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Poisson kernels on nilpotent, 3-meta-abelian groups

Richard Penney · Roman Urban

Abstract Let S be a semi-direct product $S = N \rtimes A$ where N is a connected and simply connected, nilpotent, non-meta-abelian, 3-abelian Lie group and A is isomorphic with $\mathbb{R}^k$, $k > 1$. On S, we consider a class of second-order left-invariant differential operators of the form $\mathcal{L}_\alpha = L^a + \Delta_\alpha$, where $\alpha \in \mathbb{R}^k$, and for each $a \in \mathbb{R}^k$, L^a is left-invariant second-order differential operator on N and $\Delta_\alpha = \Delta - \langle \alpha, \nabla \rangle$, where Δ is the usual Laplacian on $\mathbb{R}^k$. We prove an upper bound for the Poisson kernel for the operator $\mathcal{L}_\alpha$.

Keywords Left-invariant differential operators · Poisson kernel · Time-dependent parabolic operators · Brownian motion · Evolution kernel · Diffusion process · 3-Meta-abelian nilpotent Lie groups

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1 Introduction

1.1 Statement of the main result

Let S be a semi-direct product, $S = N \rtimes A$ where N is a connected, simply connected, 3-meta-abelian, nilpotent Lie group and A is isomorphic with $\mathbb{R}^k$. We identify A with its Lie algebra $\mathfrak{a}$. The dimension k of A is called the *rank* of S.

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Specifically, we assume that

$$
\begin{aligned}
N &= N_1 \rtimes (N_2 \rtimes N_3) \\
&\equiv N_1 \rtimes N_2 \rtimes N_3,
\end{aligned}
\tag{1.1}
$$

where N_i, $i = 1, 2, 3$, are abelian Lie groups with the corresponding Lie algebras $\mathfrak{n}_i$. To avoid trivialities, we assume that N_3 is non-central in N (Otherwise, N is meta-abelian.).

Then, there are bases $\{X_1, \dots, X_{d_1}\}$, $\{Y_1, \dots, Y_{d_2}\}$, and $\{Z_1, \dots, Z_{d_3}\}$ for $\mathfrak{n}_i$, $i = 1, 2, 3$, respectively, such that

$$
\{X_i, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}, Z_1, \dots, Z_{d_3}\}
$$

forms a Jordan–Hölder basis for the Lie algebra $\mathfrak{n}$ of N. We assume that these bases are ordered so that the matrix of ad_Z is strictly lower triangular for all $Z \in \mathfrak{n}$. We assume in addition that this basis diagonalizes the $\mathrm{ad}_\mathfrak{a}$ action on $\mathfrak{n}$. We use these bases to identify $\mathfrak{n}_1$, $\mathfrak{n}_2$, and $\mathfrak{n}_3$ with $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$, and $\mathbb{R}^{d_3}$, respectively, and we use the exponential map to identify N_i with the corresponding Lie algebras $\mathfrak{n}_i = \mathbb{R}^{d_i}$.

For $g \in S$, we write $g = x(g)a(g) = xa = (x, a)$, where $x(g) = x \in N$ and $a(g) = a \in A$ denote the components of g in $N \rtimes A$. Similarly, for $x \in N$, we write $x = m(x)v(x)w(x) = mvw = (m, v, w)$, where $m(x) = m \in N_1$, $v(x) = v \in N_2$, and $w(x) = w \in N_3$ denote the components of x in $N_1 \rtimes N_2 \rtimes N_3$.

Let

$$
\begin{aligned}
\Lambda_1 &= \{\xi_1, \dots, \xi_{d_1}\}, \\
\Lambda_2 &= \{\vartheta_1, \dots, \vartheta_{d_2}\}, \\
\Lambda_3 &= \{\psi_1, \dots, \psi_{d_3}\},
\end{aligned}
$$

be the roots of the $\mathrm{ad}_\mathfrak{a}$ action on $\mathfrak{n}_1$, $\mathfrak{n}_2$ and $\mathfrak{n}_3$, respectively, corresponding to the given bases. Let

$$
\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3.
$$

Hence, for all $H \in \mathfrak{a}$,

$$
\begin{aligned}
\mathrm{ad}_H X_i &= [H, X_i] = \xi_i(H)X_i, & 1 \le i \le d_1, \\
\mathrm{ad}_H Y_j &= [H, Y_j] = \vartheta_j(H)Y_j, & 1 \le j \le d_2, \\
\mathrm{ad}_H Z_k &= [H, Z_k] = \psi_k(H)Z_k, & 1 \le k \le d_3.
\end{aligned}
\tag{1.2}
$$

Let $d = d_1 + d_2 + d_3$. For $1 \le i \le d$, we set

$$
\lambda_i = \begin{cases}
\xi_i, & 1 \le i \le d_1, \\
\vartheta_{i-d_1}, & d_1 + 1 \le i \le d_1 + d_2, \\
\psi_{i-d_1-d_2}, & d_1 + d_2 + 1 \le i \le d.
\end{cases}
$$

We refer to the class of NA groups defined above as *3-meta-abelian NA groups*. Similarly, we can define a class of *k-meta-abelian NA groups*. This is a quite large class of NA groups. It is not difficult to see that the AN parts of classical semisimple Lie groups of type A_ℓ, B_ℓ,, and D_ℓ are all k-meta-abelian for some k.

A simple example of 3-meta-abelian NA group is the group of all upper triangular 4×4-matrices with positive diagonal.

The principal object of study in this work is the left-invariant differential operator on S,

$$\mathcal{L}_\alpha = \Delta_\alpha + \sum_{j=1}^{d_1} e^{2\xi_j(a)} X_j^2 + \sum_{j=1}^{d_2} e^{2\vartheta_j(a)} Y_j^2 + \sum_{j=1}^{d_3} e^{2\psi_j(a)} Z_j^2, \tag{1.3}$$

where, for $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$,

$$\Delta_\alpha = \sum_{i=1}^{k} (\partial_{a_i}^2 - 2\alpha_i \partial_{a_i}),$$

and the X_i, Y_j, and Z_k are considered as left-invariant differential operators on N_1, N_2, and N_3, respectively. We are particularly interested in the bounded harmonic functions for this operator, i.e., bounded functions F on S satisfying $\mathcal{L}_\alpha F = 0$.

