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Non-periodic homogenization of bending–torsion theory for inextensible rods from 3D elasticity

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Abstract We derive, by means of Γ-convergence, the equations of homogenized bending rod starting from 3D nonlinear elasticity equations. The main assumption is that the energy behaves like h^2 (after dividing by h^2, the order of vanishing volume), where h is the thickness of the body. We do not presuppose any kind of periodicity and work in the general framework. The result shows that, on a subsequence, we always obtain the equations of the same type as in bending–torsion rod theory and identifies, in an abstract formulation, the limiting quadratic form connected with that model. This result is the generalization of periodic homogenization of bending–torsion rod theory already present in the literature.

Keywords Elasticity · Dimensional reduction · Homogenization · Bending rod model

Mathematics Subject Classification 35B27 · 49J45 · 74E30 · 74Q05

Contents

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1 Introduction

This paper deals with the derivation of homogenized bending–torsion theory for rods, starting from 3D nonlinear elasticity by means of Γ-convergence. The main novelty is that we do not presuppose any kind of periodicity, but work in a general framework.

There is vast literature on deriving rod, plate and shell equations from 3D elasticity. The first work in deriving lower-dimensional models by Γ-convergence techniques was [1], where the authors derived the string model. It was well known that the obtained models depend on the relation of the external loads (i.e., the energy) with respect to the thickness of the body h. The first rigorous derivation of higher-order models was done in [8,9] for the case of bending and von Kármán plate. The key mathematical ingredient in these cases was the theorem on geometric rigidity.

After these pioneering works, there were many papers devoted to the rigorous derivation of lower-dimensional models from 3D elasticity by means of Γ-convergence. We mention only those works that refer to the derivation of rod theories.

In [17], the authors derived the bending–torsion rod theory by assuming that the stored energy density function is fixed, i.e., that there are no oscillations of the material. As customary to derivations of bending theories, they assume that the energy is of the order h^2, where h is the thickness of the body (after division with the order of vanishing volume, which is h^2). In [18] the authors derived the rod model in the so-called von Kármán regime, where the order of energy was assumed to be h^4. In [19] the authors analyzed the stationary points (i.e., the equations) in the case of bending rod and show that the limit equation is the one corresponding to the limit energy obtained by Γ-convergence. However, due to nonlinearity, it is not generally true that the global minimizers of 3D problem (even if we have their existence) satisfy the Euler–Lagrange equations from which they start the derivation (see [7,20] for details). We emphasize the fact that the techniques used here can also be adapted to the approach in [19].

It is important to notice that, although the bending theory is small strain theory, deformations are large, in contrast to von Kármán theory, where the limit deformation is a rigid deformation and the energy depends on the correctors. Thus, we can say that bending theory carries more nonlinearity. We also mention the work [24], where the author gave the full asymptotic (higher-order) theory for curved rods.

This paper deals with the effects of simultaneous homogenization and dimensional reduction. There is vast literature on the effects of simultaneous homogenization and dimensional reduction on limit equations, in different context. In [11], the authors study these effects for a linear elasticity system without periodicity assumption, adapting H-convergence to dimensional reduction. In [4], the authors study the same effects for nonlinear systems (membrane plate) by means of Γ-convergence, also without periodicity assumptions. In [6], the authors study nonlinear monotone operators in the context of simultaneous homogenization and dimensional reduction in a general framework. Much earlier, in [15], the authors study the same effects in the case of a linearized rod model, where it was assumed that the rod is homogeneous along its central line, but the microstructure is given in the cross section. We also mention the work of Arrieta on the Laplace equation and thin domains with oscillatory boundaries (see, e.g., [3]). Finally we emphasize the work [21], where the author presented the systematic approach which consisted in combining the techniques from [8,9] and two-scale convergence methods [2] to obtain the model of homogenized bending rod.

Recently, the techniques from [8,9] were combined with two-scale convergence to obtain the models of homogenized von Kármán plate (see [23,26]), homogenized von Kármán shell (see [14]) and homogenized bending plate (see [13,27]). Most of these models were derived under the assumption of periodic oscillations of material, where it was assumed that the material oscillates on the scale $\varepsilon(h)$, while the thickness of the body is h. The obtained models depend on the parameter $\gamma = \lim_{h\to 0} \frac{h}{\varepsilon(h)}$. In the case of the von Kármán plate, the situation when $\gamma = 0$ corresponds to the case in which dimensional reduction is dominated and the obtained model is the model of homogenized von Kármán plate and can be obtained as the limit case when $\gamma \to 0$. Analogously, the situation when $\gamma = \infty$ corresponds to the case when homogenization dominates and can again be obtained as the limit when $\gamma \to \infty$; this is the model of the von Kármán plate, obtained starting from homogenized energy. In the case of the von Kármán shell and the bending plate, the situation $\gamma = 0$ was more subtle and the derived models depend on the further assumption of the relation between $\varepsilon(h)$ and h. We obtained different models for the cases $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$ and $h \sim \varepsilon(h)^2$.

In this paper, we derive the bending–torsion rod model by simultaneous homogenization and dimensional reduction without any periodicity assumption. This is a generalization of the work [21], where the author derived the bending–torsion rod theory via two-scale convergence techniques, assuming that the material oscillates periodically along the central line of the rod. Our result can be interpreted as a form of a stability result: We obtain the same type of equations starting with any kind of oscillating or non-oscillating material, and the oscillations can happen in any direction (even in the cross section). Moreover, we derive an abstract variational formula for the limit energy density which covers all the possible cases and can also be used to obtain all the regimes in the periodic case. We use slight variations of standard Γ-convergence techniques for homogenization, adapted to the special case of dimensional reduction for higher-order models in elasticity. This approach has already been used in [25] to derive the model of the von Kármán plate via simultaneous homogenization and dimensional reduction techniques. Let us emphasize the fact that this kind of stability result is not expected to be valid in the case of the bending plate or even the von Kármán shell due to more complex phenomenology in the periodic case as explained above (see also the explanations in [27] and the model obtained in [22]).

The main results in this paper are given in Theorem 2.13 and Theorem 2.14, where the "lower bound" and the "upper bound" are proven, respectively. Together with Lemma 2.6 and Lemma 2.10 they imply standard Γ-compactness results, because these lemmas imply that the Assumption 2.11 is valid on a subsequence. We prove that, on a subsequence, the limit energy density is a quadratic form in the strain of the limit deformation. The limit deformation and the strain itself are standard ones for the case of the bending rod.

1.1 Notation

- By $B(x, r)$ we denote the open ball of radius $r > 0$ around $x \in \mathbb{R}^n$ in Euclidean norm;
- for $x \in \mathbb{R}$, by $\lfloor x \rfloor$ we denote the greatest integer less or equal to x;
- e_1, e_2, e_3 denotes the canonical basis in $\mathbb{R}^3$;
- ∇_h is the scaled gradient $\nabla_h = (\partial_1, \frac{1}{h}\partial_2, \frac{1}{h}\partial_3)$;
- $\mathbb{M}^{m\times n}$ is the space of matrices with m rows and n columns, while $\mathbb{M}^n$ is the space of quadratic matrices of order n. $\mathbb{M}^n_{\mathrm{sym}}$ denotes the space of symmetric matrices of order n, while $\mathbb{M}^n_{\mathrm{skw}}$ denotes the space of skew symmetric matrices of order n;

- for $A \in \mathbb{M}^n$ by sym A we denote the symmetric part of A; sym $A = \frac{1}{2}(A + A^t)$, while by skw A we denote the skew symmetric part of A; skw $A = \frac{1}{2}(A - A^t)$;
- for $A, B \in \mathbb{M}^n$ by $A \cdot B$ we denote the scalar product $\mathrm{tr}(AB^t)$;
- $\iota : \mathbb{R}^3 \to \mathbb{M}^3$ is the natural inclusion

$$\iota(m) = \sum_{i=1}^{3} m_i e_i \otimes e_1;$$

- we denote the projection of $\mathbb{R}^3$ on the x_2x_3-plane by d_ω, $d_\omega(x) = (0, x_2, x_3)^t$;
- if $O \subset \mathbb{R}^n$ open, by $W^{1,p}(O; M)$ we denote the subset of Sobolev space of functions taking values in $M \subset \mathbb{R}^m$ for a.e. $x \in O$. It is easy to see if M is a subspace of $\mathbb{R}^m$ then $W^{1,p}(O; M)$ is a subspace of $W^{1,p}(O; \mathbb{R}^m)$. If M is closed subset of $\mathbb{R}^m$, then $W^{1,p}(O; M)$ is a closed subset of $W^{1,p}(O; \mathbb{R}^m)$ in weak and strong topology;
- for $S \subset \mathbb{R}^n$, by χ_S we denote the characteristic function of S; $\chi_S : \mathbb{R}^n \to \{0, 1\}$;
- by $|S|$ we denote the Lebesgue measure of S;
- when writing sequences (h_n), (W^h) etc., we usually omit the subscripts denoting where there the indexes live ($n \in \mathbb{N}$, $h > 0$).

1.2 General framework

Let $\omega \subset \mathbb{R}^2$ be an open connected set with Lipschitz boundary. We define by $\Omega^h = [0, L] \times h\omega$, the reference configuration of a rod-like body with thickness h. When $h = 1$ we omit the superscript and write $\Omega = \Omega^1$. We may assume that the coordinate axes are chosen such that

$$\int_\omega x_2 \,\mathrm{d}x_2 \,\mathrm{d}x_3 = \int_\omega x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_3 = \int_\omega x_2 x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_3 = 0. \tag{1}$$

We denote the moments of inertia by $\mu_i = \int_\omega x_i^2 \,\mathrm{d}x_2 \,\mathrm{d}x_3$ for $i = 2, 3$.

For each $h > 0$, the elastic energy functional on the canonical domain Ω is given by

$$\int_\Omega W^h(x, \nabla_h y^h) \,\mathrm{d}x,$$

where W^h is an elastic energy density function and $\nabla_h y^h = (\partial_1 y^h, \frac{1}{h}\partial_2 y^h, \frac{1}{h}\partial_3 y^h)$ is the scaled gradient of a deformation $y^h : \Omega \to \mathbb{R}^3$. We will assume that we are in the bending regime, i.e., that there is a positive constant C independent of h such that the energy of a minimizing sequence (y^h) satisfies the inequality:

$$\int_\Omega W^h(x, \nabla_h y^h) \,\mathrm{d}x \le Ch^2. \tag{2}$$

This assumption can be replaced by the assumption on the scaling of external loads, see [9] for details. Here we state the standard assumptions on the energy densities W^h of a composite material.

Definition 1.1 (*Nonlinear material law*) Let η_1, η_2 and ρ be any positive constants such that $\eta_1 \le \eta_2$. The class $\mathcal{W}(\eta_1, \eta_2, \rho)$ consists of all measurable functions $W : \mathbb{R}^{3\times 3} \to [0, +\infty]$ that satisfy the following properties:

frame indifference (W1)

$$W(RF) = W(F) \quad \text{for all} \quad F \in \mathbb{M}^3, R \in \mathrm{SO}(3);$$

non-degeneracy (W2)

$$W(F) \geq \eta_1 \operatorname{dist}^2(F, \mathrm{SO}(3)) \quad \text{for all} \quad F \in \mathbb{M}^3;$$
$$W(F) \leq \eta_2 \operatorname{dist}^2(F, \mathrm{SO}(3)) \quad \text{for all} \quad F \in \mathbb{M}^3 \text{ with } \operatorname{dist}^2(F, \mathrm{SO}(3)) \leq \rho;$$

W is minimal at I (W3)

$$W(I) = 0;$$

W admits a quadratic expansion at I (W4)

$$W(I+G) = Q(G) + o(|G|^2), \quad \text{as } G \to 0, \ G \in \mathbb{M}^3$$

where $Q : \mathbb{M}^3 \to \mathbb{R}$ is a quadratic form.

In the following definition, we state our assumptions on the family (W^h).

