Annali di Matematica (2016) 195:845–855
DOI 10.1007/s10231-015-0493-z

Denjoy–Wolff theory for finite-dimensional bounded symmetric domains

P. Mellon[1]

Received: 20 January 2015 / Accepted: 3 April 2015 / Published online: 29 April 2015

Abstract Let B be a finite-dimensional bounded symmetric domain and $f : B \to B$ be a holomorphic map having no fixed point in B. For subsequential limits, g, of (f^n), we establish conditions, in terms of the Wolff point, ξ, of f, on which boundary components of B can contain $g(B)$. We extend Hervé's 1954 theorem on the bidisc to any finite product of bounded symmetric domains, namely if $B = B_1 \times \cdots \times B_n$ and $\xi = (\xi_1, \ldots, \xi_n)$ then there exists $d = (d_1, \ldots, d_n) \in \partial B$, satisfying $\overline{K_{d_i}} \cap \overline{K_{\xi_i}} \neq \emptyset$, such that

$$\pi_i(g(B)) \subseteq d_i + B_0(d_i),$$

where K_x denotes the affine boundary component of x, π_i is projection on the ith coordinate and $B_0(d_i)$ is a bounded symmetric subdomain of B_i. This simplifies if ξ_i is extreme, and even more so if B_i is a Hilbert ball.

Keywords Denjoy–Wolff theorem · Holomorphic mappings · Bounded symmetric domains

Mathematics Subject Classification Primary 32H50 · 32M15

1 Introduction

In the 1920s, Denjoy [11] and Wolff [29,30] proved that for a holomorphic map $f : \Delta \to \Delta$ without fixed point in Δ, the iterates (f^n) converge to a constant ξ in $\partial\Delta$. In 1963, Hervé established the result for the finite-dimensional Hilbert ball [16], while in 1984, Stachura [28] showed that the result fails for a biholomorphic map on an infinite-dimensional Hilbert ball. On the other hand, if f is compact ($\overline{f(B)}$ is compact), the result does hold on an infinite-dimensional Hilbert ball [8]. Recently, many authors [1,3,5–7,17] have established

✉ P. Mellon
pauline.mellon@ucd.ie

1 School of Mathematical Sciences, University College Dublin, Belfield, Dublin 4, Ireland

Denjoy–Wolff type results on spaces that have convexity properties rivalling those of Hilbert spaces, for example, uniformly convex, strictly convex, strictly linearly convex spaces, etc (where f is compact whenever the space is infinite dimensional).

The problem that interests us here, however, is the more common scenario of what happens when (f^n) does not converge. This is already the case for certain domains in the plane [26] and also for even the simplest of finite-dimensional balls, such as the bidisc [8]. In such cases, we seek the cluster points of (f^n) and their images. Although Hervé dealt with the bidisc in 1954 [15], progress in the general case has been slow since, with several recent publications re-examining the polydisc case [4,14]. Other recent results on bounded symmetric domains [24] show that the complications involved are not related solely to the reducibility or otherwise of the domain, but instead depend intimately on the holomorphic boundary structure.

Bounded symmetric domains are therefore an ideal context in which to tackle this problem for the following reasons. Firstly, they include large classes of domains as they characterize all homogeneous open unit balls. Secondly, and crucially, the presence of Jordan structure makes powerful algebraic techniques available, which compensates somewhat for the lack of extra convexity (apart from the Hilbert ball, these domains are not strongly convex, strictly convex or strictly linearly convex). Thirdly, there is a full algebraic description of the holomorphic boundary components of these domains. And finally, from the point of view of holomorphic dynamics, it has recently been shown in [24, Theorem 3.4] that the Hilbert ball is a natural outlier in the class of all bounded symmetric domains, suggesting that an essentially different (non-Hilbertian) approach is needed. The results presented are new for many simple finite-dimensional domains and therefore, while Jordan techniques are used throughout, key results are stated in an almost Jordan-free setting for those with non-Jordan perspectives.

As known, if f is fixed point free, then every subsequential limit, g, of (f^n) maps B into ∂B, and, in fact, maps B into a single holomorphic boundary component, K_d, in ∂B. The aim is therefore to determine which boundary components can contain such a $g(B)$, namely to find conditions on d.

We state our first main result in non-Jordan terms, using a concept of closed convex hull, $\mathrm{Ch}(x)$, introduced in [5], although its proof and later use require Jordan theory. (Convergence for holomorphic functions is uniform convergence on compact subsets of B.)

Theorem 1.1 *Let B be any finite-dimensional bounded symmetric domain, $f : B \to B$ be holomorphic and fixed-point free and ξ be the Wolff point of f. Then for any subsequential limit, g, of (f^n), there exists*

$$d \in \partial B \text{ satisfying } Ch(d) \cap Ch(\xi) \neq \emptyset \text{ such that } g(B) \subseteq K_d = d + B_0(d)$$

(and $B_0(d)$ is a bounded symmetric subdomain of B).

Our second key result is an extension of Hervé's, now classical, theorem on the bidisc [15] to any finite product of bounded symmetric domains. Projection on the ith coordinate is $\pi_i(x_1, \ldots, x_n) = x_i$, a tripotent is $x = \{x, x, x\}$, and we note that a natural partial order '$\leq$' exists on the set of all tripotents.

