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A damping term for higher-order hyperbolic equations

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Abstract We construct a *damping term* for general higher-order strictly hyperbolic homogeneous equations with constant coefficients. We derive long-time decay estimates for the solution to the Cauchy problem, and we show that no better dissipative effect can be obtained with a different *damping term*.

Keywords Higher-order equations · Damping · Decay estimates

Mathematics Subject Classification 35L30

1 Introduction

We consider a m-th-order homogeneous equation $Lu = 0$, with $m \geq 2$, in the general form

$$L \equiv \partial_t^m + \sum_{1\leq|\alpha|\leq m} b_\alpha\, \partial_t^{m-|\alpha|} \partial_x^\alpha, \tag{1}$$

where $b_\alpha \in \mathbb{R}$. We assume that the operator L is *strictly hyperbolic*, that is, for any $\xi \neq 0$, setting $\xi' = \xi/|\xi|$, its symbol satisfies

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$$P(\lambda, i\xi) \equiv \lambda^m + \sum_{1\le|\alpha|\le m} b_\alpha \lambda^{m-|\alpha|}(i\xi)^\alpha = \prod_{j=1}^m (\lambda - i|\xi|a_j(\xi')), \tag{2}$$

for some $a_j = a_j(\xi')$, real-valued, such that $a_j(\xi') \neq a_k(\xi')$ for any $j \neq k$ and $\xi' \in S^{n-1}$. In this paper, we show that the operator with symbol $cP'(\lambda, i\xi)$, where $c > 0$ is a positive constant, is a *damping term* for the operator Lu. Here and through all the paper, with the notation $P'(\lambda, i\xi)$, we mean that the derivative of the symbol is taken with respect to λ.

Our main result is the following.

Theorem 1 *Let L as* (1)*, be a* strictly hyperbolic *homogeneous operator with symbol $P(\lambda, i\xi)$ as in* (2)*, $c > 0$ be a positive constant, and M the homogeneous operator with symbol $P'(\lambda, i\xi)$. Then the solution to*

$$\begin{cases} (L + cM)u = 0, & t \ge 0, \quad x \in \mathbb{R}^n, \\ \partial_t^j u(0, x) = u_j(x), & j = 0, \dots, m-1, \end{cases} \tag{3}$$

satisfies the following long-time decay estimate:

$$\begin{aligned} \|\partial_x^\alpha \partial_t^k u(t, \cdot)\|_{L^2} \le C \sum_{j=0}^{m-2} (1+t)^{-\frac{n}{4} - \frac{|\alpha|+k-j}{2}} \|u_j\|_{L^1} \\ + C(1+t)^{-\frac{n}{4} - \frac{|\alpha|+k-(m-2)}{2}} \|u_{m-1}\|_{L^1} \\ + C\, e^{-\delta t} \sum_{j=0}^{m-1} \|u_j\|_{H^{|\alpha|+k-j}}, \end{aligned} \tag{4}$$

for any $\alpha \in \mathbb{N}^n$ and $k \ge 0$, such that

$$\frac{n}{2} + |\alpha| + k > m - 2, \tag{5}$$

and for some $C > 0$, $\delta > 0$, which do not depend on the data.

Theorem 1 extends the corresponding well-known result for the damped wave equation

$$u_{tt} - \Delta u + u_t = 0,$$

to general, *strictly hyperbolic*, operators of order $m \ge 2$. Motivated by Theorem 1, we say that cMu is a *damping term* for Lu.

The motivation to study decay estimates for the operator $L + cM$, for some $c > 0$, is based on the fact that its symbol $P(\lambda, i\xi) + cP'(\lambda, i\xi)$ is the first-order approximation of the symbol $P(\lambda + c, i\xi)$, which roots are $ia_j(\xi')|\xi| - c$. In particular, the real parts of these roots are uniformly bounded by a negative constant, as it happens for the roots of the full symbol of the damped Klein–Gordon equation,

$$u_{tt} - \Delta u + u_t + u = 0,$$

which solutions exponentially decay as $t \to \infty$.

The proof of Theorem 1 is based on the properties of the roots of the full symbol $P(\lambda, i\xi) + cP'(\lambda, i\xi)$, which we derive in Sect. 2.

The main novelty of our paper consists in the explicit construction, given any strictly hyperbolic homogeneous operator of order m, of an homogeneous operator M of order $m-1$ which produces a dissipative effect. The choice of M is, in a certain sense, optimal; indeed, we show (see Proposition 2 in Sect. 3) that no homogeneous operator of order $m-1$ can produce a stronger dissipative effect.

In the setting of long-time estimates for higher-order equations with constant coefficients, we address the interested reader to [11], where dispersive and Strichartz estimates are obtained, by assuming suitable hypotheses on the roots of the full symbol of the operator, in particular hypotheses of geometric type.

A huge literature exists for dissipative hyperbolic systems with constant coefficients, under suitable assumptions on the lower-order term and its relations with the first-order term. We address the interested reader to [12], and to [1] and the references therein, being aware that this cannot be an exhaustive list. Recently, dissipative estimates for first-order hyperbolic systems with time-dependent coefficients have been obtained in [15].

Decay estimates for the L^2 norm of the solution to a dissipative problem, obtained by using additional L^1 regularity of the data, are a very useful tool to attack semilinear problems. For the damped wave equation, decay estimates of this type have been used to derive the *sharp* critical exponent for the global existence of small data solutions to problems with power nonlinearity $|u|^p$ (see [5,6,13]) and with nonlinear memory [2]. For the wave equation with time-dependent, *effective* damping term $2c(t)u_t$, decay estimates of this type have been derived and applied to the semilinear problem in [3,4,7,10,14].

Throughout the paper, we shall use the notation $f \lesssim g$ to denote that $\exists\, C>0$ such that $f \leq Cg$. Moreover, for any $\xi \neq 0$, we shall denote $\xi/|\xi|$ by ξ'.

2 Proof of Theorem 1

Definition 1 In the following, by $\lambda_j(c,\xi)$, for $j=1,\dots,m$, $c\geq 0$ and $\xi\in\mathbb{R}^n$, we denote the roots of the full symbol

$$P(\lambda, i\xi) + cP'(\lambda, i\xi) = 0,$$

taken in a such way that λ_j are continuous with respect to c,ξ and that $\lambda_j(0,\xi) = i|\xi|a_j(\xi')$.