Definition 1.2 (*Admissible composite material*) Let η_1, η_2 and ρ be positive constants such that $\eta_1 \leq \eta_2$. We say that a family (W^h)

$$W^h : \Omega \times \mathbb{M}^3 \to [0, +\infty]$$

describes an admissible composite material of class $\mathcal{W}(\eta_1, \eta_2, \rho)$ if

(i) for each $h > 0$, W^h is almost everywhere equal to a Borel function on $\Omega \times \mathbb{M}^3$,
(ii) $W^h(x, \cdot) \in \mathcal{W}(\eta_1, \eta_2, \rho)$ for every $h > 0$ and almost every $x \in \Omega$,
(iii) there exists a monotone function $r : \mathbb{R}^+ \to (0, +\infty]$, such that $r(\delta) \to 0$ as $\delta \to 0$ and

$$\forall G \in \mathbb{M}^3, \forall h > 0 : \operatorname*{esssup}_{x \in \Omega} |W^h(x, I+G) - Q^h(x, G)| \leq r(|G|)|G|^2, \tag{3}$$

where $Q^h(x, \cdot)$ are quadratic forms defined in (W4).

Notice that Q^h inherits the measurability properties of W^h, since for each $h > 0$, it can be written as the pointwise limit

$$(x, G) \to Q^h(x, G) := \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon^2} W^h(x, Id + \varepsilon G). \tag{4}$$

Lemma 1.3 *Let (W^h) be as in Definition 1.2 and let (Q^h) be the family of the quadratic forms associated with (W^h) through the expansion (W4). Then for all $h > 0$ and almost all $x \in \Omega$ the map $Q^h(x, \cdot)$ is quadratic and satisfies*

(Q1) $\eta_1 |\operatorname{sym} G|^2 \leq Q^h(x, G) = Q^h(x, \operatorname{sym} G) \leq \eta_2 |\operatorname{sym} G|^2$, *for all* $G \in \mathbb{M}^3$*;*
(Q2) $|Q^h(x, G_1) - Q^h(x, G_2)| \leq \eta_2 |\operatorname{sym} G_1 - \operatorname{sym} G_2| \cdot |\operatorname{sym} G_1 + \operatorname{sym} G_2|$, *for all* $G_1, G_2 \in \mathbb{M}^3$.

Proof The property (Q1) is a direct consequence of (W2), while (Q2) follows from (Q1) and quadraticity. □

1.3 The strategy of the proofs

Here we briefly summarize the main steps of the proof of the lower bound and comment on the proof of the upper bound. The approach taken here is analogous to the one used by the second author in [25], where the equations of the von Kármán plate were derived. However, for the proof of the upper bound we found a simpler argument in this case.

Compactness. We use the rigidity estimate which was proved in [19, Proposition 4.1] (see Theorem 2.15 in the "Appendix"). For an arbitrary sequence (y^h), there is a sequence of rotations $(R^h) \subset C^\infty([0, L]; \mathbb{R}^{3\times 3})$ which approximate the scaled gradients $\nabla_h y^h$. We define the approximate strains by

$$G^h = \frac{(R^h)^t \nabla_h y^h - I}{h}.$$

Identifying the relaxation field. In the proof of the Theorem 2.13, we show that sym G^h is equal to

$$\underbrace{\mathrm{sym}\ \iota(Ad_\omega) + ae_1 \otimes e_1}_{\text{limiting part}} + \underbrace{\mathrm{sym}\ \iota\left((A^h)' d_\omega\right) + \mathrm{sym}\ \nabla_h v^h}_{\text{relaxation field}},$$

on a large set and up to a term which converges to zero in L^2. Here $A \in L^2([0, L]; \mathbb{M}^3_{\mathrm{skw}})$ is the limiting strain, a is the limiting variable that additionally appears, and d_ω is defined in the notation paragraph. The relaxation field consists of sequences $(A^h) \subset L^2([0, L]; \mathbb{M}^3_{\mathrm{skw}})$ and $(v^h) \subset L^2(\Omega; \mathbb{R}^3)$, which are such that $A^h \to 0$ and $v^h \to 0$ strongly in L^2 and $(A^h)'$, $(\nabla_h v^h)$ are bounded in L^2. To enable the use of the truncation argument, we replace the above-mentioned relaxation fields with equi-integrable ones (these are well-known results given in [10] and adapted for the case of dimension reduction in [5]).

Equivalence of the relaxation fields. In Sect. 2.1 by using the modified Griso's decomposition ([12]), we proved that the above relaxation field is essentially equivalent to the relaxation field sym $\nabla_h \psi^h$, where ψ^h are such that (sym $\nabla_h \psi^h$) is bounded in L^2 and $(\psi_1^h, h\psi_2^h, h\psi_3^h) \to 0$ and $\int_\omega x_3 \psi_2^h \to 0$ strongly in L^2 (or $\int_\omega x_2 \psi_3^h \to 0$ strongly in L^2). Notice also that the relaxation field sym $\nabla_h \psi^h$ is equivalent to the relaxation field $\nabla_h \psi^h$, due to the property (Q1) given above.

Construction of the limit energy density function. Section 2.2 is devoted to the definition of the limit energy density function. This construction is similar to the one in [25]. The decomposition of the limiting strain and the relaxation field naturally imposes the definition of the functional

$$\begin{aligned} K(Ad_\omega + ae_1, B(\bar{x}_1, r) \times \omega) = \inf \Big\{ \liminf_{h\to 0} \int_{B(\bar{x}_1, r)\times\omega} & Q^h\left(x, \iota(Ad_\omega + ae_1) + \nabla_h \psi^h\right) \mathrm{d}x : \\ & (\psi_1^h, h\psi_2^h, h\psi_3^h) \to 0 \text{ strongly in } L^2(B(\bar{x}_1, r)\times\omega; \mathbb{R}^3), \\ & \int_\omega x_3 \psi_2^h \to 0 \text{ strongly in } L^2(B(\bar{x}_1, r)) \Big\}. \end{aligned}$$

Then we derive the integral representation of K through the quadratic density

$$Q(\bar{x}_1, A, a) = \lim_{r\to 0} \frac{1}{2r} K(Ad_\omega + ae_1, B(\bar{x}_1, r) \times \omega).$$

The natural candidate for the limit energy density is the function Q_0

$$Q_0(\bar{x}_1, A) = \min_{a\in\mathbb{R}} Q_0(\bar{x}_1, A, a).$$

We finish the proof by invoking the truncation argument to establish the lower bound.

Construction of the recovery sequence. To prove the upper bound, we start from the representation formula for K given in Lemma 2.8. As in the proof of the lower bound, we use the equivalence of the relaxation fields to construct the sequences (A^h) and (v^h) appearing in the decomposition of the limiting strains. It is important to notice that for the proof of the upper bound we do not use any kind of additional regularity or higher integrability of the minimizing sequence (this would be non-trivial, since we have scaled gradients), but only their equi-integrability.

2 Derivation of the model

2.1 Characterization of relaxation field

As announced in the introduction, here we establish the equivalence of the relaxation field appearing in the approximating strain. First we recall a result by G.Griso, which is a special case of [12, Theorem 2.1] (given for general curved rod) obtained by applying it to the straight rod and by rescaling argument.

Lemma 2.1 *Let Ω, ω, L be as above and $1 < q < +\infty$. There exists a constant $C(\omega)$ and a number $h_0 > 0$ such that the following holds: For an arbitrary $0 < h < Lh_0$ and any given $u \in W^{1,q}(\Omega;\mathbb{R}^3)$, there are functions U_e and $\bar{u}$ such that $u = U_e + \bar{u}$, where $\bar{u} \in W^{1,q}(\Omega;\mathbb{R}^3)$ and U_e is an elementary displacement, i.e., there exist $\mathcal{U} \in W^{1,q}([0,L];\mathbb{R}^3)$ and $\mathcal{R} \in W^{1,q}([0,L];\mathbb{R}^3)$ such that*

$$U_e(x) = \mathcal{U}(x_1) + \mathcal{R}(x_1) \times (hx_2e_2 + hx_3e_3),$$

and the following estimates hold

$$\|\bar{u}\|_{L^q(\Omega;\mathbb{R}^3)} \leq C(\omega)h\,\|\mathrm{sym}\ \nabla_h u\|_{L^q(\Omega;\mathbb{R}^{3\times 3})}, \tag{5}$$

$$\|\nabla_h \bar{u}\|_{L^q(\Omega;\mathbb{R}^{3\times 3})} \leq C(\omega)\,\|\mathrm{sym}\ \nabla_h u\|_{L^q(\Omega;\mathbb{R}^{3\times 3})}, \tag{6}$$

$$\begin{aligned} h\,\|\mathcal{R}'\|_{L^q([0,L];\mathbb{R}^3)} &+ \|\mathcal{U}_1'\|_{L^q([0,L])} + \|\mathcal{U}_2' - \mathcal{R}_3\|_{L^q([0,L])} \\ &+ \|\mathcal{U}_3' + \mathcal{R}_2\|_{L^q([0,L])} \leq C(\omega)\,\|\mathrm{sym}\ \nabla_h u\|_{L^q(\Omega;\mathbb{R}^{3\times 3})}. \end{aligned} \tag{7}$$

We slightly alter this claim.

Lemma 2.2 *Let $1 < q < +\infty$. There exist a constant $C(\omega) > 0$ and a number $h_0 > 0$ such that the following is true: For an arbitrary $0 < h < Lh_0$ and any given $u \in W^{1,q}(\Omega;\mathbb{R}^3)$, there are $a \in \mathbb{R}^3$, $B \in \mathbb{M}^3_{\mathrm{skw}}$, $\varphi_\alpha \in W^{2,q}([0,L])$ for $\alpha = 1,2$, $w \in W^{1,q}([0,L])$ and $z \in W^{1,q}(\Omega;\mathbb{R}^3)$ such that*

$$u(x) = a + B(x_1, hx_2, hx_3)^t + \begin{pmatrix} -(\varphi_1)'(x_1)x_2 - (\varphi_2)'(x_1)x_3 + z_1(x) \\ \frac{1}{h}\varphi_1(x_1) + w(x_1)x_3 + z_2(x) \\ \frac{1}{h}\varphi_2(x_1) - w(x_1)x_2 + z_3(x) \end{pmatrix}, \tag{8}$$

and the following inequalities hold

$$\|\varphi_1\|_{W^{2,q}([0,L])} + \|\varphi_2\|_{W^{2,q}([0,L])} + \|w\|_{W^{1,q}([0,L])} \leq C(\omega)\|\,\mathrm{sym}\ \nabla_h u\|_{L^q(\Omega;\mathbb{R}^3)}, \tag{9}$$

$$\|z\|_{L^q(\Omega;\mathbb{R}^3)} + \|\nabla_h z\|_{L^q(\Omega;\mathbb{R}^3)} \leq C(\omega)\|\,\mathrm{sym}\ \nabla_h u\|_{L^q(\Omega;\mathbb{R}^3)}. \tag{10}$$

Proof We define the functions:

$$a = (\mathcal{U}_1(0), \mathcal{U}_2(0), \mathcal{U}_3(0)),$$

$$B = \begin{pmatrix} 0 & -\mathcal{R}_3(0) & \mathcal{R}_2(0) \\ \mathcal{R}_3(0) & 0 & -\mathcal{R}_1(0) \\ -\mathcal{R}_2(0) & \mathcal{R}_1(0) & 0 \end{pmatrix},$$

$$
\begin{aligned}
\varphi_1(x_1) &= h\left(\int_0^{x_1} \mathcal{R}_3(t)dt - x_1\mathcal{R}_3(0)\right),\\
\varphi_2(x_1) &= h\left(-\int_0^{x_1} \mathcal{R}_2(t)dt + x_1\mathcal{R}_2(0)\right),\\
w(x_1) &= -h(\mathcal{R}_1(x_1) - \mathcal{R}_1(0)),\\
z_1(x) &= \mathcal{U}_1(x_1) - \mathcal{U}_1(0) + \bar{u}_1(x),\\
z_2(x) &= \mathcal{U}_2(x_1) - \mathcal{U}_2(0) - \int_0^{x_1} \mathcal{R}_3(t)dt + \bar{u}_2(x),\\
z_3(x) &= \mathcal{U}_3(x_1) - \mathcal{U}_3(0) + \int_0^{x_1} \mathcal{R}_2(t)dt + \bar{u}_3(x).
\end{aligned}
$$

It is now straightforward to check that (8) holds. Taking into account that $\varphi_\alpha(0) = (\varphi_\alpha)'(0) = w(0) = 0$ and by using the Poincaré inequality and (6), we obtain that

$$\|\varphi_1\|_{W^{2,q}} + \|\varphi_2\|_{W^{2,q}} + \|w\|_{W^{1,q}} \le C_P h\|\mathcal{R}'\|_{L^q} \le C(\omega)\|\operatorname{sym}\nabla_h u\|_{L^q}.$$

By using the same arguments and (5 and 6), we also derive the second estimate

$$
\begin{aligned}
\|z\|_{L^q} + \|\nabla_h z\|_{L^q} &\le C_P\left(\left\|\mathcal{U}_1'\right\|_{L^q} + \left\|\mathcal{U}_2' - \mathcal{R}_3\right\|_{L^q} + \left\|\mathcal{U}_3' + \mathcal{R}_2\right\|_{L^q}\right)\\
&\quad + \|\bar{u}\|_{L^q} + \|\nabla_h \bar{u}\|_{L^q} + h\|\mathcal{R}'\|_{L^q} \le C(\omega)\left\|\operatorname{sym}\nabla_h u\right\|_{L^q}. \qquad \square
\end{aligned}
$$

The following corollary gives us the full characterization of sequences with bounded symmetrized gradients.