Theorem 1.2 *Let $B = B_1 \times \cdots \times B_n$, where each B_i is a bounded symmetric domain, $f : B \to B$ is holomorphic and fixed-point free and $\xi = (\xi_1, \ldots, \xi_n) \in \partial B$ is the Wolff point of f. Let g be any subsequential limit of (f^n). Then there is a unique tripotent $d = (d_1, \ldots, d_n) \in \partial B$ satisfying (for $1 \leq i \leq n$) $\overline{K_{d_i}} \cap \overline{K_{\xi_i}} \neq \emptyset$ such that*

$$\pi_i(g(B)) \subseteq d_i + B_0(d_i)$$

(and $B_0(d_i)$ is a bounded symmetric subdomain of B_i).

In particular, if ξ_i is extreme in $\overline{B}_i$, then $d_i \leq \xi_i$, leading to the much simpler statement that if B_i is a Hilbert ball and $\|\xi_i\| = 1$ then

$$\pi_i(g(B)) = \{\xi_i\} \quad \textit{whenever} \quad \pi_i(g(B)) \cap \partial B_i \neq \emptyset.$$

As results in this setting may appear weak in comparison with those of spaces with additional convexity, we recommend, as antidote, an elegant partial survey of Hervé's work in the bidisc, presented in section 5 of [5]. We also provide extensions to more general settings of some of the results in [5]. While certain results presented here hold also in infinite dimensions, we focus for clarity predominantly on the finite-dimensional case.

2 Notation and background

Throughout, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For X and Y complex Banach spaces, $\mathcal{L}(X, Y)$ denotes the space of continuous linear maps from X to Y, $\mathcal{L}(X) = \mathcal{L}(X, X)$ and $\mathsf{GL}(X)$ is all invertible elements in $\mathcal{L}(X)$. For domains $D \subset X$ and $\widetilde{D} \subset Y$, we denote the set of all holomorphic maps from D to $\widetilde{D}$ as $H(D, \widetilde{D})$, with $H(D) = H(D, D)$. For $f \in H(D)$, the iterates of f are $f^n := f \circ f^{n-1}, n \in \mathbb{N}, n > 1$ and $f^1 = f$.

2.1 JB^*-triples

Every homogeneous open unit ball is biholomorphically equivalent to a bounded symmetric domain, and bounded symmetric domains are classified [18] as the open unit balls of JB^*-triples. JB^*-triples include all C^*-, JB^*- and J^*-algebras.

Definition 2.1 A JB^*-triple is a complex Banach space Z with a real trilinear mapping $\{\cdot, \cdot, \cdot\} : Z \times Z \times Z \to Z$ satisfying

(i) $\{x, y, z\}$ is complex linear and symmetric in the outer variables x and z, and is complex anti-linear in y.
(ii) The map $z \mapsto \{x, x, z\}$, denoted $x \,\square\, x$, is Hermitian, $\sigma(x \,\square\, x) \geq 0$ and $\|x \,\square\, x\| = \|x\|^2$ for all $x \in Z$, where σ denotes the spectrum.
(iii) The product satisfies the following "triple identity"

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

The triple product satisfies $\|\{x, y, z\}\| \leq \|x\|\|y\|\|z\|$ so is continuous and gives rise to the linear maps: $x \,\square\, y \in \mathcal{L}(Z) : z \mapsto \{x, y, z\}$, $Q_x \in \mathcal{L}_{\mathbb{R}}(Z) : z \mapsto \{x, z, x\}$, and the geometrically significant Bergman operators

$$B(x, y) = I - 2x \,\square\, y + Q_x Q_y \in \mathcal{L}(Z).$$

2.2 Tripotents and ordering

Tripotents here replace idempotents for an algebra, namely $e \in Z$ is a tripotent if $\{e, e, e\} = e$. Every tripotent e induces a splitting of Z, as $Z = Z_0(e) \oplus Z_{\frac{1}{2}}(e) \oplus Z_1(e)$, where $Z_k(e)$ is the k eigenspace of $e \square e$ and the linear maps $P_0(e) = B(e, e)$, $P_{\frac{1}{2}}(e) = 2(e \,\square\, e - Q_e Q_e)$, and $P_1(e) = Q_e Q_e$ are mutually orthogonal projections of Z onto $Z_0(e)$, $Z_{\frac{1}{2}}(e)$, and $Z_1(e)$, respectively. $Z_0(e)$ and $Z_1(e)$ are themselves triples whose open unit balls, $B_0(e)$ and $B_1(e)$, are therefore bounded symmetric domains.

Elements $x, y \in Z$ are orthogonal, $x \perp y$, if $x \square y = 0$ (or equivalently [20] if $y \square x = 0$). In particular, if c and e are orthogonal tripotents, then $c + e$ is also a tripotent, giving a natural partial order on the set, M, of all tripotents in Z as follows.

Definition 2.2 For tripotents c and e, we say $c < e$ if $e - c \in M$ and $(e - c) \perp c$.

