\|}$. We may assume that $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^q(\partial\Omega)$. Since $\underline{u}_{\lambda_n} \in N_{\lambda_n}^+ \cup N_{\lambda_n}^0$, we have

$$0 \leq E(v_n) \leq \lambda_n C B(v_n)\|\underline{u}_{\lambda_n}\|^{q-2} \to 0, \quad n \to \infty.$$

It follows that $v_n \to v_0$ in X, v_0 is a constant, and $\|v_0\| = 1$. Since $p \leq 2^*$, the Sobolev imbedding theorem ensures that $v_n \to v_0$ in $L^p(\Omega)$. Moreover, from

$$E(v_n)\|\underline{u}_{\lambda_n}\|^{2-p} = A(v_n) + \lambda_n B(v_n)\|\underline{u}_{\lambda_n}\|^{q-p}$$

we deduce that $0 = A(v_0) = |v_0|^p \int_\Omega a < 0$, a contradiction. Thus, $\underline{u}_\lambda$ is bounded in X for $0 < \lambda < \overline{\lambda}$. By a bootstrap argument, we may assume that $\underline{u}_\lambda \to u_1$ in $C^2(\overline{\Omega})$ as $\lambda \nearrow \overline{\lambda}$. As a consequence, we infer that u_1 is a positive solution for $\lambda = \overline{\lambda}$. □

We shall consider now the Palais–Smale condition for I_λ. Let us recall that I_λ satisfies the Palais–Smale condition if any sequence such that $(I_\lambda(u_n))$ is bounded and $I'_\lambda(u_n) \to 0$ in X' has a convergent subsequence.

Proposition 4.23 *I_λ satisfies the Palais–Smale condition for any $\lambda > 0$.*

Proof Let (u_n) be a Palais–Smale sequence for I_λ. Then

$$(I_\lambda(u_n)) \text{ is bounded} \quad \text{and} \quad I'_\lambda(u_n)\phi = o(1)\|\phi\| \quad \forall\phi \in X.$$

In particular, we have

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(u_n) - \lambda\left(\frac{1}{q} - \frac{1}{p}\right) B(u_n) = I_\lambda(u_n) - \frac{1}{p} I'_\lambda(u_n)u_n \leq c + o(1)\|u_n\| \quad (4.7)$$

for some constant c. Assume that $\|u_n\| \to \infty$ and set $v_n = \frac{u_n}{\|u_n\|}$. Then we may assume that $v_n \rightharpoonup v$ in X and $v_n \to v$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. From

$$\int_\Omega \nabla u_n \nabla\phi - a(x)|u_n|^{p-2}u_n\phi - \lambda\int_{\partial\Omega} |u_n|^{q-2}u_n\phi = o(1)\|\phi\|, \quad \forall\phi \in X \quad (4.8)$$

we get, dividing it by $\|u_n\|^{p-1}$,

$$\int_\Omega a(x)|v_n|^{p-2}v_n\phi \to 0 \quad \forall\phi \in X$$

so that

$$\int_\Omega a(x)|v|^{p-2}v\phi = 0 \quad \forall\phi \in X.$$

This equality implies that $a|v|^{p-2}v = 0$ a.e. in Ω. Hence, $av \equiv 0$. Taking now $\phi = v$ in (4.8), we obtain

$$\int_\Omega \nabla v_n \nabla v - \lambda \|u_n\|^{q-2} \int_{\partial\Omega} |v_n|^{q-2} v_n v \to 0.$$

Thus,

$$\int_\Omega \nabla v_n \nabla v \to 0$$

and since $v_n \rightharpoonup v$ in X, we get $\int_\Omega |\nabla v|^2 = 0$. So v must be a constant. From $av \equiv 0$, we deduce that $v \equiv 0$. Finally, from (4.7), dividing it by $\|u_n\|^2$ we obtain $E(v_n) \to 0$. Therefore, by Lemma 4.1, we have $v_n \to 0$ in X, which contradicts $\|v_n\| = 1$.