Corollary 2.3 *Let $1 < q < +\infty$, $C(\omega)$ and h_0 the constants from Lemma 2.2 and let the sequence $(u^h) \subset W^{1,q}(\Omega, \mathbb{R}^3)$ (for $0 < h < Lh_0$) be such that $(\|\operatorname{sym}\nabla_h u^h\|_{L^q})$ is bounded, (u_1^h, hu_2^h, hu_3^h) converges to zero strongly in L^q, and $\int_\omega x_3 u_2^h$ (or $\int_\omega x_2 u_3^h$) converges to zero strongly in L^q. Take the sequences $(a^h) \subset \mathbb{R}^3$, $(B^h) \subset \mathbb{M}^3_{\mathrm{skw}}$, $(z^h) \subset W^{1,q}(\Omega; \mathbb{R}^3)$, $(\varphi_\alpha^h) \subset W^{2,q}([0, L]; \mathbb{R}^3)$, for $\alpha = 1, 2$, and $(w^h) \subset W^{1,q}([0, L]; \mathbb{R}^3)$ from Lemma 2.2. Then:*

- a. *$(a_1^h, ha_2^h, ha_3^h) \to 0$, $hB^h \to 0$, $z_1^h \to 0$ strongly in L^q, $w^h \to 0$ strongly in L^q and for $\alpha = 1, 2$, $\varphi_\alpha^h \to 0$ strongly in $W^{1,q}$ as $h \to 0$.*
- b. *For the following decomposition of z^h, $z^h = \bar{z}^h + \tilde{z}^h$, where $\bar{z}^h = \int_\omega z^h$, we have that $\bar{z}_1^h \to 0$ strongly in L^q and $\|\tilde{z}^h\|_{L^q} \le C(\omega)h\|\operatorname{sym}\nabla_h u^h\|_{L^q}$, for some $C(\omega) > 0$.*
- c. *There are sequences $(A^h) \subset W^{1,q}([0, L]; \mathbb{M}^3_{\mathrm{skw}})$ and $(v^h) \subset W^{1,q}(\Omega; \mathbb{R}^3)$, such that $A^h \to 0$ and $v^h \to 0$ strongly in L^q and the following decomposition holds*

$$\operatorname{sym}\nabla_h u^h = \operatorname{sym}\iota((A^h)'d_\omega) + \operatorname{sym}\nabla_h z^h = \operatorname{sym}\iota((A^h)'d_\omega) + \operatorname{sym}\nabla_h v^h + O(h),$$

where $\|O(h)\|_{L^q} < Ch$, for some $C > 0$. Moreover, we have that

$$\|A^h\|_{W^{1,q}} + \|v^h\|_{L^q} + \|\nabla_h v^h\|_{L^q} \le C(\omega)\|\operatorname{sym}\nabla_h u^h\|_{L^q}. \tag{11}$$

Proof Since $\int_\omega \tilde{z}^h = 0$ we conclude from the Poincaré inequality that

$$\|\tilde{z}^h\|_{L^q} \le C(\omega)h\|\nabla_h z^h\|_{L^q} \le C(\omega)h\|\operatorname{sym}\nabla_h u^h\|_{L^q}.$$

Thus, $\tilde{z}^h \to 0$ strongly in L^q. After redefining a^h and B^h, we can assume that

$$\int_\Omega z^h = \int_0^L \bar{z}^h = \int_0^L w^h = \int_0^L \varphi_\alpha^h = \int_0^L x_1\varphi_\alpha^h = 0, \text{ for } \alpha = 1, 2. \tag{12}$$

Integrating the first equation in (8) over ω and taking into account the choice of coordinate axes (1), we conclude that $a_1^h|\omega| + \bar{z}_1^h \to 0$ strongly in $L^q(\omega)$. From this, by integration over $[0, L]$, we obtain that $a_1^h \to 0$ and, consequently, $\bar{z}_1^h \to 0$ strongly in $L^q(\Omega)$. By taking into account (9) and (10), we obtain that hB_{12}^h and hB_{13}^h are bounded sequences.

We multiply the second and third equations of (8) by $h(x_1 - \frac{L}{2})$, integrate over Ω and take the limit as $h \to 0$ to obtain that $hB_{12}^h \to 0$ and $hB_{13}^h \to 0$. Again, multiplying the second and third equations of (8) with h and then integrating over Ω and taking the limit as $h \to 0$, we deduce that $ha_2^h \to 0$ and $ha_3^h \to 0$. We also obtain that $\varphi_\alpha^h \to 0$ strongly in $W^{1,q}$, since it is bounded in $W^{2,q}$.

We multiply the second equation in (8) by x_3 and integrate over ω. Using the decomposition of z^h, we conclude that $hB_{23}^h + w^h \to 0$ strongly in L^q. From this, using (12), it follows that $hB_{23}^h \to 0$ and $w^h \to 0$ strongly in L^q. This finishes the proof of (a) and (b).

To prove (c) we take a sequence $p^h = (p_2^h, p_3^h) \subset C^\infty((0, L); \mathbb{R}^2)$ such that

$$\|p^h - (\bar{z}_2^h, \bar{z}_3^h)\|_{L^q} \to 0, \quad \|p^h\|_{W^{1,q}} \leq C\|(\bar{z}_2^h, \bar{z}_3^h)\|_{W^{1,q}}, \quad h\|p^h\|_{W^{2,q}} \to 0,$$

for some $C > 0$. The sequence p^h can be constructed by mollification of $(\bar{z}_2^h, \bar{z}_3^h)$ such that the mollifiers are on a scale $r_h \gg h$. We define

$$v^h = z^h - (0, p_2^h, p_3^h)^t + (hx_2 p_2^h + hx_3 p_3^h, 0, 0)^t, \quad O(h) = (-hx_2(p_2^h)' - hx_3(p_3^h)')e_1 \otimes e_1,$$

and conclude the proof. □

It is easy to prove the other implication.

Lemma 2.4 *Let $q \geq 1$, $h > 0$ and let $A \in W^{1,q}(\Omega; \mathbb{M}^3_{\mathrm{skw}})$ and $v \in W^{1,q}(\Omega; \mathbb{R}^3)$. Then there exists $u^h \in W^{1,q}(\Omega; \mathbb{R}^3)$ such that*

$$\mathrm{sym}\ \nabla_h u^h = \mathrm{sym}\ \big(\iota(A' d_\omega)\big) + \mathrm{sym}\ \nabla_h v.$$

If, in addition, $A = 0$ and $v = 0$ in the neighborhood of $\{0, L\} \times \omega$, then $u^h = 0$ in a neighborhood of $\{0\} \times \omega$ and u^h is constant in a neighborhood of $\{L\} \times \omega$. If $(A^h) \subset W^{1,q}([0, L]; \mathbb{M}^3_{\mathrm{skw}})$ and $(v^h) \subset W^{1,q}(\Omega; \mathbb{R}^3)$ are such that $A^h \to 0$ and $v^h \to 0$ strongly in L^q, then $(u_1^h, hu_2^h, hu_3^h) \to 0$ and $\int_\omega x_3 u_2^h \to 0$ and $\int_\omega x_2 u_3^h \to 0$ strongly in L^q.

Proof The proof is easily obtained by defining

$$u^h = \Big(A_{12}(x_1)x_2 + A_{13}(x_1)x_3,\ \tfrac{1}{h}\int_0^{x_1} A_{21}(t)\,\mathrm{d}t + A_{23}(x_1)x_3,\ \tfrac{1}{h}\int_0^{x_1} A_{31}(t)\,\mathrm{d}t + A_{32}(x_1)x_2\Big)^t + v.$$

□

2.2 Definition of limit energy density

The goal here is to derive the integral representation of the limit functional. The approach and techniques here are analogous to the one used in [25] to derive von Kármán equations. Hence, we only state the necessary results and refer the reader to [25] for details.

One of the main points is to establish the claim in Lemma 2.6, which tells us that Assumption 2.7 is satisfied on a subsequence.

For any open set $O \subset [0, L]$, function m in $L^2(\Omega; \mathbb{R}^3)$ and sequence (h_n) monotonically decreasing to zero, we define

$$K^-_{(h_n)}(m, O) = \inf \Big\{ \liminf_{n\to\infty} \int_{O\times\omega} Q^{h_n}\left(x, \iota(m) + \nabla_{h_n}\psi^{h_n}\right) \mathrm{d}x :$$
$$(\psi_1^{h_n}, h_n\psi_2^{h_n}, h_n\psi_3^{h_n}) \to 0 \text{ strongly in } L^2(O\times\omega; \mathbb{R}^3), \int_\omega x_3\psi_2^{h_n} \to 0 \text{ strongly in } L^2(O) \Big\},$$
$$K^+_{(h_n)}(m, O) = \inf \Big\{ \limsup_{n\to\infty} \int_{O\times\omega} Q^{h_n}\left(x, \iota(m) + \nabla_{h_n}\psi^{h_n}\right) \mathrm{d}x :$$
$$(\psi_1^{h_n}, h_n\psi_2^{h_n}, h_n\psi_3^{h_n}) \to 0 \text{ strongly in } L^2(O\times\omega; \mathbb{R}^3), \int_\omega x_3\psi_2^{h_n} \to 0 \text{stronglyin} L^2(O) \Big\}.$$

Remark 1 By using standard diagonalization argument, it can be shown that for any (h_n) monotonically decreasing to 0 the infima are attained.

The following lemma, together with Lemma 2.8, is a key to establish the properties of the functional K.

Lemma 2.5 (continuity in m) *There exists a constant $C > 0$ dependent only on η_1 and η_2 such that for every sequence (h_n) monotonically decreasing to 0 and $A \subset [0, L]$ open set the following inequality holds*

$$\left| K^-_{(h_n)}(m_1, A) - K^-_{(h_n)}(m_2, A) \right| \le C\|m_1 - m_2\|_{L^2}\left(\|m_1\|_{L^2} + \|m_2\|_{L^2}\right), \qquad (13)$$
$$\forall m_1, m_2 \in L^2(\Omega, \mathbb{R}^3),$$

The analogous claim holds for $K^+_{(h_n)}$.

Proof The proof is identical as the one in [25, Lemma 3.4]. The only difference is that not only $(\psi_1^{h_n}, h_n\psi_2^{h_n}, h_n\psi_3^{h_n}) \to 0$, but also $\int_\omega x_3\psi_2^{h_n} \to 0$, but this condition can be handled in an analogous way. □

If A and B are subsets of $[0, L]$, we denote by $A \ll B$ if $\bar{A}$ is compact and contained in B. We recall the property of density of family of sets (see [16]). We say that a family $\mathcal{D}$ of subsets is dense in a family $\mathcal{A}$, if for every $A, B \in \mathcal{A}$, with $A \ll B$, there exists $D \in \mathcal{D}$, such that $A \ll D \ll B$.

Let $\mathcal{D}$ denote a countable family of open subsets of $[0, L]$ which is dense in the class $\mathcal{A}$ of all open subsets of $[0, L]$ and such that every $D \in \mathcal{D}$ is a finite union of open intervals which are subsets of $[0, L]$.

By using Lemma 2.5 and diagonal procedure, we can easily argument the following claim (see [25, Lemma 2.6]).

Lemma 2.6 *For every sequence (h_n) monotonically decreasing to zero, there exists a subsequence, still denoted by (h_n), such that*

$$K^+_{(h_n)}(m, D) = K^-_{(h_n)}(m, D), \quad \forall m \in L^2(\Omega, \mathbb{R}^3),\ \forall D \in \mathcal{D}.$$

We will now make an assumption on the sequence (h_n) and family (Q^{h_n}).