Then e is maximal if $Z_0(e) = 0$ and e is minimal if $Z_1(e) = \mathbb{C}e$. The set of maximal tripotents coincides with the set of extreme points and real and complex extreme points coincide. Z is said to have finite rank r if every element $z \in Z$ is contained in a subtriple of (complex) dimension $\leq r$, and r is minimal with this property. The rank 1 triples are the Hilbert spaces. For details, see [20].

2.3 Boundary structure of bounded symmetric domains

Let E be a complex Banach space with open unit ball B_E.

Definition 2.3 $A \subset \overline{B}_E, A \neq \emptyset$ is a holomorphic boundary component of B_E if A is minimal with respect to the fact that either $f(\Delta) \subset A$ or $f(\Delta) \subset \overline{B}_E \backslash A$, for all $f \in \mathcal{F} = \left\{f : \Delta \to Z \text{ holomorphic with } f(\Delta) \subset \overline{B}_E\right\}$.

The holomorphic boundary component of B_E containing a is written K_a. By replacing $\mathcal{F}$ in the above definition with the set of all complex (real) affine maps : $\Delta \to \overline{B}$, we get the definition of complex (real) affine boundary component.

For Z a finite-dimensional JB^*-triple holomorphic and affine boundary components coincide (we refer simply to boundary components) and may be described in terms of tripotents as follows.

Theorem 2.4 [20, Theorem 6.3] Let Z be a finite-dimensional JB^*-triple with open unit ball B. The following hold.

(i) Holomorphic and affine boundary components coincide and are precisely the sets

$$K_e = e + B_0(e)$$

where e is a tripotent and $B_0(e) = B \cap Z_0(e)$ is the bounded symmetric domain associated with the triple $Z_0(e)$. Moreover, the map $e \to K_e$ is a bijection between the set, M, of tripotents and the set of boundary components of B.

(ii) An element x in Z belongs to K_e if, and only if, $e = \lim_{n\to\infty} x^{2n+1}$, where $x^{2n+1} := \{x, x^{2n-1}, x\},\ n \geq 1$.

(iii) The boundary components of K_e are K_d for $d \geq e$. In particular,

$$\overline{K}_e = \bigcup_{d \geq e} K_d$$

for tripotents $e, d \in Z$.

Remarks 2.5 It follows that B is the only open boundary component, while the only closed boundary components are those singletons corresponding to extreme points. In particular, if x is not extreme then $\overline{K}_x$ is strictly larger than K_x.

2.4 A Wolff theorem for bounded symmetric domains

For a fixed-point free holomorphic self-map, f, of B, Jordan theory has been used to produce f-invariant domains [21–23] which have a simple algebraic description in terms of linear maps known as Bergman operators. See [25] for a comprehensive introduction into this Jordan approach. An infinite-dimensional version of the following result is [22, Theorem 3.8]

Theorem 2.6 [22, Theorem 3.10] *Let Z be a finite-dimensional JB^*-triple with open unit ball B and $f : B \to B$ be a fixed-point free holomorphic map. Then there exists ξ in ∂B such that for all $\lambda > 0$, there exists $c_\lambda \in B$ and $T_\lambda \in \mathsf{GL}(Z)$ such that the domain*

$$E_{\xi,\lambda} := c_\lambda + T_\lambda(B)$$

is f-invariant and is a non-empty convex affine subset of B with $\xi \in \partial E_{\xi,\lambda}$.
Moreover, for each $y \in B$, there exists $\lambda_y > 0$ such that $y \in \partial E_{\xi,\lambda_y}$. In addition, let e be the unique tripotent with $\xi \in K_e$ then

$$\lim_{\lambda\to 0^+} c_\lambda = e \ \ and \ \lim_{\lambda\to 0^+} T_\lambda = B(e,e) = P_0(e)$$

where $P_0(e)$ is the projection of Z onto the subtriple $Z_0(e)$.

We use the following well-known result, where f is called compact if $\overline{f(B)}$ is compact.

Theorem 2.7 *[17] Let D be a bounded convex domain in a complex Banach space E and $f : D \to D$ be a compact holomorphic map. Then the following are equivalent.*

(i) f has a fixed point in D;
(ii) there exists $z_0 \in D$ such that $\{f^n(z_0)\}$ is relatively compact in D;
(iii) there exists $z_0 \in D$ such that $\{f^n(z_0)\}$ has a subsequence that is relatively compact in D;
(iv) $\{f^n(z)\}$ is relatively compact in D, for all $z \in D$.

3 Results

Let Z be a finite-dimensional JB^*-triple with open unit ball B and $f : B \to B$ be a holomorphic map with no fixed point in B. Convergence for holomorphic maps refers here to uniform convergence on compact subsets of B, and hence, by Montel's theorem, each subsequence (f^{n_k}) of the sequence of iterates admits a convergent subsequence whose limit is a holomorphic map : $B \to \overline{B}$. Let $\Gamma(f)$ denote the set of subsequential limits of (f^n). Then

$$T(f) := \bigcup_{g\in\Gamma(f)} g(B)$$

is called the target set of f [5] and, by Theorem 2.7 above, $T(f) \subset \partial B$.