A fundamental result of Damek [3] implies that bounded $\mathcal{L}_\alpha$-harmonic functions exist provided the following *positivity assumption* (which we also assume) holds:

$$\lambda_i(\alpha) > 0, \quad \forall i. \tag{1.4}$$

In particular, none of the λ_i are identically 0 and the λ_i span $\mathfrak{a}^*$ (Their joint nullspace consists of vectors annihilated by $\mathrm{ad}_\mathfrak{a}$.). We set

$$A^+ = \{a \in \mathbb{R}^k : \lambda_i(a) > 0 \text{ for } 1 \le i \le d\}.$$

It also follows from [3] that under our assumptions, the bounded harmonic functions are precisely the "Poisson integrals" of $L^\infty(N)$. To describe this concept, let χ be the modular function for left-invariant Haar measure on S. Thus, for all $g \in S$,

$$\int_S f(sg)\mathrm{d}s = \chi(g)^{-1} \int_S f(s)\mathrm{d}s,$$

where $\mathrm{d}s$ is left-invariant Haar measure on S. Then,

$$\chi(g) = \det(\mathrm{Ad}(g)) = e^{\rho_0(a)}, \tag{1.5}$$

where

$$\rho_0 = \sum_{j=1}^{d} \lambda_j. \tag{1.6}$$

Assumption (1.4) together with [3] implies that there exists a *Poisson kernel* ν for $\mathcal{L}_\alpha$. That is, there is a C^∞ function ν on N such that every bounded $\mathcal{L}_\alpha$-harmonic function F on S may be written as a *Poisson integral* against a bounded function f on the quotient space $A \backslash S = N$,

$$F(g) = \int_{A \backslash S} f(gz)\nu(z)\mathrm{d}z = \int_N f(z)\check{\nu}^a(z^{-1}z_o)\mathrm{d}z, \quad g = (z_o, a_o),$$

where

$$\check{\nu}^a(z) = \nu(a^{-1}z^{-1}a)\chi(a)^{-1}, \quad \text{where} \quad \check{\nu}(z) = \nu(z^{-1}). \tag{1.7}$$

Conversely, the Poisson integral of any $f \in L^\infty(N)$ is a bounded $\mathcal{L}_\alpha$-harmonic function.

Our goal in this work is to obtain explicit estimates on the rate of decay of ν on N. To describe our results, we require some additional notation.

For $t \in \mathbb{R}^+$ and $\alpha \in A^+$, let

$$\delta_t^\alpha = \mathrm{Ad}((\log t)\alpha)\big|_N.$$

Then, $t \mapsto \delta_t^\alpha$ is a one parameter group of automorphisms of N for which the corresponding eigenvalues on $\mathfrak{n}$ are all positive. It is known [10,12] that then N has δ_t^α-homogeneous norm: A non-negative and subadditive continuous function $|\cdot|_\alpha$ on N which is homogeneous with respect to δ_t^α, i.e.,

$$|\delta_t^\alpha x|_\alpha = t|x|_\alpha,$$

and $|n|_\alpha = 0$ if and only if $n = e$.

For a subset $\Lambda_o \subseteq \Lambda$, and $a \in A$, define

$$\overline{\gamma}_{\Lambda_o}(a) = \min_{\lambda \in \Lambda_o} \lambda(a)/\|\lambda\|^2,$$

$$\gamma_{\Lambda_o}(a) = \min_{\lambda \in \Lambda_o} \lambda(a).$$

We set

$$K(1) = \{x \in N : |x|_\alpha = 1\}.$$

For every $x \in N$, there is precisely one $x_o \in K(1)$ such that $\delta_{|x|_\alpha}^\alpha x_o = x$. Thus, we have a map

$$N \ni x = (m, v, w) \mapsto x_o = (m_o, v_o, w_o) \in K(1).$$

The following is our main result.

Theorem 1.1 *Let ν be the Poisson kernel for the operator $\mathcal{L}_\alpha$, defined in* (1.3)*, with $\alpha \in A^+$. Under the above assumptions, for every $\varepsilon > 0$ there exists a constant $c = c_{\varepsilon,\alpha,\Lambda} > 0$ such that for all points $x = (m, v, w) \in N_1 \rtimes N_2 \rtimes N_3$,*

$$\nu(x) \leq \begin{cases} c(1 + |x|_\alpha)^{-\beta_1}, & \text{if } |(m_o, 0, 0)|_\alpha > \varepsilon, \\ c(1 + |x|_\alpha)^{-\beta_2}, & \text{if } |(0, v_o, 0)|_\alpha > \varepsilon, \\ c(1 + |x|_\alpha)^{-\beta_3}, & \text{if } |(0, 0, w_o)|_\alpha > \varepsilon, \end{cases} \tag{1.8}$$

where

$$\beta_1 = \overline{\gamma}_\Lambda(\alpha)\gamma_\Lambda(\alpha),$$
$$\beta_2 = \overline{\gamma}_{\Lambda_2 \cup \Lambda_3}(\alpha)\gamma_{\Lambda_2 \cup \Lambda_3}(\alpha),$$
$$\beta_3 = \overline{\gamma}_{\Lambda_3}(\alpha)\gamma_{\Lambda_3}(\alpha).$$

In particular, there is a constant $c = c_{\alpha,\Lambda} > 0$ such that

$$\nu(m, 0, 0) \leq c(1 + |(m, 0, 0)|_\alpha)^{-\beta_1}, \qquad \text{for all } m \in N_1, \tag{1.9}$$

$$\nu(0, v, 0) \leq c(1 + |(0, v, 0)|_\alpha)^{-\beta_2}, \qquad \text{for all } v \in N_2, \tag{1.10}$$

$$\nu(0, 0, w) \leq c(1 + |(0, 0, w)|_\alpha)^{-\beta_3}, \qquad \text{for all } w \in N_3, \tag{1.11}$$

Remark 1.2 Notice that $\beta_1 \leq \beta_2 \leq \beta_3$.

Remark 1.3 We note that [18, Theorem 1.1] says that on a large class of nilpotent Lie groups N containing the 3-meta-abelian groups one has that for every $q > 1$, there is $c = c_{q,\alpha,\Lambda} > 0$ such that for all $x \in N$,

$$\nu(x) \leq c(1 + |x|_\alpha)^{-\frac{2}{q}\overline{\gamma}_\Lambda(\alpha)\gamma_\Lambda(\alpha)}. \tag{1.12}$$

In many cases, the estimate in Theorem 1.1 is strictly sharper than (1.12) since clearly the inequalities

$$\frac{2}{q}\overline{\gamma}_\Lambda(\alpha)\gamma_\Lambda(\alpha) < \beta_i \quad \text{for } i = 1, 2, 3,$$

can hold for many choices of Λ and α. Also, unlike (1.12), the estimates in Theorem 1.1 differentiate between the various directions of approach to infinity.

Remark 1.4 We should also remark that in [20], we proved the analog of Theorem 1.1 in the 2-meta-abelian case. We found that there are essential difficulties if one wants to prove a similar estimate for the Poisson kernel on k-meta-abelian group with $k \geq 3$. In Sect. 1.2 below, we describe these difficulties in more details (See in particular Remarks 1.5 and 1.6.).

