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On the number of peaks of the eigenfunctions of the linearized Gel'fand problem

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Abstract We derive a second order estimate for the first m eigenvalues and eigenfunctions of the linearized Gel'fand problem associated to solutions which blow-up at m points. This allows us to determine, in some suitable situations, some qualitative properties of the first m eigenfunctions as the number of points of concentration or the multiplicity of the eigenvalue.

Keywords Gel'fand problem · Asymptotic estimates · Green's function

Mathematics Subject Classification 35J15 · 35J60 · 35J61

1 Introduction and statement of the main results

Let us consider the Gel'fand problem,

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $\lambda > 0$ is a real parameter. This problem appears in a wide variety of areas of mathematics such as the conformal embedding of a flat domain into a sphere [1], self-dual gauge field theories [13], equilibrium states of large number of vortices [3,4,14,15,19,20], stationary states of chemotaxis motion

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[21], and so forth. See [12] for more about our motivation and [22] for other background materials.

Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of positive values such that $\lambda_n \to 0$ as $n \to \infty$ and let $u_n = u_n(x)$ be a sequence of solutions of (1.1) for $\lambda = \lambda_n$. In [17], the authors studied solutions $\{u_n\}$ which blow-up at m-points (see next section for more details). This means that there is a set $\mathcal{S} = \{\kappa_1, \dots, \kappa_m\} \subset \Omega$ of m distinct points such that

(i) $\|u_n\|_{L^\infty(\omega)} = O(1)$ for any $\omega \Subset \overline{\Omega} \setminus \mathcal{S}$,
(ii) $u_n|_{\mathcal{S}} \to +\infty$ as $n \to \infty$.

In [2,8,18], and [7] some sufficient conditions which ensure the existence of this type of solutions are given. Throughout the paper, we will consider solutions u_n to (1.1) with m blow-up points and we investigate the eigenvalue problem

$$\begin{cases} -\Delta v_n^k = \mu_n^k \lambda_n e^{u_n} v_n^k & \text{in } \Omega \\ \|v_n^k\|_\infty = \max_{\overline{\Omega}} v_n^k = 1 & \\ v_n^k = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

which admits a sequence of eigenvalues $\mu_n^1 < \mu_n^2 \le \mu_n^3 \le \cdots$, where v_n^k is the kth eigenfunction of (1.2) corresponding to the eigenvalue μ_n^k. We also assume the orthogonality in Dirichlet norm,

$$\int_\Omega \nabla v_n^k \cdot \nabla v_n^{k'} = 0 \quad \text{if } k \neq k'.$$

In order to state our results, we need to introduce some notations and recall some well-known facts.

Let $R > 0$ be such that $B_{2R}(\kappa_i) \subset\subset \Omega$ for $i = 1, \dots, m$ and $B_R(\kappa_i) \cap B_R(\kappa_j) = \emptyset$ if $i \neq j$. For each $\kappa_j \in \mathcal{S}$ there exists a sequence $\{x_{j,n}\} \in B_R(\kappa_j)$ such that

$$u_n(x_{j,n}) = \sup_{B_R(x_{j,n})} u_n(x) \to +\infty \quad \text{and} \quad x_{j,n} \to \kappa_j \quad \text{as } n \to +\infty.$$

For any $j = 1, \dots, m$, we rescale u_n around $x_{j,n}$, letting

$$\tilde{u}_{j,n}(\tilde{x}) := u_n\left(\delta_{j,n}\tilde{x} + x_{j,n}\right) - u_n(x_{j,n}) \quad \text{in } B_{\frac{R}{\delta_{j,n}}}(0), \tag{1.3}$$

where the scaling parameter $\delta_{j,n}$ is determined by

$$\lambda_n e^{u_n(x_{j,n})} \delta_{j,n}^2 = 1. \tag{1.4}$$

It is known that $\delta_{j,n} \longrightarrow 0$ and for any $j = 1, \dots, m$

$$\tilde{u}_{j,n}(\tilde{x}) \to U(\tilde{x}) = \log \frac{1}{\left(1 + \frac{|\tilde{x}|^2}{8}\right)^2} \quad \text{in} \quad C^{2,\alpha}_{loc}(\mathbb{R}^2). \tag{1.5}$$

As we did for u_n, we rescale also the eigenfunctions v_n^k around $x_{j,n}$ for any $j = 1, \dots, m$. So we define

$$\tilde{v}_{j,n}^k(\tilde{x}) := v_n^k\left(\delta_{j,n}\tilde{x} + x_{j,n}\right) \quad \text{in } B_{\frac{R}{\delta_{j,n}}}(0), \tag{1.6}$$

where $\delta_{j,n}$ is as in (1.4). The rescaled eigenfunctions $\tilde{v}_{j,n}^k(\tilde{x})$ satisfy

$$\begin{cases} -\Delta \tilde{v}_{j,n}^k = \mu_n^k e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k & \text{in } B_{\frac{R}{\delta_{j,n}}}(0) \\ \|\tilde{v}_{j,n}^k\|_{L^\infty\left(B_{\frac{R}{\delta_{j,n}}}(0)\right)} \le 1. & \end{cases} \tag{1.7}$$

One of the main results of this paper concerns pointwise estimates of the eigenfunction. In particular, we are interested in the number of peaks of v_n^k for $k = 1, \dots, m$. Let us recall that, by Corollary 2.9 in [12] (see also [9]), we have that

$$v_n^k \to 0 \quad \text{in} \quad C^1\left(\overline{\Omega} \setminus \cup_{j=1}^m B_R\left(\kappa_j\right)\right)$$

This means that v_n^k can concentrate only at κ_j, $j = 1, \dots, m$. This leads to the following definition,

Definition 1 We say that an eigenfunction v_n^k concentrates at $\kappa_j \in \Omega$ if there exist a constant $C > 0$ and $\kappa_{j,n} \to \kappa_j$ such that

$$\left|v_n^k(\kappa_{j,n})\right| \geq C > 0 \quad \text{for n large.} \tag{1.8}$$

A problem that arises naturally is the following,

Question 1 *Let us suppose that u_n blows-up at the points $\{\kappa_1, \dots, \kappa_m\}$. Is the same true for the eigenfunction v_n^k, $k = 1 \dots, m$?*

A first partial answer related to this question was given in [12], where the following result was proved.