Assumption 2.7 For a given sequence (h_n) monotonically decreasing to zero, we suppose that

$$K^+_{(h_n)}(m, D) = K^-_{(h_n)}(m, D) =: K(m, D), \quad \forall m \in L^2(\Omega, \mathbb{R}^3),\ \forall D \in \mathcal{D}.$$

Although the numbers $K(m, D)$ also depend on the sequence, we will not write it, since it will be clear from the context on which sequence we are referring to.

Remark 2 As in [25, Lemma 3.8] we can see that if a sequence (h_n) satisfies the Assumption 2.7 then we have that

$$K^+_{(h_n)}(m, O) = K^-_{(h_n)}(m, O) =: K(m, O), \quad \forall m \in L^2(\Omega, \mathbb{R}^3),\ \forall O \subset [0, L] \text{ open}.$$

Lemma 2.8 *Let (h_n) be a sequence monotonically decreasing to 0 which satisfies Assumption 2.7. For $m \in L^2(\Omega, \mathbb{R}^3)$ and $O \subset \omega$ open, there exist a subsequence $(h_{n(k)})$ and $(\vartheta_k) \subset W^{1,2}(O \times \omega, \mathbb{R}^3)$ such that*

(a) $((\vartheta_k)_1, h_{n(k)}(\vartheta_k)_2, h_{n(k)}(\vartheta_k)_3) \to 0$, $\int_\omega x_3(\vartheta_k)_2 \to 0$ *strongly in* L^2,
(b) $(|\,\mathrm{sym}\ \nabla_{h_{n(k)}}\vartheta_k|^2)$ *is equi-integrable and there are sequences* $(A_k) \subset W^{1,2}([0, L]; \mathbb{M}^3_{\mathrm{skw}})$, $A_k \to 0$ *strongly in* L^2 *and* $(v_k) \subset W^{1,2}(\Omega; \mathbb{R}^3)$, $v_k \to 0$ *strongly in* L^2 *such that*

$$\mathrm{sym}\ \nabla_{h_{n(k)}}\vartheta_k = \mathrm{sym}\ \iota((A_k)'d_\omega) + \mathrm{sym}\ \nabla_{h_{n(k)}} v_k.$$

Moreover, we have that $\left(|(A_k)'|^2\right)_{k\in\mathbb{N}}$ *and* $\left(|\nabla_{h_{n(k)}} v_k|^2\right)$ *are equi-integrable. Also the following is valid*

$$\limsup_{k\to\infty} \left(\|A_k\|_{W^{1,2}(O)} + \|\nabla_{h_{n(k)}} v_k\|_{L^2(O\times\omega)}\right) \le C\left(\eta_2\|m\|^2_{L^2} + 1\right),$$

where C is independent of the domain O. For each $k \in \mathbb{N}$ we have that $A_k = 0$ in a neighborhood of ∂O and $v_k = 0$ in a neighborhood of $\partial O \times \omega$.
(c) $K(m, O) = \lim_{k\to\infty} \int_{O\times\omega} Q^{h_{n(k)}}(x, \iota(m) + \nabla_{h_{n(k)}}\vartheta_k)\,\mathrm{d}x.$

Proof The proof is analogous to the one in [25, Lemma 3.10]. Therefore, we just state the main arguments.

Firstly, it is easy to establish the claim of lemma if the set $O \subset [0, L]$ is nice, e.g., a finite union of intervals. Then we use Corollary 2.3 to write the relaxation field in the form given there. Next, by using Lemma 2.16 and Lemma 2.17, we replace this relaxation field with the equi-integrabile one (this is possible since the relaxation field is a minimizing sequence). The truncation argument on the equi-integrabile sequence can easily be made to obtain the relaxation field which is zero near the boundary.

Then this claim, together with Lemma 2.5, is used to establish some properties of the functional K ([25, Lemma 3.7]). Using the localization property, we obtain the claim of the lemma for an arbitrary $O \subset [0, L]$ open. □

The following lemma gives us the important claim that if we know the relaxation sequence for $[0, L]$, we obtain by restriction the relaxation sequence for arbitrary $O \subset [0, L]$ open (if m is fixed). Moreover, it gives a kind of uniqueness of relaxation field, up to a term converging to zero in L^2.

Lemma 2.9 *Take a sequence (h_n) monotonically decreasing to 0 that satisfies Assumption 2.7 and $m \in L^2(\Omega; \mathbb{R}^3)$. Let $(\vartheta_n) \subset W^{1,2}(\Omega, \mathbb{R}^3)$ be such that*

(a) $((\vartheta_n)_1, h_n(\vartheta_n)_2, h_n(\vartheta_n)_3) \to 0$, $\int_\omega x_3\vartheta_{n,2} \to 0$ *strongly in* L^2;
(b) $K(m, [0, L]) = \lim_{n\to\infty} \int_\Omega Q^{h_n}(x, \iota(m) + \nabla_{h_n}\vartheta_n)\,\mathrm{d}x.$

Then we have that:

(I) $(|\operatorname{sym} \nabla_{h_n} \vartheta_n|^2)$ *is equi-integrable;*
(II) for every O open subset of $[0, L]$, *we have that*

$$K(m, O) = \lim_{n\to\infty} \int_{O\times\omega} Q^{h_n}(x, \iota(m) + \nabla_{h_n}\vartheta_n)\, dx; \tag{14}$$

(III) if $(\psi_n) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ *is any other sequence that satisfies (a) and (b), then*

$$\|\operatorname{sym} \nabla_{h_n}\psi_n - \operatorname{sym} \nabla_{h_n}\vartheta_n\|_{L^2} \to 0,$$

and $(|\operatorname{sym} \nabla_{h_n}\psi_n|^2)$ *is equi-integrable.*

Proof From (Q1) and by taking the zero subsequence, we obtain the bound

$$\limsup_{n\to\infty} \|\operatorname{sym} \nabla_{h_n}\vartheta_n\|_{L^2(\Omega)} \le C\left(\eta_2\|m\|^2_{L^2} + 1\right). \tag{15}$$

From Corollary 2.3 there are sequences $(A_n) \subset W^{1,2}((0, L); \mathbb{M}^3_{\text{skw}})$ and $(v_n) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ such that $A_n \to 0$ and $v_n \to 0$ strongly in L^2 and

$$\left\|\operatorname{sym} \nabla_{h_n}\vartheta_n - \operatorname{sym} \iota((A_n)' d_\omega) - \operatorname{sym} \nabla_{h_n} v_n\right\|_{L^2} \to 0.$$

From (11) we obtain that

$$\limsup_{k\to\infty} \left(\|A_n\|_{W^{1,2}(\Omega)} + \|\nabla_{h_n} v_n\|_{L^2(\Omega)}\right) \le C(\omega)\left(\eta_2\|m\|^2_{L^2} + 1\right).$$

To prove that $(|\operatorname{sym} \nabla_{h_n}\vartheta_n|^2)$ is equi-integrable, let us assume the opposite, i.e., that there is $\varepsilon > 0$ such that for every $k > 0$ there are measurable sets (S_k) such that $|S_k| < \frac{1}{k}$ and there is an increasing function $n : \mathbb{N} \to \mathbb{N}$ such that

$$\int_{S_k} |\operatorname{sym} \nabla_{h_{n(k)}}\vartheta_{n(k)}|^2\, dx \ge \varepsilon.$$

On the other hand, by Lemma 2.16 and 2.17 there is a further subsequence, still denoted by $n(k)$ and sequences $(\tilde{A}_k) \subset W^{1,2}((0, L); \mathbb{M}^3_{\text{skw}})$ and $(\tilde{v}_k) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ such that

(i) $\displaystyle\lim_{k\to\infty} \left|\Omega \cap \left\{\tilde{A}_k \ne A_{n(k)} \text{ or } \tilde{A}'_k \ne A'_{n(k)}\right\}\right| = 0;$
(ii) $\lim_{k\to\infty} \left|\Omega \cap \left\{\tilde{v}_k \ne v_{n(k)} \text{ or } \nabla\tilde{v}_k \ne \nabla v_{n(k)}\right\}\right| = 0;$
(iii) $\tilde{A}'_k$ and $\nabla_{h_{n(k)}}\tilde{v}_k$ are equi-integrabile.

We have

$$\begin{aligned}
K(m, [0, L]) &= \liminf_{k\to\infty} \int_\Omega Q^{h_{n(k)}}\left(x, \iota(m) + \nabla_{h_{n(k)}}\vartheta_{n(k)}\right) dx \\
&> \liminf_{k\to\infty} \int_\Omega \chi_{\Omega\setminus S_k} Q^{h_{n(k)}}\left(x, \iota(m) + \nabla_{h_{n(k)}}\vartheta_{n_k}\right) dx \\
&= \liminf_{k\to\infty} \int_\Omega \chi_{\Omega\setminus S_k} Q^{h_{n(k)}}\left(x, \iota(m) + \operatorname{sym} \iota((\tilde{A}_k)' d_\omega) + \operatorname{sym} \nabla_{h_{n(k)}}\tilde{v}_k\right) dx \\
&= \liminf_{k\to\infty} \int_\Omega Q^{h_{n(k)}}\left(x, \iota(m) + \operatorname{sym} \iota((\tilde{A}_k)' d_\omega) + \operatorname{sym} \nabla_{h_{n(k)}}\tilde{v}_k\right) dx \\
&= K(m, [0, L]),
\end{aligned}$$

which is a contradiction. Therefore, $(|\operatorname{sym} \nabla_{h_n}\vartheta_n|^2)$ is equi-integrabile.

We now show that (ϑ_n) is optimal for any open set $O \in \mathcal{D}$ which is a finite union of disjoint open intervals. Otherwise, there would be a subsequence, still denoted by (h_n), such that there is a sequence $(\psi_n^1) \subset W^{1,2}(O \times \omega, \mathbb{R}^3)$ satisfying the conditions of Lemma 2.8 and

$$K(m, O) = \lim_{n\to\infty} \int_{O\times\omega} Q^{h_n}\left(x, \iota(m) + \nabla_{h_n}\psi_n^1\right) \mathrm{d}x$$
$$< \lim_{n\to\infty} \int_{O\times\omega} Q^{h_n}\left(x, \iota(m) + \nabla_{h_n}\vartheta_n\right) \mathrm{d}x.$$

On the other hand, on a further subsequence, still denoted by (h_n) we take the sequence $(\psi_n^2) \subset W^{1,2}([0, L] \setminus \bar{O}, \mathbb{R}^3)$ satisfying the conditions of Lemma 2.8 and

$$K(m, (0, L)\backslash\bar{O}) = \lim_{n\to\infty} \int_{((0,L)\backslash\bar{O})\times\omega} Q^{h_n}\left(x, \iota(m) + \nabla_{h_n}\psi_n^2\right) \mathrm{d}x$$
$$\le \lim_{n\to\infty} \int_{([0,L]\backslash\bar{O})\times\omega} Q^{h_n}\left(x, \iota(m) + \nabla_{h_n}\vartheta_n\right) \mathrm{d}x.$$

By using Lemma 2.4 we define $(\psi_n) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ such that

$$\operatorname{sym} \nabla_{h_n}\psi_n = \chi_O \operatorname{sym} \nabla_{h_n}\psi_n^1 + \chi_{[0,L]\backslash\bar{O}} \operatorname{sym} \nabla_{h_n}\psi_n^2.$$

We conclude that

$$\lim_{n\to\infty} \int_{[0,L]\times\omega} Q^{h_n}\left(x, \iota(m) + \nabla_{h_n}\psi_n\right) \mathrm{d}x < \lim_{n\to\infty} \int_{[0,L]\times\omega} Q^{h_n}\left(x, \iota(m) + \nabla_{h_n}\vartheta_n\right) \mathrm{d}x$$
$$= K(m, [0, L]),$$

which yields a contradiction with the optimality of the sequence (ϑ_n).

For any open $O \subset [0, L]$, by density, there is an increasing family of sets $(D_k) \subset \mathcal{D}$ which exhausts O. Since (ϑ_n) is optimal on each D_k and since $K(m, O) \ge K(m, D_k)$ (this can be easily seen from Lemma 2.8), we deduce from equi-integrability of $\left(|\operatorname{sym} \nabla_{h_n}\vartheta_n|^2\right)$ that

$$K(m, O) \ge \lim_{k\to\infty} K(m, D_k) = \lim_{n\to\infty} \int_{O\times\omega} Q^{h_n}\left(x, \iota(m) + \nabla_{h_n}\vartheta_n\right) \mathrm{d}x.$$

Hence (ϑ_k) is also optimal for $K(m, O)$, and (II) is proved.