1.2 Strategy of the proof

Let μ_t (resp., T_t) be the semigroups of measures (resp., operators) generated by $\mathcal{L}_\alpha$. It is known (see Sect. 6.1) that the Poisson kernel ν is equal to $\lim_{t \to \infty} \pi_N(\mu_t)$, where $\pi_N(g) = x(g)$ is a projection from S onto N. To get some information on μ_t, we use a well-known formula which express T_t as a skew-product of the diffusions on N and A. (The idea of such a decomposition goes back to [13,14,22]. In the context of NA groups with $\dim A = 1$, this decomposition was used in [4–7], and later was generalized by the authors and applied for $\dim A > 1$, see, e.g., [17,19].)

Specifically, for $\sigma \in C([0, \infty), A)$, $A = \mathbb{R}^k$, let

$$\mathcal{L}_N^\sigma = \sum_{j=1}^{d_1} e^{2\xi_j(\sigma(t))} X_j^2 + \sum_{j=1}^{d_2} e^{2\vartheta_j(\sigma(t))} Y_j^2 + \sum_{j=1}^{d_3} e^{2\psi_j(\sigma(t))} Z_j^2, \quad (1.13)$$

considered as a time-dependent left-invariant differential operator on N. It is known that then $\mathcal{L}_N^\sigma$ generates a time-inhomogeneous diffusion on N with transition kernel $P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$, $t \geq s \geq 0$, and the corresponding evolution operators $U_{s,t}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$ (see [2,21]). The skew-product formula says that

$$T_t f(x, a) = \mathbf{E}_a^\sigma U_{t,0}^{N_1 \rtimes N_2 \rtimes N_3, \sigma} f(\cdot, \sigma(t))\big|_x = \mathbf{E}_a^\sigma (f *_N P_{t,0}^{N_1 \rtimes N_2 \rtimes N_3, \sigma})(x, \sigma(t)), \quad (1.14)$$

for $f \in C_c(N \times \mathbb{R}^k)$ and $t \geq 0$, where the expectation is taken with respect to the distribution of the Brownian motion $\sigma(t) \in \mathbb{R}^k$ with drift -2α, starting from a, i.e., $\sigma(0) = a$, and generated by Δ_α. The subscript N in the convolution $*_N$ means that f is convolved with the kernel $P_{t,0}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$ with respect to the first variable in N.

The next step is to disintegrate the kernel $P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$ using [20, Theorem 1.2] (see also section 3 in [19]) which is a skew-product formula similar to formula (1.14). Specifically, from the decomposition (1.1), the time-dependent family of operators

$$\mathcal{L}_{N_2 \rtimes N_3}^\sigma = \sum_{j=1}^{d_2} e^{2\vartheta_j(\sigma(t))} Y_j^2 + \sum_{j=1}^{d_3} e^{2\psi_j(\sigma(t))} Z_j^2, \quad (1.15)$$

gives rise to an evolution on $N_2 \rtimes N_3 = \mathbb{R}^{d_2} \rtimes \mathbb{R}^{d_3}$ that is described by a kernel $P_{t,s}^{N_2 \rtimes N_3, \sigma}$. The semi-direct product $N_2 \rtimes N_3$ is a 2-meta-abelian group. Thus, we have a relatively good knowledge about the kernel $P_{t,s}^{N_2 \rtimes N_3, \sigma}$ (see [20]). Let $\eta(t) = (\eta^{N_2}(t), \eta^{N_3}(t))$ be the process generated by $\mathcal{L}_{N_2 \rtimes N_3}^\sigma$. The skew-product formula from [20] gives that

$$U_{s,t}^{N_1 \rtimes N_2 \rtimes N_3, \sigma} f(m, v, w) = \mathbf{E}_{s,(v,w)}^\eta U_{s,t}^{N_1, \sigma, \eta} f(\cdot, \eta^{N_2}(t), \eta^{N_3}(t))\big|_m, \quad (1.16)$$

where the subscript in the expectation means that $\eta(s) = (v, w)$, and $U_{s,t}^{N_1,\sigma,\eta}$ is the family of evolution operators generated by the operator

$$\mathcal{L}_{N_1}^{\sigma,\eta} = \sum_{j=1}^{d_1} e^{2\xi_j(\sigma(t))} \left(\mathrm{Ad}(\eta(t))\big|_{\mathfrak{n}_1} X_j \right)^2.$$

Formula (1.16) allows us to compute and estimate in Sects. 5 and 6 the kernel $P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$ on certain subsets of N which, it turns out, are sufficient for our purposes (See formulas (5.1), (6.3), and (6.5).).

Remark 1.5 In [20], we proved a result analogous to Theorem 1.1 for meta-abelian groups, i.e., in the 2-meta-abelian case. The generalization to the 3-meta-abelian case is complicated by the lack of an appropriate estimate for $P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$ valid on all of N. The fact that it is possible to obtain a good estimate for the Poisson kernel on all of N by piecing together estimates of $P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$ on subsets is somewhat surprising.

Remark 1.6 In order to consider k-meta-abelian NA groups with $k \geq 4$, we need a better understanding of the situation for $k = 3$. In particular, we need to invent some methods which allow us to produce a good and global Gaussian estimate for $P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$. If we try to apply methods from [20], the problem of estimating $P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$ amounts to the good knowledge of the properties of the time-inhomogeneous process $\eta(t)$ generated by the time-dependent operator (1.15). We hope that if this estimate is obtained, then we will be able to apply induction argument and eventually get estimates for the Poisson kernel for all k.

1.3 Structure of the paper

The outline of the rest of the paper is as follows. In Sect. 2, we recall some basic facts about exponential functionals of Brownian motion. In Sect. 3, we consider the evolution process $\eta(t)$ on $N_2 \rtimes N_3$ generated by the operator (1.15) and state the estimate for the corresponding transition kernels. Next, in Sect. 4, we study the evolution kernel on N_1 which is the second ingredient of the skew-product formula (1.16). In Sect. 5, we prove an estimate for the kernel $P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$. Finally in Sect. 6, we construct the Poisson kernel and prove our main theorem.

2 Preliminaries

2.1 Exponential functionals of Brownian motion

Let $b(s)$, $s \geq 0$, be the Brownian motion on $\mathbb{R}$ staring from $a \in \mathbb{R}$ and normalized so that

$$\mathbf{E}_a f(b(s)) = \frac{1}{\sqrt{4\pi s}} \int_{\mathbb{R}} f(x + a) e^{-x^2/4s} \mathrm{d}x. \qquad (2.1)$$

Hence, $b(s)$ has a normal distribution with mean $\mathbf{E}b(s) = a$ and variance $\mathrm{Var}\, b(s) = 2s$.

