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Simplicity and exceptionality of syzygy bundles over $\mathbb{P}^n$

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Abstract In this work, we will prove results that ensure the simplicity and the exceptionality of vector bundles, which are defined by the splitting of pure resolutions. We will call such objects syzygy bundles.

Keywords Syzygy bundles · Pure resolutions · Simplicity and exceptionality

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1 Introduction

The study of particular families of vector bundles over projective varieties has always taken a great part in algebraic geometry. In particular, many authors focused on the family of syzygy bundles, defined as the kernel of an epimorphism of the form

$$\phi : \oplus_{i=1}^{t} \mathcal{O}_{\mathbb{P}^n}(-d_i) \longrightarrow \mathcal{O}_{\mathbb{P}^n},$$

that have been studied in the last decades. Brenner in [4] gives combinatorial conditions for (semi)stability of the syzygy bundles on $\mathbb{P}^n$ when they are given by monomial ideals. Coanda in [5] studies stability for syzygies on $\mathbb{P}^n$ defined by polynomials of the same degree, of any possible rank for $n \geq 3$. Costa, Marques, and Miró-Roig, see [6], also study stability of syzygies on $\mathbb{P}^n$ given by polynomials of same degree and studied moduli spaces.

Ein, Lazarsfeld, and Mustopa in [8,9] extend the problem for smooth projective varieties X, studying the stability of the syzygy bundles that are given by the kernel of the evaluation map $\mathrm{eval}_L : H^0(L) \otimes_{\mathbb{K}} \mathcal{O}_X \longrightarrow L$ where L is a very ample line bundle over X.

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We define the *syzygy bundles* on $\mathbb{P}^n$ as the vector bundles coming from the splitting of pure resolutions of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_p}(-d_p) \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(-d_1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_0} \longrightarrow 0 \qquad (1)$$

into short exact sequences. Observe that the first syzygy bundle F in (1) obtained as

$$0 \longrightarrow F \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(-d_1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_0} \longrightarrow 0$$

is also a syzygy bundle in the sense of [4,5] and [6].

In the second section, we recall some notions on pure resolutions, and we introduce in detail what we will mean by *syzygy bundle* on the projective space.

In the third section, we will prove two results, see Theorems 3.1 and 3.6, which ensure the simplicity of the syzygy bundles previously defined. Recall that a vector bundle E on $\mathbb{P}^n$ is simple if $\dim \mathrm{Hom}(E, E) = 1$. The results here generalize the ones proved in Section 4 and Section 5 of [13].

In particular, we provide an answer to a question proposed by Herzog and Kühl in [11], where they wonder whether the modules coming from linear pure resolution of monomial ideals are indecomposable or not. We will be able to ensure such property under specific hypotheses, see Remark 3.9.

In the fourth section, we will show necessary and sufficient conditions to prove exceptionality of the bundles, see Theorem 4.1, and we will state a conjecture, which relates syzygy bundles with Steiner bundles. There is a long-standing conjecture saying that every exceptional bundle on $\mathbb{P}^n$ is stable. For the case $n = 2$, it was proved by Drèzet and Le Potier in [7], and for $n = 3$, it was proved by Zube in [17]. There are other results in this sense for some families of vector bundles; for example, Brambilla [1] proved that exceptional Steiner bundles S on $\mathbb{P}^n$, for $n \geq 2$, given by

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{a}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{b} \longrightarrow S \longrightarrow 0\ .$$

are stable.

In the fifth section, we will consider some classical pure resolutions, studying when the bundles defined in their splitting are simple and when exceptional.

2 Preliminaries

In this section, we fix the notation that will be used in this work, and we recall some basic definitions and results. Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and let $R = \mathbb{K}[x_0, \ldots, x_n]$ be the ring of polynomials in $n + 1$ variables. Let M be a graded R-module.

An R-module $N \neq 0$ is said to be a k-*syzygy* of M if there is an exact sequence of graded R-modules

$$0 \longrightarrow N \longrightarrow F_k \xrightarrow{\varphi_k} F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} M \longrightarrow 0$$

where the modules F_i are free R-modules.

We say that M has a *finite projective dimension* if there exist a free resolution over R

$$0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0 \qquad (2)$$

The least length s of such resolutions is called the *projective dimension* of M and denoted by $\mathrm{pd}(M)$. The resolution (2) is *minimal* if $\operatorname{im}\varphi_i \subset mF_{i-1}, \forall i$, where $m = (x_0, \dots, x_n)$ is the irrelevant ideal of R. From the Hilbert syzygy Theorem, see for example [15, Theorem 1.1.8], we have that $\mathrm{pd}(M) \leq n+1$. If M has a graded minimal free resolution

$$0 \longrightarrow \oplus_{j\in\mathbb{Z}} R^{\beta_{p,j}(M)}(-j) \longrightarrow \cdots \longrightarrow \oplus_{j\in\mathbb{Z}} R^{\beta_{1,j}(M)}(-j) \longrightarrow \oplus_{j\in\mathbb{Z}} R^{\beta_{0,j}(M)}(-j) \longrightarrow M \longrightarrow 0$$

then the integers $\beta_{i,j}(M) = \dim \mathrm{Tor}_i^R(M, \mathbb{K})_j$ are called the (i, j)-th graded Betti number of M, and $\beta_i := \sum_j \beta_{i,j}(M)$ is the ith total Betti number of M.

We say M has a *pure resolution of type* $d = (d_0, \dots, d_p)$ if it is given by

$$0 \longrightarrow R^{\beta_p}(-d_p) \longrightarrow \cdots \longrightarrow R^{\beta_1}(-d_1) \longrightarrow R^{\beta_0}(-d_0) \longrightarrow M \longrightarrow 0$$

with $d_0 < d_1 < \cdots < d_p, d_i \in \mathbb{Z}$.

We say that M has a *linear resolution* if it has a pure resolution of type $(0, 1, \dots, p)$.

Eisenbud and Schreyer [10] proved the following result conjectured by Boij and Söderberg [3].

Theorem 2.1 *For any degree sequence $d = (d_0, \dots, d_p)$, there is a Cohen–Macaulay module M with a pure resolution of type d.*

Consider $S = \mathbb{K}[x_0, \dots, x_m]$ the ring of polynomials in $m+1$ variables. Let $d = (d_0, \dots, d_p)$ be a degree sequence. Then, by Theorem 2.1, there is a Cohen–Macaulay module M with pure resolution

$$0 \longrightarrow S^{\beta_p}(-d_p) \longrightarrow \cdots \longrightarrow S^{\beta_1}(-d_1) \longrightarrow S^{\beta_0}(-d_0) \longrightarrow M \longrightarrow 0\ .$$

Let $\overline{M}$ be the Artinian reduction of M. Then, $\overline{M}$ is an Artinian module with pure resolution

$$0 \longrightarrow R^{\beta_p}(-d_p) \longrightarrow \cdots \longrightarrow R^{\beta_1}(-d_1) \longrightarrow R^{\beta_0}(-d_0) \longrightarrow \overline{M} \longrightarrow 0 \quad (3)$$

where $R = \mathbb{K}[x_0, \dots, x_n]$ with $n = p-1$.

We now want to pass from modules to vector bundles and study pure resolutions involving them. Assume we have an Artinian module $\overline{M}$ with pure resolution (3); sheafifying the complex, we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{n+1}}(-d_{n+1}) \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(-d_1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_0}(-d_0) \longrightarrow 0\ . \quad (4)$$

During this paper, we will be interested in such resolution, in particular, we will be interested in studying properties of the bundles coming by splitting the resolution in short exact sequences.

Definition 2.2 We will call *syzygy bundles* the vector bundles, which arise by splitting of resolutions of the type (4).

We conclude this section recalling the following notions and results on vector bundles.

Let E be a vector bundle on $\mathbb{P}^n$. A *resolution* of E is an exact sequence

$$0 \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0 \quad (5)$$

where every F_i splits as a direct sum of line bundles.