So (u_n) must be bounded, and up to a subsequence, $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Taking $\phi = u_n - u$ in (4.8), we get

$$\int_\Omega |\nabla u_n|^2 \to \int_\Omega |\nabla u|^2$$

and consequently $\|u_n\|^2 \to \|u\|^2$. By the uniform convexity of X, we infer that $u_n \to u$ in X. □

We prove now a multiplicity result for positive solutions of (P_λ) for $\lambda \in (0, \overline{\lambda})$. First of all, by Proposition 4.10 or Proposition 4.13, we know that $\overline{\lambda} \geq \lambda_0 > 0$. We proceed now as in [10] to obtain a solution by the variational form of the sub-supersolution method. A version of this method for a problem with Neumann boundary conditions has been proved in [14, Theorem 3]. We shall use a slightly different version of this result, namely:

Theorem 4.24 *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions such that for every $R > 0$, there exists $M = M(R) > 0$ satisfying $|f(x,s)| \leq M$ if $(x,s) \in \Omega \times [-R,R]$ and $|g(x,s)| \leq M$ if $(x,s) \in \partial\Omega \times [-R,R]$. If $\underline{u}, \overline{u} \in H^1(\Omega) \cap L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ are a weak subsolution and supersolution of (P_λ), respectively, and $\underline{u} \leq \overline{u}$ a.e. in Ω, then (P_λ) has a solution u satisfying*

$$I_\lambda(u) = \min\left\{I_\lambda(v) : \ v \in H^1(\Omega), \ \underline{u} \leq v \leq \overline{u} \text{ a.e. in } \Omega\right\}.$$

The proof of this result can be carried out following the proof of [14, Theorem 3]. As a matter of fact, the functional I_λ is not coercive but still bounded from below on the set

$$M := \{u \in H^1(\Omega) : \ \underline{u} \leq u \leq \overline{u} \ \text{ a.e. in } \Omega\}.$$

Let us pick $0 < \mu < \overline{\lambda}$ and prove that (P_μ) has two positive solutions. From the definition of $\overline{\lambda}$, we can take $\mu' \in (\mu, \overline{\lambda}]$ such that $(P_{\mu'})$ has a positive solution $u_{\mu'}$. Now, we make good use of the positive eigenfunction ϕ_1 associated with the smallest eigenvalue σ_1 of (2.1) to build up a suitable positive weak subsolution. We consider the smallest eigenvalue $\hat{\sigma}_1 := \sigma_1(\mu) < 0$ of (2.1) and the corresponding positive eigenfunction $\hat{\phi}_1 = \phi_1(\mu)$. Then $\varepsilon\hat{\phi}_1$ is a strict weak subsolution of (P_μ) if $\varepsilon > 0$ is sufficiently small. Moreover, we can choose $\varepsilon > 0$ such that $\varepsilon\hat{\phi}_1 \leq u_{\mu'}$. By Theorem 4.24 with $\underline{u} = \varepsilon\hat{\phi}_1$ and $\overline{u} = u_{\mu'}$, we obtain a solution u_0 of (P_μ) such that

$$I_\mu(u_0) = \min\left\{I_\mu(v) : \ v \in H^1(\Omega), \ \varepsilon\hat{\phi}_1 \leq v \leq u_{\mu'} \text{ a.e. in } \Omega\right\}.$$

In particular, $u_0 > 0$ in $\overline{\Omega}$. Moreover, by the strong maximum principle and the boundary point lemma, we have $\varepsilon\hat{\phi}_1 < u_0 < u_{\mu'}$ on $\overline{\Omega}$. It follows that u_0 is a local minimizer of I_μ