Remark Our normalization of the Brownian motion $b(s)$ is different than that typically used by probabilists who tend to assume that $\mathrm{Var}\, b(s) = s$.

For $d > 0$ and $\mu > 0$, we define the following exponential functional

$$I_{d,\mu} = \int_0^\infty e^{d(b(s)-\mu s)}\mathrm{d}s. \tag{2.2}$$

Theorem 2.1 (Dufresne, [9]) Let $b(0) = 0$. Then, the functional $I_{2,\mu}$ is distributed as $(4\gamma_{\mu/2})^{-1}$, where $\gamma_{\mu/2}$ denotes a gamma random variable with parameter $\mu/2$, i.e., $\gamma_{\mu/2}$ has a density $(1/\Gamma(\mu/2))x^{\frac{\mu}{2}-1}e^{-x}\mathbf{1}_{[0,+\infty)}(x)$.

As a corollary of Theorem 2.1, by scaling the Brownian motion and changing the variable, we get the following theorem (See [17, Lemma 5.4] for details.).

Theorem 2.2 Let $\sigma(u) = b(u) - 2\alpha u$ be the k-dimensional Brownian motion with a drift -2α, $d > 0$, and let $\ell \in (\mathbb{R}^k)^*$ be such that $\ell(\alpha) > 0$. Then,

$$\mathbf{E}_a f\left(\int_0^\infty e^{d\ell(\sigma(u))}du\right) = c_{d,\ell,\alpha}e^{\ell(a)}\int_0^\infty f(u)u^{-\varrho/d}\exp\left(-\frac{e^{d\ell(a)}}{2d^2\ell^2 u}\right)\frac{du}{u},$$

where $\varrho = 2\ell(\alpha)/\|\ell\|^2$.

Remark Exponential functionals of type (2.2) are called *perpetual functionals* in financial mathematics, and they play an important role there (see, e.g., [15,16,23]). In particular, the distribution of the integral over finite interval $(0, t)$ in (2.2) has many applications in Asian options (see, e.g., [1,8,11]).

2.2 Notation for exponential functionals

For a continuous function $\sigma : [0, \infty) \to A = \mathbb{R}^k$,

$$A^\sigma_{N_1,i}(s,t) = \int_s^t e^{2\xi_i(\sigma(u))}\mathrm{d}u, \quad i = 1,\ldots,d_1,$$

$$A^\sigma_{N_2,j}(s,t) = \int_s^t e^{2\vartheta_j(\sigma(u))}\mathrm{d}u, \quad j = 1,\ldots,d_2,$$

$$A^\sigma_{N_3,k}(s,t) = \int_s^t e^{2\psi_k(\sigma(u))}\mathrm{d}u, \quad k = 1,\ldots,d_3,$$

and

$$A^\sigma_{N,j}(s,t) = \int_s^t e^{2\lambda_j(\sigma(u))}\mathrm{d}u, \quad j = 1,\ldots,d.$$

We also define, for $i = 1, 2, 3$,

$$A^\sigma_{N_i,\Sigma}(s,t) = \sum_{j=1}^{d_i} A^\sigma_{N_i,j}(s,t), \qquad A^\sigma_{N_i,\Pi}(s,t) = \prod_{j=1}^{d_i} A^\sigma_{N_i,j}(s,t).$$

Finally, we put

$$A^\sigma_{N,\Sigma}(s,t) = \sum_{i=1}^{3} A^\sigma_{N_i,\Sigma}(s,t), \qquad A^\sigma_{N,\Pi}(s,t) = \prod_{i=1}^{3} A^\sigma_{N_i,\Pi}(s,t).$$

2.3 Moments of exponential functionals

The following lemma follows from Theorem 2.2.

Lemma 2.3 *The functional $A_{N,j}^\sigma(0, \infty)$, $j = 1, \ldots, d$, has a finite s-th moment (for every $a \in \mathbb{R}^k$)*

$$\mathbf{E}^\sigma A_{N,j}^\sigma(0, \infty)^s < +\infty$$

if and only if $\lambda_j(\alpha)/\|\lambda\|^2 > s$, where $\|\cdot\|$ is ℓ^2 norm on $\mathbb{R}^k$. In particular, $A_{N,j}^\sigma(0, \infty)$ has all negative moments.

2.4 Action of A on N

For $a \in \mathbb{R}^k$, $\Phi(a)$ denotes the action of a on $\mathfrak{n}$, see (1.2). By (1.2), the automorphisms $\{\Phi(a)\}_{a \in \mathbb{R}^k}$ leave $\mathfrak{n}_i$, $i = 1, 2, 3$ invariant. We identify linear transformation $\Phi(a)$ on $\mathfrak{n}$ with $d \times d$ matrix,

$$\Phi(a) = \begin{bmatrix} \Phi(a)\big|_{\mathfrak{n}_1} & 0 & 0 \\ 0 & \Phi(a)\big|_{\mathfrak{n}_2} & 0 \\ 0 & 0 & \Phi(a)\big|_{\mathfrak{n}_3} \end{bmatrix},$$

where

$$\Phi(a)\big|_{\mathfrak{n}_1} = \mathrm{diag}\left[e^{\xi_1(a)}, \ldots, e^{\xi_{d_1}(a)}\right],$$
$$\Phi(a)\big|_{\mathfrak{n}_2} = \mathrm{diag}\left[e^{\vartheta_1(a)}, \ldots, e^{\vartheta_{d_2}(a)}\right],$$
$$\Phi(a)\big|_{\mathfrak{n}_3} = \mathrm{diag}\left[e^{\psi_1(a)}, \ldots, e^{\psi_{d_3}(a)}\right].$$

Let σ be a continuous function from $[0, +\infty)$ to $A = \mathbb{R}^k$. We define

$$\Phi^\sigma(t) = \Phi(\sigma(t)). \tag{2.3}$$

3 Evolution kernel on $N_2 \rtimes N_3$

In this section, we consider time-dependent operator on $N_2 \rtimes N_3$,

$$\mathcal{L}_{N_2 \rtimes N_3}^\sigma = \sum_{j=1}^{d_2} e^{2\vartheta_j(\sigma(t))} Y_j^2 + \sum_{j=1}^{d_3} e^{2\psi_j(\sigma(t))} Z_j^2.$$

The operator $\mathcal{L}_{N_2 \rtimes N_3}^\sigma$ gives rise to an evolution on $N_2 \rtimes N_3 = \mathbb{R}^{d_2} \rtimes \mathbb{R}^{d_3}$ that is described by a kernel $P_{t,s}^{N_2 \rtimes N_3, \sigma}$ and the corresponding operator $U_{t,s}^{N_2 \rtimes N_3, \sigma}$. The semi-direct product $N_2 \rtimes N_3$ is a 2-meta-abelian group.