As the following material is required for several subsequent proofs, we present it separately here (see also [25]).

We locate the Wolff point, ξ, of f in the usual way. Choose $(\alpha_k)_k$, $0 < \alpha_k < 1$, $\alpha_k \uparrow 1$ and let $f_k := \alpha_k f$ for all k. Then f_k has a fixed point, z_k, in $\alpha_k B$ [12], and we may assume $z_k \to \xi \in \overline{B}$ and hence $\xi \in \partial B$, as otherwise it would be a fixed point of f.

Let $e = \lim_{n\to\infty} \xi^{2n+1}$. Then $\{e, e, e\} = e$ and by (ii) of Theorem 2.4 above, $K_\xi = K_e$. By Theorem 2.6, for each $\lambda > 0$, there exists a non-empty convex affine f-invariant subset

$$E_{\xi,\lambda} = c_\lambda + T_\lambda(B) \subset B$$

where $c_\lambda \in B$ and T_λ is an invertible linear operator. Invertibility of T_λ gives $\overline{E_{\xi,\lambda}} = c_\lambda + T_\lambda(\overline{B})$.

Then for $z \in E_{\xi,\lambda}$, $f^n(z) \in E_{\xi,\lambda}$ for all $n \in \mathbb{N}$ and hence $g(z) \in \overline{E_{\xi,\lambda}}$, for all $g \in \Gamma(f)$. As $g(B) \subseteq \partial B$, we have the following.

$$\text{For } \lambda > 0 \text{ and } z_\lambda \in E_{\xi,\lambda}, \text{ then } g(z_\lambda) \in \partial E_{\xi,\lambda} \cap \partial B, \quad \text{for all} \quad g \in \Gamma(f). \tag{1}$$

In general, $\partial E_{\xi,\lambda} \cap \partial B$ can be quite large (apart from the rank 1 case where $\partial E_{\xi,\lambda} \cap \partial B = \{\xi\}$ [[23] Example 4.4]). It is also not affinely connected ([22] Example 4.6) and it has recently been shown [25] that

$$\bigcap_{\lambda>0} \partial E_{\xi,\lambda} \cap \partial B = \overline{K}_\xi.$$

Consequently, $\overline{K}_\xi$ contains all constant subsequential limits of (f^n) [25, proposition 4.2]. In particular, if ξ is extreme, then it is the only possible constant subsequential limit of (f^n) [25, Corollary 4.3].

In [5], the authors introduce a concept of closed convex hull for convex domains in $\mathbb{C}^n$, defined in terms of complex supporting functionals. We recall that a complex supporting functional at $x \in \partial B$ is $\phi \in \mathcal{L}(X, \mathbb{C})$ such that $\text{Re}\,\phi(z) < \text{Re}\,\phi(x)$, for all $z \in B$. A (complex) supporting hyperplane at $x \in \partial B$ is the affine subspace $x + \ker\phi$, for ϕ a (complex) supporting functional.

Definition 3.1 [5] For $x \in \partial B$, $\text{Ch}(x)$ is the intersection of $\overline{B}$ with all supporting hyperplanes at x.

$\text{Ch}(x)$ is a closed, convex subset of ∂B, for all $x \in \partial B$. As $\text{Ch}(x)$ was introduced for its holomorphic character, we propose to use instead the holomorphic boundary component, K_x. We therefore reconcile $\text{Ch}(x)$ and K_x for a bounded symmetric domain B, as follows.

Proposition 3.2 *$Ch(x) = \overline{K}_x$, for all $x \in \partial B$.*

In particular, $Ch(x) = K_x \Leftrightarrow x$ is an extreme (and then $Ch(x) = K_x = \{x\}$).

Proof Let $x \in \partial B$ and let ϕ be any supporting functional at x. Let $A = \{z \in \partial B : \phi(z) = \phi(x)\}$ and let $f : \Delta \to \overline{B}$ be holomorphic. If $f(\Delta) \cap A \neq \emptyset$ then the Maximum Principle implies that $\phi \circ f$ is constant and hence $f(\Delta) \subseteq A$. By definition, K_x is minimal with this property and hence $K_x \subseteq A$ for all such ϕ, and hence $K_x \subseteq \text{Ch}(x)$. Since $\text{Ch}(x)$ is closed, this gives $\overline{K}_x \subseteq \text{Ch}(x)$. Let now e be the unique tripotent such that $x \in K_e$. By [20, Lemma 6.2], there exists a complex supporting hyperplane H at e such that

$$\overline{K}_e = \overline{B} \cap H$$

and therefore $\text{Ch}(e) \subseteq \overline{K}_e$. From above $\overline{K}_e \subseteq \text{Ch}(e)$ hence

$$\text{Ch}(e) = \overline{K}_e.$$

We therefore have $\text{Ch}(e) = \overline{K}_e = \overline{K}_x \subseteq \text{Ch}(x)$. On the other hand, if $y \in \text{Ch}(x)$ one sees readily that $\text{Ch}(y) \subseteq \text{Ch}(x)$ and hence $x \in \overline{K}_x = \overline{K}_e = \text{Ch}(e)$ gives $\text{Ch}(x) \subseteq \text{Ch}(e)$. Therefore, $\text{Ch}(x) = \text{Ch}(e)$. $\text{Ch}(e) = \overline{K}_e = \overline{K}_x$ gives $\text{Ch}(x) = \overline{K}_x$. From Remarks 2.5, $\text{Ch}(x) = K_x \Leftrightarrow x$ is extreme. ◻

Remarks 3.3 For $g \in \Gamma(f)$, $g(B)$ must lie inside a single boundary component (see Proposition 3.4 below), and hence this distinction between K_x (which is affinely connected) and $\mathrm{Ch}(x)$ (which is not) is significant.

