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Maximal subgroups of finite soluble groups in general position

Eloisa Detomi[1] · **Andrea Lucchini**[1]

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Abstract For a finite group G we investigate the difference between the maximum size $\mathrm{MaxDim}(G)$ of an "independent" family of maximal subgroups of G and maximum size $m(G)$ of an irredundant sequence of generators of G. We prove that $\mathrm{MaxDim}(G) = m(G)$ if the derived subgroup of G is nilpotent. However, $\mathrm{MaxDim}(G) - m(G)$ can be arbitrarily large: for any odd prime p, we construct a finite soluble group with Fitting length two satisfying $m(G) = 3$ and $\mathrm{MaxDim}(G) = p$.

Keywords Finite soluble groups · Intersection of maximal subgroups · Group generation

Mathematics Subject Classification 20F16 · 20F05 · 20F30

1 Introduction

Let G be a finite group. A sequence $(g_1, \dots, g_n)$ of elements of G is said to be *irredundant* if $\langle g_j \mid j \neq i \rangle$ is properly contained in $\langle g_1, \dots, g_n \rangle$ for every $i \in \{1, \dots, n\}$. Let $i(G)$ be the maximum size of any irredundant sequence in G and let $m(G)$ be the maximum size of any irredundant generating sequence of G [i.e. an irredundant sequence $(g_1, \dots, g_n)$ with the property that $\langle g_1, \dots, g_n \rangle = G$]. Clearly $m(G) \leq i(G) = \max\{m(H) \mid H \leq G\}$. The invariant $m(G)$ has received some attention (see, e.g., [1,2,4,5,7,9]) also because of its role in the efficiency of the product replacement algorithm [6]. In a recent paper, Fernando [3] investigates a natural connection between irredundant generating sequences of G and certain configurations of maximal subgroups of G. A family of subgroups $H_i \leq G$, indexed by a set I, is said to be in general position if for every $i \in I$, the intersection $\cap_{j \neq i} H_j$ properly contains $\cap_{j \in I} H_j$. Define $\mathrm{MaxDim}(G)$ as the size of the largest family of maximal subgroups of G in general position. It can be easily seen that $m(G) \leq \mathrm{MaxDim}(G) \leq i(G)$ (see, e.g.,

✉ Andrea Lucchini
lucchini@math.unipd.it

1 Dipartimento di Matematica, Università degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy

[3, Propositions 2 and 3]). However, the difference $\mathrm{MaxDim}(G) - m(G)$ can be arbitrarily large: for example if $G = \mathrm{Alt}(5) \wr C_p$ is the wreath product of the alternating group of degree 5 with a cyclic group of prime order p, then $\mathrm{MaxDim}(G) \geq 2p$ but $m(G) \leq 5$ [3, Proposition 12]. On the other hand, Fernando proves that $\mathrm{MaxDim}(G) = m(G)$ if G is a finite supersoluble group [3, Theorem 25], but gives also an example of a finite soluble group G with $m(G) \neq \mathrm{MaxDim}(G)$ [3, Proposition 16].

In this note we collect more information about the difference $\mathrm{MaxDim}(G) - m(G)$ when G is a finite soluble group. In this case $m(G)$ coincides with the number of complemented factors in a chief series of G (see [4, Theorem 2]). Our first result is that the equality $\mathrm{MaxDim}(G) = m(G)$ holds for a class of finite soluble groups, properly containing the class of finite supersoluble groups (see, e.g., [8, 7.2.13]).

Theorem 1 *If G is a finite group and the derived subgroup G' of G is nilpotent, then* $\mathrm{MaxDim}(G) = m(G)$.

However, already in the class of finite soluble groups with Fitting length equal to two, examples can be exhibited of groups G for which the difference $\mathrm{MaxDim}(G) - m(G)$ is arbitrarily large.

Theorem 2 *For any odd prime p, there exists a finite group G with Fitting length two such that $m(G) = 3$, $\mathrm{MaxDim}(G) = p$ and $i(G) = 2p$.*

Notice that if G is a soluble group with $m(G) \neq \mathrm{MaxDim}(G)$, then $m(G) \geq 3$. Indeed, if $m(G) \leq 2$, then a chief series of G contains at most two complemented factors and it can be easily seen that this implies that G' is nilpotent.