When $c>0$ or, respectively, $\xi\neq 0$, is fixed, we will occasionally omit the dependence of $\lambda_j(c,\xi)$ from c or, respectively, from ξ.

Lemma 1 *It holds* $\operatorname{Re}\lambda_j(c,\xi) < 0$, *for any* $j=1,\dots,m$, $c>0$ *and* $\xi\neq 0$.

Proof We fix $\xi\neq 0$. For sufficiently small c, the roots of $P(\lambda,i\xi)+cP'(\lambda,i\xi)$ are distinct. We fix one of them, say $\lambda_j(c)$. Taking the derivative with respect to c of the equation

$$P(\lambda_j(c), i\xi) + cP'(\lambda_j(c), i\xi) = 0,$$

in a neighborhood of $c=0$, we obtain

$$P'(\lambda_j(c), i\xi)\,\lambda_j'(c) + P'(\lambda_j(c), i\xi) + cP''(\lambda_j(c), i\xi)\,\lambda_j'(c) = 0.$$

In particular, at $c=0$ we get $\lambda_j'(0)=-1$. Indeed, $P'(\lambda_j(0), i\xi)\neq 0$, otherwise, $\lambda_j(0)$ would be root of both $P(\lambda,i\xi)$ and $P'(\lambda,i\xi)$, which contradicts the strict hyperbolicity assumption.

Therefore, there exists a maximal interval $(0,\bar{c}(\xi))$ such that $\operatorname{Re}\lambda_j(c,\xi) < 0$ for any $c\in(0,\bar{c}(\xi))$. By contradiction, let us assume that $\bar{c}(\xi)\neq\infty$, that is, $\operatorname{Re}\lambda_j(\bar{c},\xi)=0$.

Denoting $\lambda_j(\bar{c}, \xi) = i\tau_j$, where $\tau_j \in \mathbb{R}$, it follows that

$$0 = P(i\tau_j, i\xi) + cP'(i\tau_j, i\xi) = i^m \prod_{k=1}^{m} (\tau_j - |\xi| a_k(\xi')) + i^{m-1} \sum_{\ell=1}^{m} \prod_{k \neq \ell} (\tau_j - |\xi| a_k(\xi')).$$

One of the last two terms is real, and the other one is purely imaginary, so that both $P(i\tau_j, i\xi) = 0$ and $P'(i\tau_j, i\xi) = 0$, but this contradicts the strict hyperbolicity assumption. □

Remark 1 The strict hyperbolicity is also a necessary condition to have a negative real part for all roots of the full symbol, and for any $\xi \neq 0$. Indeed, if there exists a multiple root $a_j(\xi')$ of $P(\lambda, \xi') = 0$, for some $j = 1, \dots, m$ and $\xi' \in S^{n-1}$, then

$$P(\lambda, i\xi) + cP'(\lambda, i\xi)$$

admits a root $\lambda_j(\xi) = i|\xi| a_j(\xi')$, in particular its real part is identically zero at ξ.

Setting $\xi = 0$, we immediately see that there exists $k \in \{1, \dots, m\}$ such that

$$\lambda_k(c, 0) = -mc, \qquad \lambda_j(c, 0) = 0, \quad \text{for any } j \neq k,$$

due to $P(\lambda, 0) + cP'(\lambda, 0) = \lambda^m + cm\lambda^{m-1}$. For a fixed $c > 0$, as $|\xi| \to 0$, we need to estimate $\operatorname{Re} \lambda_j(\xi)$, for any $j \neq k$. We have the following.

Lemma 2 *Let $\lambda_j(\xi)$ be a root of the full symbol $P(\lambda, i\xi) + cP'(\lambda, i\xi) = 0$, satisfying $\lambda_j(0) = 0$. Then,*

$$|\operatorname{Im} \lambda_j(\xi)| \leq K_1 |\xi|, \qquad -\frac{K_2}{c} |\xi|^2 \leq \operatorname{Re} \lambda_j(\xi) \leq -\frac{K_3}{c} |\xi|^2, \tag{6}$$

in a neighborhood of $\xi = 0$, for some $K_1 > 0$ and $K_2 > K_3 > 0$, independent on c. Moreover, if $\lambda_\ell(\xi)$ is another root of the full symbol $P(\lambda, i\xi) + cP'(\lambda, i\xi) = 0$, i.e., $\ell \neq j$, satisfying $\lambda_\ell(0) = 0$, then

$$|\operatorname{Im} \lambda_j(\xi) - \operatorname{Im} \lambda_\ell(\xi)| \geq K_4 |\xi|, \tag{7}$$

in a neighborhood of $\xi = 0$, for some $K_4 > 0$, independent on c.

Before proving Lemma 2, for the sake of brevity, we introduce the following.

Notation 1 *We define*

$$b_j(\xi') = \sum_{|\alpha| = j} b_\alpha \, \xi'^\alpha, \qquad j = 1, \dots, m,$$

so that

$$P(\lambda, i\xi) = \lambda^m + \sum_{j=1}^{m} b_j(\xi') \, \lambda^{m-j} (i|\xi|)^j.$$

To prove Lemma 2, we need the following preliminary result.

Lemma 3 *Let P and P_1 be two homogeneous polynomials in normal form, respectively, of order m and $m-1$, namely,*

$$P(\lambda, i\xi) = \sum_{j=0}^{m} b_j(\xi') \lambda^{m-j} (i|\xi|)^j, \qquad P_1(\lambda, i\xi) = \sum_{j=0}^{m-1} d_j(\xi') \lambda^{m-1-j} (i|\xi|)^j,$$

with $b_0, d_0 \in \mathbb{R} \setminus \{0\}$. *Then, there exists one root* $\lambda_k(\xi)$ *of the full symbol* $P(\lambda, i\xi) + P_1(\lambda, i\xi) = 0$ *such that* $\lambda_k(0) = -d_0/b_0$, *whereas all the other roots satisfy* $|\lambda_j(\xi)| \lesssim |\xi|$, *for* $j \neq k$.

It is clear that it is sufficient to prove the statement of Lemma 3 in a neighborhood of $\xi = 0$, since $|\lambda_h(\xi)| \lesssim 1 + |\xi|$ for any $h = 1, \dots, m$.