The Euclidean space $\mathbb{R}^n$ is endowed with the usual scalar product $\langle x, y \rangle = x \cdot y = \sum_{i=1}^n x_i y_i$ and the corresponding ℓ^2 norm $\|x\| = \langle x, x \rangle^{1/2}$.

The following estimate is proved in [20, Theorem 4.1].

Theorem 3.1 *There are positive constants C, D and k_o such that for all $t > s \geq 0$ and all $(v, w) \in N_2 \rtimes N_3$,*

$$A^\sigma_{N_2,\Pi}(s, t)^{1/2} A^\sigma_{N_3,\Pi}(s, t)^{1/2} P^{N_2 \rtimes N_3,\sigma}_{t,s}(v, w)$$

$$\leq C(\|v\|^{\frac{1}{2k_o}} + 1) \exp\left(-\frac{D\|w\|^2}{A^\sigma_{N_3,\Sigma}(s, t)} - \frac{D\|v\|^2}{(\|v\|^{\frac{1}{2k_o}} + \|w\| + 2)^{2k_o} A^\sigma_{N_2,\Sigma}(s, t)}\right)$$

$$+ C A^\sigma_{N_3,\Sigma}(s, t)^{1/2} \exp\left(-D\frac{\|v\|^{\frac{1}{k_o}} + \|w\|^2}{A^\sigma_{N_3,\Sigma}(s, t)}\right).$$

Remark Here, k_o is the smallest non-negative integer such that

$$(\mathrm{ad}_X)^{k_o+1}\big|_{\mathfrak{n}_2} = 0, \quad \forall X \in \mathfrak{n}_3.$$

Note that $k_o > 0$ since by hypothesis N_3 is non-central.

4 Evolution kernel on N_1

Let

$$\eta(t) = \left(\eta^{N_2}(t), \eta^{N_3}(t)\right) = \left(\eta^1(t), \ldots, \eta^{d_2+d_3}(t)\right)$$

be the time-inhomogeneous Markov process generated by the operator

$$\mathcal{L}^\sigma_{N_2 \rtimes N_3} = \sum_{j=1}^{d_2} e^{2\vartheta_j(\sigma(t))} Y_j^2 + \sum_{j=1}^{d_3} e^{2\psi_j(\sigma(t))} Z_j^2$$

considered in Sect. 3.

Now on N_1, we consider time-dependent operator

$$\mathcal{L}^{\sigma,\eta}_{N_1} = \sum_{j=1}^{d_1} e^{2\xi_j(\sigma(t))} (\mathrm{Ad}(\eta(t))\big|_{\mathfrak{n}_1} X_j)^2.$$

The following notation will be useful. For a $n \times n$ invertible matrix A, we set

$$\mathcal{B}(A)(x) = 2^{-1} A^{-1} x \cdot x \quad \text{and} \quad \mathcal{D}(A) = (2\pi)^{-n/2} (\det A)^{-1/2}. \tag{4.1}$$

Since N_1 is abelian, the transition kernels $P^{N_1,\sigma,\eta}_{t,s}$ of the time-inhomogeneous process $\omega(t)$ generated by $\mathcal{L}^{\sigma,\eta}_{N_1}$ are given (see Proposition 2.10 in [19]) by

$$P^{N_1,\sigma,\eta}_{t,s}(m; m') = \mathcal{D}\left(A^{\sigma,\eta}_{N_1}(s, t)\right) e^{-\mathcal{B}\left(A^{\sigma,\eta}_{N_1}(s,t)\right)(m-m')}, \tag{4.2}$$

where

$$A^{\sigma,\eta}_{N_1}(s, t) = 2 \int_0^t \left[\mathrm{Ad}(\eta(u))\big|_{\mathfrak{n}_1} \Phi^\sigma(u)\big|_{\mathfrak{n}_1}\right] \left[\mathrm{Ad}(\eta(u))\big|_{\mathfrak{n}_1} \Phi^\sigma(u)\big|_{\mathfrak{n}_1}\right]^* du,$$

where $\Phi^\sigma(u)$ is defined in (2.3).

Exactly in the same way as Lemma 3.3 in [20], one can prove the following lemma.

Lemma 4.1 *There is a constant $C > 0$ such that*

$$\mathcal{D}(A^{\sigma,\eta}_{N_1}(s, t)) \leq C A^\sigma_{N_1,\Pi}(s, t)^{-1/2}.$$

5 Evolution kernel on $N_1 \rtimes N_2 \rtimes N_3$

To get the kernel $P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$, we apply the skew-product formula (1.16) (for the proof, see [20, Theorem 1.2]),

$$P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}(0, 0, 0; m, v, w) = \lim_{\varepsilon \to 0} \mathbf{E}_{0,(0,0)}^{\eta} P_{t,s}^{N_1, \sigma, \eta}(0; m) \psi_\varepsilon(\eta(t)), \qquad (5.1)$$

where for $(v, w) \in N_2 \rtimes N_3 = \mathbb{R}^{d_2} \rtimes \mathbb{R}^{d_3}$ given and $\varepsilon > 0$,

$$\psi_\varepsilon(v', w') = \varepsilon^{-d_2 - d_3} \mathbf{1}_{B_\varepsilon(v)}(v') \mathbf{1}_{B_\varepsilon(w)}(w')$$

is the (normalized in L^1) indicator function of the product of two ε-balls around v and w,

$$B_\varepsilon(v) = \prod_{j=1}^{d_2} B_\varepsilon^1(v_j), \qquad B_\varepsilon(w) = \prod_{j=1}^{d_3} B_\varepsilon^1(w_j)$$

and

$$B_\varepsilon^1(x) = [x - \varepsilon/2, x + \varepsilon/2].$$

The following theorem will be used in order to get the estimates for the evolution kernel $P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$ on those sets which are necessary for the upper bound for the Poisson kernel.