2 Groups whose derived subgroup is nilpotent

Definition 3 A family of subgroups $H_i \leq G$, indexed by a set I, is said to be in general position if for every $i \in I$, the intersection $\cap_{j \neq i} H_j$ properly contains $\cap_{j \in I} H_j$ (equivalently, H_i does not contain $\cap_{j \neq i} H_j$).

Note that the subgroups $\{H_i \mid i \in I\}$ are in general position if and only, whenever $I_1 \neq I_2$ are subsets of S, then $\cap_{i \in I_1} H_i \neq \cap_{i \in I_2} H_i$ (see, e.g., Definition 1 in [3]).

Lemma 4 *Let $\mathbb{F}$ be a field of characteristic p. Let V a finite dimension $\mathbb{F}$-vector space, let $H = \langle h \rangle$ where $h \in \mathbb{F}^*$ such that $\mathbb{F} = \mathbb{F}_p[h]$ and set $G = V \rtimes H$.*

If $M_1, \ldots, M_r$ is a set of maximal subgroups of G supplementing V, then

$$M_1 \cap \ldots \cap M_r = W \rtimes K$$

where W is a $\mathbb{F}$-subspace of V and K is either trivial or a conjugate H^v of H, for some $v \in V$.

Proof By induction on r we can assume that $T_1 = M_1 \cap \ldots \cap M_{r-1} = W_1 \rtimes K_1$, where W_1 is a subspace of V and $K_1 = \{1\}$ or $K_1 = H^v$, $v \in V$. The maximal subgroup M_r is a supplement of V, so we can write $M_r = W_2 \rtimes H^w$, where W_2 is a subspace of V and $w \in V$. For shortness, set $T_2 = M_r$ and $T = T_1 \cap T_2$. Since W_1 and W_2 are normal Sylow p-subgroups of T_1 and T_2, respectively, their intersection $W = W_1 \cap W_2$ is a normal Sylow p-subgroup of T. In the case where T is not a p-group, then $T = W \rtimes K$ where K is a non-trivial p'-subgroup of T. Then K is contained in some conjugates H^{v_1} and H^{v_2} of the p'-Hall subgroups of T_1 and T_2, respectively. In particular, there exists $1 \neq y \in K$ such that $y = h_1^{v_1} = h_2^{v_2}$ for some $h_1, h_2 \in H$. It follows that $1 \neq h_1 = h_2 \in C_H(v_1 - v_2)$. From $C_H(v_1 - v_2) \neq \{1\}$, we deduce that $v_1 = v_2$. Thus we have $T_1 = W_1 \rtimes H^{v_1}$, $T_2 = W_2 \rtimes H^{v_1}$ and $T = W \rtimes H^{v_1}$. □

Corollary 5 *In the hypotheses of Lemma 4, if $M_1, \ldots, M_r$ are in general position, then*

(1) $r \leq \dim(V) + 1$;
(2) *if $r = \dim(V) + 1$, then, for a suitable permutation of the indices, $\bigcap_{i=1}^{r-1} M_i = H^v$ for some $v \in V$, and $\bigcap_{i=1}^{r} M_i = \{1\}$.*

Proof Let $n = \dim V$. Since the subgroups $M_1, \ldots, M_r$ are in general position, the set of the intersections $T_j = \cap_{i=1}^{j} M_i$, for $j = 1, \ldots, r$, is a strictly decreasing chain of subgroups. By Lemma 4, $T_i = W_i \rtimes K_i$, where W_i is a $\mathbb{F}$-subspace of W_{i-1} and K_i is either trivial or a conjugate of H. Note that $n - 1 = \dim W_1 \geq \dim W_i \geq \dim W_{i+1}$. Moreover, if $\dim W_i = \dim W_{i+1}$ for some index i, then $W_i = W_{i+1}$ and, since $T_i \neq T_{i+1}$, we have that

- $K_1, \ldots, K_i$ are non-trivial;
- $K_{i+1} = \cdots = K_r = \{1\}$.

