Annali di Matematica (2016) 195:29–39
DOI 10.1007/s10231-014-0450-2

The rank of fusion systems

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Received: 3 October 2013 / Accepted: 2 September 2014 / Published online: 28 September 2014

Abstract In the early 1990s, Puig created his theory of fusion systems as a tool in modular representation theory. Later, Broto, Levi and Oliver used this theory to provide a formal setting for and prove results about the p-completed classifying spaces of finite groups. Aschbacher also started a program to establish a local theory of fusion systems similar to the local theory of finite groups. In this paper, we define the notion of ranks for fusion systems which imitates the notion of p-local ranks for finite groups and prove some results about weakly normal subsystems and factor systems.

Keywords Rank · Fusion system · Control fusion

Mathematics Subject Classification 20C20 · 20D15

1 Introduction

Puig thoroughly built up the common framework for fusion systems of finite groups and p-blocks in the early 1990s, though Puig calls them Frobenius categories rather than fusion systems, see [12,13]. With applications to algebraic topology, Broto et al. [5] introduced the notion of p-local finite groups using fusion systems as the underlying algebraic structure. Comparing with finite group theory, there are some theorems about finite groups which have more attractive statements and proofs in the language of fusion systems. This theory has attracted considerable and growing attention. Aschbacher [1] started a program to establish

Supported by National 973-Program (2011CB808003) and NSFC (11131001, 11371044).

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a local theory of fusion systems. With these applications in modular representation theory, homotopy theory and finite group theory, the theory of fusion systems is becoming one of the crucial parts of algebra.

For a finite group G and any pair of subgroups $H, K \leq G$, we define $\mathrm{Hom}_G(H, K)$ to be the set of all monomorphisms from H to K induced by conjugation in G. Let S be a Sylow p-subgroup of G. The standard example of a fusion system is the *fusion category* of G over S, as named in [4], which is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and which has morphism sets $\mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q)$. A subgroup H of G is said to control p-fusion in G if H contains a Sylow p-subgroup S of G and the canonical inclusion of H in G induces an equivalence of categories between $\mathcal{F}_S(H)$ and $\mathcal{F}_S(G)$. A subgroup Q of S is said to *control p-fusion* in G if its normalizer $N_G(Q)$ controls p-fusion in G. Let $\mathcal{F}$ be a fusion system over a finite p-group S. A subgroup Q of S is said to be *normal* in $\mathcal{F}$, denoted by $Q \trianglelefteq \mathcal{F}$, if its normalizer $N_{\mathcal{F}}(Q)$ is equal to $\mathcal{F}$. There are several necessary and sufficient conditions (see [6] for further references) for a subgroup to be normal in a saturated fusion system. There exists a unique maximal normal subgroup, denoted by $O_p(\mathcal{F})$, in $\mathcal{F}$. In this paper, we will define the rank of a fusion system which might give an indication to show how far a fusion system is from the case that $\mathcal{F}$ possesses a normal p-group S.

The paper is organized as follows. The second section contains foundational material about fusion systems, including the most basic definitions, notation, concepts and lemmas. Then, in the third section, we definite the notion of ranks for fusion systems. In the fourth section, we prove the equivalence of two definitions. Finally, various properties of fusion systems are described in terms of the rank in the last section. Throughout this paper, p denotes a prime and all groups are finite.

2 Background material on fusion systems

Fusion systems were originally introduced by Puig [11], where he calls such objects "full Frobenius systems." Broto et al. [5] had a different definition of fusion systems which they have proved to be equivalent to Puig's definition. For more background, see [4,9]. Let G be a group and g an element of G. Denote by c_g, the map induced by conjugation by g.

Definition 2.1 [5,12] Let S be a p-group. A *fusion system* over S is a category $\mathcal{F}$, whose objects are the subgroups of S and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ consist, for any two subgroups P and Q of S, of injective group homomorphisms $P \to Q$, with the composition in $\mathcal{F}$ as the usual composition of group homomorphisms, satisfying the following axioms:

- for each $g \in S$ with ${}^gP \leq Q$, the associated map $c_g : P \to Q$ is in $\mathrm{Hom}_{\mathcal{F}}(P, Q)$;
- for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, the induced isomorphism $P \to \varphi(P)$ and its inverse are morphisms in $\mathcal{F}$.