One can show that every vector bundle on $\mathbb{P}^n$ admits resolution of the form (5), see [12, Proposition 5.3]. The minimal number d of such resolution is called *homological dimension* of E, and it is denoted by $\mathrm{hd}(E)$. For a coherent sheaf F on $\mathbb{P}^n$, let us denote the graded R-module $\oplus_{j\in\mathbb{Z}} H^q(F(j)) = H^q_*(F)$. Bohnhorst and Spindler proved the following two results, [2, Proposition 1.4] and [2, Corollary 1.7], respectively.

Proposition 2.3 *Let E be a vector bundle on $\mathbb{P}^n$. Then,*

$$\mathrm{hd}(E) \leq d \iff H^q_*(E) = 0, \forall\ 1 \leq q \leq n - d - 1.$$

Proposition 2.4 *Let E be a nonsplitting vector bundle on $\mathbb{P}^n$. Then,*

$$\mathrm{rk}(E) \geq n + 1 - \mathrm{hd}(E).$$

Recall that a vector bundle E on $\mathbb{P}^n$ is *simple* if $\dim \mathrm{Hom}(E, E) = 1$, and it is *exceptional* if it is simple and $\mathrm{Ext}^i(E, E) = 0$, for $i \geq 1$. We also recall the notion of *cokernel bundles* and *Steiner bundles*, as defined, respectively, in [1] and [16].

Definition 2.5 Let E_0 and E_1 be two vector bundles on $\mathbb{P}^n$, with $n \geq 2$. A cokernel bundle of type (E_0, E_1) on $\mathbb{P}^n$ is a vector bundle C defined by the following short exact sequence

$$0 \longrightarrow E_0^a \longrightarrow E_1^b \longrightarrow C \longrightarrow 0$$

where $b\,\mathrm{rk}\,E_1 - a\,\mathrm{rk}\,E_0 \geq n$, with $a, b \in \mathbb{N}$, and E_0, E_1 satisfy the following conditions:

- E_0 and E_1 are simple;
- $\mathrm{Hom}(E_1, E_0) = 0$;
- $\mathrm{Ext}^1(E_1, E_0) = 0$;
- the bundle $E_0^\vee \otimes E_1$ is globally generated;
- $W = \mathrm{Hom}(E_0, E_1)$ has dimension $w \geq 3$.

If, moreover,

$$\mathrm{Ext}^i(E_1, E_0) = 0, \quad \text{for each } i \geq 2$$

and

$$\mathrm{Ext}^i(E_0, E_1) = 0, \quad \text{for each } i \geq 1,$$

the pair (E_0, E_1) is called strongly exceptional, and the bundle C is called a Steiner bundle of type (E_0, E_1) on $\mathbb{P}^n$.

3 Simplicity of syzygy bundles

In this section, we will give some results that ensure the simplicity of the syzygy bundles of the following pure resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{n+1}}(-d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_n}(-d_n) \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(-d_1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_0} \longrightarrow 0 \tag{6}$$

with $d_1 < d_2 < \ldots < d_{n+1}$ and $d_i > 0$ for each i, which splits in short exact sequences

$$\begin{gathered} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{n+1}}(-d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_n}(-d_n) \longrightarrow G_1 \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow G_i \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{n-i}}(-d_{n-i}) \longrightarrow G_{i+1} \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow G_{n-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(-d_1) \longrightarrow \mathcal{O}_{\mathbb{P}_n}^{\beta_0} \longrightarrow 0 \end{gathered} \tag{7}$$

We will also consider the dual resolution of (6) and tensor it by $\mathcal{O}_{\mathbb{P}^n}(-d_{n+1})$, obtaining

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_{n+1})^{\beta_0} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(d_1 - d_{n+1}) \longrightarrow \cdots \longrightarrow \\ \mathcal{O}_{\mathbb{P}^n}^{\beta_n}(d_n - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{n+1}} \longrightarrow 0 \end{aligned} \tag{8}$$

which splits as

$$\begin{gathered} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_0}(-d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(d_1 - d_{n+1}) \longrightarrow F_1 \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow F_j \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{j+1}}(d_{j+1} - d_{n+1}) \longrightarrow F_{j+1} \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow F_{n-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_n}(d_n - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}_n}^{\beta_{n+1}} \longrightarrow 0 \end{gathered} \tag{9}$$

Let us notice that we have supposed, without loss of generality, that $d_0 = 0$; else we can tensor the resolution (4) by $\mathcal{O}_{\mathbb{P}^n}(-d_0)$ in order to obtain a new resolution as in (6). Let us prove now some results which ensure the simplicity of the bundles F_i, for i from 1 to $n-1$; in particular, the next theorem tell us when the syzigies are simple only looking at the first or the last Betti number.

Theorem 3.1 *Consider a pure resolution as in* (6). *If* $\beta_0 = 1$ *or* $\beta_{n+1} = 1$, *then all bundles* F_i, *for i from 1 to* $n-1$, *are simple.*

Proof Let us consider first the case $\beta_0 = 1$, whose importance will be explained by Corollary 3.2.

Let us prove first that the bundle F_1 is simple.

Consider the exact sequence, obtained by (9),

$$0 \longrightarrow (F_1^\vee)^{\beta_0}(-d_{n+1}) \longrightarrow (F_1^\vee)^{\beta_1}(d_1 - d_{n+1}) \longrightarrow F_1^\vee \otimes F_1 \longrightarrow 0$$

which induces the long exact sequence in cohomology

$$\begin{aligned} 0 \longrightarrow H^0((F_1^\vee)^{\beta_0}(-d_{n+1})) \longrightarrow H^0((F_1^\vee)^{\beta_1}(d_1 - d_{n+1})) \\ \longrightarrow H^0(F_1^\vee \otimes F_1) \longrightarrow H^1((F_1^\vee)^{\beta_0}(-d_{n+1})) \longrightarrow \cdots \end{aligned}$$

Taking again the short exact sequences obtained by dualizing (9) and tensoring by $\mathcal{O}_{\mathbb{P}^n}(d_1 - d_{n+1})$

$$0 \longrightarrow F_{i+1}^\vee(d_1 - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{i+1}}(d_1 - d_{i+1}) \longrightarrow F_i^\vee(d_1 - d_{n+1}) \longrightarrow 0,$$

for i from 1 to $n-2$, we obtain the following chain of isomorphisms

$$H^0(F_1^\vee(d_1 - d_{n+1})) \simeq H^1(F_2^\vee(d_1 - d_{n+1})) \simeq \cdots \simeq H^{n-2}(F_{n-1}^\vee(d_1 - d_{n+1})) = 0,$$

and the isomorphisms are true (especially the first one) because $d_1 < d_{n+1}$ and the vanishing comes from the short exact sequences of (9). Combining these two results, we get an injective map

$$H^0(F_1^\vee \otimes F_1) \hookrightarrow H^1((F_1^\vee)^{\beta_0}(-d_{n+1})).$$

Consider the short exact sequence

$$0 \longrightarrow F_1^\vee(-d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(-d_1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_0} \longrightarrow 0 \tag{10}$$

from which it is straightforward to obtain $\beta_0 = h^1\left(F_1^\vee(-d_{n-1})\right)$; this implies, in the case $\beta_0 = 1$, that the F_1 is a simple bundle.

Let us prove now that each bundle F_i is simple, for i from 2 to $n-1$.