In particular there exists at most one index i such that $\dim W_i = \dim W_{i+1}$. As $\dim W_1 = n - 1$, it follows that we can have at most $n + 1$ subgroups T_i, hence $r \leq n + 1$.

In the case where $r = n+1$, we actually have that $\dim W_i = \dim W_{i+1}$ for at least one, and precisely one, index i. This implies that $W_i = W_{i+1}$ and, setting $J = \{1, \ldots, n+1\} \setminus \{i+1\}$ and $T = \cap_{l \in J} M_l$, we get that W_{n+1} coincides with the Sylow p-subgroup of T. Since $\dim W_{n+1} = 0$ and $T \neq 1$ we deduce that $T = H^v$, for some $v \in V$. Finally, $T \cap M_{i+1} = \{1\}$. □

A proof of the following lemma is implicitly contained in Sect. 1 of [3], but, for the sake of completeness, we sketch a direct proof here.

Lemma 6 *Let H be an abelian finite group. The size of a set of subgroups in general position is at most $m(H)$.*

Proof The proof is by induction on the order of H. Let $\Omega = \{A_1, \ldots, A_r\}$ be a set of subgroups of H in general position. Without loss of generality we can assume that $\cap_{i=1}^{r} A_i = \{1\}$. If $m = m(H)$, then H decomposes as a direct product of m cyclic groups of prime-power order. Let B be one of these factors, and let X be the unique minimal normal subgroup of B. Since $\cap_{i=1}^{r} A_i = \{1\}$, there exists at least an integer i such that X is not contained in A_i. It follows that $A_i \cap B = \{1\}$, hence $A_i \cong A_i B/B \leq H/B$ and

$$m(A_i) \leq m\,(H/B) = m - 1.$$

Now, the set of subgroups of A_i

$$\Omega^* = \{A_j \cap A_i \mid j \neq i,\ 1 \leq j \leq r\}$$

is in general position, hence, by inductive hypothesis, $|\Omega^*| = r - 1 \leq m(A_i)$. Therefore, $r \leq m$. □

Proof of Theorem 1 Since

$$m\,(G) = m\,(G/\operatorname{Frat}(G)) \text{ and } \operatorname{MaxDim}(G) = \operatorname{MaxDim}(G/\operatorname{Frat}(G)),$$

without loss of generality we can assume that $\mathrm{Frat}(G) = 1$. In this case the Fitting subgroup $\mathrm{Fit}(G)$ of G is a direct product of minimal normal subgroups of G, it is abelian and complemented. Let K be a complement of $\mathrm{Fit}(G)$ in G; note that, being G' nilpotent by assumption, K is abelian. Let F be a complement of $Z(G)$ in $\mathrm{Fit}(G)$ and let $H = Z(G) \times K$. We have $G = F \rtimes H$ and we can write F as a product of nontrivial H-irreducible modules

$$F = V_1^{n_1} \times \cdots \times V_r^{n_r}$$

where $V_1, \ldots, V_r$ are irreducible H-modules, pairwise not H-isomorphic.

By [4, Theorem 2] $m(G)$ coincides with the number of complemented factors in a chief series of G, hence

$$m(G) = \sum_{i=1}^{r} n_i + m(H).$$

Let $\mathcal{M}$ be a family of maximal subgroups of G in general position.

Let $M_{0,1}, \ldots, M_{0,\nu_0}$ the elements of $\mathcal{M}$ containing F. We can write

$$M_{0,i} = F \rtimes Y_i$$

where Y_i is a maximal subgroup of H. Note that $Y_1, \ldots, Y_{\nu_0}$ are maximal subgroups of H in general position, hence, by Lemma 6, $\nu_0 \leq \mathrm{MaxDim}(H) \leq m(H)$.