Note that all inclusions, $\iota : P \hookrightarrow Q$, for $P \leq Q \leq S$ are contained in $\mathcal{F}$ by the first axiom. Motivated by properties of fusion in finite groups, additional conditions are needed for fusion systems. This leads to the concept of what we call a "saturated fusion system". The following version is due to Roberts and Shpectorov [14].

Definition 2.2 Let $\mathcal{F}$ be a fusion system over a p-group S.

- Two subgroups $P, Q \leq S$ are *$\mathcal{F}$-conjugate* if they are isomorphic as objects of the category $\mathcal{F}$. Let $P^{\mathcal{F}}$ denote the set of all subgroups of S which are $\mathcal{F}$-conjugate to P.

- A subgroup $P \leq S$ is *fully automised* in $\mathcal{F}$ if $Aut_S(P) \in Syl_p(Aut_{\mathcal{F}}(P))$, where $Aut_S(P) = \mathrm{Hom}_S(P, P)$ and $Aut_{\mathcal{F}}(P) = \mathrm{Hom}_{\mathcal{F}}(P, P)$.
- A subgroup $P \leq S$ is *receptive* in $\mathcal{F}$ if it has the following property: for each $Q \leq S$ and each isomorphism $\varphi : Q \to P$, if we set

$$N_\varphi = \left\{x \in N_S(Q) | \text{there exists } y \in N_S(P) \text{ such that } \varphi({}^x g) = {}^y\varphi(g), \quad \forall g \in Q\right\},$$

  then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_Q = \varphi$.
- A fusion system $\mathcal{F}$ over a p-group S is *saturated* if each subgroup of S is $\mathcal{F}$-conjugate to a subgroup which is fully automised and receptive.

It is well known from the literature that the fusion category $\mathcal{F}_S(G)$ is a saturated fusion system (see for instance [5]).

We list some important classes of subgroups in a given fusion system, which are modeled on analogous definitions for p-subgroups of groups.

Definition 2.3 Let $\mathcal{F}$ be a fusion system over a p-group S.

- A subgroup $Q \leq S$ is *fully centralized* in $\mathcal{F}$ if $|C_S(Q)| \geq |C_S(R)|$ for all $R \in Q^{\mathcal{F}}$.
- A subgroup $Q \leq S$ is *fully normalized* in $\mathcal{F}$ if $|N_S(Q)| \geq |N_S(R)|$ for all $R \in Q^{\mathcal{F}}$.
- A subgroup $Q \leq S$ is *$\mathcal{F}$-centric* if $C_S(R) = Z(R)$ for any subgroup $R \in Q^{\mathcal{F}}$.
- A subgroup $Q \leq S$ is *$\mathcal{F}$-radical* if $O_p(Aut_{\mathcal{F}}(Q)/Aut_Q(Q)) = 1$.
- A subgroup $Q \leq S$ is *$\mathcal{F}$-essential* if Q is $\mathcal{F}$-centric, fully normalized in $\mathcal{F}$ and $Aut_{\mathcal{F}}(Q)/Aut_Q(Q)$ has a strongly p-embedded subgroup M (that is, M contains a Sylow p-subgroup T of $Aut_{\mathcal{F}}(Q)/Aut_P(Q)$ such that ${}^\varphi T \cap T = \{1\}$ for every $\varphi \in Aut_{\mathcal{F}}(Q)/Aut_Q(Q) \backslash M$).
- A subgroup $Q \leq S$ is *weakly $\mathcal{F}$-closed*, if for every morphism $\varphi : Q \to S$ in $\mathcal{F}$ we have $\varphi(Q) = Q$.
- A subgroup $Q \leq S$ is *strongly $\mathcal{F}$-closed*, if for every subgroup $P \leq Q$, each $R \in \mathcal{F}$ and each morphism $\varphi : P \to R$ in $\mathcal{F}$, we have $\varphi(P) \leq Q$.

As in group theory, one can define normalizers and centralizers in fusion systems.