Notice that, for every $h > 0$, there is a measurable mapping $\mathcal{L}^h : \Omega \times \mathbb{M}^3 \to \mathbb{M}^3_{\mathrm{sym}}$ such that for a.e. $x \in \Omega$, $\mathcal{L}^h(x, \cdot)$ is a positive semidefinite linear operator and

$$Q^h(x, M) = \mathcal{L}^h(x, M) \cdot M$$

holds for all $M \in \mathbb{M}^3$. Notice also that

$$\mathcal{L}^h(x, M) = \mathcal{L}^h(x, \operatorname{sym} M), \quad \|\mathcal{L}^h\|_{L^\infty} \le \eta_2. \tag{16}$$

To prove (III) we first show that

$$\lim_{n\to\infty} \int_{\Omega} \mathcal{L}^h(x, \iota(m) + \nabla_{h_n}\vartheta_n) \cdot \nabla_{h_n}\tilde{\psi}_n = 0, \tag{17}$$

for every $(\tilde{\psi}_n) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ that satisfies (a) and such that $(|\operatorname{sym} \nabla_{h_n}\tilde{\psi}_n|)$ is bounded in L^2.

To prove this we take $\varepsilon > 0$, and for n large enough we derive:

$$
\begin{aligned}
0 &\leq \int_\Omega Q^{h_n}(x, \iota(m) + \nabla_{h_n}\vartheta_n + \varepsilon\nabla_{h_n}\tilde{\psi}_n)\,\mathrm{d}x - \int_\Omega Q^{h_n}(x, \iota(m) + \nabla_{h_n}\vartheta_n)\,\mathrm{d}x \\
&= \int_\Omega \mathcal{L}^h(x, \iota(m) + \nabla_{h_n}\vartheta_n + \varepsilon\nabla_{h_n}\tilde{\psi}_n) \cdot (\iota(m) + \nabla_{h_n}\vartheta_n + \varepsilon\nabla_{h_n}\tilde{\psi}_n)\,\mathrm{d}x \\
&\quad - \int_\Omega \mathcal{L}^h(x, \iota(m) + \nabla_{h_n}\vartheta_n) \cdot (\iota(m) + \nabla_{h_n}\vartheta_n)\,\mathrm{d}x \\
&= 2\varepsilon \int_\Omega \mathcal{L}^h(x, \iota(m) + \nabla_{h_n}\vartheta_n) \cdot (\nabla_{h_n}\tilde{\psi}_n)\,\mathrm{d}x + \varepsilon^2 \int_\Omega \mathcal{L}^h(x, \nabla_{h_n}\tilde{\psi}_n) \cdot (\nabla_{h_n}\tilde{\psi}_n)\,\mathrm{d}x \\
&\leq 2\varepsilon \int_\Omega \mathcal{L}^h(x, \iota(m) + \nabla_{h_n}\vartheta_n) \cdot (\nabla_{h_n}\tilde{\psi}_n)\,\mathrm{d}x + \varepsilon^2 \eta_2 |\operatorname{sym}\nabla_{h_n}\tilde{\psi}_n|^2 \\
&= 2\varepsilon \int_\Omega \mathcal{L}^h(x, \iota(m) + \operatorname{sym}\nabla_{h_n}\vartheta_n) \cdot (\operatorname{sym}\nabla_{h_n}\tilde{\psi}_n)\,\mathrm{d}x + \varepsilon^2 \eta_2 |\operatorname{sym}\nabla_{h_n}\tilde{\psi}_n|^2.
\end{aligned}
$$

If (17) did not hold, we would choose ε (by taking the appropriate sign) such that the linear term dominates and the inequality is violated. Thus, we deduce (17), by the contradiction. To finish the proof, we take two sequences $(\vartheta_n) \subset W^{1,2}(\Omega;\mathbb{R}^3)$, $(\psi_n) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ that satisfy (a) and (b). We have, using (17)

$$
\begin{aligned}
\eta_1 \|\operatorname{sym}\nabla_{h_n}(\psi_n - \vartheta_n)\|_{L^2}^2 &\leq \int_\Omega \mathcal{L}^{h_n}(x, \nabla_{h_n}(\psi_n - \vartheta_n)) \cdot \nabla_{h_n}(\psi_n - \vartheta_n)\,\mathrm{d}x \\
&= \int_\Omega \mathcal{L}^{h_n}(x, \iota(m) + \nabla_{h_n}\psi_n) \cdot \nabla_{h_n}(\psi_n - \vartheta_n)\,\mathrm{d}x \\
&\quad - \int_\Omega \mathcal{L}^{h_n}(x, \iota(m) + \nabla_{h_n}\vartheta_n) \cdot \nabla_{h_n}(\psi_n - \vartheta_n)\,\mathrm{d}x \to 0.
\end{aligned}
$$

□

The following lemma proves the compactness result we need.

Lemma 2.10 *For every sequence (h_n) that satisfy the Assumption 2.7, there exists a subsequence, still denoted by (h_n) such that for each $m \in L^2(\Omega;\mathbb{R}^3)$ there exists $\left(\vartheta_n^m\right) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ which satisfies*

(a) $\left((\vartheta_n^m)_1, h_n(\vartheta_n^m)_2, h_n(\vartheta_n^m)_3\right) \to 0$, $\int_\omega x_3(\vartheta_n^m)_2 \to 0$ *strongly in* L^2,

(b)

$$
K(m,[0,L]) = \lim_{n\to\infty} \int_\Omega Q^{h_n}(x, \iota(m) + \nabla_{h_n}\vartheta_n^m)\,\mathrm{d}x.
$$

Proof Let $\mathcal{M} \subset L^2(\Omega;\mathbb{R}^3)$ be a countable dense family. By diagonalization procedure, it is possible to construct the subsequence, still denoted by (h_n), such that for each $m \in \mathcal{M}$ there is a sequence $(\vartheta(m)_n)$ for which (a) and (b) holds. Now we take the sequence $(m_n) \subset \mathcal{M}$ such that $m_n \to m$ in L^2 as $n \to \infty$ and define the strictly increasing function $k : \mathbb{N} \to \mathbb{N}$ in a way that for every $n_0 \in \mathbb{N}$ we have

$$
\begin{aligned}
\left| K(m_{n_0},[0,L]) - \int_\Omega Q^{h_n}\left(x, \iota(m_{n_0}) + \nabla_{h_n}\vartheta_n^{m_{n_0}}\right)\mathrm{d}x \right| &< \frac{1}{n_0}, \quad \text{for every } n \geq k(n_0), \\
\left\| \left((\vartheta_n^{m_{n_0}})_1, h_n(\vartheta_n^{m_{n_0}})_2, h_n(\vartheta_n^{m_{n_0}})_3\right) \right\|_{L^2} &< \frac{1}{n_0}, \quad \text{for every } n \geq k(n_0), \\
\left\| \int_\omega x_3(\vartheta_n^{m_{n_0}})_2 \right\|_{L^2} &< \frac{1}{n_0}, \quad \text{for every } n \geq k(n_0).
\end{aligned}
$$

For every $i \in \mathbb{N}$ and $j \in [k(i), k(i+1))$ take $\vartheta_j^m := \vartheta_j^{m_{k(i)}}$ and use Lemma 2.5 to show (b). $\square$

We make the assumption on the sequence (Q^{h_n}) of quadratic forms.

Assumption 2.11 For given (h_n) monotonically decreasing to zero, we suppose that for every $m \in L^2(\Omega; \mathbb{R}^3)$ there exists $(\vartheta_n) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ (we omit the superscript m) such that

(a) $((\vartheta_n)_1, h_n(\vartheta_n)_2, h_n(\vartheta_n)_3) \to 0$, $\int_\omega x_3(\vartheta_n)_2 \to 0$ strongly in L^2,
(b) for every $O \subset [0, L]$ open we have

$$K(m, O) = \lim_{n\to\infty} \int_{O\times\omega} Q^{h_n}(x, \iota(m) + \nabla_{h_n}\vartheta_n)\,\mathrm{d}x.$$

Here

$$\begin{aligned} K(m, O) = \min\Big\{ &\liminf_{n\to\infty} \int_{O\times\omega} Q^{h_n}\left(x, \iota(m) + \nabla_{h_n}\psi^{h_n}\right)\,\mathrm{d}x : \\ &(\psi_1^{h_n}, h_n\psi_2^{h_n}, h_n\psi_3^{h_n}) \to 0 \text{ strongly in } L^2(O\times\omega; \mathbb{R}^3) \\ &\int_\omega x_3\psi_2^{h_n} \to 0 \text{ strongly in } L^2(O)\Big\} \\ = \min\Big\{ &\limsup_{n\to\infty} \int_{O\times\omega} Q^{h_n}\left(x, \iota(m) + \nabla_{h_n}\psi^{h_n}\right)\,\mathrm{d}x : \\ &(\psi_1^{h_n}, h_n\psi_2^{h_n}, h_n\psi_3^{h_n}) \to 0 \text{ strongly in } L^2(O\times\omega; \mathbb{R}^3), \\ &\int_\omega x_3\psi_2^{h_n} \to 0 \text{ strongly in } L^2(O)\Big\}. \end{aligned}$$

We define the mapping $m : L^2([0, L]; \mathbb{M}^3_{\mathrm{skw}}) \times L^2([0, L]) \to L^2([0, L]; \mathbb{R}^3)$ by $m(A, a) = A(0, x_2, x_3)^t + ae_1$. Finally, we derive the integral representation of K.

Proposition 2.12 *Let (h_n) be a sequence monotonically decreasing to zero for which the Assumption 2.11 is valid. Then there exists a measurable function $Q : [0, L] \times \mathbb{M}^3_{\mathrm{skw}} \times \mathbb{R} \to [0, +\infty\rangle$ possibly depending on this sequence such that for every $O \subset [0, L]$ open and every $A \in L^2([0, L]; \mathbb{M}^3_{\mathrm{skw}})$ we have*

$$K(m(A, a), O) = \int_O Q(x_1, A(x_1), a(x_1))\,\mathrm{d}x_1. \tag{18}$$

Moreover, Q satisfies the following property

(Q'1) for almost all $x_1 \in [0, L]$ the map $Q(x_1, \cdot, \cdot)$ is a quadratic form and there is a positive constant C_ω, independent of x_1, such that

$$\begin{aligned} &C_\omega(|A|^2 + |a|^2) \le Q(x_1, A, a) \\ &\le \eta_2\left(\max\{\mu_2, \mu_3\}|A|^2 + |a|^2\right) \textit{ for all } (A, a) \in \mathbb{M}^3_{\mathrm{skw}} \times \mathbb{R}. \end{aligned} \tag{19}$$

Proof The existence of Q and the proof of (18) is identical as in [25, Proposition 2.9]. Therefore, we will only prove the boundedness and coercivity property. The function Q is defined via:

$$Q(\bar{x}_1, A, a) = \lim_{r\to 0} \frac{1}{2r} K\left(m(A, a), B(\bar{x}_1, r)\right), \text{ for a.e. } \bar{x}_1 \in [0, L]. \tag{20}$$

The upper bound in (19) is easily obtained by taking the zero subsequence $\vartheta_n = 0$ and by using (Q1) and (1) to deduce

$$|Q(\bar{x}_1, A, a)| \leq \eta_2 \left(|\operatorname{sym} \iota(ae_1 + Ad_\omega)|^2\right) \leq \eta_2 \left(\max\{\mu_2, \mu_3\}|A|^2 + |a|^2\right),$$

for a.e. $\bar{x}_1 \in (0, L)$.