Theorem 5.1 *There is a constant $C > 0$ such that for every $t > s \geq 0$, and for every $(m, v, w) \in N_1 \rtimes N_2 \rtimes N_3$,*

$$P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}(0, 0, 0; m, v, w) \leq C A_{N_1, \Pi}^\sigma(s, t)^{-1/2} P_{t,s}^{N_2 \rtimes N_3, \sigma}(0, 0; v, w).$$

Proof It follows from (4.2) and Lemma 4.1 that

$$P_{t,s}^{N_1, \sigma, \eta}(0; m) \leq P_{t,s}^{N_1, \sigma, \eta}(0; 0) = \mathcal{D}\left(A_{N_1}^{\sigma, \eta}(s, t)\right) \leq C A_{N_1, \Pi}^\sigma(s, t)^{-1/2}.$$

Then, from (5.1)

$$P_{t,s}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}(0, 0, 0; m, v, w) \leq C A_{N_1, \Pi}^\sigma(s, t)^{-1/2} \lim_{\varepsilon \to 0} \mathbf{E}_{0,(0,0)}^{\eta} \psi_\varepsilon(\eta(t))$$

Clearly,

$$\lim_{\varepsilon \to 0} \mathbf{E}_{0,(0,0)}^{\eta} \psi_\varepsilon(\eta(t)) = P_{t,s}^{N_2 \rtimes N_3, \sigma}(0, 0; v, w)$$

and the lemma follows. $\square$

6 Poisson kernel on $N_1 \rtimes N_2 \rtimes N_3$

In this section, we will give the proof of Theorem 1.1.

6.1 Construction of the Poisson kernel

Here, we recall the construction of the Poisson kernel ν—the principle object of our study. At the same time, this construction provides convenient formula for the Poisson kernel. Let μ_t be the semigroup of probability measures on $S = N \rtimes \mathbb{R}^k$ generated by $\mathcal{L}_\alpha$. It is known [5] that

$$\lim_{t \to \infty} (\pi_N(\check{\mu}_t), f) = (\nu, f),$$

where π_N denotes the projection from S onto N and $(\check{\mu}, f) = (\mu, \check{f})$, $\check{f}(x) = f(x^{-1})$. Let $a \in \mathbb{R}^k$ and let μ be a measure on N. We define

$$(\mu^a, f) = (\mu, f \circ \mathrm{Ad}(a)).$$

For $a \in \mathbb{R}^k$, we have

$$\nu^a(x) = \nu(a^{-1}xa)\chi(a)^{-1}, \quad x \in N, \tag{6.1}$$

where χ is as in (1.5).

We will need the following fact (see [17, Lemma 4.1] for a proof).

Lemma 6.1 *We have*

$$(\nu^a, f) = \lim_{t \to \infty} (\pi_N(\check{\mu}_t)^a, f) = \lim_{t \to \infty} (\mathbf{E}_a^\sigma \check{P}_{t,0}^{N_3 \rtimes N_2 \rtimes N_1, \sigma}(0, 0, 0; \cdot, \cdot, \cdot), f),$$

where $P_{t,0}^{N_3 \rtimes N_2 \rtimes N_1, \sigma}$ is the evolution kernel for the operator $\mathcal{L}_N^\sigma$ defined in (1.13).

6.2 Upper bound for $\nu^{s\alpha}$

Our main aim in this subsection is to obtain an upper bound for

$$\nu^{s\alpha}(x) = \nu((s\alpha)^{-1}x(s\alpha))\chi(s\alpha)^{-1}$$

for all $s < 0$, where $\alpha \in A^+$ is a drift vector of the operator (1.3). Then, in Sect. 6.4, in order to get an upper bound for ν we will apply a simple homogeneity argument together with some comparison results about evolution kernels.

By Lemma 6.1,

$$\nu^{s\alpha}(m, v, w) = \lim_{t \to \infty} \mathbf{E}_{s\alpha}^\sigma \check{P}_{t,0}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}(0, 0, 0; m, v, w). \tag{6.2}$$

Hence, in order to estimate $\nu^{s\alpha}$ we need to estimate $P_{t,0}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}$. The main results of this section are the following upper bounds for $\nu^{s\alpha}$.

Theorem 6.2 *Let $\alpha \in A^+$, and let K_3 be a compact subset of N_3 such that $0 \notin N_3$. Then, there is a constant $c = c_{\Lambda, K_3, \alpha} > 0$ such that for every $s < 0$, and all $w \in K_3 \subset N_3$,*

$$\nu^{s\alpha}(0, 0, w) \le ce^{-\rho_0(s\alpha)}e^{s\overline{\gamma}_{\Lambda_3}(\alpha)\gamma_{\Lambda_3}(\alpha)}.$$

Theorem 6.3 *Let $\alpha \in A^+$, and let K_2 be a compact subset of N_2 such that $0 \notin N_2$. Then, there is a constant $c = c_{\Lambda, K_2, \alpha} > 0$ such that for every $s < 0$, and all $v \in K_2 \subset N_2$,*

$$\nu^{s\alpha}(0, v, 0) \le ce^{-\rho_0(s\alpha)}e^{s\overline{\gamma}_{\Lambda_2 \cup \Lambda_3}(\alpha)\gamma_{\Lambda_2 \cup \Lambda_3}(\alpha)}.$$

6.3 Proofs of Theorems 6.2 and 6.3

We start with three lemmas. In all of them, the exponential functionals are on the interval $(0, \infty)$, i.e., $A_{\star,\star}^\sigma$ denotes $A_{\star,\star}^\sigma(0, \infty)$.

Lemma 6.4 *There is a constant $C > 0$ such that for all $s < 0$ and $\alpha \in A^+$,*

$$\mathbf{E}_{s\alpha}^\sigma(A_{N,\Pi}^\sigma)^{-1} \le Ce^{-2\rho_0(s\alpha)}$$

and

$$\mathbf{E}_{s\alpha}^\sigma(A_{N,\Pi}^\sigma)^{-1}A_{N_3,\Sigma}^\sigma \le Ce^{-2\rho_0(s\alpha)}.$$

Proof We have

$$\mathbf{E}_{s\alpha}^{\sigma}(A_{N,\Pi}^{\sigma})^{-1} = e^{-2\rho_0(s\alpha)}\mathbf{E}_0^{\sigma}(A_{N,\Pi}^{\sigma})^{-1}.$$

The expected value $\mathbf{E}_0^{\sigma}(A_{N,\Pi}^{\sigma})^{-1}$ is finite. This follows by applying Cauchy–Schwarz inequality succesively and the fact that exponential functionals $A_{N,j}^{\sigma}$ have negative moments (Lemma 2.3).