Consider the following exact sequence obtained from (9)

$$0 \longrightarrow F_{i-1} \otimes F_i^\vee \longrightarrow (F_i^\vee)^{\beta_i}(d_i - d_{n+1}) \longrightarrow F_i \otimes F_i^\vee \longrightarrow 0 \tag{11}$$

Taking the exact sequences of type

$$0 \longrightarrow F_{i+1}^\vee(d_i - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{i+1}}(d_i - d_{i+1}) \longrightarrow F_i^\vee(d_i - d_{n+1}) \longrightarrow 0$$

for i from 1 to $n-2$, and their induced long exact sequence in cohomology, we get a chain of isomorphisms of type

$$H^0(F_i^\vee(d_i - d_{n+1})) \simeq H^1(F_{i+1}^\vee(d_i - d_{n+1})) \simeq \cdots \simeq H^{n-1-i}(F_{n-1}^\vee(d_i - d_{n+1}) = 0, \tag{12}$$

again because $d_i < d_{n+1}$ and the vanishing comes from short exact sequences of (9). Therefore, inducing the long exact sequence in cohomology of (11), we have an inclusion of type

$$H^0(F_i \otimes F_i^\vee) \hookrightarrow H^1(F_{i-1} \otimes F_i^\vee).$$

Proceeding step by step, lowering by one the value of i, and using similar isomorphisms as in (12) that are consequence of the short exact sequences in (9), we manage to obtain the following inclusions

$$H^1(F_{i-1} \otimes F_i^\vee) \hookrightarrow H^2(F_{i-2} \otimes F_i^\vee) \hookrightarrow \cdots \hookrightarrow H^{i-1}(F_1 \otimes F_i^\vee) \hookrightarrow H^i(F_i^\vee(-d_{n+1})).$$

In order to compute the last cohomology group, we consider, as before, the exact sequences of the following form

$$0 \longrightarrow F_{i+1}^\vee(-d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{i+1}}(-d_{i+1}) \longrightarrow F_i^\vee(-d_{n+1}) \longrightarrow 0$$

for i from 1 to $n-2$, obtaining, by our hypothesis $\beta_0 = 1$,

$$H^i(F_i^\vee(-d_{n+1})) \simeq H^{i-1}(F_{i-1}^\vee(-d_{n+1})) \simeq \cdots \simeq H^1(F_1^\vee(-d_{n+1})) \simeq \mathbb{C}.$$

This proves that the bundle F_i is simple.

The case $\beta_{n+1} = 1$ can be proved, by duality, applying the same technique. Indeed, we can define $\tilde{d}_i = d_{n+1} - d_{n+1-i}$, and dualizing the resolution (8) and tensoring by $\mathcal{O}_{\mathbb{P}_n}(-\tilde{d}_{n+1})$, we obtain a new resolution of the form

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-\tilde{d}_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_n}(\tilde{d}_1 - \tilde{d}_{n+1}) \longrightarrow \cdots \longrightarrow \\ \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(\tilde{d}_n - \tilde{d}_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_0} \longrightarrow 0 \end{aligned} \tag{13}$$

where, as before, the integers $\tilde{d}_i$ satisfy $\tilde{d}_{n+1} > \tilde{d}_n > \ldots > \tilde{d}_1 > 0$, and we apply the previous technique. □

As a corollary, we have the following.

Corollary 3.2 *Consider a quotient ring $A = R/I$ where I is Artinian module and its pure resolution. Then, each vector bundle F_i, arising from the splitting of the resolution in short exact sequences, is simple.*

Proof Since A is a quotient, we have that $\beta_0 = 1$, then we can apply the previous theorem and obtain that all the bundles F_i are simple. □

With the next lemmas, we give an explicit description and boundaries for the Betti number arising in the resolution we are considering, which will be useful to prove a different theorem about simplicity of the syzygies.

Lemma 3.3 *The syzygies in the short exact sequences (9) satisfies*

$$h^0(F_{i-1}^{\vee}(d_i - d_{n+1})) = \beta_i, \ \ for\ i = 2, \ldots, n.$$

Proof Twisting the short exact sequences, we have

$$0 \longrightarrow F_i^{\vee}(d_i - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_i} \longrightarrow F_{i-1}^{\vee}(d_i - d_{n+1}) \longrightarrow 0\ .$$

One can check that

$$h^0(F_i^{\vee}(d_i - d_{n+1})) = h^1(F_i^{\vee}(d_i - d_{n+1}))$$

and the result follows. □

Lemma 3.4 *Consider the syzygies G_i and F_i from the short exact sequences (7) and (9). Then, hd(G_i) =hd$(F_i) = i$, for $i = 1, \ldots, n-1$.*

Proof Let us prove for F_i. The case of G_i is analogous. We prove it by induction on i. From the short exact sequences (9), it is clear that $\mathrm{hd}(F_1) = 1$. Let us suppose that $\mathrm{hd}(F_{i-1}) = i-1$. We know that $\mathrm{hd}(F_i) \leq i$. Suppose $\mathrm{hd}(F_i) \leq i-1$. By Proposition 2.3,

$$H_*^q(F_i) = 0, \quad \forall\ 1 \leq q \leq n-i.$$

Since $H_*^{n-i}(F_i) \simeq H_*^{n-i+1}(F_{i-1})$ from the sequences (9), and $\mathrm{hd}(F_{i-1}) = i-1$, by induction, there exists $t \in \mathbb{Z}$ such that $H^{n-i+1}(F_{i-1}(t)) \neq 0$. Therefore, $\mathrm{hd}(F_i) = i$. □

Lemma 3.5 *The Betti numbers β_i from the sequence (6) satisfy the inequalities*

$$\begin{aligned}
&\beta_1 - \beta_0 \geq n \\
&\beta_i \geq 2n - 2i + 3, \ for\ \ 2 \leq i \leq \tfrac{n+1}{2} \\
&\beta_i \geq 2i + 1, \qquad for\ \ \tfrac{n+1}{2} \leq i \leq n-1 \\
&\beta_n - \beta_{n+1} \geq n
\end{aligned}$$

In particular, $\beta_i \geq 3$, for $2 \leq i \leq n-1$.

Proof We have by Lemma 3.4 that $\mathrm{hd}(G_i)$ $=\mathrm{hd}(F_i) = i$, $1 \leq i \leq n-1$. With Proposition 2.4

$$\mathrm{rk}(E) \geq n + 1 - \mathrm{hd}(E)$$

and using the short exact sequences of F_i for $1 \leq i \leq \frac{n+1}{2}$ and the sequences of G_i for $\frac{n+1}{2} \leq i \leq n-1$, we prove the inequalities. □

We are now ready to state the second theorem on the simplicity of the syzygy bundles.

Theorem 3.6 *Consider the pure resolution* (8) *and the syzygies given by the short exact sequences. Then, if* F_1 *or* F_{n-1} *are simple, all the syzygies are simple.*

Proof Suppose F_1 is simple. Let us prove by induction hypothesis that F_i is simple for $i = 1, \ldots, n-1$. Consider an injective generic map

$$\alpha_i : F_{i-1} \to \mathcal{O}_{\mathbb{P}^n}^{\beta_i}(d_i - d_{n+1}).$$

We have the following properties:

(i) $F_{i-1}, \mathcal{O}_{\mathbb{P}^n}(d_i - d_{n+1})$ are simple, the first bundle by induction hypothesis;
(ii) $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(d_i - d_{n+1}), F_{i-1}) = 0$. It follows from

$$H^0(F_{i-1}(d_{n+1} - d_i)) \simeq H^j(F_{i-j-1}(d_{n+1} - d_i)),\ 0 \leq j \leq i-2$$

and $H^{i-2}(F_1(d_{n+1} - d_i)) = H^{i-1}(\mathcal{O}_{\mathbb{P}^n}(-d_i)) = 0$, by the sequence (9);
(iii) $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^n}(d_i - d_{n+1}), F_{i-1}) = 0$. In fact, by the above sequence (9)

$$H^1(F_{i-1}(d_{n+1} - d_i)) \simeq H^j(F_{i-j}(d_{n+1} - d_i)),\ 0 \leq j \leq i-1$$

and $H^{i-1}(F_1(d_{n+1} - d_i)) = H^i(\mathcal{O}_{\mathbb{P}^n}(-d_i)) = 0$;
(iv) $F_{i-1}^{\vee} \otimes \mathcal{O}_{\mathbb{P}^n}(d_i - d_{n+1})$ is globally generated. This is clear from the short exact sequences also in (9).
(iv) $\dim \mathrm{Hom}(F_{i-1}, \mathcal{O}_{\mathbb{P}^n}(d_i - d_{n+1})) \geq 3$. Follows from Lemmas 3.3 and 3.5.