Theorem 1.1 *For each $k \in \{1, \dots, m\}$ we have that $\mu_n^k \to 0$ and there exists a vector*

$$\mathfrak{c}^k = (c_1^k, \dots, c_m^k) \in [-1, 1]^m \subset \mathbb{R}^m, \quad \mathfrak{c}^k \neq \mathbf{0} \tag{1.9}$$

such that for each $j \in \{1, \dots, m\}$, there exists a sub-sequence satisfying

$$\tilde{v}_{j,n}^k(x) \to c_j^k \quad \textit{in } C_{loc}^{2,\alpha}\left(\mathbb{R}^2\right) \tag{1.10}$$

$$\mathfrak{c}^k \cdot \mathfrak{c}^h = 0 \qquad \textit{if } h \neq k \tag{1.11}$$

and

$$\frac{v_n^k}{\mu_n^k} \to 8\pi \sum_{j=1}^m c_j^k G(\cdot, \kappa_j) \quad \textit{in } C_{loc}^{2,\alpha}\left(\overline{\Omega} \setminus \{\kappa_1, \dots, \kappa_m\}\right). \tag{1.12}$$

Here $G(x, y)$ denotes the Green function of $-\Delta$ in Ω with Dirichlet boundary condition, i.e.,

$$G(x, y) = \frac{1}{2\pi} \log |x - y|^{-1} + K(x, y), \tag{1.13}$$

$K(x, y)$ is the regular part of $G(x, y)$ and $R(x) = K(x, x)$ the Robin function. A consequence of Theorem 1.1 and Proposition 2.11 of [12] is that

$$v_n^k \text{ concentrates at } \kappa_j \text{ if} \quad \text{and} \quad \text{only if } c_j^k \neq 0. \tag{1.14}$$

In this paper, we characterize the values c_j^k in term of the Green function and this will allow us to determine whether c_j^k is equal to 0 or not.

Theorem 1.2 *For each $k \in \{1, \dots, m\}$, we have that*

(i) The vector $\mathfrak{c}^k = (c_1^k, \dots, c_m^k) \in [-1, 1]^m \subset \mathbb{R}^m \backslash \{\mathbf{0}\}$ *is a kth eigenvector of the matrix*

$$h_{ij} = \begin{cases} R(\kappa_i) + 2\sum_{\substack{1 \le h \le m \\ h \ne i}} G(\kappa_h, \kappa_i) & \text{if } i = j, \\ -G(\kappa_i, \kappa_j) & \text{if } i \ne j, \end{cases} \tag{1.15}$$

(ii) A sub-sequence of $\{v_n^k\}$ *satisfies*

$$\tilde{v}_{j,n}^k(\tilde{x}) = v_n^k(x_{j,n}) + \mu_n^k c_j^k U(\tilde{x}) + o\left(\mu_n^k\right) \quad in \quad C_{loc}^{2,\alpha}\left(\mathbb{R}^2\right) \tag{1.16}$$

for each $j \in \{1, \dots, m\}$, *where* $U(\tilde{x})$ *is as defined in* (1.5).

Let us observe that (1.16) is a second order estimates for v_n^k. We stress that this is new even for the case of one-peak solutions ($k = 1$). From Theorem 1.2, we can deduce the answer to the Question 1,

Corollary 1.3 *Let* $\mathfrak{c}^k = (c_1^k, \dots, c_m^k)$ *be the kth eigenvector of the matrix* (h_{ij}) *associated to a simple eigenvalue. Then if* $c_j^k \ne 0$ v_n^k *concentrates at* κ_j.

Our next aim is to understand better when $c_j^k \ne 0$. The following proposition gives some information in this direction.

Theorem 1.4 *Let* $k \in \{1, \dots, m\}$ *and* v_n^k *be the corresponding eigenfunction.*
Then we have that,

(i) v_n^1 *concentrates at m points* $\kappa_1, \dots, \kappa_m$,
(ii) any v_n^k *concentrates at least at two points* κ_i, κ_j *with* $i, j \in \{1, \dots, m\}$, $i \ne j$.

However, there are other interesting questions. One is the following:

Question 2 *Let us suppose that* μ_n^k *is a multiple eigenvalue of* (1.2). *What about its multiplicity?*

We will give an answer to this question in the case where Ω is an annulus.

Let us fix an integer $m > 2$. In [18], there was constructed a m-mode solution u_n to (1.1), i.e., a solution which is invariant with respect to a rotation of $\frac{2\pi}{m}$ in $\mathbb{R}^2$,

$$u(r, \theta) = u\left(r, \theta + \frac{2\pi}{m}\right). \tag{1.17}$$

Reasoning as in [8] one can construct, in an annulus, an m-mode solution verifying (2.1) with the symmetry properties (1.17).

Theorem 1.5 *Let* Ω *be an annulus and* u_n *be the m-mode solutions of* (1.1) *that verify* (1.17). *Let* V_n^k *the eigenspace associated to* μ_n^k *and* $dim\left(V_n^k\right)$ *denote its dimension. Then,*

- *if m is odd then* $dim\left(V_n^k\right) \ge 2$ *for any* $k \ge 2$.
- *If m is even and* μ_n^h *is simple for* $h \ge 2$ *then the limiting eigenvector* $\mathfrak{c}^h$ *verifies* $\mathfrak{c}^h = (-1, 1, -1, 1, \dots, -1, 1)$. *All the other eigenvalues satisfy* $dim\left(V_n^k\right) \ge 2$ *for any* $k \ge 2$, $k \ne h$.