with respect to the $\mathcal{C}^1(\overline{\Omega})$ topology. We may then argue as in [11, Lemma 6.4] to deduce that u_0 is a local minimizer of I_μ with respect to the $H^1(\Omega)$ topology. Now we use an argument from [10]: let $\delta > 0$ such that u_0 minimizes I_μ in $B(u_0, \delta)$ and $0 \notin B(u_0, \delta)$. If u_0 is not a strict minimizer, then there exists $v_0 \in B(u_0, \delta)$, $v_0 \not\equiv 0$ such that $I_\mu(v_0) = I_\mu(u_0)$, in which case v_0 is also a local minimizer of I_μ, and consequently a solution of (P_μ). Now, if u_0 is a strict minimizer, then by [9, Theorem 5.10], we infer that for $r > 0$ sufficiently small we have

$$I_\mu(u_0) < \inf \left\{ I_\mu(u) : \ u \in H^1(\Omega), \|u - u_0\| = r \right\},$$

so that I_μ has the mountain-pass geometry (note that if $w \in A^+$, then $I_\mu(tw) \to -\infty$ as $t \to \infty$). Finally, by Proposition 4.23, I_μ satisfies the Palais–Smale condition, and since I_μ is even, the mountain-pass theorem provides a second positive solution of (P_μ).

5 Existence of a smooth positive solution curve

In this section, we discuss the existence of a smooth curve of positive solutions of (P_λ) containing the minimal positive solution $\underline{u}_\lambda$ for $\lambda \in (0, \overline{\lambda})$. To this end, we consider (P_λ) in the framework of Hölder spaces in the following way: let $U \subset \mathcal{C}^{2+\alpha}(\overline{\Omega})$ be an open neighbourhood of a function positive on $\overline{\Omega}$ such that any $v \in U$ is positive on $\overline{\Omega}$. We set

$$\begin{aligned} \mathcal{G} : (0, \infty) \times U &\longrightarrow \mathcal{C}^\alpha(\overline{\Omega}) \times \mathcal{C}^{1+\alpha}(\partial\Omega), \\ (\lambda, u) &\longmapsto \left(-\Delta u - a u^{p-1}, \ \frac{\partial u}{\partial \mathbf{n}} - \lambda u^{q-1} \right), \end{aligned}$$

so that u is a positive solution of (P_λ) if and only if $\mathcal{G}(\lambda, u) = 0$. We recall that the minimal positive solution $\underline{u}_\lambda$ is weakly stable, i.e. $\gamma_1(\lambda, \underline{u}_\lambda) \geq 0$. Moreover, we know that $\underline{u}_\lambda$ is increasing and left-continuous in $(0, \overline{\lambda}]$, i.e. $\underline{u}_\mu < \underline{u}_\lambda$ on $\overline{\Omega}$ if $\mu < \lambda$, and $\lim_{\mu \nearrow \lambda} \underline{u}_\mu = \underline{u}_\lambda$, see [1, Theorem 20.3].

For our procedure, we prove the following lemma.

Lemma 5.1 *Let u_λ be a positive solution of (P_λ) such that $\gamma_1(\lambda, u_\lambda) = 0$. Then the solution set around (λ, u_λ) is exactly given by a $\mathcal{C}^\infty$-curve $(\lambda(s), u(s)) \in \mathbb{R} \times \mathcal{C}^{2+\alpha}(\overline{\Omega})$ of positive solutions, parametrized by $s \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ and such that $(\lambda(0), u(0)) = (\lambda, u_\lambda)$, $\lambda'(0) = 0$, $\lambda''(0) < 0$, and $u(s) = u_\lambda + s\phi_1 + z(s)$, where ϕ_1 is a positive eigenfunction associated with $\gamma_1(\lambda, u_\lambda)$, and $z(0) = z'(0) = 0$. Moreover, the lower branch $(\lambda(s), u(s))$, $s \in (-\varepsilon, 0)$, is asymptotically stable, i.e. $\gamma_1(\lambda(s), u(s)) > 0$, whereas the upper branch $(\lambda(s), u(s))$, $s \in (0, \varepsilon)$, is unstable, i.e. $\gamma_1(\lambda(s), u(s)) < 0$.*