Proof Being $P(\lambda, 0) + P_1(\lambda, 0) = (b_0\lambda + d_0)\lambda^{m-1}$, the first part of the statement is clear. We may write

$$R(\lambda, \xi) := \frac{P(\lambda, i\xi) + P_1(\lambda, i\xi)}{\lambda - \lambda_k(\xi)} = \sum_{\ell=0}^{m-1} g_\ell(\xi)\lambda^{m-1}(i|\xi|)^\ell,$$

which is a polynomial with respect to the variable λ, with coefficients $g_\ell(\xi)(i|\xi|)^\ell$, $\ell = 0, \dots, m-1$. In particular, $g_0 = b_0$. We claim that $g_\ell(\xi)$ are bounded functions of ξ, in a neighborhood of $\xi = 0$. In turn, it follows that the roots of $R(\lambda, \xi)$ are bounded by $C|\xi|$, for some $C > 0$.

To prove our claim, we consider $P + P_1$ as a polynomial with respect to the variable λ. Therefore, from

$$\begin{aligned}
&b_0\lambda^m + \sum_{j=1}^{m} \lambda^{m-j}(b_j(\xi')i|\xi| + d_{j-1}(\xi'))(i|\xi|)^{j-1} \\
&\quad = P(\lambda, i\xi) + P_1(\lambda, i\xi) = (\lambda - \lambda_k(\xi))R(\lambda, \xi) \\
&\quad = b_0\lambda^m + \sum_{\ell=1}^{m-1} \lambda^{m-\ell}(g_\ell(\xi)i|\xi| - \lambda_k(\xi)g_{\ell-1}(\xi))(i|\xi|)^{\ell-1} \\
&\qquad - \lambda_k(\xi)g_{m-1}(\xi)(i|\xi|)^{m-1},
\end{aligned}$$

we derive

$$\begin{aligned}
g_\ell(\xi)i|\xi| - \lambda_k(\xi)g_{\ell-1}(\xi) &= b_\ell(\xi')i|\xi| + d_{\ell-1}(\xi'), \qquad \ell = 1, \dots, m-2 \\
-\lambda_k(\xi)g_{m-1}(\xi) &= b_m(\xi')i|\xi| + d_{m-1}(\xi').
\end{aligned}$$

We may now prove our claim by finite induction. There exists a neighborhood of $\xi = 0$ such that $|\lambda_k(\xi)| \geq |d_0|/(2|b_0|)$. Then, we may estimate

$$|g_{m-1}(\xi)| = \frac{|b_m(\xi')i|\xi| + d_{m-1}(\xi')|}{|\lambda_k(\xi)|} \leq \frac{2|b_0|}{|d_0|}|b_m(\xi')i|\xi| + d_{m-1}(\xi')| \leq C,$$

for some $C > 0$. Let us assume that $g_{m-h}(\xi)$ is bounded for some $h \in \{1, \dots, m-2\}$. Then

$$|g_{m-(h+1)}(\xi)| = \frac{|b_{m-h}(\xi')i|\xi| + d_{m-h-1}(\xi') - g_{m-h}(\xi)i|\xi||}{|\lambda_k(\xi)|} \leq C',$$

as well. □

Proof (Lemma 2) Let us fix $\xi' \in S^{n-1}$, and let $\xi = \xi'\rho$, for $\rho > 0$. For any $\rho \in (0, \varepsilon)$, we define $\eta_j := \lambda_j/(i\rho)$. Then $|\eta_j| \leq C$, by virtue of Lemma 3. We may write the polynomials P and P' in the form

$$P(\lambda, i\xi) = (i\rho)^m \sum_{j=0}^{m} b_j(\xi')\eta^{m-j} \equiv (i\rho)^m Q_0(\eta), \tag{8}$$

$$P'(\lambda, i\xi) = (i\rho)^{m-1} \sum_{j=0}^{m-1} (m-j) b_j(\xi') \eta^{m-1-j} \equiv (i\rho)^{m-1} Q_1(\eta). \tag{9}$$

Being $|\eta_j| \leq C$, also $Q_0(\eta_j)$ and $Q_1(\eta_j)$ are bounded, with respect to ρ. Moreover, $Q_0(\eta_j) \neq 0$ and $Q_1(\eta_j) \neq 0$, since Q_0 and Q_1 have distinct roots. Now, we consider

$$i\rho Q_0(\eta_j) + c Q_1(\eta_j) = 0.$$

Due to

$$|Q_1(\eta_j)| = \frac{\rho}{c} |Q_0(\eta_j)|,$$

it follows $Q_1(\eta_j) \to 0$, as $\rho \to 0$, that is, η_j tends to a root $\bar{\eta}$ of Q_1. We recall that $\bar{\eta} \in \mathbb{R}$, being Q_1 hyperbolic. We may write

$$Q_1(\eta_j) = (\eta_j - \bar{\eta}) \tilde{Q}_1(\eta_j),$$

therefore,

$$\eta_j - \bar{\eta} = -i \frac{\rho}{c} \frac{Q_0(\eta_j)}{\tilde{Q}_1(\eta_j)};$$

that is,

$$\lambda_j = i\rho\eta_j = i\rho\bar{\eta} + \frac{\rho^2}{c} \frac{Q_0(\eta_j)}{\tilde{Q}_1(\eta_j)}.$$

We notice that $\bar{\eta}$ may be zero, in general, unless $b_{m-1}(\xi') \neq 0$, but, recalling that $Q_0(\bar{\eta}) \neq 0$ and $\tilde{Q}_1(\bar{\eta}) \neq 0$, due to the strict hyperbolicity assumption, we get

$$\operatorname{Re} \left(\frac{Q_0(\eta_j)}{\tilde{Q}_1(\eta_j)} \right) \neq 0,$$

for sufficiently small ρ, thanks to $\eta_j \to \bar{\eta} \in \mathbb{R}$. By the compactness of S^{n-1}, it follows that

$$|\operatorname{Im} \lambda_j(\xi)| \leq K_1 |\xi|, \qquad \frac{K_3}{c} |\xi|^2 \leq |\operatorname{Re} \lambda_j(\xi)| \leq \frac{K_2}{c} |\xi|^2;$$

recalling that $\operatorname{Re} \lambda_j < 0$, this concludes the proof of (6). Now, we prove (7). We define $\eta_\ell = \lambda_\ell/(i\rho)$ as we did for η_j. As $\rho \to 0$, η_j and η_ℓ tend to two different roots $\bar{\eta}_j$, $\bar{\eta}_\ell$ of Q_1. Being Q_1 a strictly hyperbolic polynomial, it follows, from the previous representation, that

$$|\operatorname{Im} \lambda_j - \operatorname{Im} \lambda_\ell| = \rho |\operatorname{Re} \eta_j - \operatorname{Re} \eta_\ell| = \rho |\bar{\eta}_j - \bar{\eta}_\ell| + O(\rho^2) \geq C'\rho.$$

Using the compactness of S^{n-1}, we conclude the proof of (7). □

Finally, we need to estimate the behavior of $\lambda_j(\xi)$ as $|\xi| \to \infty$. We have the following.