The following result for (not necessarily finite dimensional) Banach spaces is a consequence of Definition 2.3 and will be used implicitly hereafter. By Proposition 3.2, it refines [5, Proposition 1] in the case of finite-dimensional bounded symmetric domains.

Proposition 3.4 *Let D be a domain in any complex Banach space and E be a complex Banach space with open unit ball B_E. For every holomorphic map $h : D \to E$ such that $h(D) \subset \overline{B}_E$ then*

$$h(D) \subseteq \bigcap_{z\in D} K_{h(z)}.$$

If, in addition, $h(D) \cap \partial B_E \neq \emptyset$ then $h(D) \subseteq \bigcap_{z\in D} K_{h(z)} \subset \partial B_E$.

In particular, if f is a fixed-point free compact holomorphic self-map of B_E and g is any subsequential limit of (f^n) for the topology of local uniform convergence on B_E, then

$$g(B_E) \subseteq \bigcap_{z\in B_E} K_{g(z)} \subset \partial B_E.$$

Proof It follows from Definition 2.3 above that $h(D)$ must lie inside precisely one boundary component, K_d say, so $h(z) \in K_d$, and hence $K_{h(z)} = K_d$, for all $z \in D$ giving $h(D) \subseteq \bigcap_{z\in D} K_{h(z)}$.

If now $h(D) \cap \partial B_E \neq \emptyset$, then $K_d \cap \partial B_E \neq \emptyset$ and hence $K_d \subseteq \partial B_E$.

Let now f be as given and g be any subsequential limit of (f^n) for the topology of local uniform convergence on B_E. Theorem 2.7 gives that $g(B_E) \subseteq \partial B_E$ and the result follows. □

We return now to the finite-dimensional bounded symmetric domain B and, in the spirit of [5], define

$$K_W = \bigcup_{z\in W} K_z, \quad \text{for } W \subseteq \overline{B},$$

emphasizing, however, that K_W is no longer any kind of boundary component. The following refines [5, Lemma 6] for bounded symmetric domains.

Proposition 3.5 *Let B be a bounded symmetric domain, $f : B \to B$ be holomorphic and fixed-point free and ξ be the Wolff point of f. Then*

$$T(f) \subseteq \bigcap_{\lambda>0} K_{\partial E_{\xi,\lambda}\cap\partial B}.$$

Proof Fix $\lambda > 0$ and choose $z_\lambda \in E_{\xi,\lambda}$. Let $g \in \Gamma(f)$. From (1) above, $g(z_\lambda) \in \partial E_{\xi,\lambda} \cap \partial B$ and hence $K_{g(z_\lambda)} \subseteq K_{\partial E_{\xi,\lambda}\cap\partial B}$. Proposition 3.4 gives

$$g(B) \subseteq \bigcap_{\lambda>0} K_{g(z_\lambda)} \subseteq \bigcap_{\lambda>0} K_{\partial E_{\xi,\lambda}\cap\partial B}.$$

□

Let $g \in \Gamma(f)$. Our first main result, relating $g(B)$ to the Wolff point ξ, is key to everything that follows and, in particular, to the promised extension of Hervé's work. It allows us to confine $T(f)$ to certain boundary components of B, namely to place restrictions on those tripotents d for which $g(B) \subseteq K_d$.

Theorem 3.6 *Let B be a bounded symmetric domain, $f : B \to B$ be holomorphic and fixed-point free and ξ be the Wolff point of f. Let $g \in \Gamma(f)$. Then there is a unique tripotent $d \in \partial B$ satisfying*

$$\overline{K_d} \cap \overline{K_\xi} \neq \emptyset \text{ such that } g(B) \subseteq K_d = d + B_0(d).$$

(Note that $B_0(d)$ is itself a bounded symmetric domain.)

Proof Fix $g \in \Gamma(f)$. Theorems 2.7 and 2.4 above give $g(B) \subseteq K_d \subseteq \partial B$, for a unique tripotent $d \in \partial B$. Fix now $\lambda > 0$ and $z_\lambda \in E_{\xi,\lambda}$. Then $K_d = K_{g(z_\lambda)}$. From (1) $g(z_\lambda) \in \overline{E_{\xi,\lambda}} = c_\lambda + T_\lambda(\overline{B})$. Therefore, $g(z_\lambda) = c_\lambda + T_\lambda(w_\lambda) \in K_d$, for some $w_\lambda \in \overline{B}$ and hence $c_\lambda + T_\lambda(w_\lambda) = d + x_\lambda$, for some $x_\lambda \in B_0(d) = B \cap Z_0(d)$ by Theorem 2.4. If now e is the unique tripotent with $\xi \in K_e$, Theorem 2.6 above gives

$$\lim_{\lambda \to 0^+} c_\lambda = e \text{ and } \lim_{\lambda \to 0^+} T_\lambda = P_0(e).$$