Proof Since $\gamma_1(\lambda, u_\lambda) = 0$, it follows from [8, Theorem 3.2] that we have a $\mathcal{C}^\infty$-curve $(\lambda(s), u(s))$ of positive solutions which satisfies the assertions of this lemma except $\lambda''(0) < 0$. Let us prove that $\lambda''(0) < 0$. We take $(\lambda, u) = (\lambda(s), u(s))$ and differentiate (P_λ) with respect to s to obtain

$$\begin{cases} -\Delta u' = (p-1) a u^{p-2} u' & \text{in } \Omega, \\ \frac{\partial u'}{\partial \mathbf{n}} = \lambda' u^{q-1} + \lambda (q-1) u^{q-2} u' & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Differentiating (5.1) with respect to s once more, we have

$$\begin{cases} -\Delta u'' = (p-1)(p-2)au^{p-3}(u')^2 + (p-1)au^{p-2}u'' & \text{in } \Omega, \\ \frac{\partial u''}{\partial \mathbf{n}} = \lambda'' u^{q-1} + 2\lambda'(q-1)u^{q-2}u' + \lambda(q-1)(q-2)u^{q-3}(u')^2 + \lambda(q-1)u^{q-2}u'' & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Putting $s = 0$ in (5.1) and (5.2), we have respectively

$$\begin{cases} -\Delta \phi_1 = (p-1)au_\lambda^{p-2}\phi_1 & \text{in } \Omega, \\ \frac{\partial \phi_1}{\partial \mathbf{n}} = \lambda(q-1)u_\lambda^{q-2}\phi_1 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta \psi = (p-1)(p-2)au_\lambda^{p-3}\phi_1^2 + (p-1)au_\lambda^{p-2}\psi & \text{in } \Omega, \\ \frac{\partial \psi}{\partial \mathbf{n}} = \lambda''(0)u_\lambda^{q-1} + \lambda(q-1)(q-2)u_\lambda^{q-3}\phi_1^2 + \lambda(q-1)u_\lambda^{q-2}\psi & \text{on } \partial\Omega, \end{cases}$$

where $u''(0) = \psi$. Let

$$\mathcal{L}_\lambda = -\Delta - (p-1)au_\lambda^{p-2}, \quad \mathcal{B}_\lambda = \frac{\partial}{\partial \mathbf{n}} - \lambda(q-1)u_\lambda^{q-2}.$$

Then we note that

$$\begin{cases} \mathcal{L}_\lambda \phi_1 = 0 & \text{in } \Omega, \\ \mathcal{B}_\lambda \phi_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \mathcal{L}_\lambda \psi = (p-1)(p-2)au_\lambda^{p-3}\phi_1^2 & \text{in } \Omega, \\ \mathcal{B}_\lambda \psi = \lambda''(0)u_\lambda^{q-1} + \lambda(q-1)(q-2)u_\lambda^{q-3}\phi_1^2 & \text{on } \partial\Omega. \end{cases}$$

It follows that

$$\int_\Omega \left\{(p-1)(p-2)au_\lambda^{p-3}\phi_1^2\right\}\phi_1 + \int_{\partial\Omega} \left\{\lambda''(0)u_\lambda^{q-1} + \lambda(q-1)(q-2)u_\lambda^{q-3}\phi_1^2\right\}\phi_1 = 0,$$

and thus that

$$\lambda''(0)\int_{\partial\Omega} u_\lambda^{q-1}\phi_1 = -(p-1)(p-2)\int_\Omega au_\lambda^{p-3}\phi_1^3 - \lambda(q-1)(q-2)\int_{\partial\Omega} u_\lambda^{q-3}\phi_1^3. \tag{5.3}$$

On the other hand, we have by a direct computation

$$\sum_j \frac{\partial}{\partial x_j} u_\lambda^2 \frac{\partial}{\partial x_j}\left(\frac{\phi_1}{u_\lambda}\right) = \Delta\phi_1 u_\lambda - \phi_1 \Delta u_\lambda = (2-p)au_\lambda^{p-1}\phi_1. \tag{5.4}$$