Hence, we have that F_{i-1} and $\mathcal{O}_{\mathbb{P}^n}(d_i - d_{n+1})$ satisfies the conditions of cokernel bundles (see Definition 2.5); therefore, $\overline{F}_i = \mathrm{coker}\alpha_i$ is a cokernel bundle and since $1 + {\beta_i}^2 - h^0(F_{i-1}^{\vee}(d_i - d_{n+1}))\beta_i = 1$ by Lemma 3.3, we have that $\overline{F_i}$ is simple, see [1, Theorem 4.3].

Notice that we have two short exact sequences of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{i-1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{\beta_i}(d_i - d_{n+1}) & \longrightarrow & F_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda I} & \searrow^{g_i} & \downarrow{\scriptstyle h_i} & & \downarrow & & \\ 0 & \longrightarrow & F_{i-1} & \xrightarrow{\alpha_i} & \mathcal{O}_{\mathbb{P}^n}^{\beta_i}(d_i - d_{n+1}) & \longrightarrow & \overline{F_i} & \longrightarrow & 0 \end{array}$$

Having a 1:1 correspondence between α_i and h_i, we can always get h_i an isomorphism and therefore, $\overline{F_i} \simeq F_i$ and F_i is simple.

Regarding the other case, i.e., supposing that F_{n-1} is simple, we can define new coefficients $\tilde{d}_i$, in the same way as in the last part of the proof of Theorem 3.1, and take the dual of (8), tensor it by $\mathcal{O}_{\mathbb{P}_n}(-\tilde{d}_{n+1})$ and apply the same technique. □

We would like to find conditions to grant simplicity for every syzygy bundle in the resolution; therefore, in the next results, we ask for conditions which give us either F_1 or F_{n-1} simple bundles.

Corollary 3.7 *Consider the complex* (6). *If* $\beta_n - \beta_{n+1} = n$ *or* $\beta_1 - \beta_0 = n$, *then all the syzygies are simple.*

Proof Under this hypothesis, it follows from [2, Theorem 2.7] that either F_{n-1} is stable or F_1 is stable, then all the syzygies are simple. □

Corollary 3.8 *Consider the complex (6). If the injective map*

$$\alpha_{n+1} : \mathcal{O}_{\mathbb{P}^n}^{\beta_{n+1}}(-d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_n}(-d_n)$$

is generic and ${\beta_{n+1}}^2 + \beta_n^2 - h^0(\mathcal{O}_{\mathbb{P}^n}(d_{n+1} - d_n))\beta_{n+1}\beta_n \leq 1$, *or the injective map*

$$\alpha_1^\vee : \mathcal{O}_{\mathbb{P}^n}^{\beta_0} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(d_1)$$

is generic and ${\beta_0}^2 + \beta_1^2 - h^0(\mathcal{O}_{\mathbb{P}^n}(d_1))\beta_0\beta_1 \leq 1$, *then all the syzygies are simple.*

Proof If we have the hypothesis above, coker$\alpha_{n+1} = F_{n-1}^\vee(-d_{n+1})$ or coker$\alpha_1^\vee = F_1$ are simple cokernel bundles, see [1, Theorem 4.3], and the previous theorem applies. □

We conclude this part with the following observation.

Remark 3.9 Consider the syzygy modules N_i, for i from 1 to $p-2$, which are obtained by the pure resolution

$$\begin{array}{ccccccccccc} 0 \longrightarrow R^{\beta_p}(-d_p) \longrightarrow \cdots \longrightarrow & R^{\beta_1}(-d_1) & \longrightarrow R^{\beta_0}(-d_0) \longrightarrow \overline{M} \longrightarrow 0 \\ & \nearrow & \\ N_1 & & \\ \nearrow & & \\ 0 & & \end{array}$$

Recalling the equivalence of category between modules and their sheafifications, we get that, if the vector bundles F_i are simple, the modules N_i are indecomposable.

4 Exceptionality of syzygy bundles

In this section, we state and prove sufficient and necessary conditions to ensure the exceptionality of the syzygy bundles F_i. We obtain the following result.

Theorem 4.1 *Consider the syzygy bundles F_i as defined in (9), for i from 1 to $n-1$. Suppose also that F_i are simple for each i, then every F_i is exceptional if and only if each one of the following conditions hold*

(i) $\beta_0^2 + \beta_1^2 - \binom{d_1+n}{n}\beta_0\beta_1 = 1$;

(ii) $d_1 \leq n$;

(iii)

$$\begin{cases} H^{n-i+1}(F_{i-1}(d_{n+1} - d_i)) = H^i(F_i^\vee(d_i - d_{n+1})) = 0 \text{ if } n \text{ is even;} \\ H^{n-i+1}(F_{i-1}(d_{n+1} - d_i)) = H^i(F_i^\vee(d_i - d_{n+1})) = 0 \text{ if } n \text{ is odd and } i \neq \frac{n+1}{2}; \\ H^{n-i+1}(F_{i-1}(d_{n+1} - d_i)) \overset{H^i(\varphi)}{\simeq} H^i(F_i^\vee(d_i - d_{n+1})) \quad \text{if } n \text{ is odd and } i = \frac{n+1}{2}. \end{cases}$$

If n is odd and $i = \frac{n+1}{2}$, *then* $H^{n-i+1}(F_{i-1}(d_{n+1} - d_i)) \simeq H^i(F_{i-1} \otimes F_i^\vee)$ *and the morphism* $H^i(\varphi) : H^i(F_i^\vee \otimes F_{i-1}) \to H^i((F_i^\vee)^{\beta_i}(d_i - d_{n+1}))$ *is the one obtained by the short exact sequence*

$$0 \longrightarrow F_i^\vee \otimes F_{i-1} \xrightarrow{\varphi} (F_i^\vee)^{\beta_i}(d_i - d_{n+1}) \longrightarrow F_i^\vee \otimes F_i \longrightarrow 0$$

considering the long exact induced in cohomology.

Proof We will first look for conditions, which are equivalent to the exceptionality of the bundle F_1; therefore, we must compute the cohomology of the bundle $F_1^\vee \otimes F_1$. Consider the short exact sequence

$$0 \longrightarrow (F_1^\vee)^{\beta_0}(-d_{n+1}) \longrightarrow (F_1^\vee)^{\beta_1}(d_1 - d_{n+1}) \longrightarrow F_1^\vee \otimes F_1 \longrightarrow 0. \tag{14}$$

The first step consists now in calculating the cohomology of $F_1^\vee(-d_{n+1})$. We now consider the sequence

$$0 \longrightarrow F_1^\vee(-d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(-d_1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_0} \longrightarrow 0$$

obtaining that

$$\begin{aligned} &H^i(F_1^\vee(-d_{n+1})) = 0 &&\text{for } i = 0, 2, 3, \dots, n-2, n-1 \\ &H^1(F_1^\vee(-d_{n+1}) \simeq H^0(\mathcal{O}_{\mathbb{P}^n}^{\beta_0}) \simeq \mathbb{K}^{\beta_0} \\ &H^n(F_1^\vee(-d_{n+1}) \simeq H^n(\mathcal{O}_{\mathbb{P}^n}^{\beta_1}(-d_1)) \end{aligned}$$