The previous results rely on the next theorem which is a refinement up the second order of some estimates proved of [12]. In our opinion, this result is interesting in itself.

Theorem 1.6 *For each $k \in \{1, \dots, m\}$, it holds that*

$$\mu_n^k = -\frac{1}{2}\frac{1}{\log \lambda_n} + \left(2\pi \Lambda^k - \frac{3\log 2 - 1}{2}\right)\frac{1}{(\log \lambda_n)^2} + o\left(\frac{1}{(\log \lambda_n)^2}\right) \tag{1.18}$$

as $n \to +\infty$, where Λ^k is the kth eigenvalue of the $m \times m$ matrix (h_{ij}) defined in (1.15) assuming $\Lambda^1 \le \cdots \le \Lambda^m$.

So the effect of the domain Ω on the eigenvalues μ_n^k appears in the second order term of the expansion of μ_n^k.

The paper is organized as follows: in Sect. 2, we give some definitions and we recall some known facts. In Sect. 3, we prove Theorem 1.6 and some results on the vector $\mathfrak{c}^k$ introduced in Theorem 1.2. In Sect. 4, we complete the proof of Theorem 1.2 and prove Theorem 1.4 and Theorem 1.5.

2 Preliminaries and known facts

Let us recall some results about the asymptotic behavior of $u_n = u_n(x)$ as $n \to +\infty$. In [17], the authors proved that, along a sub-sequence,

$$\lambda_n \int_\Omega e^{u_n}\,\mathrm{d}x \to 8\pi m \tag{2.1}$$

for some $m = 0, 1, 2, \dots, +\infty$. Moreover

- If $m = 0$ the pair $(\lambda_n, u_{\lambda_n})$ converges to $(0, 0)$ as $\lambda_n \to 0$.
- If $m = +\infty$ the entire blow-up of the solutions $\{u_n\}$ occurs , i.e. $\inf_K u_n \to +\infty$ for any $K \Subset \Omega$.
- If $0 < m < \infty$ the solutions $\{u_n\}$ blow-up at m-points. Thus there is a set $\mathcal{S} = \{\kappa_1, \dots, \kappa_m\} \subset \Omega$ of m distinct points such that $\|u_n\|_{L^\infty(\omega)} = O(1)$ for any $\omega \Subset \overline{\Omega} \setminus \mathcal{S}$,

$$u_n|_{\mathcal{S}} \to +\infty \quad \text{as} \quad n \to \infty,$$

  and

$$u_n \to \sum_{j=1}^m 8\pi\, G(\cdot, \kappa_j) \quad \text{in} \quad C^2_{\text{loc}}(\overline{\Omega} \setminus \mathcal{S}). \tag{2.2}$$

In [17], it is also proved that the blow-up points $\mathcal{S} = \{\kappa_1, \dots, \kappa_m\}$ satisfy

$$\nabla H^m(\kappa_1, \dots, \kappa_m) = 0, \tag{2.3}$$

where

$$H^m(x_1, \dots, x_m) = \frac{1}{2}\sum_{j=1}^m R(x_j) + \frac{1}{2}\sum_{\substack{1 \le j,h \le m \\ j \ne h}} G(x_j, x_h).$$

Here H^m is the *Hamiltonian* function of the theory of vortices with equal intensities, see [3,4,14,15,20] and references therein.

As we did in the introduction, let $R > 0$ be such that $B_{2R}(\kappa_i) \subset\subset \Omega$ for $i = 1, \dots, m$ and $B_R(\kappa_i) \cap B_R(\kappa_j) = \emptyset$ if $i \ne j$ and $x_{j,n}$, u_n, $\tilde{u}_{j,n}$, and $\delta_{j,n}$ as in (1.3), (1.4). In [11], Corollary 4.3, it is shown that there exists a constant $d_j > 0$ such that

$$\delta_{j,n} = d_j \lambda_n^{\frac{1}{2}} + o\left(\lambda_n^{\frac{1}{2}}\right) \tag{2.4}$$

as $n \to \infty$ for a sub-sequence, and in particular, $\delta_{j,n} \longrightarrow 0$. In [11], the exact value of d_j was not computed, but for our aim, it is crucial to have it. We will give it in (3.10). From (1.4) and (2.4), we have

$$u_n(x_{j,n}) = -2\log\lambda_n - 2\log d_j + o(1) \tag{2.5}$$

as $n \to \infty$ for any $j = 1, \dots, m$.

The function $\tilde{u}_{j,n}$ defined in the Introduction satisfies

$$\begin{cases} -\Delta\tilde{u}_{j,n} = e^{\tilde{u}_{j,n}} & \text{in } B_{\frac{R}{\delta_{j,n}}}(0) \\ \tilde{u}_{j,n} \le \tilde{u}_{j,n}(0) = 0 & \text{in } B_{\frac{R}{\delta_{j,n}}}(0). \end{cases}$$

Using the result of [5], it is easy to see that, for any $j = 1, \dots, m$

$$\tilde{u}_{j,n}(\tilde{x}) \to U(\tilde{x}) = \log\frac{1}{\left(1 + \frac{|\tilde{x}|^2}{8}\right)^2} \quad \text{in} \quad C^{2,\alpha}_{loc}(\mathbb{R}^2). \tag{2.6}$$

Moreover, it holds

$$\left|\tilde{u}_{j,n}(\tilde{x}) - U(\tilde{x})\right| \le C \quad \forall \tilde{x} \in B_{\frac{R}{\delta_{j,n}}}(0) \tag{2.7}$$

for any $j = 1, \dots, m$ for a suitable positive constant C, see [16].