If M is a maximal subgroup supplementing F, then M contains the subgroup $U_i = \prod_{j \neq i} V_j^{n_j}$ for some index i. In particular $M = (U_i \times W_i) \rtimes H^v$ for some $v \in V_i^{n_i}$ and some maximal H-submodule W_i of $V_i^{n_i}$. Set $C_i = C_H(V_i)$ and $H_i = H/C_i$. Then $\mathbb{F}_i = \mathrm{End}_{H_i}(V_i)$ is a field and V_i is an absolutely irreducible $\mathbb{F}_i H_i$-module. Since H_i is abelian, $\dim_{\mathbb{F}_i} V_i = 1$, that is $V_i \cong \mathbb{F}_i$, and hence H_i is isomorphic to a subgroup of $\mathbb{F}_i^*$ generated by a primitive element. In particular we can apply Corollary 5 to the group $V_i^{n_i} \rtimes H_i$. Let $M_{i,1}, \ldots, M_{i,\nu_i}$ the maximal subgroups in $\mathcal{M}$ containing U_i; say

$$M_{i,l} = \big(U_i \times W_{i,l}\big) \rtimes H^{v_{i,l}},$$

where $v_{i,l} \in V_i^{n_i}$. Note that the subgroups $\overline{M}_{i,l} = W_{i,l} \rtimes H_i^{v_{i,l}}$, for $l \in \{1, \ldots, \nu_i\}$, are maximal subgroups of $V_i^{n_i} \rtimes H_i$ in general position, hence, by Corollary 5,

$$\nu_i \leq n_i + 1.$$

If $\nu_i \leq n_i$ for every $i \neq 0$, then

$$|\mathcal{M}| = \sum_{i=1}^{r} \nu_i + \nu_0 \leq \sum_{i=1}^{r} n_i + m(H) = m(G),$$

and the result follows.

Otherwise let J be the set of the integers $i \in \{1, \ldots, r\}$ such that $\nu_i = n_i + 1$. By Corollary 5, we can assume that, for some $v_i \in V_i^{n_i}$,

$$\bigcap_{l=1}^{n_i} M_{i,l} = U_i \rtimes H^{v_i},$$

$$\bigcap_{l=1}^{n_i+1} M_{i,l} = U_i \rtimes C_i.$$

Recall that the $M_{0,j} = F \rtimes Y_j$, for $j = 1, \dots, \nu_0$, are the elements of $\mathcal{M}$ containing F. Our next task is to prove that

$$\Omega = \{C_i \mid i \in J\} \cup \{Y_j \mid j = 1, \dots, \nu_0\}$$

is a set of subgroups of H in general position.

Assume, by contradiction, that for example $C_1 \geq \left(\cap_{i \neq 1} C_i\right) \cap \left(\cap_{j=1}^{\nu_0} Y_j\right)$; then

$$M_{1,n_1+1} \geq U_1 \rtimes C_1 \geq \left(\cap_{l=1}^{n_1} M_{1,l}\right) \cap \left(\cap_{i \neq 1} \left(\cap_{l=1}^{n_i+1} M_{i,l}\right)\right) \cap \left(\cap_{j=1}^{\nu_0} M_{0,j}\right)$$

against the fact that $\mathcal{M}$ is in general position. Similarly, if $Y_1 \geq (\cap_{i \in J} C_i) \cap \left(\cap_{j \neq 1} Y_j\right)$, then

$$M_{0,1} = F \rtimes Y_1 \geq \left(\cap_{i \in J} \left(\cap_{l=1}^{n_i+1} M_{i,l}\right)\right) \cap \left(\cap_{j \neq 1} M_{0,j}\right),$$

a contradiction.