Definition 2.4 Let $\mathcal{F}$ be a fusion system over a p-group S. For any $Q \leq S$ and any group of automorphisms $K \leq Aut(Q)$, set $N_S^K(Q) = \{x \in N_S(Q) | c_x \in K\}$. Let $N_{\mathcal{F}}^K(Q) \subseteq \mathcal{F}$ be the fusion system over $N_S^K(Q)$ where for $P, R \leq N_S^K(Q)$,

$$\begin{aligned}&\mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P, R)\\ &\quad = \left\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, R) | \, \exists\, \psi \in \mathrm{Hom}_{\mathcal{F}}(QP, QR) \text{ with } \psi|_P = \varphi \text{ and } \psi|_Q \in K\right\}.\end{aligned}$$

As special cases, set $N_{\mathcal{F}}(Q) = N_{\mathcal{F}}^{Aut(Q)}(Q)$ and $C_{\mathcal{F}}(Q) = N_{\mathcal{F}}^{\{1\}}(Q)$: the normalizer and centralizer, respectively, of Q in $\mathcal{F}$.

Q is said to be *normal* in $\mathcal{F}$, if $\mathcal{F} = N_{\mathcal{F}}(Q)$, i.e., for all $P, R \leq S$ and all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, R)$, φ extends to a morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(PQ, RQ)$ such that $\psi(Q) = Q$. If Q is normal in $\mathcal{F}$, we write $Q \trianglelefteq \mathcal{F}$.

It follows from the definition that if $Q_1, Q_2 \leq S$ are normal in $\mathcal{F}$, then so is Q_1Q_2. So there is a unique maximal normal subgroup, denoted by $O_p(\mathcal{F})$, in $\mathcal{F}$.

Fix a fusion system $\mathcal{F}$ over a p-group S. Let $Q \leq S$ and $K \leq Aut(Q)$. For any monomorphism $\varphi \in \mathrm{Hom}(Q, R)$, we write ${}^\varphi K = \{{}^\varphi\sigma | \sigma \in K\} \leq Aut(\varphi(Q))$. We can generalize the notion "fully normalized" as follows:

Definition 2.5 Let $\mathcal{F}$ be a fusion system over a p-group S. For any $Q \leq S$ and any group of automorphisms $K \leq Aut(Q)$, Q is *fully K-normalized* in $\mathcal{F}$ if $\left|N_S^K(Q)\right| \geq \left|N_S^{\varphi K}(\varphi(Q))\right|$ for all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$.

In general, $N_{\mathcal{F}}^K(Q)$ is not saturated, but it is true if Q is fully K-normalized.

Theorem 2.6 [5,12] *Fix a saturated fusion system $\mathcal{F}$ over a p-group S. Assume $Q \leq S$ and $K \leq Aut(Q)$ are such that Q is fully K-normalized in $\mathcal{F}$. Then, $N_{\mathcal{F}}^K(Q)$ is a saturated fusion system over $N_S^K(Q)$.*

For a fusion system $\mathcal{F}$ over a p-group S, we define the quotient $\overline{\mathcal{F}} = \mathcal{F}/T$ of $\mathcal{F}$ by a normal subgroup $T \trianglelefteq S$.

Definition 2.7 Let $\mathcal{F}$ be a fusion system over a p-group S, and T a normal subgroup of S. By the factor system $\mathcal{F}/T$, we mean the category, whose objects are all subgroups of S/T and for any two subgroups P, Q of S containing T, $\mathrm{Hom}_{\mathcal{F}/T}(P/T, Q/T)$ is the set of homomorphisms φ induced from $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ such that $\varphi(T) = T$.

Traditionally, in the definition above the subgroup T is strongly $\mathcal{F}$-closed, but this is not necessary for the following result.

Theorem 2.8 [6,9] *Let $\mathcal{F}$ be a saturated fusion system over a p-group S, and T a weakly $\mathcal{F}$-closed subgroup of S. Then, the fusion system $\mathcal{F}/T$ is saturated.*

There are several different definitions of normal fusion systems. Here, we introduce the definition of "weakly normal" by Linckelmann [8] and Oliver [10] since "normal" refers to the definition of normality given by Aschbacher in [1].

Definition 2.9 Let $\mathcal{F}$ be a fusion system over a p-group S, and Q a subgroup of S. A *subsystem* $\mathcal{E}$ of $\mathcal{F}$, over Q, is a subcategory of $\mathcal{F}$ that is a fusion system over Q. We say that $\mathcal{E}$ is *$\mathcal{F}$-invariant* if Q is strongly $\mathcal{F}$-closed and, for each $R \leq T \leq Q$, $\phi \in \mathrm{Hom}_{\mathcal{E}}(R, T)$, and $\psi \in \mathrm{Hom}_{\mathcal{F}}(T, S)$, we have that $\psi\phi\psi^{-1}$ is a morphism in $\mathrm{Hom}_{\mathcal{E}}(\psi(R), S)$. If in addition $\mathcal{E}$ is saturated, we say that $\mathcal{E}$ is *weakly normal* in $\mathcal{F}$. We denote weak normality by $\mathcal{E} \prec \mathcal{F}$.