From Assumption 2.11 and Corollary 2.3 we deduce that there are bounded sequences $(A^{h_n}) \subset W^{1,2}([0, L]; \mathbb{M}^3_{\mathrm{skw}})$ and $(v^{h_n}) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ such that $A^{h_n} \to 0$ and $v^{h_n} \to 0$ strongly in L^2 and

$$\begin{aligned} &K(m(A, a), B(\bar{x}_1, r)) \\ &\quad = \lim_{n\to\infty} \int_{B(\bar{x}_1, r)\times\omega} Q^{h_n}\left(x, \iota(m) + \operatorname{sym} \iota((A^{h_n})' d_\omega) + \operatorname{sym} \nabla_{h_n} v^{h_n}\right) \mathrm{d}x. \end{aligned}$$

for some $C > 0$. We can assume, by the density argument, that v^{h_n} and A^{h_n} are smooth functions. Using the property (Q1), we have

$$K(m(A, a), B(\bar{x}_1, r)) \geq \eta_1(I_1 + I_2),$$

where I_1 and I_2 are defined by:

$$I_1 = \lim_{h\to 0} \int\limits_{B(\bar{x}_1, r)\times\omega} \left(a + A_{12}x_2 + A_{13}x_3 + (A^{h_n}_{12})' x_2 + (A^{h_n}_{13})' x_3 + \partial_1 v^{h_n}_1\right)^2 dx$$

$$\begin{aligned} I_2 = &\frac{1}{2} \lim_{h\to 0} \left\{ \int\limits_{B(\bar{x}_1, r)\times\omega} \left(A_{23}x_3 + (A^{h_n}_{23})' x_3 + \partial_1 v^{h_n}_2 + \frac{\partial_2 v^{h_n}_1}{h}\right)^2 dx \right. \\ &\left. + \int\limits_{B(\bar{x}_1, r)\times\omega} \left(-A_{23}x_2 - (A^{h_n}_{23})' x_2 + \partial_1 v^{h_n}_3 + \frac{\partial_3 v^{h_n}_1}{h_n}\right)^2 dx \right\} \end{aligned}$$

From the choice of the coordinate axis (1), we have that for every $x_1 \in B(\bar{x}_1, r)$

$$\int\limits_{\{x_1\}\times\omega} a A_{12} x_2 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = \int\limits_{\{x_1\}\times\omega} a A_{13} x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = \int\limits_{\{x_1\}\times\omega} A_{13} A_{12} x_2 x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = 0.$$

Thus, we derive that

$$\begin{aligned} I_1 \geq &\int_{B(\bar{x}_1, r)\times\omega} \left(|a|^2 + x_2^2 A_{12}^2 + x_3^2 A_{13}^2\right) \mathrm{d}x \\ &+ 2 \lim_{n\to\infty} \int_{B(\bar{x}_1, r)\times\omega} (a + A_{12}x_2 + A_{13}x_3) \left((A^{h_n}_{12})' x_2 + (A^{h_n}_{13})' x_3 + \partial_1 v^{h_n}_1\right) \mathrm{d}x. \end{aligned}$$

Since $(A_{h_n})' \rightharpoonup 0$ and $\partial_1 v_1{}^{h_n} \rightharpoonup 0$ weakly in L^2, the mixed term vanishes as $n \to \infty$. Hence, we obtain that

$$I_1 \geq 2r\left(|a|^2 + \mu_2|A_{12}|^2 + \mu_3|A_{13}|^2\right). \tag{21}$$

To obtain the lower bound for I_2, we look for a solution of the minimum problem

$$\min_{\psi\in H^1(\omega)} \int_\omega |u - \nabla\psi|^2 \, \mathrm{d}x.$$

The solution of this problem is unique up to constant and satisfies the variational equation

$$\int_{\omega} (\nabla \varphi_u - u) \cdot \nabla \psi \ \, \mathrm{d}x = 0, \tag{22}$$

for every $\psi \in H^1(\omega)$. The solution corresponds to L^2 projection on the space

$$G(\omega) = \left\{ w \in L^2(\omega; \mathbb{R}^2) : w = \nabla p, \text{ for some } p \in H^1(\omega) \right\},$$

which is a closed subspace in $L^2(\omega; \mathbb{R}^2)$. We denote by $Pu = u - \nabla \varphi_u$.

Denote also by

$$\Psi^h(x) = \left(A_{23} + (A_{23}^h)' \right) \begin{pmatrix} x_3 \\ -x_2 \end{pmatrix} + \begin{pmatrix} \partial_1 v_2^{h_n} \\ \partial_1 v_3^{h_n} \end{pmatrix} + \frac{1}{h_n} \begin{pmatrix} \partial_2 v_1^{h_n} \\ \partial_3 v_1^{h_n} \end{pmatrix}.$$

Since P is a projection, we conclude

$$I_2 = \int\limits_{B(\bar{x}_1, r) \times \omega} |\Psi^{h_n}|^2 dx \geq \int\limits_{B(\bar{x}_1, r) \times \omega} |P \Psi^{h_n}|^2 dx,$$

where $P\Psi^h$ equals

$$P(\Psi^{h_n}(x)) = \left(A_{23} + (A_{23}^{h_n})' \right) P \begin{pmatrix} x_3 \\ -x_2 \end{pmatrix} + P \begin{pmatrix} \partial_1 v_2^{h_n} \\ \partial_1 v_3^{h_n} \end{pmatrix}.$$

Notice that the projection is taken for every $x_1 \in [0, L]$. This yields that:

$$\begin{aligned} I_2 \geq & \bar{C}_\omega \lim_{n \to \infty} \left(\int_{B(\bar{x}_1, r)} |A_{23}|^2 + 2 \int_{B(x_1, r)} A_{23} (A_{23}^{h_n})' dx_1 \right) \\ & + 2 \lim_{n \to \infty} \int_{B(\bar{x}_1, r) \times \omega} A_{23} P \begin{pmatrix} x_3 \\ -x_2 \end{pmatrix} \cdot P \begin{pmatrix} \partial_1 v_2^{h_n} \\ \partial_1 v_3^{h_n} \end{pmatrix} dx, \end{aligned}$$

where the constant $\bar{C}_\omega$ equals

$$\bar{C}_\omega = \int_\omega \left| P \begin{pmatrix} x_3 \\ -x_2 \end{pmatrix} \right|^2 \mathrm{d}x_2 \, \mathrm{d}x_3. \tag{23}$$

Since $A_h' \rightharpoonup 0$ in L^2, the second term converges to zero. Since P is the projection, we have that

$$\int\limits_{B(\bar{x}_1, r) \times \omega} A_{23} P \begin{pmatrix} x_3 \\ -x_2 \end{pmatrix} \cdot P \begin{pmatrix} \partial_1 v_2^{h_n} \\ \partial_1 v_3^{h_n} \end{pmatrix} \mathrm{d}x = \int\limits_{B(\bar{x}_1, r) \times \omega} A_{23} P \begin{pmatrix} x_3 \\ -x_2 \end{pmatrix} \cdot \begin{pmatrix} \partial_1 v_2^{h_n} \\ \partial_1 v_3^{h_n} \end{pmatrix} \mathrm{d}x \to 0,$$

since $\partial_1 v^{h_n} \rightharpoonup 0$ weakly in L^2. We obtain that

$$I_2 \geq 2r \bar{C}_\omega A_{23}^2.$$

Combing this with (20) and (21) and taking the limit as $r \to 0$ yield the coercivity of Q. $\square$

For a given sequence (h_n) monotonically decreasing to zero for which the Assumption 2.11 is satisfied, we also define the function $Q_0 : [0, L] \times \mathbb{M}^3_{\mathrm{skw}} \to \mathbb{R}$ such that

$$Q_0(x_1, A) = \min_{a \in \mathbb{R}} Q(x_1, A, a), \tag{24}$$

and mapping $a_{\min} : [0, L] \times \mathbb{M}^3_{\text{skw}} \to \mathbb{R}$ that satisfies

$$Q_0(x_1, A) = Q(x_1, A, a_{\min}(x_1, A)). \tag{25}$$

It is easy to see that Q_0 satisfies the following property.

($Q_0'1$) For almost all $x_1 \in [0, L]$ the map $Q_0(x_1, \cdot)$ is a quadratic form and satisfies

$$C_1(\eta_1, \eta_2, \mu_1, \mu_2, \bar{C}_\omega)|A|^2 \\ \leq Q_0(x_1, A) \leq C_2(\eta_1, \eta_2, \mu_1, \mu_2, \bar{C}_\omega)|A|^2, \quad \text{for all} \quad A \in \mathbb{M}^3_{\text{skw}},$$

where $\bar{C}_\omega$ is defined in (23) and C_1, C_2 depend only on the constants in the bracket.

The mapping $a_{\min}$ is well defined, linear in A and for some $C_a = C_a(\eta_1, \eta_2, \mu_1, \mu_2, \bar{C}_\omega) > 0$ we have

$$|a_{\min}(x_1, A)| \leq C_a|A|, \quad \text{for} \quad \text{a.e.} x_1 \in [0, L].$$

The function Q_0 will be the energy density of the limit functional.

2.3 Identification of limit equations

We will state and prove liminf and limsup inequality.

Theorem 2.13 *Let the family (W^h) describe an admissible composite material in the sense of the Definition 1.2. Let (h_n) be a sequence monotonically decreasing to zero such that the Assumption 2.11 is valid. Let $(y^{h_n}) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ be a sequence of deformations such that*

$$\int_\Omega W^{h_n}(x, \nabla_{h_n} y^{h_n})\, dx \leq Ch_n^2, \text{ for some } C > 0.$$

Then, there is a subsequence (still denoted by (h_n)) such that $\nabla_{h_n} y^{h_n} \to R$ strongly in L^2, where $R \in W^{1,2}([0, L]; \mathrm{SO}(3))$. Moreover we have

$$\liminf_{n\to\infty} \tfrac{1}{h_n^2} \int_\Omega W^{h_n}(x, \nabla_{h_n} y^{h_n}) \geq \int_{[0,L]} Q_0(R^t R')dx_1.$$

Proof By Theorem 2.15 there is a sequence $(R^{h_n}) \subset C^\infty([0, L]; \mathbb{R}^{3\times 3})$ such that $R^{h_n}(x_1) \in \mathrm{SO}(3)$ for a.e. $x_1 \in [0, L]$ and R^h satisfies (41) and (42). From (42) we conclude that on a subsequence $R^{h_n} \rightharpoonup R$ weakly in $W^{1,2}([0, L]; \mathbb{R}^3)$ and thus also in $C([0, L]; \mathbb{R}^3)$. We take a further subsequence, without relabeling, on which lim inf is accomplished. We define the sequence v^h by the following decomposition

$$y^{h_n} = \tfrac{1}{|\omega|} \int_{\{x_1\}\times\omega} y^{h_n} + h_n x_2 R^{h_n} e_2 + h_n x_3 R^{h_n} e_3 + h_n v^{h_n}. \tag{26}$$

Integrating over ω and using (1) yield:

$$\int_{\{x_1\}\times\omega} v^{h_n} dx = 0, \tag{27}$$

for a.e. $x_1 \in [0, L]$. Note that

$$\nabla_{h_n} v^{h_n} = \tfrac{1}{h_n}(\nabla_{h_n} y^{h_n} - R^{h_n}) - \left(p^{h_n} + x_2(R^{h_n})' e_2 + x_3(R^{h_n})' e_3|0|0\right),$$

where

$$p^{h_n} = \tfrac{1}{h_n|\omega|} \int_{\{x_1\}\times\omega} \left(\partial_1 y^{h_n} - R^{h_n} e_1\right).$$

From (41) we obtain that $\|p^{h_n}\|_{L^2}$ is bounded. Hence, there is $p \in L^2([0, L]; \mathbb{R}^3)$ such that $p^{h_n} \rightharpoonup p$ weakly in L^2 (on a subsequence). By using (41) and (42), we deduce that there exists a constant $C > 0$ such that

$$\|\nabla_{h_n} v^{h_n}\|_{L^2} \le C. \tag{28}$$

Using (27) and the Poincaré inequality, we conclude that for some $C > 0$

$$\|v^{h_n}\|_{L^2} \le Ch_n. \tag{29}$$

Define the approximate strain by

$$G^{h_n} = \frac{(R^{h_n})^T \nabla_{h_n} y^{h_n} - I}{h_n}. \tag{30}$$

From (41) we conclude that (G^{h_n}) is bounded in L^2. It can be easily checked that

$$G^{h_n} = \iota\Big((A + A^{h_n})d_\omega\Big) + ((R^{h_n})^t p^{h_n}|0|0) + (R^{h_n})^t \nabla_{h_n} v^{h_n}, \tag{31}$$

where $A = R^t R'$, $A^{h_n} = (R^{h_n})^t (R^{h_n})' - R^t R'$. Take any sequence $(r_n) \subset C^1([0, L]; \mathbb{R}^3)$ such that

$$r_n \to R^t p \qquad \text{and} \qquad h_n(r_n)' \to 0,$$

strongly in L^2 and define the functions:

$$\begin{aligned}
\tilde{p}^{h_n} &= \int_0^{x_1} \Big((R^{h_n})^t p^{h_n} - R^t p\Big),\\
\tilde{v}^{h_n} &= (R^{h_n})^t v^{h_n} + (h_n x_2 (r_n)_2 + h_n x_3 (r_n)_3, 0, 0)^t + \tilde{p}^{h_n},\\
o^{h_n} &= (R^{h_n})^t \nabla_{h_n} v^{h_n} - \nabla_{h_n}((R^{h_n})^t v^{h_n}) - (h_n x_2 (r_n)_2' + h_n x_3 (r_n)_3') e_1 \otimes e_1\\
&\quad + \sum_{i=2,3} \big((R^t p)_i - (r_n)_i\big)\, e_i \otimes e_1.\\
\tilde{A}^{h_n} &= \int_0^{x_1} A^{h_n}.
\end{aligned}$$