We must now compute the cohomology of the second bundle appearing in (14), and we will do it using the short exact sequence

$$0 \longrightarrow F_1^\vee(d_1 - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_0}(d_1) \longrightarrow 0$$

from which we obtain that

$$H^0(F_1^\vee(d_1 - d_{n+1})) \simeq \cdots \simeq H^{n-2}(F_{n-1}^\vee(d_1 - d_{n+1})) = 0,$$

that we have already computed in the proof of Theorem 3.1, and moreover

$$\begin{aligned} h^1(F_1^\vee(d_1 - d_{n+1})) = \dim H^1(F_1^\vee(d_1 - d_{n+1})) = \beta_0\binom{d_1+n}{n} - \beta_1, \\ H^2(F_1^\vee(d_1 - d_{n+1})) \simeq \cdots \simeq H^n(F_1^\vee(d_1 - d_{n+1})) = 0. \end{aligned}$$

From the cohomology we have already calculated, we get that

$$H^2(F_1^\vee \otimes F_1) \simeq \cdots \simeq H^{n-2}(F_1^\vee \otimes F_1) \simeq H^n(F_1^\vee \otimes F_1) = 0.$$

Recall that we supposed F_1 to be simple, hence we have the following exact sequence in cohomology

$$0 \longrightarrow \mathbb{K} \longrightarrow H^1((F_1^\vee)^{\beta_0}(-d_{n+1})) \longrightarrow H^1((F_1^\vee)^{\beta_1}(d_1 - d_{n+1})) \longrightarrow H^1(F_1^\vee \otimes F_1) \longrightarrow 0.$$

Therefore, $H^1(F_1^\vee \otimes F_1)$ vanishes if and only if

$$\beta_0^2 + \beta_1^2 - \binom{d_1+n}{n}\beta_0\beta_1 = 1.$$

The other cohomology which we need to vanish is given by

$$H^{n-1}(F_1^\vee \otimes F_1) \simeq H^n(F_1^\vee(-d_{n+1}) \simeq H^n(\mathcal{O}_{\mathbb{P}^n}^{\beta_1}(-d_1)),$$

that is equal to zero if and only if $d_1 \leq n$.

Let us suppose the bundle F_{i-1} to be exceptional, and let us find conditions ensuring the exceptionality of F_i. In order to do so, consider an i fixed from 2 to $n-1$ and consider also the following short exact sequences

$$0 \longrightarrow F_i^\vee \otimes F_{i-1} \longrightarrow (F_i^\vee)^{\beta_i}(d_i - d_{n+1}) \longrightarrow F_i^\vee \otimes F_i \longrightarrow 0 \tag{15}$$

$$0 \longrightarrow F_i^\vee \otimes F_{i-1} \longrightarrow F_{i-1}^{\beta_i}(d_{n+1} - d_i) \longrightarrow F_{i-1}^\vee \otimes F_{i-1} \longrightarrow 0 \tag{16}$$

Recall that by induction hypothesis, we have that

$$H^0(F_{i-1}^\vee \otimes F_{i-1}) \simeq \mathbb{K} \quad \text{and} \quad H^k(F_{i-1}^\vee \otimes F_{i-1}) = 0 \text{ for each } k > 0.$$

Let us compute the cohomology of the bundle $F_i^\vee(d_i - d_{n+1})$; using the usual short exact sequences defined in (9), we have that:

- if $k < i$ then $H^k(F_i^\vee(d_i - d_{n+1})) \simeq H^{k+n-i-1}(F_{n-1}^\vee(d_i - d_{n+1})) = 0$, because $k+n-1-i \leq n-2$;
- similarly, if $k > i$ then $H^k(F_i^\vee(d_i - d_{n+1})) \simeq H^{k+1-i}(F_1^\vee(d_i - d_{n+1})) = 0$, because $k+1-i \geq 2$;

hence, we notice that the only possible nonvanishing cohomology of the bundle is given exactly by $H^i(F_i^\vee(d_i - d_{n+1}))$.

Using Serre duality and a similar argument, we obtain that

$$H^k(F_{i-1}(d_{n+1} - d_i)) = 0, \text{ if } k \neq n-i+1$$

the only possible nonvanishing cohomology of the bundle is given exactly by $H^{n-i+1}(F_{i-1}(d_{n+1} - d_i))$. Considering this and the induction hypothesis, we get from the exact sequence (16), that

$$H^1(F_i^\vee \otimes F_{i-1}) \simeq H^0(F_{i-1}^\vee \otimes F_{i-1}) \simeq \mathbb{K}$$

and

$$H^k(F_i^\vee \otimes F_{i-1}) \simeq H^k(F_{i-1}(d_{n+1} - d_i)) \ \forall\, i = 2, \dots, n.$$

Notice that the following cohomology groups are isomorphic

$$H^i(F_i^\vee(d_i - d_{n+1})) \simeq H^1(F_1^\vee(d_i - d_{n+1})) \simeq H^{n-1}(F_{n-1}^\vee(d_i - d_{n+1})).$$

Let us first look for an explicit expression of the group $H^1(F_1^\vee(d_i - d_{n+1}))$ and in order to do so, take the sequence

$$0 \longrightarrow F_1^\vee(d_i - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(d_i - d_1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_0}(d_i) \longrightarrow 0$$

and we are interested in the following part of the induced sequence in cohomology

$$\begin{aligned} 0 &\longrightarrow H^0(F_1^\vee(d_i - d_{n+1})) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}^{\beta_1}(d_i - d_1)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}^{\beta_0}(d_i)) \\ &\longrightarrow H^1(F_1^\vee(d_i - d_{n+1})) \longrightarrow 0 \end{aligned}$$

The problem now moves to the computation of the dimension of the vector space $H^0(F_1^\vee(d_i - d_{n+1}))$.

Consider the exact sequences

$$\begin{gathered} 0 \longrightarrow F_2^\vee(d_i - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_2}(d_i - d_2) \longrightarrow F_1^\vee(d_i - d_{n+1}) \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow F_{i-1}^\vee(d_i - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{i-1}}(d_i - d_{i-1}) \longrightarrow F_{i-2}^\vee(d_i - d_{n+1}) \longrightarrow 0 \\ 0 \longrightarrow F_i^\vee(d_i - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_i} \longrightarrow F_{i-1}^\vee(d_i - d_{n+1}) \longrightarrow 0 \\ 0 \longrightarrow F_{i+1}^\vee(d_i - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{i+1}}(\underbrace{d_i - d_{i+1}}_{<0}) \longrightarrow F_i^\vee(d_i - d_{n+1}) \longrightarrow 0 \end{gathered}$$

We have already proven that $H^0(F_i^\vee(d_i - d_{n+1})) = H^1(F_i^\vee(d_i - d_{n+1})) = 0$ and also $H^1(F_j^\vee(d_i - d_{n+1})) = 0$ for each $j = 2, \dots, i+1$; hence, we obtain that

$$h^i(F_i^\vee(d_i - d_{n+1})) = h^1(F_1^\vee(d_i - d_n + 1)) = \sum_{k=0}^{i}(-1)^k \beta_k \binom{d_i - d_k + n}{n}.$$

Let us now "go to the other side", arriving to $H^{n-1}(F_{n-1}^\vee(d_i - d_{n+1}))$.

Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{n+1}}(d_i - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_n}(d_i - d_n) \longrightarrow F_{n-1}^\vee(d_i - d_{n+1}) \longrightarrow 0$$

from which we induce the following part induced in cohomology

$$\begin{aligned} &0 \longrightarrow H^{n-1}(F_{n-1}^\vee(d_i - d_{n+1})) \longrightarrow H^n(\mathcal{O}_{\mathbb{P}^n}^{\beta_{n+1}}(d_i - d_{n+1})) \longrightarrow H^n(\mathcal{O}_{\mathbb{P}^n}^{\beta_n}(d_i - d_n)) \\ &\longrightarrow H^n(F_{n-1}^\vee(d_i - d_{n+1})) \longrightarrow 0. \end{aligned}$$

As before, take

$$\begin{gathered} 0 \longrightarrow F_{n-1}^\vee(d_i - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{n-1}}(d_i - d_{n+1}) \longrightarrow F_{n-2}^\vee(d_i - d_{n+1}) \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow F_{i+2}^\vee(d_i - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{i+2}}(d_i - d_{i+2}) \longrightarrow F_{i+1}^\vee(d_i - d_{n+1}) \longrightarrow 0 \\ 0 \longrightarrow F_{i+1}^\vee(d_i - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{i+1}}(d_i - d_{i+1}) \longrightarrow F_i^\vee(d_i - d_{n+1}) \longrightarrow 0 \\ 0 \longrightarrow F_i^\vee(d_i - d_{n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_i} \longrightarrow F_{i-1}^\vee(d_i - d_{n+1}) \longrightarrow 0 \end{gathered}$$

Suppose that $i < n-1$ (or else the computation comes directly considering only the first exact sequence and we will obtain the same result), we have that

$$H^n(F_i^\vee(d_i - d_{n+1})) \simeq H^{n-1}(F_i^\vee(d_i - d_{n+1})) = 0$$

and also

$$H^{n-1}(F_j^\vee(d_i - d_{n+1})) = 0, \quad \text{for each } j = i+1, \dots, n-2.$$

We can conclude that

$$h^i(F_i^\vee(d_i - d_{n+1})) = h^{n-1}(F_{n-1}^\vee(d_i - d_{n+1})) = \begin{cases} \sum_{k=i+1}^{n+1}(-1)^{k+1}\beta_k\binom{d_k - d_i - 1}{n} & \text{for } n \text{ even} \\ \sum_{k=i+1}^{n+1}(-1)^{k}\beta_k\binom{d_k - d_i - 1}{n} & \text{for } n \text{ odd.} \end{cases}$$

Let us focus now on the cohomology of the bundle $F_{i-1}(d_{n+1} - d_i)$.

We obtain by Serre duality that

$$H^k(F_{i-1}(d_{n+1} - d_i)) \simeq H^{n-k}(F_{i-1}^\vee(d_i - d_{n+1} - n - 1));$$

therefore, we already know that

- $H^k(F_{i-1}(d_{n+1} - d_i)) = 0$ if $k \neq n - i + 1$,
- $H^{n-i+1}(F_{i-1}(d_{n+1} - d_i)) \simeq H^{i-1}(F_{i-1}^\vee(d_i - d_{n+1} - n - 1))$.

As before, we have the isomorphisms

$$\begin{aligned} H^{i-1}(F_{i-1}^\vee(d_i - d_{n+1} - n - 1)) &\simeq H^1(F_1^\vee(d_i - d_{n+1} - n - 1)) \\ &\simeq H^{n-1}(F_{n-1}^\vee(d_i - d_{n+1} - n - 1)). \end{aligned}$$

Using the same techniques as before, if we focus on the first isomorphism, then we have to consider the exact sequences

$$\begin{array}{c}
0 \longrightarrow F_1^\vee(d_i - d_{n+1} - n - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_1}(d_i - d_1 - n - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_0}(d_i - n - 1) \longrightarrow 0 \\
0 \longrightarrow F_2^\vee(d_i - d_{n+1} - n - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_2}(d_i - d_2 - n - 1) \longrightarrow F_1^\vee(d_i - d_{n+1} - n - 1) \longrightarrow 0 \\
\vdots \\
0 \longrightarrow F_{i-1}^\vee(d_i - d_{n+1} - n - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{i-1}}(d_i - d_{i-1} - n - 1) \longrightarrow F_{i-2}^\vee(d_i - d_{n+1} - n - 1) \longrightarrow 0 \\
0 \longrightarrow F_i^\vee(d_i - d_{n+1} - n - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_i}(-n - 1) \longrightarrow F_{i-1}^\vee(d_i - d_{n+1} - n - 1) \longrightarrow 0
\end{array}$$

and knowing that if $i \geq 3$ (or else, as before, we only consider the first short exact sequence and obtain the same result), we have $H^0(F_{i-1}^\vee(d_i - d_{n+1} - n - 1)) = H^1(F_{i-1}^\vee(d_i - d_{n+1} - n - 1)) = 0$, and therefore,

$$h^{n-i+1}(F_{i-1}(d_{n+1} - d_i)) = h^1(F_1^\vee(d_i - d_{n+1} - n - 1)) = \sum_{k=0}^{i-1}(-1)^k \beta_k \binom{d_i - d_k - 1}{n}.$$

Let us focus now on the other isomorphism, computing $H^{n-1}(F_{n-1}^\vee(d_i - d_{n+1} - n - 1))$. Take the sequences

$$\begin{array}{l}
0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{n+1}}(d_i - d_{n+1} - n - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_n}(d_i - d_n - n - 1) \longrightarrow F_{n-1}^\vee(d_i - d_{n+1} - n - 1) \longrightarrow 0 \\
0 \longrightarrow F_{n-1}^\vee(d_i - d_{n+1} - n - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{n-1}}(d_i - d_{n+1} - n - 1) \longrightarrow F_{n-2}^\vee(d_i - d_{n+1} - n - 1) \longrightarrow 0 \\
\vdots \\
0 \longrightarrow F_{i+2}^\vee(d_i - d_{n+1} - n - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{i+2}}(d_i - d_{i+2} - n - 1) \longrightarrow F_{i+1}^\vee(d_i - d_{n+1} - n - 1) \longrightarrow 0 \\
0 \longrightarrow F_{i+1}^\vee(d_i - d_{n+1} - n - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_{i+1}}(d_i - d_{i+1} - n - 1) \longrightarrow F_i^\vee(d_i - d_{n+1} - n - 1) \longrightarrow 0 \\
0 \longrightarrow F_i^\vee(d_i - d_{n+1} - n - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_i}(-n - 1) \longrightarrow F_{i-1}^\vee(d_i - d_{n+1} - n - 1) \longrightarrow 0
\end{array}$$

and, being $i - 1 < n - 1$ we can state that $H^{n-1}(F_{i-1}^\vee(d_i - d_{n+1} - n - 1)) = H^n(F_{i-1}^\vee(d_i - d_{n+1} - n - 1)) = 0$ and also that

$$H^{n-1}(F_j^\vee(d_i - d_{n+1} - n - 1)) = 0 \text{ for each } j = i, \ldots, n-2.$$

We obtain that

$$\begin{aligned}
h^{n-i+1}(F_{i-1}(d_{n+1} - d_i)) &= h^{n-1}(F_{n-1}^\vee(d_i - d_{n+1} - n - 1)) \\
&= \begin{cases} \sum_{k=i}^{n+1}(-1)^{k+1}\beta_k\binom{d_k - d_i + n}{n} & \text{for } n \text{ even} \\ \sum_{k=i}^{n+1}(-1)^{k}\beta_k\binom{d_k - d_i + n}{n} & \text{for } n \text{ odd} \end{cases}
\end{aligned}$$

Let us fix some notation, for each i fixed, we will call

$$\begin{aligned}
\Sigma_{i,1} &= h^i(F_i^\vee(d_i - d_{n+1})) \\
\Sigma_{i,2} &= h^{n-i+1}(F_{i-1}(d_{n+1} - d_i)).
\end{aligned}$$

We have learned that for each i fixed from 2 to $n-1$ the cohomology group of $F_i^\vee(d_i - d_{n+1})$ which may not vanish is the i-th group; hence, the important part of the exact sequence induced in cohomology by (15) is

$$\begin{array}{l}
\longrightarrow \underbrace{H^{i-1}((F_i^\vee)^{\beta_i}(d_i - d_{n+1}))}_{=0} \longrightarrow H^{i-1}(F_i^\vee \otimes F_i) \longrightarrow H^i(F_{i-1}^{\beta_i}(d_{n+1} - d_i)) \\
\longrightarrow \underbrace{H^i((F_i^\vee)^{\beta_i}(d_i - d_{n+1}))}_{\text{dimension } \beta_i\Sigma_{i,1}} \longrightarrow\longrightarrow H^i(F_i^\vee \otimes F_i) \longrightarrow H^{i+1}(F_{i-1}^{\beta_i}(d_{n+1} - d_i)) \\
\qquad \longrightarrow \underbrace{H^{i+1}((F_i^\vee)^{\beta_i}(d_i - d_{n+1}))}_{=0} \longrightarrow
\end{array} \tag{17}$$

We now need to check out how the nonvanishing group in cohomology, associated with the bundle $F_{i-1}(d_{n+1}-d_i)$, relates to the first group, we can have the following situations.