In addition, the divergence theorem yields

$$\begin{aligned} \int_\Omega \left(\frac{\phi_1}{u_\lambda}\right)^2 \sum_j \frac{\partial}{\partial x_j} u_\lambda^2 \frac{\partial}{\partial x_j}\left(\frac{\phi_1}{u_\lambda}\right) &= -\int_\Omega u_\lambda^2 2\left(\frac{\phi_1}{u_\lambda}\right)\left|\nabla \frac{\phi_1}{u_\lambda}\right|^2 + \int_{\partial\Omega} \phi_1^2 \frac{\partial}{\partial \mathbf{n}}\left(\frac{\phi_1}{u_\lambda}\right) \\ &= -C + \lambda(q-2)\int_{\partial\Omega} u_\lambda^{q-3}\phi_1^3, \end{aligned} \tag{5.5}$$

where C is a positive constant. Combining (5.4) and (5.5), we deduce that

$$(2-p)\int_\Omega au_\lambda^{p-3}\phi_1^3 = -C + \lambda(q-2)\int_{\partial\Omega} u_\lambda^{q-3}\phi_1^3. \tag{5.6}$$

We combine (5.3) and (5.6) to get rid of $\int_\Omega a u_\lambda^{p-3}\phi_1^3$, so that

$$\lambda''(0)\int_{\partial\Omega} u_\lambda^{q-1}\phi_1 = -(p-1)(p-2)\left\{\frac{C}{p-2} + \lambda\frac{2-q}{p-2}\int_{\partial\Omega} u_\lambda^{q-3}\phi_1^3\right\} - \lambda(q-1)(q-2)\int_{\partial\Omega} u_\lambda^{q-3}\phi_1^3 = -C(p-1) - \lambda(2-q)(p-q)\int_{\partial\Omega} u_\lambda^{q-3}\phi_1^3 < 0,$$

as desired. □

Based on Lemma 5.1, we can prove the following result:

Proposition 5.2 *Assume* (1.8). *Then the following assertions hold:*

(1) $\underline{u}_\lambda$ *is asymptotically stable for each* $\lambda \in (0, \overline{\lambda})$, *that is,* $\gamma_1(\lambda, \underline{u}_\lambda) > 0$.
(2) $\lambda \mapsto \underline{u}_\lambda$ *is* C^∞ *in* $(0, \overline{\lambda})$.
(3) $\underline{u}_\lambda \to 0$ *in* $C^{2+\alpha}(\overline{\Omega})$ *as* $\lambda \to 0^+$.
(4) *If* (P_λ) *has a positive solution, then it has at most one weakly stable positive solution.*

Proof The argument is similar as in [13]. First we prove assertion (1). If we assume $\gamma_1(\lambda, \underline{u}_\lambda) = 0$ for some $\lambda \in (0, \overline{\lambda})$, then by the left continuity, Lemma 5.1 provides that for some $\varepsilon > 0$ there holds $\gamma_1(\mu, \underline{u}_\mu) > 0$ for $\mu \in (\lambda - \varepsilon, \lambda)$, and $\mu \mapsto \underline{u}_\mu$ is continuous in $(\lambda - \varepsilon, \lambda]$ and C^∞ in $(\lambda - \varepsilon, \lambda)$. Since $\underline{u}_\mu$ is increasing, we deduce that $\mu \mapsto \underline{u}_\mu$ is continuous in $(0, \lambda]$, C^∞ in $(0, \lambda)$, and $\|\underline{u}_\mu\|_{C(\overline{\Omega})}$ is bounded for $\mu \in (0, \lambda)$ using the implicit function theorem and Lemma 5.1 repeatedly. By elliptic regularity, we deduce that for $r > N$, $\|\underline{u}_\mu\|_{W^{1,r}(\Omega)}$ is bounded in $(0, \lambda)$. By the Sobolev imbedding and a compactness argument, $\underline{u}_\mu \to u_0$ in $C^\theta(\overline{\Omega})$ for some $\theta \in (0, 1)$ as $\mu \searrow 0$. Note that $u_0 \geq 0$, and u_0 satisfies (3.4) with $\lambda = 0$. Hence, if $u_0 \not\equiv 0$, then u_0 is a positive solution of (P_λ) with $\lambda = 0$ by a bootstrap argument. Moreover, by continuity, $\gamma_1(0, u_0) \geq 0$. However, it is easy to verify that any positive solution of (P_λ) with $\lambda = 0$ is unstable, which provides a contradiction. Hence, $u_0 \equiv 0$.