Then

$$\lim_{\lambda \to 0^+} c_\lambda + T_\lambda(w_\lambda) = \lim_{\lambda \to 0^+} d + x_\lambda,$$

so that for $w = \lim_\lambda w_\lambda \in \overline{B}$ and $x = \lim_\lambda x_\lambda \in \overline{B_0(d)}$ (passing to a subnet where necessary) we have

$$e + P_0(e)(w) = d + x.$$

Now $e + P_0(e)(w) \in e + P_0(e)(\overline{B}) \subset e + \overline{P_0(e)(B)} = e + \overline{B_0(e)} = \overline{K_e}$. This gives

$$e + P_0(e)(w) = d + x \in \overline{K}_e \cap \overline{K}_d.$$

As $\overline{K}_e = \overline{K}_\xi$ we are done. □

Proposition 3.2 allows us to restate the above in an almost Jordan-free form.

Corollary 3.7 *Let B be a bounded symmetric domain, $f : B \to B$ be holomorphic and fixed-point free and ξ be the Wolff point of f. Let $g \in \Gamma(f)$. Then there is a unique tripotent $d \in \partial B$ satisfying*

$$Ch(d) \cap Ch(\xi) \neq \emptyset$$

such that $g(B) \subseteq K_d = d + B_0(d)$.

In particular, we get the following refinement.

Corollary 3.8 *If the Wolff point ξ is extreme then for $g \in \Gamma(f)$ there is a tripotent $d \in \partial B$ satisfying $d \leq \xi$ such that*

$$g(B) \subseteq K_d = d + B_0(d).$$

Proof Fix $g \in \Gamma(f)$. As ξ is extreme, $\overline{K}_\xi = \{\xi\}$ and hence, from Theorem 3.6, there is $d \in \partial B$ with $g(B) \subseteq K_d$ and $\xi \in \overline{K_d}$. Theorem 2.4 (iii) gives

$$\overline{K}_d = \bigcup_{q \geq d} K_q$$

and hence $\xi \in K_q$, for some tripotent $q \geq d$. On the other hand, ξ extreme means $\xi = q \geq d$ and we are done. □

Theorem 3.6 also provides a short alternative proof of Hervé's Denjoy–Wolff theorem for the finite-dimensional Hilbert ball, as follows.

Corollary 3.9 *[16] Let B be a finite-dimensional Hilbert ball and $f : B \to B$ be a fixed-point free holomorphic map. Then there exists $\xi \in \partial B$ such that (f^n) converges to the constant map ξ uniformly on compact subsets of B.*

Proof For B a Hilbert ball, $K_x = \{x\}$, for all $x \in \partial B$, hence, for ξ the Wolff point of f, $\overline{K_d} \cap \overline{K_\xi} \neq \emptyset \Leftrightarrow d = \xi$ so Theorem 3.6 thus gives $g(B) = \{\xi\}$, for all $g \in \Gamma(f)$. In other words, all subsequential limits of (f^n) are ξ, and it follows readily that (f^n) itself must therefore converge to ξ, giving the result. □

In the following example, we probe the role played by ξ in Theorem 3.6 by explicitly calculating all non-zero tripotents d satisfying

$$\overline{K_d} \cap \overline{K_\xi} \neq \emptyset.$$

Example 3.10 Let $B = \Delta^3$, $g \in \Gamma(f)$ and $d = (d_1, d_2, d_3)$ be the unique tripotent with $g(B) \subseteq K_d$. Since $d \in \partial B$ is a tripotent, $|d_i| \in \{0, 1\}$ for $1 \leq i \leq 3$ and $\max_i |d_i| = 1$.

(i) If $\xi = (1, 1, 1)$ then $d_i \in \{0, 1\}$ for $1 \leq i \leq 3$.
(ii) If $\xi = (1, 1, \frac{1}{2})$ then $d_i \in \{0, 1\}$ for $1 \leq i \leq 2$.
(iii) If $\xi = (1, \frac{1}{2}, \frac{1}{3})$ then $d_1 \in \{0, 1\}$.

We notice in the above example that ξ exerts control over component d_i of d only if $\|\xi_i\| = 1$. This is also true in a more general context. In fact, the above now suggests a way to use Theorem 3.6 to develop an analog of Hervé's theorem on the bidisc, [15], for any finite product of bounded symmetric domains.

Let $Z_1, \ldots, Z_n$ be JB*-triples. Hereafter, $Z := Z_1 \times \cdots \times Z_n$ is a JB*-triple for the triple product defined coordinatewise and norm given by $\|(z_1, \ldots, z_n)\| = \max_{1 \leq i \leq n} \|z_i\|$. We write $B := B_Z = B_1 \times \cdots \times B_n$ where $B_i := B_{Z_i}$.

Since boundary components are defined in terms of tripotents and the triple product, the following is a simple consequence of Theorem 2.4.