Lemma 4 *The roots of the full symbol $P(\lambda, i\xi) + cP'(\lambda, i\xi) = 0$ satisfy*

$$\lim_{|\xi| \to \infty} \operatorname{Re} \lambda_j(\xi) = -c, \quad \lim_{|\xi| \to \infty} \left(|\xi|^{-1} \operatorname{Im} \lambda_j(\xi) - a_j(\xi') \right) = 0, \quad j = 1, \ldots, m. \tag{10}$$

Proof As in the proof of Lemma 2, we fix $\xi' \in S^{n-1}$ and we set $\xi = \xi'\rho$, where $\rho > 0$. For any $\rho \geq M > 1$, we define again $\eta_j := \lambda_j/(i\rho)$. Clearly, $|\eta_j| \leq C$. We write again the polynomials P and P' in the form (8)–(9) and we consider

$$Q_0(\eta_j) - i\,\frac{c}{\rho}\,Q_1(\eta_j) = 0.$$

As $\rho \to \infty$, $Q_0(\eta_j) \to 0$, that is, $\eta_j \to a_j(\xi')$. Let $Q_0(\eta_j) = (\eta_j - a_j)\tilde{Q}_0(\eta_j)$. We obtain

$$\eta - a_j = i\,\frac{c}{\rho}\,\frac{Q_1(\eta_j)}{\tilde{Q}_0(\eta_j)}\,,$$

so that

$$\lambda_j = i\rho\eta_j = i\rho a_j(\xi') - c\,\frac{Q_1(\eta_j)}{\tilde{Q}_0(\eta_j)}\,. \tag{11}$$

Multiplying (11) by ρ^{-1}, we immediately obtain the second part of (10), for $\rho \to \infty$. On the other hand, we see that $Q_1(a_j) = \tilde{Q}_0(a_j)$, since

$$\tilde{Q}_0(\eta) = \prod_{\ell \neq j}(\eta - a_\ell), \qquad Q_1(\eta) = \tilde{Q}_0(\eta) + (\eta - a_j)\sum_{\ell \neq j}\prod_{\kappa \neq \ell, j}(\eta - a_\kappa),$$

so that the first part of (10) follows from (11) as $\rho \to \infty$. □

In particular, from Lemmas 1 and 4, it follows that

$$\forall\,\varepsilon > 0\ \exists\,c_\varepsilon > 0\ :\ \mathrm{Re}\,\lambda_j(\xi) \leq -c_\varepsilon \quad \forall\,\xi : |\xi| \geq \varepsilon\,. \tag{12}$$

We are now ready to prove Theorem 1; from now on, we shall denote by $\hat{u}(t, \xi)$ the (partial) Fourier transform of $u(t, x)$ with respect to the x variable.

Proof (Theorem 1) Assume first that all $\lambda_j(\xi)$ are distinct for $\xi \neq 0$. After performing the Fourier transform of the equation in (3), we may write

$$\hat{u}(t, \xi) = \sum_{j=1}^{m} e^{\lambda_j(\xi)t}\,\Delta_j(\xi)\sum_{h=0}^{m-1}\sigma_{m-1-h,j}(\xi)\,\hat{u_h}(\xi),$$

where

$$\Delta_j(\xi) = \prod_{k \neq j}\frac{1}{\lambda_j(\xi) - \lambda_k(\xi)}$$

and

$$\sigma_{0,j} = 1, \quad \sigma_{1,j} = -\sum_{k \neq j}\lambda_k, \qquad \sigma_{2,j} = \sum_{k<l}\lambda_k\lambda_l + \lambda_j\sigma_{1,j},$$
$$\sigma_{3,j} = -\sum_{k<l<p}\lambda_k\lambda_l\lambda_p + \lambda_j\sigma_{2,j}, \ldots$$

By Plancherel's theorem, we want to estimate the L^2 norm of

$$|\xi|^{|\alpha|}\partial_t^k\hat{u}(t, \xi).$$

Let $\varepsilon > 0$. By (12) we know that $\operatorname{Re}\lambda_j \leq -c_\varepsilon$ for any $\xi : |\xi| > \varepsilon$ and $j = 1, \ldots, m$. Moreover, for large $|\xi|$, taking into account Lemma 4 and the assumption of strict hyperbolicity, we get $|\Delta_j(\xi)| \leq |\xi|^{-(m-1)}$ and $|\sigma_{m-1-h,j}(\xi)| \leq |\xi|^{m-1-h}$. We also remark that

$$|\partial_t^k \hat{u}(t,\xi)| \lesssim |\xi|^k \sum_{j=1}^m e^{\operatorname{Re}\lambda_j(\xi)t} |\Delta_j(\xi)| \sum_{h=0}^{m-1} |\sigma_{m-1-h,j}(\xi)| \, |\hat{u_h}(\xi)|,$$

for large $|\xi|$, being $|\lambda_j(\xi)| \lesssim |\xi|$. Therefore,

$$\sup_{|\xi|>\varepsilon} |\xi|^{|\alpha|} |\partial_t^k \hat{u}(t,\xi)| \lesssim e^{-\delta t} \, |\xi|^{|\alpha|+k} \sum_{h=0}^{m-1} |\xi|^{-h} \, |\hat{u}_h(\xi)|,$$

for some $\delta > 0$. By applying Plancherel's theorem on the initial data, it immediately follows that

$$\|\hat{u}(t,\cdot)\|_{L^2(|\xi|\geq\varepsilon)} \lesssim e^{-\delta t} \sum_{h=0}^{m-1} \|u_h\|_{H^{|\alpha|+k-h}}.$$