Let us consider the eigenfunction v_n^k defined in (1.2) and recall the following result:

Theorem 2.1 ([12]) *For $\lambda_n \to 0$, it holds that*

$$\mu_n^k = -\frac{1}{2}\frac{1}{\log\lambda_n} + o\left(\frac{1}{\log\lambda_n}\right) \quad \textit{for } 1 \le k \le m, \tag{2.8}$$

$$\mu_n^k = 1 - 48\pi\eta^{2m-(k-m)+1}\lambda_n + o(\lambda_n) \quad \textit{for } m+1 \le k \le 3m, \tag{2.9}$$

and

$$\mu_n^k > 1 \quad \textit{for} \quad k \ge 3m+1, \tag{2.10}$$

where η^k $(k = 1, \dots, 2m)$ is the kth eigenvalue of the matrix $D(\text{Hess}H^m)D$ at $(\kappa_1, \dots, \kappa_m)$. Here $D = (D_{ij})$ is the diagonal matrix diag$[d_1, d_1, d_2, d_2, \dots, d_m, d_m]$ *(see* (2.4) *for the definition of the constants d_j and* (3.10) *for the precise value of it).*

One of the purposes of this paper is to refine (2.8) (see Theorem 1.6 in the introduction).

3 Fine behavior of eigenvalues

We start from the following proposition, which plays a crucial role in our argument.

Proposition 3.1 *For any $k = 1, \ldots, m$ we have*

$$\left\{\frac{1}{\mu_n^k} - u_n(x_{j,n})\right\} \lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k \, dx$$
$$= (8\pi)^2 \sum_{\substack{1 \le i \le m \\ i \ne j}} (c_i^k - c_j^k) G(\kappa_j, \kappa_i) - 16\pi c_j^k + o(1). \quad (3.1)$$

Proof From (1.1) and (1.2), we have

$$\begin{aligned}\int_{\partial B_R(x_{j,n})} \left\{\frac{\partial u_n}{\partial \nu}\frac{v_n^k}{\mu_n^k} - u_n \frac{\partial}{\partial \nu}\left(\frac{v_n^k}{\mu_n^k}\right)\right\} \mathrm{d}\sigma &= \int_{B_R(x_{j,n})} \left\{\Delta u_n \frac{v_n^k}{\mu_n^k} - u_n \Delta \frac{v_n^k}{\mu_n^k}\right\} \mathrm{d}x \\ &= -\frac{1}{\mu_n^k}\lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k \, \mathrm{d}x \\ &\quad + \lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k u_n \, \mathrm{d}x \\ &= -\frac{1}{\mu_n^k}\lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k \, \mathrm{d}x \\ &\quad + u_n(x_{j,n})\lambda_n \int_{B_R(x_{j,n})} e^{u_n} v_n^k \, \mathrm{d}x \\ &\quad + \int_{B_{\frac{R}{\delta_{j,n}}}(0)} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \tilde{u}_{j,n} \, \mathrm{d}\tilde{x} \end{aligned} \quad (3.2)$$

and

$$\int_{B_{\frac{R}{\delta_{j,n}}}(0)} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \tilde{u}_{j,n} \, \mathrm{d}x \to \int_{\mathbb{R}^2} e^U c_j^k U \, \mathrm{d}x = -16\pi c_j^k. \quad (3.3)$$

On the other hand, from (2.2) and (1.12), we have

$$\int_{\partial B_R(x_{j,n})} \left\{\frac{\partial u_n}{\partial \nu}\frac{v_n^k}{\mu_n^k} - u_n \frac{\partial}{\partial \nu}\left(\frac{v_n^k}{\mu_n^k}\right)\right\} \mathrm{d}\sigma$$
$$\to (8\pi)^2 \sum_{i=1}^m \sum_{h=1}^m c_h^k \int_{\partial B_R(\kappa_j)} \left\{\frac{\partial}{\partial \nu} G(x,\kappa_i) G(x,\kappa_h) - G(x,k_i)\frac{\partial}{\partial \nu} G(x,\kappa_h)\right\} \mathrm{d}\sigma. \quad (3.4)$$

We let

$$I_{i,h} = \int_{\partial B_R(\kappa_j)} \left\{\frac{\partial}{\partial \nu} G(x,\kappa_i) G(x,\kappa_h) - G(x,\kappa_i)\frac{\partial}{\partial \nu} G(x,\kappa_h)\right\} \mathrm{d}\sigma.$$

Then we have

case 1 $i = h$

$$I_{i,h} = 0.$$

case 2 $i \neq h$ In this case, we have

$$\begin{aligned} I_{i,h} &= \int_{B_R(\kappa_j)} \left\{\Delta G(x,\kappa_i) G(x,\kappa_h) - G(x,\kappa_i)\Delta G(x,\kappa_h)\right\} \mathrm{d}\sigma \\ &= -G(\kappa_j,\kappa_h)\delta_i^j + G(\kappa_j,\kappa_i)\delta_j^h, \end{aligned}$$

where $\delta_a^b = 1$ if $a = b$ and $\delta_a^b = 0$ otherwise.

Therefore, from (3.4) we have

$$\begin{aligned}
&\int_{\partial B_R(x_{j,n})} \left\{ \frac{\partial u_n}{\partial \nu} \frac{v_n^k}{\mu_n^k} - u_n \frac{\partial}{\partial \nu} \left(\frac{v_n^k}{\mu_n^k} \right) \right\} \mathrm{d}\sigma \\
&= (8\pi)^2 \sum_{i=1}^{m} \sum_{\substack{1 \le h \le m \\ h \ne i}} c_h^k \left\{ -G(\kappa_j, \kappa_h)\delta_i^j + G(\kappa_j, \kappa_i)\delta_j^h \right\} + o(1) \\
&= (8\pi)^2 \Big\{ - \sum_{\substack{1 \le h \le m \\ h \ne j}} c_h^k G(\kappa_j, \kappa_h) + \sum_{\substack{1 \le i \le m \\ i \ne j}} c_j^k G(\kappa_j, \kappa_i) \Big\} + o(1) \\
&= -(8\pi)^2 \sum_{\substack{1 \le i \le m \\ i \ne j}} \left(c_i^k - c_j^k \right) G(\kappa_j, \kappa_i) + o(1).
\end{aligned} \tag{3.5}$$

The proof follows from (3.2), (3.3), and (3.5). □

Next we are going to get the precise value of d_j in (2.5). For this purpose we need to strengthen (2.5).