Case 1 If $i \neq \frac{n+1}{2}$ and $i \neq \frac{n}{2}$, which means that $n-i \neq i-1$ and $n-i-1 \neq i-1$, then the two groups belong to two different exact sequences of type (17), and we have

$$H^{n-i}(F_i^\vee \otimes F_i) \simeq H^{n-i+1}(F_{i-1}^{\beta_i}(d_{n+1}-d_i))$$

and

$$H^i(F_i^\vee \otimes F_i) = H^i((F_i^\vee)^{\beta_i}(d_i - d_{n+1}))$$

hence F_i is exceptional if and only if $H^{n-i+1}(F_{i-1}(d_{n+1}-d_i)) = H^i(F_i^\vee(d_i-d_{n+1})) = 0$.

Case 2 If $i = \frac{n}{2}$, so only in the even cases, we are in the following situation

$$0 \longrightarrow \underbrace{H^i((F_i^\vee)^{\beta_i}(d_i-d_{n+1}))}_{\text{dimension } \beta_i\Sigma_{i,1}} \longrightarrow H^i(F_i^\vee \otimes F_i) \longrightarrow \underbrace{H^{i+1}(F_{i-1}^{\beta_i}(d_{n+1}-d_i))}_{\beta_i\Sigma_{i,2}} \longrightarrow 0.$$

Being $\Sigma_{i,p} \geq 0$ for $p = 1, 2$, we can state that F_i is exceptional if and only if

$$H^i((F_i^\vee)^{\beta_i}(d_i-d_{n+1})) = H^{i+1}(F_{i-1}^{\beta_i}(d_{n+1}-d_i)) = 0.$$

Case 3 If $i = \frac{n+1}{2}$, so only in the odd cases, we are in the following situation

$$0 \longrightarrow H^{i-1}(F_i^\vee \otimes F_i) \longrightarrow \underbrace{H^i(F_{i-1}^{\beta_i}(d_{n+1}-d_i))}_{\text{dimension } \beta_i\Sigma_{i,2}} \xrightarrow{H^i(\varphi)} \underbrace{H^i((F_i^\vee)^{\beta_i}(d_i-d_{n+1}))}_{\text{dimension } \beta_i\Sigma_{i,1}}$$
$$\longrightarrow H^i(F_i^\vee \otimes F_i) \longrightarrow 0,$$

where $H^i(\varphi)$ is the morphism induced in cohomology by $\varphi : F_i^\vee \otimes F_{i-1} \to (F_i^\vee)^{\beta_i}(d_i - d_{n+1})$. Therefore F_i is exceptional if and only if $H^{i-1}(F_i^\vee \otimes F_i) = H^i(F_i^\vee \otimes F_i) = 0$ if and only if $H^i(\varphi)$ is an isomorphism.

This concludes the proof. □

Corollary 4.2 *If each bundle F_i for $i = 2, \ldots, n-1$, defined as*

$$0 \longrightarrow F_{i-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\beta_i}(d_i - d_{n+1}) \longrightarrow F_i \longrightarrow 0,$$

is a Steiner bundle of type $(F_{i-1}, \mathcal{O}_{\mathbb{P}^n}(d_i-d_{n+1}))$ and $\beta_0^2+\beta_1^2-\binom{d_1+n}{n}\beta_0\beta_1 = 1$; then all bundles F_i are exceptional, for $i = 1, \ldots, n-1$.

Proof The cohomological vanishings appearing in the definition of strongly exceptional pairs, used to define Steiner bundles (recall Definition 2.5), imply the hypothesis (iii) of Theorem 4.1. □

We would like to know if the viceversa of the previous result holds, but at the moment, we are only able to state the following.

Conjecture 1 *The syzygy bundles F_i are Steiner if and only if they are also exceptional.*

The conjecture would be true if, considering n odd and $i = \frac{n+1}{2}$, we prove that the two cohomology groups

$$H^{n-i+1}(F_{i-1}(d_{n+1}-d_i)) \quad \text{and} \quad H^i(F_i^\vee(d_i-d_{n+1}))$$

are isomorphic if and only if they are zero.

As for the results implying simplicity, also for the last theorem, we have a correspondent result obtained considering the dual resolution. Recall that the bundles F_i are simple or exceptional if and only if the bundles G_i are.

Theorem 4.3 *Consider the syzygy bundles G_i as defined in (7), for i from 1 to $n-1$. Suppose also that G_i are simple for each i; then G_i, for $i = 1, \dots, n-1$, is exceptional if and only if each one of the following conditions hold*

(i) $\beta_{n+1}^2 + \beta_n^2 - \binom{d_{n+1}-d_n+n}{n}\beta_{n+1}\beta_n = 1$;

(ii) $d_{n+1} - d_n \leq n$;

(iii)

$$\begin{cases} H^{n-i+1}(G_{i-1}(d_{n+1-i})) = H^i(G_i^\vee(-d_{n+1-i})) = 0 \text{ if } n \text{ is even;} \\ H^{n-i+1}(G_{i-1}(d_{n+1-i})) = H^i(G_i^\vee(-d_{n+1-i})) = 0 \text{ if } n \text{ is odd and } i \neq \frac{n+1}{2}; \\ H^{n-i+1}(G_{i-1}(d_{n+1-i})) \overset{H^i(\varphi)}{\simeq} H^i(G_i^\vee(-d_{n+1-i})) \quad \text{if } n \text{ is odd and } i = \frac{n+1}{2}. \end{cases}$$

where, if n is odd and $i = \frac{n+1}{2}$, we get $H^{n-i+1}(G_{i-1}(d_{n+1-i})) \simeq H^i(G_{i-1} \otimes G_i^\vee)$ and the morphism $H^i(\varphi) : H^i(G_i^\vee \otimes G_{i-1}) \to H^i((G_i^\vee)^{\beta_i}(-d_{n+1-i}))$ is the one obtained by the short exact sequence

$$0 \longrightarrow G_i^\vee \otimes G_{i-1} \xrightarrow{\varphi} (G_i^\vee)^{\beta_{n+1-i}}(-d_{n+1-i}) \longrightarrow G_i^\vee \otimes G_i \longrightarrow 0$$

considering the long exact induced in cohomology.

5 Examples

In this section, we present some famous pure resolutions, and we will apply the results obtained to determine whenever the syzygies are simple or exceptional. Some of these resolutions were studied by [13].