Lemma 3.11 *Let $d = (d_1, \ldots, d_n)$ be a tripotent in $Z = Z_1 \times \cdots \times Z_n$. Then*

$$K_d = K_{d_1} \times \cdots \times K_{d_n}.$$

Lemma 3.12 *Let $\xi = (\xi_1, \ldots, \xi_n)$ be the Wolff point of f and $e = (e_1, \ldots, e_n)$ be the unique tripotent determining K_ξ. Then*

$$\|\xi_i\| < 1 \Leftrightarrow e_i = 0, \text{ for } 1 \leq i \leq n.$$

Proof By Lemma 3.11, $K_e = K_{e_1} \times \cdots \times K_{e_n}$ so that $\xi \in K_e$ gives $\xi_i \in K_{e_i}$, for $1 \leq i \leq n$. Since the boundary component of 0 in B_i is B_i itself we have

$$K_{e_i} \cap B_i \neq \emptyset \Leftrightarrow K_{e_i} = B_i = K_0 \Leftrightarrow e_i = 0.$$

In particular, if $\|\xi_i\| < 1$ then $K_{e_i} \cap B_i \neq \emptyset$ so $e_i = 0$. In the opposite direction, if $e_i = 0$ then $\xi_i \in K_{e_i} = B_i$. □

We now state the desired extension of Hervé's bidisc result [15] to a product of bounded symmetric domains. The result is new whenever $n > 1$ and at least one B_i has rank ≥ 2. Projection onto the ith co-ordinate is denoted by π_i.

Theorem 3.13 *Let $B = B_1 \times \cdots \times B_n$ be a finite product of bounded symmetric domains, $f : B \to B$ be a holomorphic map with no fixed point in B, $\xi = (\xi_1, \ldots, \xi_n) \in \partial B$ be the Wolff point of f and g be any subsequential limit of (f^n).*

Then there is a unique tripotent $d = (d_1, \ldots, d_n) \in \partial B$ such that $g(B) \subset d + B_0(d) \subset \partial B$ and for $1 \leq i \leq n$ the following hold.

(i) $\pi_i(g(B)) \subseteq K_{d_i} = d_i + B_0(d_i)$ and d_i satisfies $\overline{K_{d_i}} \cap \overline{K_{\xi_i}} \neq \emptyset$.
(ii) In particular, if ξ_i is extreme in $\overline{B}_i$, then $d_i \leq \xi_i$.
(iii) In particular, if B_i is a Hilbert ball and $\|\xi_i\| = 1$ then

$$\pi_i(g(B)) = \{\xi_i\} \text{ whenever } \pi_i(g(B)) \cap \partial B_i \neq \emptyset.$$

(Note that if $\|\xi_i\| < 1$, $K_{\xi_i} = B_i$ and the condition on d_i in (i) is thus trivial.)

Proof By definition $g(B)$ lies in a single boundary component, which, by Theorems 2.4 and 2.7 above, is determined by a tripotent $d = (d_1, \ldots, d_n) \in \partial B$ so that $g(B) \subset K_d = d + B_0(d)$. Lemma 3.11 then gives $\pi_i(g(B)) \subseteq K_{d_i} = d_i + B_0(d_i)$.

By Theorem 3.6 above d must satisfy

$$\overline{K_d} \cap \overline{K_\xi} \neq \emptyset$$

and hence from Lemma 3.11 $\overline{K_{d_i}} \cap \overline{K_{\xi_i}} \neq \emptyset$ for $1 \leq i \leq n$, giving (i).

If now ξ_i is extreme in $\overline{B_i}$, then $\overline{K_{\xi_i}} = \{\xi_i\}$ and from (i) $\xi_i \in \overline{K_{d_i}}$. Part (iii) of Theorem 2.4 then gives $\xi_i \geq d_i$ and we have (ii).
For (iii), assume that B_i is a Hilbert ball and $\|\xi_i\| = 1$. As each point in ∂B_i is extreme, then $d_i \leq \xi_i$ by (ii). Since d_i is itself a tripotent, this means either $d_i = 0$ or $\|d_i\| = 1$. On the other hand, $\|d_i\| = 1$ means d_i is extreme so $d_i \leq \xi_i$ gives $d_i = \xi_i$. In other words, $d_i \in \{0, \xi_i\}$, namely, either $K_{d_i} = B_i$ or $K_{d_i} = \{\xi_i\}$. Since from (i) $\pi_i(g(B)) \subseteq K_{d_i}$, it follows in particular that if $\pi_i(g(B)) \cap \partial B_i \neq \emptyset$ then $\pi_i(g(B)) = \{\xi_i\}$ and we are done. □

In the case that each B_i $(1 \leq i \leq n)$ is a Hilbert ball, we get a much simplified version of Theorem 3.13 as follows.

Corollary 3.14 *Let $B = B_1 \times \cdots \times B_n$, where each B_i is a Hilbert ball. Let $f : B \to B$ be a holomorphic map with no fixed point in B, $\xi = (\xi_1, \ldots, \xi_n)$ be the Wolff point of f and $g \in \Gamma(f)$.*

If $\|\xi_i\| = 1$ then

$$\pi_i(g(B)) = \{\xi_i\} \text{ whenever } \pi_i(g(B)) \cap \partial B_i \neq \emptyset.$$

We note that Corollary 3.14 is proved via alternative methods in [10].