The intersection of two subsystems is defined in the obvious way: If $\mathcal{G}$ and $\mathcal{H}$ are subsystems of $\mathcal{F}$ over Q and R, respectively, then $\mathcal{G} \cap \mathcal{H}$ is the fusion system over $Q \cap R$ consisting of all morphisms of $\mathcal{F}$ that are in both $\mathcal{G}$ and $\mathcal{H}$.

3 Definitions

In this section, we will define the notion of ranks for fusion systems. Firstly, we define a class of subgroups.

Definition 3.1 Let $\mathcal{F}$ be a fusion system over a p-group S. A subgroup $R \trianglelefteq S$ is called a *modified $\mathcal{F}$-radical*, or $\mathfrak{m}$-*radical* for short, subgroup of $\mathcal{F}$ if $R = O_p(N_{\mathcal{F}}(R))$. When R is $\mathfrak{m}$-radical in $\mathcal{F}$, we call $N_{\mathcal{F}}(R)$ a *parabolic* subsystem of $\mathcal{F}$ if $N_{\mathcal{F}}(R)$ is properly contained in $\mathcal{F}$.

Remark Let G be a finite group, S a Sylow p-subgroup of G. A p-subgroup R of G is said to be *radical* if $N_G(R)/R$ is p-reduced, namely, if $R = O_p(N_G(R))$, where $O_p(H)$ denotes the unique maximal normal p-subgroup of H. Comparing the notions, radical, $\mathcal{F}$-radical in

Definition 2.3 and $\mathfrak{m}$-radical, it is clear that these three notions are not equivalent, though they are related and defined to satisfy similar conditions. For $\mathcal{F} = \mathcal{F}_S(G)$, a subgroup $R \leq S$ is $\mathcal{F}$-radical if and only if $O_p(N_G(R)/RC_G(R)) = 1$ while R is radical in G if and only if $O_p(N_G(R)/R) = 1$. For example, let G be a finite group with a non-normal abelian Sylow p-subgroup S which is a T. I. set in G. The radical p-subgroups of G are 1 and S; each subgroup of S is $\mathcal{F}$-radical and the unique $\mathfrak{m}$-radical subgroup of $\mathcal{F}$ is S itself, where $\mathcal{F} = \mathcal{F}_S(G)$.

Definition 3.2 Let $\mathcal{F}$ be a fusion system over a p-group S. We define the *rank*, $\mathfrak{rank}(\mathcal{F})$, of $\mathcal{F}$ recursively as follows:

1. if $O_p(\mathcal{F}) = S$, then $\mathfrak{rank}(\mathcal{F}) = 0$;
2. if $O_p(\mathcal{F}) \neq S$, then $\mathfrak{rank}(\mathcal{F})$ is by definition equal to

$$1 + \max\{\mathfrak{rank}(\mathcal{P}) \mid \mathcal{P} \text{ is a parabolic subsystem of } \mathcal{F}\}.$$

4 An alternate definition and the equivalence

In [15], Robinson introduced the notion of p-local ranks for finite groups. There are two equivalent definitions of p-local rank for a finite group G, namely by induction and by radical p-chains. In the followings, we will similarly definite the rank of a fusion system by using p-chains.

Let $\mathcal{F}$ be a fusion system over a p-group S. Given a chain of p-subgroups $\sigma : Q_0 < Q_1 < \cdots < Q_n$ of S, define the length $|\sigma| = n$, the initial subgroup $V_\sigma = Q_0$, the k-th initial sub-chain $\sigma_k : Q_0 < Q_1 < \cdots < Q_k$. It would be natural to define the normalizer of σ to be $N_\mathcal{F}(\sigma)_I := N_\mathcal{F}(Q_0) \cap N_\mathcal{F}(Q_1) \cap \cdots \cap N_\mathcal{F}(Q_n)$. But for a saturated fusion system, the intersection of two saturated subsystems in general is not saturated. So for a special p-chain $\sigma : Q_0 < Q_1 < \cdots < Q_n$ in $\mathcal{F}$ where each of Q_i's is normal in S, it seems better to define the normalizer $N_\mathcal{F}(\sigma)$ recursively as follows to guarantee that is saturated.