On the other hand, for sufficiently small $\varepsilon > 0$, for any $\xi : |\xi| \leq \varepsilon$, we distinguish two cases:

- if $\lambda_j(0) = -mc$, then $|\Delta_j(\xi)| \leq C$ and $|\sigma_{m-1-h,j}(\xi)| \lesssim |\xi|^{m-1-h}$, thanks to (6);
- if $\lambda_j(0) = 0$, then $|\Delta_j(\xi)| \lesssim |\xi|^{-(m-2)}$, thanks to (7), and $|\sigma_{m-1-h,j}(\xi)| \lesssim |\xi|^{m-2-h}$, for any $h \leq m-2$, thanks to (6), whereas we recall that $\sigma_{0,j} = 1$. We also remark that $|\lambda_j(\xi)| \lesssim |\xi|$ thanks to (6).

Our plan is to estimate

$$\begin{aligned}
&\left\| |\xi|^{|\alpha|} (\lambda_j(\xi))^k \, e^{\lambda_j(\xi)t} \, \Delta_j(\xi) \sum_{h=0}^{m-1} \sigma_{m-1-h,j}(\xi) \, \hat{u_h}(\xi) \right\|_{L^2(|\xi|\leq\varepsilon)} \\
&\quad \leq \sum_{h=0}^{m-1} \|K_{j,h}(\xi) \, e^{\lambda_j(\xi)t}\|_{L^2(|\xi|\leq\varepsilon)} \, \|\hat{u_h}\|_{L^\infty},
\end{aligned}$$

where we set

$$K_{j,h}(\xi) := |\xi|^{|\alpha|} \, (\lambda_j(\xi))^k \, \Delta_j(\xi) \, \sigma_{m-1-h,j}(\xi).$$

We remark that $\|\hat{u_h}\|_{L^\infty} \lesssim \|u_h\|_{L^1}$. For j such that $\lambda_j(0) = -mc$, we immediately obtain

$$|K_{j,h}(\xi)| \lesssim |\xi|^{|\alpha|+m-1-h},$$

which clearly is in $L^2(|\xi| \leq \varepsilon)$; hence,

$$\|K_{j,h}(\xi) \, e^{\lambda_j(\xi)t}\|_{L^2(|\xi|\leq\varepsilon)} \lesssim e^{-\delta t},$$

for some $\delta > 0$. Now, let j be such that $\lambda_j(0) = 0$. In this case, we obtain

$$|K_{j,h}(\xi)| \lesssim \begin{cases} |\xi|^{|\alpha|+k-h} & \text{if } \; h = 0, \ldots, m-2, \\ |\xi|^{|\alpha|+k-(m-2)} & \text{if } \; h = m-1. \end{cases} \tag{13}$$

In particular, $K_{j,h}$ are in $L^2(|\xi| \leq \varepsilon)$ for any $h = 0, \ldots, m-1$, thanks to (5) [incidentally, we remark that condition (5) may be relaxed if $u_{m-1} = u_{m-2} = \cdots = u_{m-\ell} = 0$ for some $\ell \geq 2$ in (3)].

If $t \le 1$, we simply estimate the $L^2(|\xi| \le \varepsilon)$ norm of $K_{j,h}e^{\lambda_j t}$ by a constant. Let $t \ge 1$. By the change of variable $\theta = \sqrt{t}\xi$, we immediately derive

$$\int_{|\xi|\le\varepsilon} |\xi|^{2(|\alpha|+k-h)} e^{-2\frac{K_2}{c}|\xi|^2 t}\, d\xi \lesssim t^{-\frac{n}{2}-|\alpha|-k+h} \int_{\mathbb{R}^n} |\theta|^{2(|\alpha|+k-h)} e^{-2\frac{K_2}{c}|\theta|^2}\, d\theta.$$

Since the last integral is bounded, we obtain

$$\begin{aligned} \|K_{j,h}e^{\lambda_j t}\|_{L^2(|\xi|\le\varepsilon)} &\lesssim (1+t)^{-\frac{n}{4}-\frac{|\alpha|+k-h}{2}}, \qquad h = 0,\dots,m-2\,; \\ \|K_{j,m-1}e^{\lambda_j t}\|_{L^2(|\xi|\le\varepsilon)} &\lesssim (1+t)^{-\frac{n}{4}-\frac{|\alpha|+k-(m-2)}{2}}. \end{aligned}$$

By gluing the estimates obtained for $\|\hat{u}(t,\cdot)\|_{L^2(|\xi|\ge\varepsilon)}$ and $\|\hat{u}(t,\cdot)\|_{L^2(|\xi|\le\varepsilon)}$, we conclude the proof of (4).

We now remove the assumption that $\lambda_j(\xi)$ are distinct for any $\xi \neq 0$. By Lemmas 2 and 4, we know that all possible zeros of the discriminant are contained in a compact subset of $\mathbb{R}^n \setminus \{|\xi| \le \varepsilon\}$. In any bounded neighborhood $V \subset \mathbb{R}^n \setminus \{|\xi| \le \varepsilon\}$ of a zero of the discriminant, it still holds true

$$\||\xi|^{|\alpha|}\partial_t^k \hat{u}(t,\xi)\|_{L^2(V)} \le C t^{C_1} e^{-c_\varepsilon t} \sum_{h=0}^{m-1} \|\hat{u}_h(\xi)\|_{L^2(V)}$$

for sufficiently large $C, C_1 > 0$, uniformly for all zeros; hence, estimate (4) remains true. □

3 Additional remarks

Remark 2 In the proof of Theorem 1, the L^1 regularity of the initial data only came into play at low frequencies, whereas the $H^{|\alpha|+k-j}$ regularity of u_j, $j = 0,\dots,m-1$, only came into play at high frequencies. More precisely, estimate (4) may be replaced by

$$\begin{aligned} \|\partial_x^\alpha \partial_t^k u(t,\cdot)\|_{L^2} \le C &\sum_{j=0}^{m-2} (1+t)^{-\frac{n}{4}-\frac{|\alpha|+k-j}{2}} \|\hat{u}_j\|_{L^\infty(|\xi|\le\varepsilon)} \\ &+ C(1+t)^{-\frac{n}{4}-\frac{|\alpha|+k-(m-2)}{2}} \|\hat{u}_{m-1}\|_{L^\infty(|\xi|\le\varepsilon)} \\ &+ C\, e^{-\delta t} \sum_{j=0}^{m-1} \||\xi|^{|\alpha|+k-j}\hat{u}_j\|_{L^2(|\xi|>\varepsilon)}, \end{aligned} \tag{14}$$

for some $\varepsilon > 0$, $C > 0$ and $\delta > 0$, which do not depend on the data.