Now we can apply Lemma 6 to get that $|\Omega| \leq m(H)$. Therefore, we conclude that

$$|\mathcal{M}| = \sum_{i=1}^{r} \nu_i + \nu_0 \leq \sum_{i=1}^{r} n_i + |J| + \nu_0 = \sum_{i=1}^{r} n_i + |\Omega| \leq \sum_{i=1}^{r} n_i + m(H) = m(G),$$

and the proof is complete. □

3 Finite soluble groups with $m(G) = 3$ and $\mathrm{MaxDim}(G) \geq p$

In this section we will assume that p and q are two primes and that p divides $q - 1$. Let $\mathbb{F}$ be the field with q elements and let $C = \langle c \rangle$ be the subgroup of order p of the multiplicative group of $\mathbb{F}$. Let $V = \mathbb{F}^p$ be a p-dimensional vector space over $\mathbb{F}$ and let $\sigma = (1, 2, \dots, p) \in \mathrm{Sym}(p)$. The wreath group $H = C \wr \langle \sigma \rangle$ has an irreducible action on V defined as follows: if $v = (f_1, \dots, f_p) \in V$ and $h = (c_1, \dots, c_p)\sigma \in H$, then $v^h = (f_{1\sigma^{-1}} c_{1\sigma^{-1}}, \dots, f_{p\sigma^{-1}} c_{p\sigma^{-1}})$. We will concentrate our attention on the semidirect product

$$G_{q,p} = V \rtimes H.$$

Proposition 7 $m(G_{q,p}) = 3$.

Proof Since V is a complemented chief factor of $G_{q,p}$, by [4, Theorem 2], we have $m(G_{q,p}) = 1 + m(H) = 1 + m(H/\mathrm{Frat}(H)) = 1 + m(C_p \times C_p) = 3$. □

Proposition 8 $i(G_{q,p}) = 2p$.

Proof Let $B \cong C^p$ be the base subgroup of H and consider $K = V \rtimes B \cong (\mathbb{F} \rtimes C)^p$. A composition series of K has length $2p$, and all its factors are indeed complemented chief factors, so $m(K) = 2p$. Now by definition $i(G_{q,p}) = \max\{m(X) \mid X \leq G_{q,p}\} \geq m(K) = 2p$. On the other hand, $m(G_{q,p}) = 3$ and, if $X < G_{q,p}$, then $|X|$ is a proper divisor of $|G| = (pq)^p p$ and the composition length of X is at most $2p$, so $m(X) \leq 2p$. Therefore, $i(G_{q,p}) \leq 2p$, and consequently $i(G_{q,p}) = m(K) = 2p$. □

Lemma 9 $\mathrm{MaxDim}(G_{q,p}) \geq p$.

Proof Let $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_p = (0, 0, \ldots, 1) \in V$ and let $h_1 = (c, 1, \ldots, 1), h_2 = (1, c, \ldots, 1), \ldots, h_p = (1, 1, \ldots, c) \in C^p \leq H$. For any $1 \leq i, j \leq p$, we have

$$h_i^{e_j} = h_i \text{ if } \quad i \neq j, \quad h_i^{e_i} = ((1/c - 1)\, e_i)\, h_i.$$

But then, for each $i \in \{1, \ldots, p\}$, we have

$$h_i \in \cap_{j \neq i} H^{e_j}, \quad h_i \notin H^{e_i},$$

hence $H^{e_1}, \ldots, H^{e_p}$ is a family of maximal subgroups of $G_{q,p}$ in general position. □

In order to compute the precise value of $\mathrm{MaxDim}(G_{q,p})$, the following lemma is useful.

Lemma 10 *Let* $v_1 = (x_1, \ldots, x_p)$ *and* $v_2 = (y_1, \ldots, y_p)$ *be two different elements of* $V = \mathbb{F}^p$ *and let* $\Delta(v_1, v_2) = \{i \in \{1, \ldots, p\} \mid x_i = y_i\}$. *Then*

- *if* $|\Delta(v_1, v_2)| = 0$, *then* $|H^{v_1} \cap H^{v_2}| \leq p$;
- *if* $|\Delta(v_1, v_2)| = u \neq 0$, *then* $|H^{v_1} \cap H^{v_2}| = p^u$.