It is straightforward to check that

$$\text{sym } G^{h_n} = \text{sym } \iota(A d_\omega) + (R^t p)_1 e_1 \otimes e_1 + \text{sym } \iota((\tilde{A}^{h_n})' d_\omega) + \text{sym } \nabla_{h_n} \tilde{v}^{h_n} + \text{sym } o^{h_n}. \tag{32}$$

By the Sobolev embedding theorem, we deduce from (42) that $\|h_n (R^{h_n})'\|_{L^\infty} \to 0$. By combining this and (29), we obtain that

$$(R^h)^t \nabla_{h_n} v^{h_n} - \nabla_{h_n}((R^{h_n})^t v^{h_n}) = -((R^{h_n})' v^{h_n}|0|0) \longrightarrow 0, \text{ strongly in} L^2.$$

From that it follows that

$$o^{h_n} \to 0 \text{ strongly in } L^2. \tag{33}$$

It also easily follows that

$$\tilde{A}^{h_n} \to 0,\ \ \tilde{v}^{h_n} \to 0, \text{ strongly in } L^2, \quad \|\nabla_{h_n} \tilde{v}^{h_n}\|_{L^2} \le C, \tag{34}$$

for some $C > 0$.

Using Lemma 2.16 and Lemma 2.17, we take a subsequence $(h_{n(k)})$ such that there exist sequences $(\bar{A}_k) \subset W^{1,2}([0, L]; M^3_{\text{skw}})$, $\bar{v}_k \subset W^{1,2}(\Omega; \mathbb{R}^3)$ which satisfy

(i) $\lim_{k\to\infty} |\{\bar{v}_k \neq \tilde{v}^{h_{n(k)}} \text{ or } \nabla\bar{v}_k \neq \nabla\tilde{v}^{h_{n(k)}}\}| = 0$,
$\lim_{k\to\infty} |\{\bar{A}_k \neq \tilde{A}^{h_{n(k)}} \text{ or } \bar{A}_k' \neq (\tilde{A}^{h_{n(k)}})'\}| = 0$;
(ii) $|\bar{A}_k'|^2$ and $(|\nabla_{h_{n(k)}}\bar{v}_k|^2)$ are equi-integrable.

It can be easily seen that $\bar{A}_k \to 0$, $\bar{v}_k \to 0$, strongly in L^2 (i.e., weakly in $W^{1,2}$). By using Lemma 2.4, we obtain a sequence $(\psi_k) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that

$$\begin{aligned} &\operatorname{sym}\,\nabla_{h_{n(k)}}\psi_k = \operatorname{sym}\,\iota((\bar{A}_k)'d_\omega) + \operatorname{sym}\,\nabla_{h_{n(k)}}\bar{v}_k, \\ &((\psi_k)_1, h_{n(k)}(\psi_k)_2, h_{n(k)}(\psi_k)_3) \to 0,\ \int_\omega x_3(\psi_k)_2 \to 0 \text{ strongly in} L^2, \end{aligned} \tag{35}$$

and the sequence $(|\operatorname{sym}\,\nabla_{h_{n(k)}}\psi_k|^2)$ is equi-integrable. We define the sets

$$C^{h_n} = \left\{x \in \Omega : |G^{h_n}| \leq \tfrac{1}{\sqrt{h_n}}\right\}.$$

From the boundedness of the sequence (G^{h_n}) in L^2, we conclude that $|\Omega\backslash C^{h_n}| \to 0$ as $h \to 0$. Using (Q1), the decomposition of G^{h_n}, (35) and the equi-integrability of $(|\operatorname{sym}\,\nabla_{h_{n(k)}}\psi_k|^2)$, we deduce that

$$\begin{aligned} &\liminf_{k\to\infty} \int_\Omega Q^{h_{n(k)}}(x, \chi_{C^{h_{n(k)}}} G^{h_n}) \\ &\quad = \liminf_{k\to\infty} \int_\Omega Q^{h_{n(k)}}\left(x, \chi_{C^{h_{n(k)}}}\left(\iota(Ad_\omega) + (R^t p)_1 e_1 \otimes e_1 + \iota((\tilde{A}^{h_n})'d_\omega) + \nabla_{h_n}\tilde{v}^{h_n}\right)\right) \\ &\quad = \liminf_{k\to\infty} \int_\Omega Q^{h_{n(k)}}\left(x, \iota(Ad_\omega) + (R^t p)_1 e_1 \otimes e_1 + \nabla_{h_{n(k)}}\psi^{h_{n(k)}}\right). \end{aligned} \tag{36}$$

Using frame indifference property (W1), we have that $W^{h_n}(x, \nabla_h y^h) = W^{h_n}(x, I + hG^h)$. From (3), by integrating, we conclude that

$$\begin{aligned} &\limsup_{n\to\infty} \left| \frac{1}{{h_n}^2} \int_\Omega W^{h_n}(\cdot, I + h_n \chi_{C^{h_n}} G^{h_n}) - \int_\Omega Q^{h_n}(\cdot, \chi_{C^{h_n}} G^{h_n}) \right| \\ &\quad \leq r(\sqrt{h_n}) \int_\Omega |\chi_{C^{h_n}} G^{h_n}|^2 \to 0. \end{aligned} \tag{37}$$

Finally, we conclude, using (36), (37) and the definition of K and Q_0

$$\begin{aligned} \liminf_{k\to\infty} \frac{1}{h_{n(k)}^2} \int_\Omega W^{h_{n(k)}}(x, \nabla_{h_{n(k)}} y^{h_{n(k)}}) &\geq \liminf_{k\to\infty} \frac{1}{h_{n(k)}^2} \int_\Omega \chi_{C^{h_{n(k)}}} W^{h_{n(k)}}(x, \nabla_{h_{n(k)}} y^{h_{n(k)}}) \\ &= \liminf_{k\to\infty} \int_\Omega Q^{h_{n(k)}}(x, \chi_{C^{h_{n(k)}}} G^{h_n}) \\ &\geq K(m(A, (R^t p)_1, [0, L]) \\ &\geq \int_{[0,L]} Q_0\left(R^t R'(x_1)\right) dx_1. \end{aligned}$$

□

The next theorem gives the construction of the recovery sequence. Note that the statement of the Theorem 2.14 is not the classical upper bound statement, since the subsequence depends on R, but we can find one subsequence on which the convergences are true for all R, by using the density argument. However, Theorem 2.13 and Theorem 2.14 do imply in their form the convergence of minimizers on every subsequence of (h_n), on which they are converging. Thus, Theorem 2.14 can also be seen as an upper bound statement.

Theorem 2.14 *Let the family (W^h) describe an admissible composite material in the sense of the Definition 1.2. Let (h_n) be a sequence monotonically decreasing to zero for which the Assumption 2.11 is valid. Then for every $R \in W^{1,2}([0, L]; \mathrm{SO}(3))$ there exists a subsequence, still denoted by (h_n), such that*

a. *there exists $(y_n) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ such that $y_n \to \int_0^{x_1} Re_1$ strongly in $W^{1,2}$ and $\nabla_{h_n} y_n \to R$ strongly in L^2.*
b. *$\lim_{n\to\infty} \frac{1}{h_n^2} \int_\Omega W^{h_n}(x, \nabla_{h_n} y_n) = \int_{[0,L]} Q_0 \left(R^t R'(x_1)\right) \mathrm{d}x_1$.*

Proof It is easy to see that smooth rotations are dense in $W^{1,2}([0, L]; \mathrm{SO}(3))$. This can be seen by approximating with smooth maps taking values in $\mathbb{M}^3$ and then projecting on SO(3) (by Sobolev embedding weak $W^{1,2}$ implies strong convergence in L^∞ and we can project from tubular neighborhood of SO(3)). Therefore, without the loss of generality we may assume that $R \in C^2([0, L]; \mathrm{SO}(3))$, since in the general case we can use the diagonal procedure.

Take any $a \in C([0, L])$ and define $A \in C^1([0, L]; \mathbb{M}^3_{\mathrm{skw}})$ by $A = R^t R'$. Now we take $m = m(R^t R', a)$ and the sequence $(\vartheta_n) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ which satisfies (a) and (b) of the Assumption 2.11. From Lemma 2.9 we have that the sequence $(|\operatorname{sym} \nabla_{h_n} \vartheta_n|^2)$ is bounded (see (15)) and equi-integrabile. By Corollary 2.3, there are sequences $(A_n) \subset W^{1,2}([0, L]; \mathbb{M}^3_{\mathrm{skw}})$ and $(v_n) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ such that $A_n \to 0$, $v_n \to 0$ strongly in L^2 and

$$\|\operatorname{sym} \nabla_{h_n} \vartheta_n - \operatorname{sym} \iota(A_n' d_\omega) - \operatorname{sym} \nabla_{h_n} v_n\|_{L^2} \to 0.$$

Moreover, we have that

$$\sup \|A_n\|_{W^{1,2}} + \sup \left(\|v_n\|_{L^2} + \|\nabla_{h_n} v_n\|_{L^2}\right) < \infty. \tag{38}$$

Choose a subsequence $(h_{n(k)})$ such that $kh_{n(k)} \to 0$. Using Lemma 2.16 and Lemma 2.17, we conclude that there exist sequences $(\tilde{A}_k) \subset W^{1,\infty}([0, L]; \mathbb{M}^3_{\mathrm{skw}})$ and $(\tilde{v}_k) \subset W^{1,\infty}(\Omega; \mathbb{R}^3)$ such that for some $C > 0$ we have (on a further subsequence; not relabeled)

(i) $|\tilde{A}_k'| \le Ck$, for a.e. $x_1 \in [0, L]$, $|\nabla_{h_{n(k)}} \tilde{v}_k| \le Ck$ for a.e. $x \in \Omega$;
(ii) $\lim_{k\to\infty} |\{\tilde{A}_k \ne A_{n(k)} \text{ or } \tilde{A}_k' \ne A_{n(k)}'\}| = 0$;
$\lim_{k\to\infty} |\{\tilde{v}_k \ne v_{n(k)} \text{ or } \nabla\tilde{v}_k \ne \nabla v_{n(k)}\}| = 0$.
(iii) the sequences $(|\tilde{A}_k'|^2)$, $(|\nabla_{h_{n(k)}} \tilde{v}_k|^2)$ are equi-integrable.