Remark 3 The constants C and δ in Theorem 1 depend on the positive constant $c > 0$ in (3), and in particular, they cannot be uniformly fixed for any $c \in (0,\infty)$. However, it is easy to see that estimate (4) may be replaced by

$$\begin{aligned} \|\partial_x^\alpha \partial_t^k u(t,\cdot)\|_{L^2} \le C_1 &\sum_{j=0}^{m-2} (c^{-2}+c^{-1}t)^{-\frac{n}{4}-\frac{|\alpha|+k-j}{2}} \|u_j\|_{L^1} \\ &+ C_1(c^{-2}+c^{-1}t)^{-\frac{n}{4}-\frac{|\alpha|+k-(m-2)}{2}} \|u_{m-1}\|_{L^1} \end{aligned}$$

$$+ C_1\, e^{-c\delta_1 t} \sum_{j=0}^{m-1} \|u_j\|_{H^{|\alpha|+k-j}}, \tag{15}$$

where $C_1, \delta_1 > 0$ do not depend on the positive constant $c > 0$, neither on the data.

In order to prove (15), let $\tilde{u}$ be a solution to $L\tilde{u} + M\tilde{u} = 0$. Then, for any $c > 0$, $u(t,x) = \tilde{u}(ct, cx)$ solves $Lu + cMu = 0$. In particular, it solves (3) with initial data $u_j(x) = c^j \partial_t^j \tilde{u}(0, cx)$, for $j = 0, \ldots, m-1$.

Applying Theorem 1 to $\tilde{u}$ and performing the change of variables into (14), straightforward calculations lead to (15).

It is easy to see that the decay terms $C_1(c^{-2}+c^{-1}t)^{-\frac{n}{4}-\frac{|\alpha|+k-j}{2}}$ in (15) blows up as $c \to \infty$, whereas the decay term $C_1 e^{-c\delta_1 t}$ reduces to the constant C_1 at $c = 0$. The first case hints to some kind of *overdamping* phenomenon: if the damping coefficient is larger, the decay estimate becomes worse. On the other hand, the second case is related to the fact that removing the damping term we may no longer expect decay of the energy, in general.

Remark 4 If we drop the assumption of additional L^1 regularity for the data, we may easily modify the proof of Theorem 1, getting the following

Proposition 1 *Let L as* (1)*, be a* strictly hyperbolic *operator with symbol $P(\lambda, i\xi)$ as in* (2)*, $c > 0$ be a positive constant, and M the operator with symbol $P'(\lambda, i\xi)$. Then, the solution to* (3) *satisfies the following long-time decay estimate:*

$$\|\partial_x^\alpha \partial_t^k u(t,\cdot)\|_{L^2} \leq C \sum_{j=0}^{m-2} (1+t)^{-\frac{|\alpha|+k-j}{2}} \|u_j\|_{H^{|\alpha|+k-j}} + C(1+t)^{-\frac{|\alpha|+k-(m-2)}{2}} \|u_{m-1}\|_{L^2}$$

for any $\alpha \in \mathbb{N}^n$ and $k \geq 0$, such that $|\alpha| + k \geq m-2$.

In particular, taking $|\alpha| + k = m-1$ in Proposition 1, we may derive the following decay estimate for the $(m-1)$-th-order energy of the solution to (3):

$$\sum_{k=0}^{m-1} \|\partial_t^k u(t,\cdot)\|_{H^{m-1-k}}^2 \leq C\,(1+t)^{-1} \sum_{k=0}^{m-1} \|u_k\|_{H^{m-1-k}}^2.$$

Remark 5 To prove that the choice of M in Theorem 1 is optimal, we show that a different homogeneous operator of order $m-1$ may not produce a stronger dissipative effect. Namely, we have the following.

Proposition 2 *Let $P(\lambda, i\xi)$ be a m-th-order homogeneous strictly hyperbolic symbol as in* (2)*, and let*

$$P_1(\lambda, i\xi) = \sum_{j=0}^{m-1} d_j(\xi')\lambda^{m-1-j}(i|\xi|)^j$$

be a $(m-1)$-th-order homogeneous symbol with $d_0 > 0$ and $d_j(\xi') \in \mathbb{R}$, $j = 1, \ldots, m-1$. Then, the following properties hold:

(a) If P_1 is a hyperbolic polynomial whose roots are at most double, and all the double roots are (simple) roots of P, then all roots $\lambda_j(\xi)$ of $P(\lambda, i\xi) + P'(\lambda, i\xi) = 0$, but one, satisfy

$$|\mathrm{Re}\,\lambda_j(\xi)| \lesssim |\xi|^2$$

in a neighborhood of $\xi = 0$*, whereas the other one verifies* $\lambda_j(0) = -d_0$.

(b) *If* P_1 *does not fall in the previous case, then* $P(\lambda, i\xi) + P_1(\lambda, i\xi)$ *has, for some* $\xi \neq 0$*, at least one root with strictly positive real part.*

Proof We fix $\xi' \in S^{n-1}$ and for any $\rho > 0$, we set $\xi = \rho\xi'$.