References

1. Abate, M.: Horospheres and iterates of holomorphic maps. Math. Z. **198**, 225–238 (1988)
2. Abate, M.: Iteration Theory of Holomorphic Maps on Taut Manifolds, Research and Lecture Notes in Mathematics. Mediterranean Press, Italy (1989)
3. Abate, M.: Iteration theory, compactly divergent sequences and commuting holomorphic maps. Ann. Scuola Norm. Sup. Pisa **18**, 167–191 (1991)
4. Abate, M.: The Julia-Wolff-Carathéodory theorem in polydiscs. J. Anal. Math **74**, 275–306 (1999)
5. Abate, M., Raissy, J.: Wolff-Denjoy theorems in non-smooth convex domains, Ann. Mat. Pura Appl. doi:10.1007/s10231-013-0341-y
6. Budzyńska, M., Kuczumov, T., Reich, S.: Theorems of Denjoy-Wolff type. Ann. Mat. Pura Appl. **192**, 621–648 (2013)

7. Budzyńska, M., Kuczumov, T., Reich, S.: A Denjoy-Wolff theorem for compact holomorphic mappings in complex Banach spaces. Ann. Acad. Sci. Fenn. Math. **38**, 747–756 (2013)
8. Chu, C.H., Mellon, P.: Iteration of compact holomorphic maps on a Hilbert ball. Proc. Am. Math. Soc. **125**(6), 1771–1777 (1997)
9. Chu, C.H., Mellon, P.: Jordan structures in Banach spaces and symmetric manifolds. Expos. Math. **16**(2), 157–180 (1998)
10. Chu, C.H., Rigby, M.: Iteration of self-maps on a product of Hilbert balls. J. Math. Anal. Appl. **411**(2), 773–786 (2014)
11. Denjoy, A.: Sur l'itération des fonctions analytiques. C. R. Acad. Sci. Paris **182**, 255–257 (1926)
12. Earle, C.J., Hamilton, R.S.: A fixed point theorem for holomorphic mappings. Proc. Symp. Pure Math. **16**, 61–65 (1969)
13. Goebel, K., Reich, S.: Iterating holomorphic self-mappings of the Hilbert ball. Proc. Jpn. Acad. **58**, 349–352 (1982)
14. Frosini, C.: Busemann functions and the Julia-Wolff-Carathéodory theorem for polydiscs. Adv. Geom. **10**(3), 435–463 (2010)
15. Hervé, M.: Itération des transformations analytiques dans le bicercle-unité. Ann. Sci. Éc. Norm. Supér. **71**, 1–28 (1954)
16. Hervé, M.: Quelques proprietés des application analytiques d'une boule à m dimensions dans elle-meme. J. Math. Pures Appl. **42**, 117–147 (1963)
17. Kapeluszny, J., Kuczumow, T., Reich, S.: The Denjoy-Wolff Theorem in the open unit ball of a strictly convex Banach space. Adv. Math. **143**, 111–123 (1999)
18. Kaup, W.: A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. Math. Z. **138**, 503–529 (1983)
19. Kaup, W., Sauter, J.: Boundary structure of bounded symmetric domains. Manuscr. Math. **101**(3), 351–360 (2000)
20. Loos, O.: Bounded Symmetric domains and Jordan Pairs, Lecture Notes, University of California at Irvine, (1977)
21. Mellon, P.: Another look at results of Wolff and Julia type for J^*-algebras. J. Math. Anal. Appl. **198**, 444–457 (1996)
22. Mellon, P.: Holomorphic invariance on bounded symmetric domains. J. Reine. Angew. Math. **523**, 199–223 (2000)
23. Mellon, P.: A general Wolff theorem for arbitrary Banach spaces. Math. Proc. R. Ir. Acad. **104A**(2), 127–142 (2004)
24. Mellon, P.: Dynamics of biholomorphic self-maps on bounded symmetric domains. Math. Scand., To appear. (https://mathsci.ucd.ie/docserve)
25. Mellon, P.: A Jordan approach to iteration theory for bounded symmetric domains. Cont. Proc. Am. Math. Soc., To appear. (https://mathsci.ucd.ie/docserve)
26. Poggi-Corradini, P.: On the failure of a generalized Denjoy-Wolff Theorem. Conform. Geom. Dyn. **6**, 13–32 (2002)
27. Reich, S., Shoikhet, D.: The Denjoy-Wolff Theorem. Ann. Univ. Mariae Curie-Skłodowska **51A**, 219–240 (1997)
28. Stachura, A.: Iterations of holomorphic self-maps of the unit ball in Hilbert space. Proc. Am. Math. Soc. **93**, 88–90 (1985)
29. Wolff, J.: Sur l'iteration des fonctions bornées. C. R. Acad. Sci. Paris **182**, 200–201 (1926)
30. Wolff, J.: Sur une généralisation d'un thèorém de Schwarz. C. R. Acad. Sci. Paris **182**, 918–920 (1926)