It is easy to argument that $\tilde{A}_k \to 0$, $\tilde{v}_k \to 0$ strongly in L^2 (i.e., weakly in $W^{1,2}$). We define the sequence $(R_k) \subset W^{1,\infty}([0, L]; \mathbb{M}^3)$ as the solutions of the following Cauchy problem

$$\begin{cases} R_k' &= R_k(A + \tilde{A}_k'), \\ R_k(0) &= R(0). \end{cases} \tag{39}$$

Since the right-hand side of the first equation in (39) is Lipschitz function, this system has a unique solution. Moreover, since it is tangential to SO(3) it can be easily argumented that we have $R_k(x_1) \in SO(3)$ for every $x_1 \in [0, L]$ (this can be done, e.g., by approximating A_k with smooth fields and then using the standard theorem for the solutions of ODE system whose right-hand side is tangential to some smooth manifold). Notice also that $R_k \rightharpoonup R$ weakly in $W^{1,2}$ and thus, by Sobolev embedding, strongly in L^∞. Define for every $k \in \mathbb{N}$; $\bar{v}_k = \tilde{v}_k - \int_\Omega \tilde{v}_k$ to accomplish $\|\bar{v}_k\|_{W^{1,\infty}} \le Ck$, which follows by the Poincaré inequality. By the equi-integrability property, we obtain that

$$\|\operatorname{sym} \nabla_{h_{n(k)}} \vartheta_{n(k)} - \operatorname{sym} \iota(\tilde{A}_k' d_\omega) - \operatorname{sym} \nabla_{h_{n(k)}} \bar{v}_k\|_{L^2} \to 0. \tag{40}$$

Define the recovery sequence with the formulae

$$y_k = \int_0^{x_1} R_k e_1 + h_{n(k)} x_2 R_k e_2 + h_{n(k)} x_3 R_k e_3 + h_{n(k)} R \bar{v}_k + h_{n(k)} \int_0^{x_1} a R e_1.$$

Define also the approximate strain by

$$G_k = \frac{R_k^t \nabla_{h_{n(k)}} y_k - I}{h_{n(k)}}.$$

The following properties can easily be verified:

(i) $\|y_k - \int_0^{x_1} R e_1\|_{L^\infty} \to 0$, $\|\nabla_{h_{n(k)}} y_k - R_k\|_{L^\infty} \to 0$;
(ii) $\|h_{n(k)} G_k\|_{L^\infty} \to 0$;
(iii) $\left\| \text{sym } G_k - a e_1 \otimes e_1 - \text{sym } \iota((A + \tilde{A}_k') d_\omega) - \text{sym } \nabla_{h_{n(k)}} \bar{v}_k \right\|_{L^2} \to 0$.

This proves (a). To prove (b) notice that from the frame indifference property (W1) of Definition 1.1, we have that $W^{h_{n(k)}}(x, \nabla_{h_{n(k)}} y_k) = W^{h_{n(k)}}(x, I + h_{n(k)} G_k)$, for a.e. $x \in \Omega$. Using property (iii) of Definition 1.2 and the property (ii) of G_k, we conclude that

$$\left| \frac{1}{h_{n(k)}^2} \int_\Omega W^{h_{n(k)}}(x, \nabla_{h_{n(k)}} y_k) - \int_\Omega Q^{h_{n(k)}}(x, G_k) \right| \to 0.$$

From (40) we have that

$$\left\| \text{sym } G_k - a e_1 \otimes e_1 - \text{sym } \iota(A d_\omega) - \text{sym } \nabla_{h_{n(k)}} \vartheta_{n(k)} \right\|_{L^2} \to 0.$$

Using this we obtain that

$$\left| \lim_{k\to\infty} \int_\Omega Q^{h_{n(k)}}(x, G_k) - \int_{[0,L]} Q(x_1, A, a) \right| \to 0.$$

Finally, we approximate $a_{\min}(\cdot, A(\cdot)) \in L^\infty([0, L])$, defined in (25), with continuous maps in L^2 norm and use the diagonalizing procedure to prove (b). □

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Appendix

Here we state some auxiliary results that we use. First is a variant of the rigidity estimate for rods proved in [19].

Theorem 2.15 *Assume that the sequence of deformations* $y^h \in W^{1,2}(\Omega; \mathbb{R}^3)$ *satisfies*

$$\int_\Omega \text{dist}^2(\nabla_h y^h, \text{SO}(3))\, \text{d}x \le C_1 h^2,$$

for some $C_1 > 0$*, independent of* h*. Then there exist a constant* $C > 0$ *and a sequence* $(R^h) \subset C^\infty([0, L]; \mathbb{R}^{3\times 3})$*, such that* $R^h(x_1) \in \text{SO}(3)$*, for every* $x_1 \in [0, L]$*, and*

$$\|\nabla_h y^h - R^h\|_{L^2} \le Ch, \tag{41}$$
$$\|(R^h)'\|_{L^2} + \|h(R^h)''\|_{L^2} \le C. \tag{42}$$

From the estimate (42), we conclude that there is a subsequence (R^h) which converges weakly in $W^{1,2}([0,L];\mathbb{R}^{3\times 3})$ and, by the Sobolev embedding theorem, strongly in $L^\infty([0,L];\mathbb{R}^{3\times 3})$.

We also state two lemmas that enable us to pass to equi-integrabile sequences.

Lemma 2.16 *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz set and $p > 1$. Let (w_n) be a bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$. There exists a subsequence $(w_{n(k)})$ such that for every $k \in \mathbb{N}$ there exists $z_k \in W^{1,p}(\Omega;\mathbb{R}^m)$ which satisfies*

(i) $|\nabla z_k| \leq C(N)k$, fora.e.$x \in \Omega$;
(ii) $\lim_{n\to\infty} |\Omega \cap \{z_k \neq w_{n(k)}$ or $\nabla z_k \neq \nabla w_{n(k)}\}| = 0$;
(iii) $(|\nabla z_k|^p)$ *is equi-integrable.*

Proof The proof is implicitly contained in the proof of Lemma 1.2. (decomposition lemma) in [10] when the authors use a truncation argument. We shall skip it here. □

Lemma 2.17 *Let $\omega \subset \mathbb{R}^2$ be a set with Lipschitz boundary, let $\Omega = [0,L] \times \omega$, and let $p > 1$. Let (w^{h_n}) be a sequence bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$, and let us additionally assume that the sequence $(\nabla_{h_n} w^{h_n})$ is bounded in L^p. Then there exists a subsequence $(w^{h_{n(k)}})$ such that for every $k \in \mathbb{N}$ there exists $z_k \in W^{1,p}(\Omega;\mathbb{R}^m)$ which satisfies*

(i) $|\nabla_{h_{n(k)}} z_k| \leq C(N)k$, fora.e.$x \in \Omega$;
(ii) $\lim_{k\to\infty} |\Omega \cap \{z_k \neq w^{h_{n(k)}}$ or $\nabla z_k \neq \nabla w^{h_{n(k)}}\}| = 0$;
(iii) $(|\nabla_{h_{n(k)}} z_k|^p)$ *is equi-integrable.*

Proof In [5] the authors provide a general proof for the function space $W^{1,p}(\omega_\alpha \times \omega_\beta;\mathbb{R}^m)$ where $\omega_\alpha \subset \mathbb{R}^n$ and $\omega_\beta \subset \mathbb{R}^l$ and $\{n,m,l\}$ are arbitrary space dimensions. For completeness we give the proof for our case.

By de la Vallée Poussin's criterion, a sequence $(\zeta_k) \subset L^1(\Omega;\mathbb{R}^m)$ is equi-integrabile if and only if there exists a nonnegative Borel function $\varphi : [0,\infty) \to [0,\infty]$ such that

$$\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty \quad \text{and} \quad \sup_k \int_\Omega \varphi(|\zeta_k|) < +\infty.$$

By translation and dilatation, we can assume without loss of generality that $\omega \subset Q^2$, where $Q^2 = (0,1)^2$. Let (w^{h_n}) be a given bounded sequence in $W^{1,p}(\Omega,\mathbb{R}^3)$ such that $(\nabla_{h_n} w^{h_n})$ is also bounded in L^p. By using standard extension techniques, we extend the definition of w^{h_n} to $W^{1,p}((0,L)\times Q^2;\mathbb{R}^3)$ (the extension is done for every fixed $x_1 \in (0,L)$), while keeping the boundness properties. We separate the proof into several steps.

1. Define the functions $\hat{w}^{h_n}(x) := w^{h_n}\left(x_1, \frac{x'}{h_n}\right)$ on a strip $(0,L)\times(0,h_n)^2$. Then $\hat{w}^{h_n}$ is in $W^{1,p}((0,L)\times(0,h_n)^2;\mathbb{R}^3)$, and from the boundness of w^{h_n} and $\nabla_{h_n} w^{h_n}$, by rescaling the integrals on the new domain we obtain that there is a constant $C > 0$ such that
$$\frac{1}{h_n^2}\int_{(0,L)\times(0,h_n)^2} |\hat{w}^{h_n}|^p\,\mathrm{d}x + \frac{1}{h_n^2}\int_{(0,L)\times(0,h_n)^2} \left(|\partial_1 \hat{w}^{h_n}|^p + |\nabla' \hat{w}^{h_n}|^p\right)\mathrm{d}x \leq C. \tag{43}$$
2. Next, define $\tilde{w}^{h_n}$ on $(0,L)\times(-h_n,h_n)^2$ by reflecting the functions $\hat{w}^{h_n}$ with respect to the x_2 and x_3 variable $\tilde{w}^{h_n}(x) = \hat{w}^{h_n}(x_1,|x_2|,|x_3|)$. We define the functions $\bar{w}^{h_n}(x) = \tilde{w}^{h_n}(x_1, x' - (2ih_k, 2jh_k)), i,j \in \mathbb{Z}$ on $(0,L)\times\mathbb{R}^2$ by periodically extending $\tilde{w}^{h_n}$. From the construction of $\tilde{w}^{h_n}$, it is easy to see that $\bar{w}^{h_n} \in W^{1,p}_{loc}((0,L)\times\mathbb{R}^2;\mathbb{R}^3)$. Since

$(0, L) \times Q^2$ is contained in $(\lfloor \frac{1}{2h_n} \rfloor + 2)^2$ cubes and since $\tilde{w}^{h_n}$ is symmetric with respect to x_2 and x_3 axes, we derive that for n large enough:

$$\int_{(0,L)\times Q^2} |\bar{w}^{h_n}|^p \, \mathrm{d}x \le 4\left(2 + \left\lfloor \frac{1}{2h_n} \right\rfloor\right)^2 \int_{(0,L)\times(0,h_n)^2} |\hat{w}^{h_n}|^p \, \mathrm{d}x$$
$$\le \frac{4}{h_n^2} \int_{(0,L)\times(0,h_n)^2} |\hat{w}^{h_n}|^p \, \mathrm{d}x.$$

Thus, from (43) we deduce that $\tilde{w}^{h_n}$ is bounded with respect to n. Using the same arguments, the gradients $\nabla \bar{w}^{h_n}$ are also bounded with respect to n.

3. Since the sequences $(\bar{w}^{h_n})$ satisfy the assumptions of the lemma (2.16), there is a sequence $(v_k) \subset W^{1,p}((0, L) \times Q^2)$ such that $|\nabla v_k| < C(N)k$ *a.e.* on $(0, L) \times Q^2$ and

$$\lim_{k\to\infty} \left|(0, L) \times Q^2 \cap \{v_k \ne w^{h_{n(k)}} \text{ or } \nabla v_k \ne \nabla w^{h_{n(k)}}\}\right| = 0,$$

and $(|\nabla v_k|^p)$ is equi-integrable on $(0, L) \times Q^2$. By de la Vallée Poussin's criterion, there is a positive Borel function $\varphi : [0, \infty) \to [0, \infty]$ such that

$$\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty \quad \text{and} \quad \sup_k \int_{(0,L)\times Q^2} \varphi(|\nabla v_k|^p) < +\infty.$$

We denote with

$$M_k = \int_{(0,L)\times Q^2} \varphi(|\nabla v_k|^p),$$
$$m_k = \left|(0, L) \times Q^2 \cap \{v_k \ne w^{h_{n(k)}} \text{ or } \nabla v_k \ne \nabla w^{h_{n(k)}}\}\right|,$$

and (by Lemma 2.16) we have that $\sup_k M_k < \infty$ and $\lim_{k\to\infty} m_k = 0$.

4. It is easy to argument that for k large enough there exists a part of the domain $S_{ij}^{h_{n(k)}} \subset (0, L) \times Q^2$ of the form

$$S_{ij}^{h_{n(k)}} = (0, L) \times (ih_{n(k)}, (i+1)h_{n(k)}) \times (jh_{n(k)}, (j+1)h_{n(k)}),$$

such that

$$\int_{S_{ij}^{h_{n(k)}}} \varphi(|\nabla v_k|^p) \le 3h_{n(k)}^2 M_k,$$
$$\left|S_{ij}^{h_{n(k)}} \cap \{v_k \ne w^{h_{n(k)}} \text{ or } \nabla z_k \ne \nabla w^{h_{n(k)}}\}\right| \le 3h_{n(k)}^2 m_k.$$

5. Finally, we define the functions $\tilde{z}_k = v_k|_{S_j^{h_{n(k)}}}$ and the functions $z_k \in W^{1,p}((0, L) \times Q^2; \mathbb{R}^3)$ by translation, dilatation in x_2, x_3 variable and possible reflection of the functions $\tilde{z}_k$.

□

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