Similarly, since $s < 0$ and $\alpha \in A^+$,

$$\begin{aligned}
\mathbf{E}_{s\alpha}^{\sigma}(A_{N,\Pi}^{\sigma})^{-1}A_{N_3,\Sigma}^{\sigma} &= \sum_{j=1}^{d_3}\mathbf{E}_{s\alpha}^{\sigma}\prod_{\substack{k=1\\k\neq j}}^{d}(A_{N,k}^{\sigma})^{-1}\\
&= \sum_{j=1}^{d_3}e^{-2\rho_0(s\alpha)+\psi_j(s\alpha)}\mathbf{E}_0^{\sigma}\prod_{\substack{k=1\\k\neq j}}^{d}(A_{N,k}^{\sigma})^{-1}\\
&\leq e^{-2\rho_0(s\alpha)}\sum_{j=1}^{d_3}\mathbf{E}_0^{\sigma}\prod_{\substack{k=1\\k\neq j}}^{d}(A_{N,k}^{\sigma})^{-1}.
\end{aligned}$$

Again, by Lemma 2.3, the expectations $\mathbf{E}_0^{\sigma}\prod_{\substack{k=1\\k\neq j}}^{d}(A_{N,k}^{\sigma})^{-1}$ for $j = 1, \ldots, d_3$ are finite. $\square$

The next two lemmas follows immediately from the proof of [17, Lemma 6.2] and the inequality (6.3) on p. 269 in [17].

Lemma 6.5 *Let $\alpha \in A^+$. For every $\beta > 0$, there is a constant $c = c_{\Lambda_3,\alpha,\beta} > 0$ such that for every $s < 0$,*

$$\mathbf{E}_{s\alpha}^{\sigma}e^{-\beta/A_{N_3,\Sigma}^{\sigma}} \leq ce^{2s\overline{\gamma}_{\Lambda_3}(\alpha)\gamma_{\Lambda_3}(\alpha)}.$$

Lemma 6.6 *Let $\alpha \in A^+$. For every $\beta > 0$, there is a constant $c = c_{\Lambda_2,\Lambda_3,\alpha,\beta} > 0$ such that for every $s < 0$,*

$$\mathbf{E}_{s\alpha}^{\sigma}e^{-\beta/(A_{N_2,\Sigma}^{\sigma}+A_{N_3,\Sigma}^{\sigma})} \leq ce^{2s\overline{\gamma}_{\Lambda_2\cup\Lambda_3}(\alpha)\gamma_{\Lambda_2\cup\Lambda_3}(\alpha)}.$$

Proof of Theorem 6.2 By Theorem 5.1,

$$P_{t,0}^{N_1\rtimes N_2\rtimes N_3,\sigma}(0,0,0;0,0,w) \leq CA_{N_1,\Pi}^{\sigma}(0,t)^{-1/2}P_{t,0}^{N_2\rtimes N_3,\sigma}(0,w).$$

From Theorem 3.1, since $0 \notin K_3$ there is a constant $c > 0$ such that for all $w \in K_3$,

$$\begin{aligned}
P_{t,0}^{N_2\rtimes N_3,\sigma}(0,w) &\leq\\
CA_{N_2,\Pi}^{\sigma}(0,t)^{-1/2}A_{N_3,\Pi}^{\sigma}&(0,t)^{-1/2}\left(1+A_{N_3,\Sigma}^{\sigma}(0,t)^{1/2}\right)e^{-c/A_{N_3,\Sigma}^{\sigma}(0,t)}.
\end{aligned}$$

Consequently, for $w \in K_3$,

$$\begin{aligned}
P_{t,0}^{N_1\rtimes N_2\rtimes N_3,\sigma}&(0,0,0;0,0,w)\\
&\leq CA_{N_3,\Pi}^{\sigma}(0,t)^{-1/2}\left(1+A_{N_3,\Sigma}^{\sigma}(0,t)^{1/2}\right)e^{-c/A_{N_3,\Sigma}^{\sigma}(0,t)}.
\end{aligned}\tag{6.3}$$

Using (6.2), we get that for all $w \in K_3$, (note that below the range of integration in all functionals is $(0,+\infty)$)

$$\begin{aligned}
\nu^{s\alpha}(0,0,w) &\leq C\mathbf{E}_{s\alpha}^{\sigma}(A_{N,\Pi}^{\sigma})^{-1/2}e^{-c/A_{N_3,\Sigma}^{\sigma}}\\
&\quad + C\mathbf{E}_{s\alpha}^{\sigma}(A_{N,\Pi}^{\sigma})^{-1/2}(A_{N_3,\Sigma}^{\sigma})^{1/2}e^{-c/A_{N_3,\Sigma}^{\sigma}}.
\end{aligned}\tag{6.4}$$

By the Cauchy–Schwarz inequality

$$\nu^{s\alpha}(0,0,w) \le C \left(\mathbf{E}^\sigma_{s\alpha}(A^\sigma_{N,\Pi})^{-1}\right)^{1/2} \left(\mathbf{E}^\sigma_{s\alpha}e^{-2c/A^\sigma_{N_3,\Sigma}}\right)^{1/2}$$

$$+ C\left(\mathbf{E}^\sigma_{s\alpha}(A^\sigma_{N,\Pi})^{-1}A^\sigma_{N_3,\Sigma}\right)^{1/2} \left(\mathbf{E}^\sigma_{s\alpha}e^{-2c/A^\sigma_{N_3,\Sigma}}\right)^{1/2}.$$

Now Theorem 6.2 follows from Lemma 6.4 and Lemma 6.5.   □

Proof of Theorem 6.3 By Theorem 5.1,

$$P^{N_1\rtimes N_2\rtimes N_3,\sigma}_{t,0}(0,0,0;0,v,0) \le CA^\sigma_{N_1,\Pi}(0,t)^{-1/2}P^{N_2\rtimes N_3,\sigma}_{t,0}(v,0). \tag{6.5}$$

From Theorem 3.1 with $w = 0$, since $0 \notin K_2$ there is a constant $c > 0$ such that for all $v \in K_2$,

$$A^\sigma_{N_2,\Pi}(0,t)^{1/2}A^\sigma_{N_3,\Pi}(0,t)^{1/2}P^{N_2\rtimes N_3,\sigma}_{t,0}(v,0)$$

$$\le Ce^{-c_1/A^\sigma_{N_2,\Sigma}(0,t)} + CA^\sigma_{N_3,\Sigma}(0,t)^{1/2}e^{-c_2/A^\sigma_{N_3,\Sigma}(0,t)}$$

$$\le C(1 + A^\sigma_{N_3,\Sigma}(0,t)^{1/2})e^{-c/(A^\sigma_{N_2,\Sigma}(0,t)+A^\sigma_{N_3,\Sigma}(0,t))}.$$

As in the proof of Theorem 6.2, we conclude that on K_2, (the range of integration in all functionals below is $(0,+\infty)$)

$$\nu^{s\alpha}(0,v,0) \le C\mathbf{E}^\sigma_{s\alpha}(A^\sigma_{N,\Pi})^{-1/2}e^{-c/(A^\sigma_{N_2,\Sigma}(0,t)+A^\sigma_{N_3,\Sigma}(0,t))}$$

$$+ C\mathbf{E}^\sigma_{s\alpha}(A^\sigma_{N,\Pi})^{-1/2}(A^\sigma_{N_3,\Sigma})^{1/2}e^{-c/(A^\sigma_{N_2,\Sigma}(0,t)+A^\sigma_{N_3,\Sigma}(0,t))}.$$

This together with Lemma 6.4 and Lemma 6.6 finish the proof.   □

6.4 Upper bound for the Poisson kernel ν

Having Theorem 5.1, Theorem 6.2, and Theorem 6.3, we are ready to prove the estimate for the Poisson kernel.