Proof Clearly $|H^{v_1} \cap H^{v_2}| = |H \cap H^{v_2 - v_1}| = |C_H(v_2 - v_1)|$. If $\Delta(v_1, v_2) = \varnothing$, then $C_H(v_2 - v_1) \cap C^p = \{1\}$, hence $|C_H(v_2 - v_1)| \leq p$. If $|\Delta(v_1, v_2)| = u \neq 0$, then

$$C_H(v_2 - v_1) = \{(c_1, \ldots, c_p) \in C^p \mid c_i = 1 \text{ if } \quad i \notin \Delta(v_1, v_2)\} \cong C^u$$

has order p^u. □

Proposition 11 *If* $p \neq 2$, *then* $\mathrm{MaxDim}(G_{q,p}) = p$.

Proof By Lemma 9 it suffices to prove that $\mathrm{MaxDim}(G_{q,p}) \leq p$. Assume that $\mathcal{M}$ is a family of maximal subgroups of $G = G_{q,p}$ in general position and let $t = |\mathcal{M}|$. Let $M \in \mathcal{M}$. One of the following two possibilities occurs:

(1) M is a complement of V in G : hence $M = H^v$ for some $v \in V$.
(2) M contains V : hence $M = V \rtimes X$ for some maximal subgroup X of H.

If M_1 and M_2 are two different maximal subgroups of type (2), then $M_1 \cap M_2 = V \rtimes \mathrm{Frat}(X)$ is contained in any other maximal subgroup of type (2). Hence, $\mathcal{M}$ cannot contain more then two maximal subgroups of type (2). Now we prove the following claim: if $\mathcal{M}$ contains at least three different complements of V in G, then $t \leq p$. In order to prove this claim, assume, by contradiction that $t > p$. This implies in particular that in the intersection X of any two subgroups of $\mathcal{M}$, the subgroup lattice $\mathcal{L}(X)$ must contain a chain of length at least $p - 1$.

Assume that $H^{v_1}, H^{v_2}, H^{v_3}$ are different maximal subgroups in $\mathcal{M}$. It is not restrictive to assume $v_1 = (0, \ldots, 0)$. Let $v_2 = (x_1, \ldots, x_p)$ and $v_3 = (y_1, \ldots, y_p)$. For $i \in \{2, 3\}$, it must $|H \cap H^{v_i}| \geq p^{p-1}$, hence, by Lemma 10, $|\Delta(0, v_2)| = |\Delta(0, v_3)| = p - 1$, i.e. there exists $i_1 \neq i_2$ such that $x_{i_1} \neq 0, x_j = 0$ if $j \neq i_1, y_{i_2} \neq 0, y_j = 0$ if $j \neq i_2$. But then $|\Delta(v_2, v_3)| = p - 2$, hence $|H^{v_2} \cap H^{v_3}| = p^{p-2}$, a contradiction. We have so proved that either $t \leq p$ or $\mathcal{M}$ contains at most two maximal subgroups of type (1) and at most two maximal subgroups of type (2), and consequently $t \leq 4$. It remains to exclude the possibility that $t = 4$ and $p = 3$. By the previous considerations it is not restrictive to assume $\mathcal{M} = \{H, H^v, V \rtimes X_1, V \rtimes X_2\}$ where X_1 and X_2 are maximal subgroups of H and $|\Delta(0, v)| = 2$. In particular we would have $H \cap H^v \leq C^3$: this excludes $C^3 \in \{X_1, X_2\}$ but then $X_1 \cap C^3 = X_2 \cap C^3 = \mathrm{Frat}\ H = \{(c_1, c_2, c_3) \mid c_1 c_2 c_3 = 1\}$, hence $H \cap H^v \cap X_1 = H \cap H^v \cap X_2$, a contradiction. □

Proposition 12 $\mathrm{MaxDim}(G_{q,2}) = 3$.

Proof By Lemma 7, $\mathrm{MaxDim}(G_{q,2}) \geq m(G_{q,2}) = 3$. Assume now, by contradiction, that M_1, M_2, M_3, M_4 are a family of maximal subgroups of $G_{q,2}$. As in the proof of the previous proposition, at least two of these maximal subgroups, say M_1 and M_2, are complements of V in $G_{q,2}$. But then, by Lemma 10, $|M_1 \cap M_2| \leq 2$, hence $M_1 \cap M_2 \cap M_3 = 1$, a contradiction. □

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