Proposition 3.2 (cf. Estimate D in [6]) *Let u_n be a solution of (1.1) corresponding to λ_n, and let $x_{j,n}$ and R be as in Sect. 1. Then, for any $j = 1, \dots, m$ we have*

$$u_n(x_{j,n}) = -\frac{\sigma_{j,n}}{\sigma_{j,n} - 4\pi} \log \lambda_n - 8\pi \Big\{ R(x_{j,n}) + \sum_{\substack{1 \le i \le m \\ i \ne j}} G(x_{j,n}, x_{i,n}) \Big\} + 6 \log 2 + o(1), \tag{3.6}$$

where

$$\sigma_{j,n} = \lambda_n \int_{B_R(x_{j,n})} e^{u_n}\, dx \to 8\pi. \tag{3.7}$$

Proof Using the Green representation formula, from (1.1), we have

$$\begin{aligned}
u_n(x_{j,n}) &= \int_\Omega G(x_{j,n}, y)\lambda_n e^{u_n(y)}\, dy \\
&= \frac{1}{2\pi} \int_{B_R(x_{j,n})} \log |x_{j,n} - y|^{-1} \lambda_n e^{u_n(y)}\, \mathrm{d}y \\
&\quad + \int_{B_R(x_{j,n})} K(x_{j,n}, y)\lambda_n e^{u_n(y)}\, \mathrm{d}y \\
&\quad + \sum_{\substack{1 \le i \le m \\ i \ne j}} \int_{B_R(x_{i,n})} G(x_{j,n}, y)\lambda_n e^{u_n(y)}\, \mathrm{d}y \\
&\quad + \int_{\Omega \setminus \bigcup_{i=1}^m B_R(x_{i,n})} G(x_{j,n}, y)\lambda_n e^{u_n(y)}\, \mathrm{d}y \\
&= -\frac{\sigma_{j,n}}{2\pi} \log \delta_{j,n} + \frac{1}{2\pi} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \log |\tilde{y}|^{-1} e^{\tilde{u}_{j,n}(\tilde{y})}\, \mathrm{d}\tilde{y}
\end{aligned}$$

$$+ 8\pi \left\{ R(x_{j,n}) + \sum_{\substack{1 \le i \le m \\ i \ne j}} G(x_{j,n}, x_{i,n}) \right\} + o(1).$$

Using the estimate (2.7), we get here

$$\frac{1}{2\pi} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \log |\tilde{y}|^{-1} e^{\tilde{u}_{j,n}(\tilde{y})} \, \mathrm{d}\tilde{y} \to \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |\tilde{y}|^{-1} e^{U(\tilde{y})} \, \mathrm{d}\tilde{y} = -6 \log 2. \tag{3.8}$$

Then the conclusion follows by (1.4) and (3.7). □

Here we recall a fine behavior of the local mass $\sigma_{j,n}$ defined in (3.7).

Proposition 3.3 *For any $j \in \{1, \ldots, m\}$, we have*

$$\sigma_{j,n} = 8\pi + o(\lambda_n) \tag{3.9}$$

Proof see (3.56) of [6]. □

Using Proposition 3.2 and Proposition 3.3, we get the precise value of d_j given in (2.4).

Proposition 3.4 *For any $j = 1, \ldots, k$ it holds,*

$$\mathrm{d}_j = \frac{1}{8} \exp \left\{ 4\pi R(\kappa_j) + 4\pi \sum_{\substack{1 \le i \le m \\ i \ne j}} G(\kappa_j, \kappa_i) \right\}. \tag{3.10}$$

Proof From (3.6), we get

$$\begin{aligned} u_n(x_{j,n}) = &- 2 \log \lambda_n + \frac{\sigma_{j,n} - 8\pi}{\sigma_{j,n} - 4\pi} \log \lambda_n \\ &- 8\pi \left\{ R(\kappa_j) + \sum_{\substack{1 \le i \le m \\ i \ne j}} G(\kappa_j, \kappa_i) \right\} + 6 \log 2 + o(1). \end{aligned} \tag{3.11}$$

From (3.9), it follows that $\frac{\sigma_{j,n} - 8\pi}{\sigma_{j,n} - 4\pi} \log \lambda_n = o(1)$. Therefore, the claim follows from (2.5). □

As a consequence of (2.5) and Proposition 3.4, we get, using (3.1)

$$\begin{aligned} &\left\{ \frac{1}{\mu_n^k} + 2 \log \lambda_n \right\} \int_{B_R(x_{j,n})} \lambda_n e^{u_n} v_n^k \, \mathrm{d}x = (8\pi)^2 \sum_{\substack{1 \le i \le m \\ i \ne j}} c_i^k G(\kappa_j, \kappa_i) \\ &- (8\pi)^2 c_j^k \left\{ R(\kappa_j) + 2 \sum_{\substack{1 \le i \le m \\ i \ne j}} G(\kappa_j, \kappa_i) \right\} + 48\pi c_j^k \log 2 - 16\pi c_j^k + o(1) \\ &= -(8\pi)^2 \sum_{i=1}^{m} h_{ji} c_i^k + 16\pi c_j^k (3 \log 2 - 1) + o(1), \end{aligned} \tag{3.12}$$

(see the definition of the matrix (h_{ij}) in (1.15).