5.1 Pure linear resolution

Let $R = \mathbb{K}[x_0, \dots, x_n]$ be the ring of polynomials and $I = (x_0, \dots, x_n)$ be the ideal generated by the coordinate variables . The Koszul complex $K(x_0, \dots, x_n)$ is given by

$$0 \longrightarrow R(-n-1)) \longrightarrow R^{\binom{n+1}{n}}(-n) \longrightarrow \cdots \longrightarrow R^{n+1}(-1) \longrightarrow$$
$$R \longrightarrow R/I \longrightarrow 0$$

Sheafifying we get the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n-1)) \longrightarrow \mathcal{O}_{\mathbb{P}^n}{}^{\binom{n+1}{n}}(-n) \longrightarrow \cdots \longrightarrow$$
$$\mathcal{O}_{\mathbb{P}^n}{}^{n+1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0\,. \quad (18)$$

Proposition 5.1 *The syzygy bundles arising from the complex (18) are all simple and exceptional.*

Proof It is a simple computation that the complex satisfies the hypothesis of Theorem 3.1 and of Theorem 3.6 for simplicity, and the hypothesis of Theorem 4.1 for the exceptionality. □

5.2 Compressed Gorenstein Artinian graded algebras

Let $I = (f_1, \dots, f_{\alpha_1})$ be an ideal generated by α_1 forms of degree $t+1$, such that the algebra $A = R/I$ is a compressed Gorenstein Artinian graded algebra of embedding dimension $n+1$ and socle degree $2t$. Thus, by Proposition 3.2 of [14], the minimal free resolution of A is

$$0 \longrightarrow R(-2t-n-1) \longrightarrow R^{\alpha_n}(-t-n) \longrightarrow R^{\alpha_{n-1}}(-t-n+1) \longrightarrow \cdots$$
$$\cdots \longrightarrow R^{\alpha_p}(-t-p) \longrightarrow \cdots \longrightarrow R^{\alpha_2}(-t-2) \longrightarrow$$
$$R^{\alpha_1}(-t-1) \longrightarrow R \longrightarrow A \longrightarrow 0$$

where

$$\alpha_i = \binom{t+i-1}{i-1}\binom{t+n+1}{n+1-i} - \binom{t+n-i}{n+1-i}\binom{t+n}{i-1}, \quad \text{for } i = 1, \dots, n.$$

Sheafifying the complex above, we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2t-n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\alpha_n}(-t-n) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\alpha_{n-1}}(-t-n+1) \longrightarrow \cdots \tag{19}$$
$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\alpha_p}(-t-p) \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\alpha_2}(-t-2) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\alpha_1}(-t-1)$$
$$\xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

where β is the map given by the α_1 forms of degree $t+1$.

Remark 5.2 By applying Theorem 3.1 , we have that the syzygies F_i of the complex (19) are simple vector bundles. Moreover, $\mathrm{hd}(F_i) = n - i$, $h^0(F_i^*(-t-i)) = \alpha_i$ for $1 \le i \le n-1$. If we take $t = 1$, then we get the linear resolution and we already know that all syzygies are exceptional. Nevertheless, it is easy to loose the exceptionality. For instance, if we take t such that $t > n-1$, the second condition of Theorem 4.1 is not satisfied. Moreover, being $\beta_0 = 1$, the first condition of Theorem 4.1 is equivalent to prove that

$$\binom{t+n+1}{n} - \binom{t+n-1}{n} = \beta_1 = \binom{d_1+n}{n} = \binom{t+n+1}{n},$$

which are not equal if $t \ge 1$. Hence, for this example, the only exceptional bundles come from the linear resolution.

5.3 Generalized Koszul complex

The reference for this section is [15].

Definition 5.3 Let $\mathcal{A}$ be a $p \times q$ matrix with entries in R. We say that $\mathcal{A}$ is a t-homogeneous matrix if the minors of size $j \times j$ are homogeneous polinomials for all $j \le t$. The matrix $\mathcal{A}$ is an *homogeneous matrix* if their minors of any size are homogeneous.

Let $\mathcal{A}$ be an homogeneous matrix. We denote by $I(\mathcal{A})$ the ideal of R generated by the maximal minors of $\mathcal{A}$. Le $\mathcal{A}$ be a t-homogeneous matrix. For all $j \le t$, we denote by $I_j(\mathcal{A})$ the ideal of R generated by the minors of size j of $\mathcal{A}$.

Note that to any homogeneous $p \times q$ matrix $\mathcal{A}$, we have a morphism $\varphi : F \to G$ of free graded R-modulos of ranks p and q, respectively. We write $I(\varphi) = I(\mathcal{A})$.

An homogeneous ideal $I \subset R$ is called *determinantal ideal* if

(1) there exists a r-homogeneous matrix $\mathcal{A}$ of size $p \times q$ with entries in R such that $I = I_r(\mathcal{A})$ and
(2) $ht(I) = (p - r + 1)(q - r + 1)$, where $ht(I)$ is the height of I.

An homogeneous determinantal ideal $I \subset R$ is called *standard determinantal ideal* if $r = \max\{p, q\}$. That is, an homogeneous ideal $I \subset R$ of codimension c is called standard determinantal ideal if $I = I_r(\mathcal{A})$ for some homogeneous matrix $\mathcal{A}$ of size $r \times (r + c - 1)$.

Let $X \subset \mathbb{P}^{n+c}$, and $\mathcal{A}$ homogeneous matrix associated with X. Let $\varphi : F \to G$ be a morphism of free graded R-modules of ranks t and $t + c - 1$, respectively, defined by $\mathcal{A}$. The generalized Koszul complex $C_i(\varphi^*)$ is given by

$$0 \longrightarrow \wedge^i G^* \otimes S_0(F^*) \longrightarrow \wedge^{i-1} G^* \otimes S_1(F^*) \longrightarrow \cdots \longrightarrow \wedge^0 G^* \otimes S_i(F^*) \longrightarrow 0$$

From this complex, we have the complex $D_i(\varphi^*)$

$$\begin{aligned} 0 \longrightarrow \wedge^{t+c-1} G^* \otimes S_{c-i-1}(F) \otimes \wedge^t F &\longrightarrow \wedge^{t+c-2} G^* \otimes S_{c-i-2}(F) \otimes \wedge^t (F) \longrightarrow \\ \cdots \qquad \cdots \longrightarrow \wedge^{t+i} G^* \otimes S_0(F) \otimes \wedge F &\longrightarrow \wedge^i G^* \otimes S_0(F^*) \longrightarrow \\ \wedge^{i-1} G^* \otimes S_1(F^*) \longrightarrow \cdots\cdots &\longrightarrow \wedge^0 G^* \otimes S_i F^* \longrightarrow 0 \end{aligned}$$

where $D_0(\varphi^*)$ is called *Eagon–Northcott complex* and $D_1(\varphi^*)$ is called *Buchsbaum–Rim complex*.

Let $\varphi : R(-d)^a \to R^{a+n}$ be a map, let M be the matrix associated with the map, and $I = I_a(M)$ be the ideal generated by the maximal minors of M. The Eagon–Northcott complex $D_0(\varphi^*)$ gives us a minimal free resolution of R/I

$$\begin{aligned} 0 \longrightarrow R^{\binom{n+a-1}{n-1}}(-d(n+a)) &\longrightarrow R^{(n+a)\binom{n+a-2}{a-1}}(-d(n+a-1)) \longrightarrow \cdots \\ \cdots \longrightarrow R^{\binom{a+n}{a}}(-da) &\longrightarrow R \longrightarrow R/I \longrightarrow 0 \end{aligned}$$

Sheafifying, we get the complex

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n+a-1}{a-1}}(-d(n+a)) &\longrightarrow \mathcal{O}_{\mathbb{P}^n}^{(n+a)\binom{n+a-2}{a-1}}(-d(n+a-1)) \longrightarrow \cdots \\ \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{a+n}{a}}(-da) &\longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0 \end{aligned} \quad (20)$$

Remark 5.4 Applying Theorem 3.1, all syzygies F_i of the complex (20) are simple. If we take $d = a = 1$, then we get the linear resolution, and we already know that all syzygies are exceptional. We obtain exceptionality, for example, also for $n = 3$, $d = 1$, and $a = 2$. Nevertheless, it is easy to loose the exceptionality. For instance, if we take d, a such that $da > n$, the second condition of Theorem 4.1 is not satisfied. Moreover, if we consider $n = 3$, $d = 2$, and $a = 1$, the syzygy bundles are not exceptional because the first condition of Theorem 4.1 is not satisfied.

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