First, let $a_j(\xi')$ be a (real) root of $P(\lambda, \xi') = 0$, which is also a root of $P_1(\lambda, \xi') = 0$. Then, $\lambda_j(\xi) = ia_j(\xi')\rho$ is a purely imaginary root of $P(\lambda, i\xi) + P_1(\lambda, i\xi) = 0$. For any root $\lambda_j(\xi)$ with this property, we may divide both $P(\lambda, i\xi)$ and $P_1(\lambda, i\xi)$ by $\lambda - \lambda_j(\xi)$; hence, it is not restrictive to assume that $P(\lambda, \xi') = 0$ and $P_1(\lambda, \xi') = 0$ have no common roots. Therefore, the two cases (a)–(b) reduce, respectively, to:

(i) If $P_1(\lambda, \xi')$ is a strictly hyperbolic polynomial whose roots are all different from the roots of $P(\lambda, \xi')$, then all roots $\lambda_j(\xi)$ of $P(\lambda, i\xi) + P'(\lambda, i\xi) = 0$, but one, satisfy

$$|\mathrm{Re}\,\lambda_j(\xi)| \lesssim |\xi|^2$$

in a neighborhood of $\xi = 0$, whereas the other one verifies $\lambda_j(0) = -d_0$.

(ii) Let $P_1(\lambda, \xi')$ be a polynomial whose roots are all different from the roots of $P(\lambda, \xi')$. Moreover, let us assume that at least one root of $P_1(\lambda, \xi') = 0$ is multiple, if all its roots are real-valued. Then, $P(\lambda, i\xi) + P_1(\lambda, i\xi)$ has, for some $\xi \neq 0$, at least one root with strictly positive real part.

By virtue of Lemma 3, there exists one root $\lambda_k(\xi)$, of $P(\lambda, i\xi) + P_1(\lambda, i\xi) = 0$, satisfying $\lambda_k(0) = -d_0$, whereas $|\lambda_j(\xi)| \lesssim |\xi|$, for any $j \neq k$. Recalling that we fixed $\xi' \in S^{n-1}$, we define $\eta_j := \lambda_j/(i\rho)$, which satisfy $|\eta_j| \leq C\varepsilon$, for $j \neq k$, as we did in the proof of Lemma 2. We define $Q_\beta(\eta) = (i\rho)^{-(m-\beta)}P_\beta(\eta, \xi')$, $\beta = 0, 1$, as in (8)–(9), and we consider the equation

$$i\rho Q_0(\eta_j) + cQ_1(\eta_j) = 0.$$

As $\rho \to 0$, if follows $Q_1(\eta_j) \to 0$, that is, η_j tends to a root $\bar{\eta}$ of Q_1.

We first consider case (i). Then, Q_1 has $m - 1$ distinct, real roots, which are not roots of Q_0. Then, we may closely follow the proof of Lemma 2, obtaining, in particular

$$|\mathrm{Re}\,\lambda_j(\rho)| \lesssim \rho^2, \qquad \text{for any } j \neq k,$$

in a neighborhood of $\rho = 0$.

When we consider case (ii), we distinguish two cases.

First, let assume that Q_1 has real-valued roots and that $\bar{\eta}$ is a root of Q_1 with multiplicity $\ell \geq 2$. Then, there exists a set $I_\ell \subset \{1, \ldots, m\}$, with ℓ indexes, such that $\eta_j \to \bar{\eta}$ for any $j \in I_\ell$, as $\rho \to 0$. We may write

$$i\rho Q_0 + (\eta_j - \bar{\eta})^\ell \check{Q}_1 = 0, \qquad \text{for any } j \in I_\ell,$$

where $\check{Q}_1 = Q_1/(\eta_j - \bar{\eta})^\ell$ is a polynomial of order $m - 1 - \ell$ such that $\check{Q}_1(\bar{\eta}) \neq 0$. Then, in a neighborhood of $\rho = 0$, it holds $Q_0(\eta_j), \check{Q}_1(\eta_j) \neq 0$ and we may write

$$\eta_j = \bar{\eta} + \phi_j \rho^{\frac{1}{\ell}}, \qquad \text{for any } j \in I_\ell,$$

where ϕ_j is a ℓ-th-order complex-valued root of $-iQ_0(\eta_j)/\check{Q}_1(\eta_j)$. In particular,

$$\mathrm{Im}\,\eta_j = \mathrm{Im}\,\phi_j \rho^{\frac{1}{\ell}},$$

since $\bar{\eta} \in \mathbb{R}$. Due to $-i Q_0(\bar{\eta})/\check{Q}_1(\bar{\eta}) \in i\mathbb{R} \setminus \{0\}$, it follows that

$$\operatorname{Im} \phi_j < 0, \qquad \text{for some } j \in I_\ell,$$

in a neighborhood of $\rho = 0$. Therefore,

$$\operatorname{Re} \lambda_j = -\operatorname{Im} \phi_j \rho^{1+\frac{1}{\ell}} > 0,$$

for some $j \in I_\ell$ and for some $\xi \neq 0$.

Finally, let us assume that Q_1 has a nonreal-valued root. Since all the roots are conjugated, we may find a root $\bar{\eta}$ satisfying $\operatorname{Im} \bar{\eta} < 0$. It follows that

$$\operatorname{Im} \eta_j \to \operatorname{Im} \bar{\eta} > 0,$$

for some j, in particular, $\operatorname{Re} \lambda_j(\xi) > 0$ for some $\xi \neq 0$. □

Remark 6 It is well known that the solution to the Cauchy problem for the damped wave equation

$$\begin{cases} u_{tt} - \Delta u + 2cu_t = 0 \qquad t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0(x) \\ u_t(0, x) = u_1(x) \end{cases} \tag{16}$$

with initial data in the energy space and with additional L^1 regularity, satisfies the following decay estimates (see [8], Lemma 1):

$$\begin{aligned} \|\partial_x^\alpha \partial_t^k u(t, \cdot)\|_{L^2} \leq C(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}-k} \big(\|u_0\|_{L^1} + \|u_1\|_{L^1}\big) \\ + Ce^{-\delta t} \big(\|u_0\|_{H^{|\alpha|+k}} + \|u_1\|_{H^{|\alpha|+k-1}}\big), \end{aligned} \tag{17}$$

where $C > 0$ and $\delta > 0$ do not depend on the data.

An important difference between (4) and (17) is that each time derivative of u only brings an additional $(1+t)^{-\frac{1}{2}}$ decay rate in (4), whereas it brings an additional $(1+t)^{-1}$ decay rate in (17). This difference is related to the fact that oscillations are excluded at low frequencies for the solution to (16), i.e., the roots of the full symbol of the damped wave equation are real-valued for small values of $|\xi|$. This property cannot hold for equations of the form $(L + cM)u = 0$ when $m \geq 3$, due to (7) in Lemma 2, namely for any $\xi' \in S^{n-1}$ there exists at least one root with nonzero imaginary part for small values of $|\xi|$.