Proposition 3.5 *For any $j, h \in \{1, \dots, m\}$, it holds that*

$$c_h^k \sum_{i=1}^m h_{ji} c_i^k = c_j^k \sum_{i=1}^m h_{hi} c_i^k \tag{3.13}$$

Proof Multiplying $\int_{B_R(x_{h,n})} \lambda_n e^{u_n} v_n^k \, dx$ to (3.12) and $\int_{B_R(x_{j,n})} \lambda_n e^{u_n} v_n^k \, dx$ to (3.12) with $j = h$, and then subtracting the latter from the former, we get the conclusion from (1.5) and (1.10). □

Proposition 3.6 *The vector $\mathfrak{c}^k$, defined in (1.9), is an eigenvector of (h_{ij}).*

Proof First we assume that there are $c_j^k \neq 0$ and $c_h^k \neq 0$ for $j \neq h$. Then (3.13) gives

$$\frac{1}{c_j^k} \sum_{i=1}^m h_{ji} c_i^k = \frac{1}{c_h^k} \sum_{i=1}^m h_{hi} c_i^k = \Lambda^k. \tag{3.14}$$

Then Λ^k is an eigenvalue of (h_{ij}) if $c_j^k \neq 0$ for all $j = 1, \dots, m$.

On the other hand, for $j \in \{1, \dots, m\}$ satisfying $c_j^k = 0$, we can choose $c_h^k \neq 0$ (see (1.9)) so that

$$\sum_{i=1}^m h_{ji} c_i^k = 0 \quad \text{if } c_j^k = 0. \tag{3.15}$$

From (3.14) and (3.15), we get that $\mathfrak{c}^k$ is an eigenvector of (h_{ij}) if there are at least two j satisfying $c_j^k \neq 0$.

The last case is that there is only one j satisfying $c_j^k \neq 0$, but this never happens. Indeed, in this case (3.13) becomes

$$\sum_{i=1}^m h_{hi} c_i^k = h_{hj} c_j^k = 0 \quad (j \neq h)$$

which contradicts $h_{hj} = -G(\kappa_h, \kappa_j) \neq 0$. □

Proof of Theorem 1.6 Take $c_j^k \neq 0$. Then Proposition 3.6 implies that $\sum_{i=1}^m h_{ji} c_i^k = \Lambda^k c_j^k$ and therefore (3.12) implies that

$$\frac{1}{\mu_n^k} = -2 \log \lambda_n - 8\pi \Lambda^k + 2(3 \log 2 - 1) + o(1). \tag{3.16}$$

Indeed, letting $L = -8\pi \Lambda^k + 2(3 \log 2 - 1)$, (3.16) leads that

$$\mu_n^k = \frac{1}{-2 \log \lambda_n + L + o(1)} = -\frac{1}{2 \log \lambda_n} - \frac{L}{4} \cdot \frac{1}{(\log \lambda_n)^2} + o\left(\frac{1}{(\log \lambda_n)^2}\right). \tag{3.17}$$

Therefore (1.18) follows.

The formula (1.18) gives $\Lambda^1 \leq \cdots \leq \Lambda^m$, since $\mu_n^1 < \mu_n^2 \leq \cdots \leq \mu_n^m$. Consequently, we get Λ^k is the kth eigenvalue. Since Λ^k depends only on (h_{ij}) then equation (1.18) holds without taking a sub-sequence. □

4 Fine behavior of eigenfunctions

We start this section with the following

Proposition 4.1 *For any $k, j \in \{1, \ldots, m\}$, we have*

$$\frac{v_n^k(x_{j,n})}{\mu_n^k} = \frac{1}{2\pi} \log \delta_{j,n}^{-1} \int_{B_R(x_{j,n})} \lambda_n e^{u_n(y)} v_n^k(y)\, dy + 8\pi \left\{ c_j^k R(\kappa_j) + \sum_{\substack{1 \le i \le m \\ i \ne j}} c_i^k G(\kappa_j, \kappa_i) \right\} - 6 c_j^k \log 2 + o(1). \tag{4.1}$$

Proof Using the Green representation formula and (1.2), we have, as in the proof of the Proposition 3.2

$$\begin{aligned} \frac{v_n^k(x_{j,n})}{\mu_n^k} &= \int_\Omega G(x_{j,n}, y) \lambda_n e^{u_n(y)} v_n^k(y)\, dy \\ &= \frac{1}{2\pi} \log \delta_{j,n}^{-1} \int_{B_R(x_{j,n})} \lambda_n e^{u_n(y)} v_n^k(y)\, dy \\ &\quad + \frac{1}{2\pi} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \log |\tilde{y}|^{-1} e^{\tilde{u}_{j,n}(\tilde{y})} \tilde{v}_{j,n}^k(\tilde{y})\, d\tilde{y} \\ &\quad + \left\{ 8\pi c_j^k R(\kappa_j) + 8\pi \sum_{\substack{1 \le i \le m \\ i \ne j}} c_i^k G(\kappa_j, \kappa_i) \right\} + o(1) \end{aligned}$$

and the claim follows. □

Remark 4.2 From (4.1), (1.4), (3.6), and Proposition 3.3, we get

$$\begin{aligned} \frac{1}{\mu_n^k} \int_{B_R(x_{j,n})} \lambda_n e^{u_n} v_n^k(x_{j,n})\, dx &= \frac{\sigma_{j,n} v_n^k(x_{j,n})}{\mu_n^k} \\ &= -2 \log \lambda_n \int_{B_R(x_{j,n})} \lambda_n e^{u_n} v_n^k\, dx - (8\pi)^2 \sum_{i=1}^m h_{ji} c_i^k \\ &\quad + 48\pi c_j^k \log 2 + o(1). \end{aligned} \tag{4.2}$$

Proposition 4.3 *For any $k, j \in \{1, \ldots, m\}$ we have*

$$\lambda_n \int_{B_R(x_{j,n})} e^{u_n} \frac{v_n^k(x) - v_n^k(x_{j,n})}{\mu_n^k}\, dx = -16\pi c_j^k + o(1).$$

Proof Subtracting (3.12) by (4.2) we get the claim. □

Proof of Theorem 1.2 Set

$$\tilde{z}_n := \frac{\tilde{v}_{j,n}^k - v_n^k(x_{j,n})}{\mu_n^k} \quad \text{in } B_{\frac{R}{\delta_{j,n}}}(0).$$