Proof of Theorem 1.1 By continuity of ν, there is a constant $C_\alpha > 0$ such that for all $x \in N$ with the norm $|x|_\alpha \le 1$, we have $\nu(x) \le C_\alpha$.

Consider $x \in N$ with $|x|_\alpha > 1$. Let $\delta^\alpha_t = \mathrm{Ad}((\log t)\alpha)$. Then, $|\delta^\alpha_t x|_\alpha = t|x|_\alpha$. We write x as $x = \delta^\alpha_{\exp(-s)}x_o$ with $|x_o|_\alpha = 1$ and $s < 0$. Then, $|x|_\alpha = e^{-s} > 1$. Let $K(1) = \{x_o : |x_o|_\alpha = 1\}$. By definition (6.1) of $\nu^{s\alpha}$, we get

$$\nu(x) = \nu(\delta^\alpha_{\exp(-s)}x_o) = \nu((s\alpha)^{-1}x_o(s\alpha)) = e^{\rho_0(s\alpha)}\nu^{s\alpha}(x_o), \tag{6.6}$$

where $\rho_0 = \sum_{j=1}^d \lambda_j$. Now, estimates (1.10) and (1.11) follow from Theorems 6.3 and 6.2, respectively, if we apply (6.6) to $x = (0,v,0)$ and $x = (0,0,w)$, respectively. Then, $x_o = (0,v_o,0)$ and $x_o = (0,0,w_o)$. Estimate (1.9) is a consequence of (1.12). Finally, in order to prove (1.8), we proceed as follows.

Let $\mathcal{U}^\sigma_t(v,w)$, $(v,w) \in N_2 \rtimes N_3$, be the estimate for the evolution kernel $P^{N_2\rtimes N_3,\sigma}_{t,0}(0,0;v,w)$ given by Theorem 3.1.

By Theorem 5.1,

$$P^{N_1\rtimes N_2\rtimes N_3,\sigma}_{t,0}(0,0,0;m,v,w) \le CA^\sigma_{N_1,\Pi}(0,t)^{-1/2}P^{N_2\rtimes N_3,\sigma}_{t,0}(v,w)$$

$$\le CA^\sigma_{N_1,\Pi}(0,t)^{-1/2}\mathcal{U}^\sigma_t(v,w). \tag{6.7}$$

Note that $\check{P}_{t,0}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}(0, 0, 0; m, v, w)$ has the same estimate. Now we consider (m, v, w) in a compact sets $K(1)$ (clearly, $0 \notin K(1)$). By Lemma 6.1 and (6.7),

$$\nu^{s\alpha}(m, v, w) = \lim_{t \to \infty} \mathbf{E}_{s\alpha}^{\sigma} \check{P}_{t,0}^{N_1 \rtimes N_2 \rtimes N_3, \sigma}(0, 0, 0; m, v, w)$$

$$\leq C \mathbf{E}_{s\alpha}^{\sigma} A_{N_1, \Pi}^{\sigma}(0, \infty)^{-1/2} \mathcal{U}_{\infty}^{\sigma}(v, w). \qquad (6.8)$$

Notice that it follows from Theorem 3.1 that there is $c > 0$ such that for all $(u, w) \in K(1) \cap (N_2 \rtimes N_3)$, and all $t > 0$,

$$\mathcal{U}_t^{\sigma}(v, w) \leq c \mathcal{U}_t^{\sigma}(0, w) \quad \text{and} \quad \mathcal{U}_t^{\sigma}(v, w) \leq c \mathcal{U}_t^{\sigma}(v, 0).$$

Applying the above inequalities to (6.8), we get

$$\nu^{s\alpha}(m, v, w) \leq \mathbf{E}_{s\alpha}^{\sigma} A_{N_1, \Pi}^{\sigma}(0, \infty)^{-1/2} \mathcal{U}_{\infty}^{\sigma}(0, w)$$

$$\text{and } \nu^{s\alpha}(m, v, w) \leq \mathbf{E}_{s\alpha}^{\sigma} A_{N_1, \Pi}^{\sigma}(0, \infty)^{-1/2} \mathcal{U}_{\infty}^{\sigma}(v, 0).$$

In fact, the quantities

$$\mathbf{E}_{s\alpha}^{\sigma} A_{N_1, \Pi}^{\sigma}(0, \infty)^{-1/2} \mathcal{U}_{\infty}^{\sigma}(0, w) \quad \text{and} \quad A_{N_1, \Pi}^{\sigma}(0, \infty)^{-1/2} \mathbf{E}_{s\alpha}^{\sigma} \mathcal{U}_{\infty}^{\sigma}(v, 0)$$

are estimated in the proofs of Theorem 6.2 and Theorem 6.3, respectively. Therefore, we have that for every $\varepsilon > 0$, there exists a constant $c = c_{\varepsilon, \Lambda, \alpha}$ such that for all $(m, v, w) \in K(1)$,

$$\nu^{s\alpha}(m, v, w) \leq c e^{-\rho_0(s\alpha)} e^{s \overline{\gamma}_{\Lambda_3}(\alpha) \gamma_{\Lambda_3}(\alpha)} \text{ if } \|w\| \geq \varepsilon > 0$$

and

$$\nu^{s\alpha}(m, v, w) \leq c e^{-\rho_0(s\alpha)} e^{s \overline{\gamma}_{\Lambda_2 \cup \Lambda_3}(\alpha) \gamma_{\Lambda_2 \cup \Lambda_3}(\alpha)} \text{ if } \|v\| \geq \varepsilon > 0.$$

Thus by the homogeneity (6.6),

$$\nu(m, v, w) \leq c(1 + |(m, v, w)|_{\alpha})^{-\overline{\gamma}_{\Lambda_3}(\alpha) \gamma_{\Lambda_3}(\alpha)} \text{ if } \|w_o\| \geq \varepsilon > 0$$

and

$$\nu(m, v, w) \leq c(1 + |(m, v, w)|_{\alpha})^{-\overline{\gamma}_{\Lambda_2 \cup \Lambda_3}(\alpha) \gamma_{\Lambda_2 \cup \Lambda_3}(\alpha)} \text{ if } \|v_o\| \geq \varepsilon > 0.$$

The inequality (1.8) follows.   □

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