However, if we only consider second-order equation, and we assume that $b_1(\xi') = 0$ for any $\xi' \in S^{n-1}$, then we may improve estimate (4).

Proposition 3 *Let*

$$Lu \equiv u_{tt} + \sum_{|\alpha|=2} b_\alpha \partial_x^\alpha u,$$

be a strictly hyperbolic operator, i.e.,

$$b_2(\xi') := \sum_{|\alpha|=2} b_\alpha (\xi')^\alpha < 0,$$

for any $\xi' \in S^{n-1}$. Then, for any $c > 0$, the solution to

$$\begin{cases} Lu + 2cu_t = 0, \qquad t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0(x) \\ u_t(0, x) = u_1(x) \end{cases} \tag{18}$$

satisfies the decay estimate (17).

The result in Proposition 3 covers the classical result for the damped wave equation, obtained for $b_2(\xi')=-1$.

Proof It is sufficient to follow the proof of Theorem 1, having in mind that the roots $\lambda_\pm(c,\xi)$ of the full symbol are real-valued for small values of $|\xi|$, namely,

$$\lambda_\pm(\xi)=\begin{cases}-c\pm\sqrt{c^2+b_2(\xi')|\xi|^2} & \text{if } c^2\geq -b_2(\xi')|\xi|^2,\\ -c\pm i\sqrt{b_2(\xi')|\xi|^2-c^2} & \text{if } c^2\leq -b_2(\xi')|\xi|^2.\end{cases} \tag{19}$$

In particular, $|\lambda_+(\xi)|\lesssim -b_2(\xi')|\xi|^2/c$, in a neighborhood of $\xi=0$, therefore, (13) is replaced by

$$|K_{j,h}(\xi)|\lesssim |\xi|^{|\alpha|+2k}. \tag{20}$$

With this modification, estimate (17) immediately follows. □

Remark 7 The asymptotic profile of the solution to (18) is the same of the solution to the Cauchy problem for the heat-type equation

$$\begin{cases} v_t+\frac{1}{2c}\sum_{|\alpha|=2}b_\alpha\partial_x^\alpha v=0, & t\geq 0,\quad x\in\mathbb{R}^n,\\ v(0,x)=v_0(x)\end{cases} \tag{21}$$

where $v_0=u_0+(2c)^{-1}u_1$ (see, for instance, [9]). More precisely, we get extra decay rate $(1+t)^{-1}$ for $u-v$, with respect to the corresponding estimate for u and v, i.e., we have the following:

$$\begin{aligned}\|\partial_x^\alpha\partial_t^k\big(u(t,\cdot)-v(t,\cdot)\big)\|_{L^2}\leq{}& C(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}-k-1}\big(\|u_0\|_{L^1}+\|u_1\|_{L^1}\big)\\ &+Ce^{-\delta t}\big(\|u_0\|_{H^{|\alpha|+k}}+\|u_1\|_{H^{|\alpha|+k-1}}\big).\end{aligned}$$

This effect is called *diffusion phenomenon*, and it depends on the fact that the roots of the full symbol are real-valued at low frequencies.

References

1. Beauchard, K., Zuazua, E.: Large time asymptotics for partially dissipative hyperbolic systems. Arch. Ration. Mech. Anal. **199**(1), 177–227 (2011). doi:10.1007/s00205-010-0321-y
2. D'Abbicco, M.: The influence of a nonlinear memory on the damped wave equation. Nonlinear Anal. **95**, 130–145 (2014). doi:10.1016/j.na.2013.09.006
3. D'Abbicco, M.: The threshold of effective damping for semilinear wave equations. Math. Methods Appl. Sci. doi:10.1002/mma.3126
4. D'Abbicco, M., Lucente, S., Reissig, M.: Semilinear wave equations with effective damping. Chin. Ann. Math. **34B**(3), 345–380 (2013). doi:10.1007/s11401-013-0773-0
5. Ikehata, R., Mayaoka, Y., Nakatake, T.: Decay estimates of solutions for dissipative wave equations in $\mathbb{R}^N$ with lower power nonlinearities. J. Math. Soc. Jpn. **56**(2), 365–373 (2004)
6. Ikehata, R., Tanizawa, K.: Global existence of solutions for semilinear damped wave equations in R^N with noncompactly supported initial data. Nonlinear Anal. **61**, 1189–1208 (2005)
7. Lin, J., Nishihara, K., Zhai, J.: Critical exponent for the semilinear wave equation with time-dependent damping. Discrete Contin. Dyn. Syst. **32**(12), 4307–4320 (2012). doi:10.3934/dcds.2012.32.4307
8. Matsumura, A.: On the asymptotic behavior of solutions of semi-linear wave equations. Publ. RIMS. **12**, 169–189 (1976)
9. Nishihara, K.: Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping. J. Differ. Equ. **137**(2), 384–395 (1997). doi:10.1006/jdeq.1997.3268

10. Nishihara, K.: Asymptotic behavior of solutions to the semilinear wave equation with time-dependent damping. Tokyo J. Math. **34**, 327–343 (2011)
11. Ruzhansky, M., Smith, J.: Dispersive and Strichartz Estimates for Hyperbolic Equations with Constant Coefficients. MSJ Memoirs 22. Mathematical Society of Japan, Tokyo (2010)
12. Shizuta, Y., Kawashima, S.: Systems of equations of hyperbolic–parabolic type with applications to the discrete Boltzmann equation. Hokkaido Math. J. **14**(2), 249–275 (1985). doi:10.14492/hokmj/1381757663
13. Todorova, G., Yordanov, B.: Critical exponent for a nonlinear wave equation with damping. J. Differ. Equ. **174**, 464–489 (2001)
14. Wakasugi, Y.: Critical exponent for the semilinear wave equation with scale invariant damping, Fourier analysis. In: Ruzhansky, M., Turunen, V. (eds.) Trends in Mathematics, pp. 375–390. Springer, Basel (2014)
15. Wirth, J.: Diffusion phenomena for partially dissipative hyperbolic systems. J. Math. Anal. Appl. **414**(2), 666–677 (2014). doi:10.1016/j.jmaa.2014.01.034