Then

$$-\Delta \tilde{z}_n = \mu_n^k e^{\tilde{u}_{j,n}} \tilde{z}_n + v_n^k(x_{j,n}) e^{\tilde{u}_{j,n}}. \tag{4.3}$$

The claim follows from elliptic estimates once we prove that

$$\tilde{z}_n = c_j^k U(\tilde{x}) + o(1) \quad \text{locally uniformly in } \mathbb{R}^2. \tag{4.4}$$

Using again the Green representation formula for (1.2), we have for $x \in \omega \subset\subset B_R(x_{j,n})$

$$\begin{aligned}\frac{v_n^k(x)}{\mu_n^k} =& \lambda_n \int_\Omega G(x,y) e^{u_n(y)} v_n^k(y)\,\mathrm{d}y \\ =& \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \frac{1}{2\pi} \log \frac{1}{|x-(\delta_{j,n}\tilde{y}+x_{j,n})|} e^{\tilde{u}_n} \tilde{v}_{j,n}^k \,\mathrm{d}\tilde{y} \\ &+ 8\pi c_j^k K(x,\kappa_j) + 8\pi \sum_{\substack{1\le i\le m \\ i\ne j}} c_i^k G(x,\kappa_i) + o(1).\end{aligned}$$

Therefore, letting $x = \delta_{j,n}\tilde{x} + x_{j,n}$, we have for every $\tilde{x} \in \tilde{\omega} \subset\subset \mathbb{R}^2$ that

$$\begin{aligned}\frac{\tilde{v}_{j,n}^k(\tilde{x})}{\mu_n^k} =& \frac{1}{2\pi} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \log \frac{1}{|\delta_{j,n}\tilde{x} + x_{j,n} - \delta_{j,n}\tilde{y} - x_{j,n}|} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \,\mathrm{d}\tilde{y} \\ &+ 8\pi c_j^k K(\delta_{j,n}\tilde{x} + x_{j,n}, \kappa_j) + 8\pi \sum_{\substack{1\le i\le m \\ i\ne j}} c_i^k G(\delta_{j,n}\tilde{x} + x_{j,n}, \kappa_i) + o(1) \\ =& \frac{1}{2\pi} \log \frac{1}{\delta_{j,n}} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \,\mathrm{d}\tilde{y} + \frac{1}{2\pi} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \log \frac{1|\tilde{x}-\tilde{y}|^{\tilde{u}_{j,n}}}{e} \tilde{v}_{j,n}^k \,\mathrm{d}\tilde{y} \\ &+ 8\pi c_j^k R(\kappa_j) + 8\pi \sum_{\substack{1\le i\le m \\ i\ne j}} c_i^k G(\kappa_j,\kappa_i) + o(1) \quad \text{(using (4.1))} \\ =& \frac{v_n^k(x_{j,n})}{\mu_n^k} + \frac{1}{2\pi} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \log \frac{1}{|\tilde{x}-\tilde{y}|} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \, d\tilde{y} + 6c_j^k \log 2 + o(1)\end{aligned}$$

Then recalling the definition of $\tilde{z}_n$, we have

$$\tilde{z}_n = \frac{1}{2\pi} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} \log \frac{1}{|\tilde{x}-\tilde{y}|} e^{\tilde{u}_{j,n}} \tilde{v}_{j,n}^k \,\mathrm{d}\tilde{y} + 6c_j^k \log 2 + o(1)$$

so that

$$\tilde{z}_n = \frac{1}{2\pi} c_j^k \int_{\mathbb{R}^2} \log \frac{1}{|\tilde{x}-\tilde{y}|} e^U \,\mathrm{d}\tilde{y} + 6c_j^k \log 2 + o(1)$$

locally uniformly with respect to $\tilde{x}$ since $e^{\tilde{u}_{j,n}} = O(|\tilde{x}|^{-4})$ uniformly as $|\tilde{x}| \to \infty$.

Observe that

$$\tilde{\Psi}(\tilde{x}) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|\tilde{x}-\tilde{y}|} e^U \,\mathrm{d}\tilde{y}$$

satisfy

$$-\Delta\tilde{\Psi} = e^U \quad \text{in} \quad \mathfrak{D}'(\mathbb{R}^2).$$

and it is a radially symmetric function. Then, since $-\Delta U = e^U$ and $U(0) = 0$, we have $\tilde{\Psi} - \tilde{\Psi}(0) = U$, where $\tilde{\Psi}(0) = -6\log 2$, see (3.8). Therefore, $\tilde{\Psi} = U - 6\log 2$. This implies that $\tilde{z}_n \to c_j^k U$ locally uniformly and this proves (4.4). Finally, by Proposition 3.6, we have that the proof of Theorem 1.2 is complete. □

5 Proof of Theorems 1.4 and 1.5

Proof of Theorem 1.4 The final part of the proof of Proposition 3.5 shows that, for any vector $\mathfrak{c}^k$, we have that at least *two* components of $\mathfrak{c}^k$ are different from zero. This shows ii). Now we are going to prove i).

We can assume that $v_n^1 > 0$ and then $c_j^1 \geq 0$ for any $j = 1, \ldots, m$. We want to prove that $c_j^1 > 0$ for any $j = 1, \ldots, m$ and so, by contradiction, let us assume that $c_1^1 = 0$ (the generic case is analogous). By (3.13), we deduce that $c_h^1 \sum_{i=2}^m h_{1i} c_i^1 = 0$. Since $\mathfrak{c}^1 \neq \mathbf{0}$, there exists $h \geq 2$ such that $c_h^1 \neq 0$. Moreover $h_{1i} < 0$ for any $i \geq 2$ and this gives a contradiction. □

Proof of Theorem 1.5 Without loss of generality, we may assume that $\Omega = \{x \in \mathbb{R}^2$ such that $0 < a < |x| < 1\}$. The m blow-up points $\kappa_1, \cdots, \kappa_m$ are located on a circle concentric with the annulus and are vertices of a regular polygon with m sides. So we can assume that $\kappa_1 = (r_0, 0)$, $\kappa_2 = r_0\left(\cos\frac{2\pi}{m}, \sin\frac{2\pi}{m}\right), \ldots, \kappa_m = r_0\left(\cos\frac{2(m-1)\pi}{m}, \sin\frac{2(m-1)\pi}{m}\right)$ for some $r_0 \in (a, 1)$.