Now, from the above argument we can pick a minimal positive solution $\underline{u}_\sigma$ of (P_σ) for some $\sigma \in (\lambda, \overline{\lambda})$ such that $\gamma_1(\sigma, \underline{u}_\sigma) > 0$. Using the implicit function theorem and Lemma 5.1 again, we can extend a C^∞-positive solution curve $\{(\mu, v_\mu)\}$ of (P_μ) to the left step by step such that $\gamma_1(\mu, v_\mu) > 0$. In addition, we see that

$$\begin{cases} \underline{\mathcal{L}}_\mu \dfrac{d\underline{u}_\mu}{d\mu} = 0 & \text{in } \Omega, \\ \underline{\mathcal{B}}_\mu \dfrac{d\underline{u}_\mu}{d\mu} = \underline{u}_\mu^{q-1} & \text{on } \partial\Omega. \end{cases}$$

Here

$$\underline{\mathcal{L}}_\mu = -\Delta - (p-1)a\underline{u}_\mu^{p-2}, \quad \underline{\mathcal{B}}_\mu = \frac{\partial}{\partial \mathbf{n}} - \mu(q-1)\underline{u}_\mu^{q-2}.$$

Since $\gamma_1(\mu, \underline{u}_\mu) > 0$ we deduce that

$$\begin{cases} \underline{\mathcal{L}}_\mu \phi_1 = \gamma_1(\mu, \underline{u}_\mu)\phi_1 > 0 & \text{in } \Omega, \\ \underline{\mathcal{B}}_\mu \phi_1 = \gamma_1(\mu, \underline{u}_\mu)\phi_1 > 0 & \text{on } \partial\Omega, \end{cases}$$

where ϕ_1 is a positive eigenfunction associated with $\gamma_1(\mu, \underline{u}_\mu)$. It follows from [17, Theorem 13, Chapter 2] that $\frac{d\underline{u}_\mu}{d\mu} \geq 0$. Hence, we can deduce that $v_\mu \to 0$ in $C^\theta(\overline{\Omega})$ as $\mu \to 0^+$

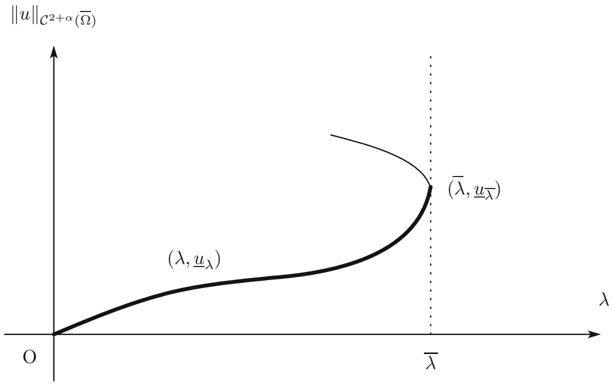


Fig. 4 A smooth positive solution curve in the case (1.1), (1.8), and the condition $\overline{\lambda} < \infty$ are satisfied

in the same way. Here we note that this curve never meets $\{(\mu, \underline{u}_\mu) : \mu \in (0, \lambda]\}$. To sum up, we infer that $\underline{u}_\mu$, v_μ both converge to 0 in X as $\mu \to 0^+$ by elliptic regularity. However, this is contradictory with Proposition 4.19. Assertion (1) has been verified.

Assertion (2) is a direct consequence of Assertion (1) and an application of the implicit function theorem.