1. If $|\sigma| = 0$, then $N_\mathcal{F}(\sigma) = N_\mathcal{F}(Q_0)$;
2. If $|\sigma| = n$, $N_\mathcal{F}(\sigma)$ is by definition equal to $N_{N_\mathcal{F}(\sigma_{n-1})}(Q_n)$.

Note that $N_\mathcal{F}(\sigma)$ is a subsystem of $N_\mathcal{F}(\sigma)_I$ and we will prove that they are equal in some cases.

Lemma 4.1 *Let $\mathcal{F}$ be a fusion system over S, R a weakly $\mathcal{F}$-closed subgroup in $\mathcal{F}$ and $R \leq T \leq S$. Then, $N_\mathcal{F}(R) \cap N_\mathcal{F}(T) = N_{N_\mathcal{F}(R)}(T)$.*

Proof It is obviously $N_{N_\mathcal{F}(R)}(T) \subseteq N_\mathcal{F}(R) \cap N_\mathcal{F}(T)$ and both systems are over $N_S(T)$. Let $P, Q \leq N_S(T)$ and $\varphi \in \mathrm{Hom}_{N_\mathcal{F}(R)\cap N_\mathcal{F}(T)}(P, Q)$. There exists $\psi \in \mathrm{Hom}_\mathcal{F}(TP, TQ)$ such that $\psi|_P = \varphi$ and $\psi(T) = T$. We only need to show $\psi \in N_\mathcal{F}(R)$. But this follows as $R \leq T$ and R is weakly $\mathcal{F}$-closed, so $\psi(R) = R$. Since $\varphi \in N_\mathcal{F}(R)$, we have $\varphi \in N_{N_\mathcal{F}(R)}(T)$. So $N_\mathcal{F}(R) \cap N_\mathcal{F}(T) = N_{N_\mathcal{F}(R)}(T)$ as required. □

Corollary 4.2 *Let $\mathcal{F}$ be a fusion system over S and R a weakly $\mathcal{F}$-closed subgroup of S. If $R \leq U \leq V \leq S$ and U is weakly $N_\mathcal{F}(R)$-closed, then $N_\mathcal{F}(R) \cap N_\mathcal{F}(U) \cap N_\mathcal{F}(V) = N_{N_\mathcal{F}(R)}(U) \cap N_\mathcal{F}(V) = N_{N_\mathcal{F}(R)\cap N_\mathcal{F}(U)}(V)$.*

Proof In fact, we have that

$$\begin{aligned} N_{\mathcal{F}}(R) \cap N_{\mathcal{F}}(U) \cap N_{\mathcal{F}}(V) &= (N_{\mathcal{F}}(R) \cap N_{\mathcal{F}}(U)) \cap (N_{\mathcal{F}}(R) \cap N_{\mathcal{F}}(V)) \\ &= N_{N_{\mathcal{F}}(R)}(U) \cap N_{N_{\mathcal{F}}(R)}(V) \\ &= N_{N_{N_{\mathcal{F}}(R)}(U)}(V) \\ &= N_{N_{\mathcal{F}}(R) \cap N_{\mathcal{F}}(U)}(V). \end{aligned}$$

So we are done. □

We say that a p-chain σ is $\mathfrak{m}$-*radical* if Q_0 is an $\mathfrak{m}$-radical subgroup of $\mathcal{F}$ and Q_i is an $\mathfrak{m}$-radical subgroup of $N_{\mathcal{F}}(\sigma_{i-1})$ for each $i \geq 1$. Write $\mathscr{R} = \mathscr{R}(\mathcal{F})$ for the set of $\mathfrak{m}$-radical p-chains in $\mathcal{F}$ and write $\mathscr{R}(\mathcal{F}|Q) = \{\sigma \in \mathscr{R}(\mathcal{F}) | V_\sigma = Q\}$. Let $\mathfrak{rank}_C(\mathcal{F})$ denote the length of a longest chain in $\mathscr{R}(\mathcal{F})$. We will prove that for a fusion system $\mathcal{F}$ over a p-group S, $\mathfrak{rank}(\mathcal{F}) = \mathfrak{rank}_C(\mathcal{F})$. As the corresponding notion, p-local rank, in the block theory of finite groups, it was proved in [17]. Though we can follow approaches in group theory, for fusion systems we must establish some basic results, the correspondences of which are well known in the category of finite groups.