Observe that since $G(x, \kappa_1)$ is symmetric with respect to the x_1-axis, (see Lemma 2.1 in [10]), we get $G(\kappa_j, \kappa_1) = G(\kappa_{m-j+2}, \kappa_1)$, $j = 2, \ldots, m$. Similarly the value $G(\kappa_i, \kappa_j)$ depends only on the distance between κ_i and κ_j. For example, $G(x, \kappa_2) = G(L_{-\frac{2\pi}{m}} x, \kappa_1)$ and consequently $G(\kappa_{i+1}, \kappa_2) = G(\kappa_i, \kappa_1)$, where L_θ denotes the rotation operator around 0 with angle θ. Similarly $G(\kappa_{i+k}, \kappa_{1+k}) = G(\kappa_i, \kappa_1)$. Note also that, if Ω is an annulus, the Robin function $R(x)$ is radial, so that $R(\kappa_1) = \cdots = R(\kappa_m) = R$.

Here we set $G(\kappa_i, \kappa_1) = G_i$ and $R_l = R + 4\sum_{h=2}^l G_h$.

The matrix (h_{ij}) becomes

if $m = 2l$ $(l = 1, 2, \ldots)$,

$$(h_{ij}) = \begin{pmatrix} R_l + 2G_{l+1} & -G_2 & -G_3 \ldots -G_{l+1} \ldots & -G_2 \\ -G_2 & R_l + 2G_{l+1} & -G_2 \ldots \quad \ldots \quad \ldots & -G_3 \\ & \ldots & & \\ -G_2 & -G_3 & \ldots \ \ldots \quad \ldots \quad \ldots & R_l + 2G_{l+1} \end{pmatrix},$$

and for $m = 2l + 1$ $(l = 1, 2, \ldots)$,

$$(h_{ij}) = \begin{pmatrix} R_l & -G_2 & -G_3 \ldots -G_l & -G_l & \ldots & -G_2 \\ -G_2 & R_l & -G_2 \ldots \ \ldots & \ldots & \ldots & -G_3 \\ & \ldots & & & & \\ -G_2 & -G_3 & \ldots \ \ldots \ \ldots & \ldots & -G_2 & R_l \end{pmatrix},$$

A straightforward computation shows that the first eigenvalue of (h_{ij}) is $\Lambda^1 = R + 2\sum_{h=2}^l G_h + G_{l+1}$ for $m = 2l$ and $R + 2\sum_{h=2}^l G_h$ for $m = 2l + 1$ which is simple. It is easy to see that the eigenspace corresponding to Λ^1 is spanned by $\mathfrak{c}^1 = (1, 1, \ldots, 1)$.

Now consider separately the cases where m is odd and m is even.

Case 1 m is odd.

Let v_n^k be an eigenfunction related to the eigenvalue μ_n^k with $k \geq 2$ and rotate it by an angle of $\frac{2\pi}{m}$. By the symmetry of the problem, we get that the rotated function $\bar{v}_n^k(r, \theta) =$

$v_n^k\left(r,\theta+\frac{2\pi}{m}\right)$ is still an eigenfuction related to the same eigenvalue μ_n^k. If by contradiction the eigenvalue μ_n^k is simple we have that

$$\bar{v}_n^k=\alpha_n v_n^k, \tag{5.1}$$

for some $\alpha_n\neq 0$ with $\alpha_n\leq 1$.

Let $\mathfrak{c}^k$ and $\bar{\mathfrak{c}}^k$ be the limiting eigenvectors given by (1.15) associated to v_n^k and $\bar{v}_n^k$, respectively. Denoting by $\mathfrak{c}^k=(c_1^k,\ldots,c_m^k)$, we have, by the definition of $\bar{v}_n^k$,

$$\bar{\mathfrak{c}}^k=\left(c_2^k,c_3^k\ldots,c_m^k,c_1^k\right). \tag{5.2}$$

By (5.1) and (5.2), we derive that

$$\alpha c_i^k=c_{i+1}^k \quad \text{for } i=1,\ldots,m, \text{ meaning that } c_{m+1}=c_1, \tag{5.3}$$

where $\alpha=\lim_{n\longrightarrow\infty}\alpha_n$. From (5.3), we get that $c_i^k=\alpha^m c_i^k$. Since $\mathfrak{c}^k\neq 0$, we get $\alpha^m=1$ and since m is odd we derive that $\mathfrak{c}^k=(1,1,\ldots,1)=\mathfrak{c}^1$. This gives a contradiction since $k\geq 2$.

Case 2 m is even.

Let v_n^k be an eigenfunction related to the eigenvalue μ_n^k with $k\geq 2$ and define $\bar{v}_n^k$ as in the previous case. Repeating step by step the proof, assuming that μ_n^k is a simple eigenvalue, we again deduce that $\alpha^m=1$. However, since in this case m is even, we have the solution $\alpha=-1$ and by (5.3), we get $\mathfrak{c}^k=(-1,1,-1,1,\ldots,-1,1)$. □

Remark 5.1 When $m=2l$, the eigenvalue Λ^k corresponding to $\mathfrak{c}^k=(-1,1,-1,1,\ldots,-1,1)$ is given by

$$\Lambda^k=R+\left(2+(-1)^{\frac{m+2}{2}}\right)G_{\frac{m+2}{2}}+2\sum_{h=2}^{l}\left(2+(-1)^h\right)G_h.$$

A direct computation proves that for $m=4$ the eigenvalue Λ^k is simple if $G_2\neq G_3$. For $m>4$, similar conditions hold. Anyway these conditions are not easy to check because we do not know explicitly the Green function of the annulus. For this reason, we will not investigate further.

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