Assertion (3) is a consequence of Proposition 4.19 and Proposition 3.3. Finally, Assertion (4) can be verified in the same way as Assertion (1). ◻

The following result is derived from Assertion (4) in Proposition 5.2.

Corollary 5.3 *The second positive solution of* (P_λ) *for* $\lambda \in (0, \overline{\lambda})$ *provided by Theorem* 4.24 *is unstable.*

Lastly, using Lemma 5.1 we provide some features of the positive solution set around $(\overline{\lambda}, \underline{u}_\lambda)$:

Proposition 5.4 *Assume* (1.1) *and* (1.8). *If* $\overline{\lambda} < \infty$, *then the solution set around* $(\overline{\lambda}, \underline{u}_{\overline{\lambda}})$ *consists of a* C^∞*-curve* $(\lambda(s), u(s)) \in \mathbb{R} \times C^{2+\alpha}(\overline{\Omega})$ *of positive solutions, which is parametrized by* $s \in (-\varepsilon, \varepsilon)$, *for some* $\varepsilon > 0$, *and such that* $(\lambda(0), u(0)) = (\overline{\lambda}, \underline{u}_{\overline{\lambda}})$, $\lambda'(0) = 0$, $\lambda''(0) < 0$, *and* $u(s) = \underline{u}_{\overline{\lambda}} + s\phi_1 + z(s)$, *where* ϕ_1 *is a positive eigenfunction associated with* $\gamma_1(\overline{\lambda}, \underline{u}_{\overline{\lambda}})$, *and* $z(0) = z'(0) = 0$. *Moreover, the lower branch* $(\lambda(s), u(s))$, $s \in (-\varepsilon, 0)$, *is asymptotically stable, whereas the upper branch* $(\lambda(s), u(s))$, $s \in (0, \varepsilon)$, *is unstable.*

Remark 5.5 Propositions 5.2 and 5.4 suggest a bifurcation diagram of positive solutions as in Fig. 4.

We conclude now the proof of our main results.

Proof of Theorem 1.1 Assertion (1) is derived from Propositions 3.3 and 2.3.

Assertion (i) in (2)(a) is a direct consequence of the general theory for minimal positive solutions, see [1, Theorem 20.3], whereas assertion (iv) in (2)(a) is derived from Proposition 5.2(3) and a combined argument of Proposition 3.3 and Corollary 4.21. The remaining assertions in (2)(a) follow from Propositions 3.6 and 5.2.

Assertion (2)(b) is a consequence Proposition 5.4; Assertion (2)(c) follows from Propositions 4.19 and Corollary 4.21; Assertion (2)(d) follows from Proposition 3.7. ◻

Proof of Theorem 1.3 In (1), the uniqueness result follows from Proposition 4.3, whereas the asymptotical stability of the unique positive solution is verified by Theorem 1.1(2)(a)(ii). Assertion (2) is derived from Propositions 4.3 and 4.6. □

Proof of Theorem 1.4 Assertion (1) is derived from Proposition 3.8. In (2), the existence of a second positive solution is provided by the argument in Sect. 4.2 based on Theorem 4.24; the ordering property of the second positive solution is derived from a combined argument of Theorem 1.1(2)(a) and an application of the strong maximum principle and the boundary point lemma; the instability result follows from Corollary 5.3; lastly, the asymptotic behaviour is provided by Proposition 4.17. □

References

1. Amann, H.: Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. SIAM Rev. **18**, 620–709 (1976)
2. Amann, H.: Nonlinear elliptic equations with nonlinear boundary conditions, New developments in differential equations. In: Proceedings of 2nd Scheveningen Conference on Scheveningen, 1975. North-Holland Math. Studies, vol. 21, pp. 43–63. North-Holland, Amsterdam (1976)
3. Ambrosetti, A., Brezis, H., Cerami, G.: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. **122**, 519–543 (1994)
4. Bandle, C., Pozio, A.M., Tesei, A.: Existence and uniqueness of solutions of nonlinear Neumann problems. Math. Z. **199**(2), 257–278 (1988)
5. Brown, K.J.: The Nehari manifold for a semilinear elliptic equation involving a sublinear term. Calc. Var. Partial Differ. Equ. **22**, 483–494 (2005)
6. Brown, K.J., Wu, T.-F.: A fibering map approach to a semilinear elliptic boundary value problem. Electron. J. Differ. Equ. **2007**(69), 9 (2007)
7. Brown, K.J., Zhang, Y.: The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function. J. Differ. Equ. **193**(2), 481–499 (2003)
8. Crandall, M.G., Rabinowitz, P.H.: Bifurcation, perturbation of simple eigenvalues and linearized stability. Arch. Ration. Mech. Anal. **52**, 161–180 (1973)
9. de Figueiredo, D.G.: Lectures on the Ekeland variational principle with applications and detours. Lectures Mathematical Physics, vol. 81. Tata Institute of Fundamental Research. Springer (1989)
10. de Figueiredo, D.G., Gossez, J.-P., Ubilla, P.: Multiplicity results for a family of semilinear elliptic problems under local superlinearity and sublinearity. J. Eur. Math. Soc. **8**(2), 269–286 (2006)
11. Garcia-Azorero, J., Peral, I., Rossi, J.D.: A convex–concave problem with a nonlinear boundary condition. J. Differ. Equ. **198**, 91–128 (2004)
12. García-Melián, J., Rossi, J.D., Sabina de Lis, J.C.: A bifurcation problem governed by the boundary condition. I. NoDEA Nonlinear Differ. Equ. Appl. **14**(5–6), 499–525 (2007)
13. García-Melián, J., Morales-Rodrigo, C., Rossi, J.D., Suárez, A.: Nonnegative solutions to an elliptic problem with nonlinear absorption and a nonlinear incoming flux on the boundary. Ann. Mat. Pura Appl. **187**(4), 459–486 (2008)
14. García-Melián, J., Rossi, J.D., Sabina de Lis, J.C.: Limit cases in an elliptic problem with a parameter-dependent boundary condition. Asymptot. Anal. **73**, 147–168 (2011)
15. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order, 2nd edn. Springer, Berlin (1983)
16. Morales-Rodrigo, C., Suárez, A.: Uniqueness of solution for elliptic problems with non-linear boundary conditions. Comm. Appl. Nonlinear Anal. **13**, 69–78 (2006)
17. Protter, M.H., Weinberger, H.F.: Maximum Principles in Differential Equations. Springer, New York (1984)
18. Ramos Quoirin, H., Umezu, K.: The effects of indefinite nonlinear boundary conditions on the structure of the positive solutions set of a logistic equation. J. Differ. Equ. **257**, 3935–3977 (2014)
19. Ramos Quoirin, H., Umezu, K.: Positive steady states of an indefinite equation with a nonlinear boundary condition: existence, multiplicity and asymptotic profiles. arXiv:1509.01753
20. Ramos Quoirin, H., Umezu, K.: Bifurcation for a logistic elliptic equation with nonlinear boundary conditions: a limiting case. J. Math. Anal. Appl. **428**(2), 1265–1285 (2015)

21. Rossi, J.D.: Elliptic problems with nonlinear boundary conditions and the Sobolev trace theorem. Stationary Partial Differential Equations. Handbook of Differential Equations, vol. II, pp. 311–406. Elsevier/North-Holland, Amsterdam (2005)
22. Tarfulea, N.: Existence of positive solutions of some nonlinear Neumann problems. An. Univ. Craiova Ser. Mat. Inform. **23**, 9–18 (1998)
23. Umezu, K.: Global bifurcation results for semilinear elliptic boundary value problems with indefinite weights and nonlinear boundary conditions. NoDEA Nonlinear Differ. Equ. Appl. **17**, 323–336 (2010)
24. Wu, T.-F.: A semilinear elliptic problem involving nonlinear boundary condition and sign-changing potential. Electron. J. Differ. Equ. **2006**(131), 15 (2006)