The group theoretic version of the following lemma is well known.

Lemma 4.3 *Let $\mathcal{F}$ be a fusion system over S. Assume that $R \leq S$ is an $\mathfrak{m}$-radical subgroup of $\mathcal{F}$ and $U \leq S$. If $N_{\mathcal{F}}(R) \subseteq N_{\mathcal{F}}(U)$, then $U \leq R$.*

Proof We will show that U is normal in $N_{\mathcal{F}}(R)$. Note that R is $\mathfrak{m}$-radical in $\mathcal{F}$. So we have $N_S(R) = S$ and hence $U \trianglelefteq S$ as $N_{\mathcal{F}}(R) \subseteq N_{\mathcal{F}}(U)$ over S. For any $P, Q \leq S$, $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}(R)}(P, Q)$, there is a morphism $\overline{\psi} \in \mathrm{Hom}_{N_{\mathcal{F}}(R)}(RP, RQ)$ such that $\overline{\psi}|_P = \varphi$ and $\overline{\psi}(R) = R$. Since $N_{\mathcal{F}}(R) \subseteq N_{\mathcal{F}}(U)$, there is a morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(URP, URQ)$ such that $\psi|_{RP} = \overline{\psi}$ and $\psi(U) = (U)$. Let $\overline{\varphi} = \psi|_{UP}$. Then, by the construction, $\overline{\varphi} \in \mathrm{Hom}_{N_{\mathcal{F}}(R)}(UP, UQ)$ is an extension of φ such that $\overline{\varphi}(U) = \psi(U) = U$. So U is normal in $N_{\mathcal{F}}(R)$. Since $R = O_p(N_{\mathcal{F}}(R))$, we have $U \leq R$ as required. □

Recall that if R is a radical p-subgroup of a finite group G, then $R \geq O_p(G)$. In a fusion system $\mathcal{F}$, we have a similar result that is a direct corollary of Lemma 4.3.

Lemma 4.4 *Let $\mathcal{F}$ be a fusion system over S. If $R \leq S$ is $\mathfrak{m}$-radical, then $O_p(\mathcal{F}) \leq R$.*

Lemma 4.5 *Let $\mathcal{F}$ be a fusion system over S, $\mathcal{N} = N_{\mathcal{F}}(R)$ a parabolic subsystem of $\mathcal{F}$, where R is $\mathfrak{m}$-radical in $\mathcal{F}$. Then, $\mathfrak{rank}(\mathcal{N})_C < \mathfrak{rank}(\mathcal{F})_C$.*

Proof Take $\sigma : Q_0 < \cdots < Q_s$ to be one of the longest $\mathfrak{m}$-radical chains of $\mathcal{N}$. By Lemma 4.4, we have $Q_0 \geq R$, since $R = O_p(\mathcal{N})$. In fact, $R = Q_0$ since σ has the greatest length. Note that $\mathcal{N}$ is a parabolic subsystem which is properly contained in $\mathcal{F}$. So $R > O_p(\mathcal{F})$. Then $\overline{\sigma} : O_p(\mathcal{F}) < Q_0 < \cdots < Q_s$ is an $\mathfrak{m}$-radical chain of $\mathcal{F}$. By the definition, we have $\mathfrak{rank}(\mathcal{N})_C < \mathfrak{rank}(\mathcal{F})_C$. □

Theorem 4.6 *Let $\mathcal{F}$ be a fusion system over S. Then, $\mathfrak{rank}(\mathcal{F}) = \mathfrak{rank}(\mathcal{F})_C$.*

Proof We prove our assertion by induction.

Note that S is $\mathfrak{m}$-radical in $\mathcal{F}$. It is easy to verify that $\mathfrak{rank}(\mathcal{F})_C = 0$ if and only if $S \trianglelefteq \mathcal{F}$. In fact, if $\mathfrak{rank}(\mathcal{F})_C = 0$, then $O_p(\mathcal{F}) = S$, or $O_p(\mathcal{F}) < S$ is an $\mathfrak{m}$-radical chain of length 1, which is a contradiction. By Lemma 4.4, if $S \trianglelefteq \mathcal{F}$, then S is the unique $\mathfrak{m}$-radical subgroup; hence, $\mathfrak{rank}(\mathcal{F})_C = 0$.

We claim that for any fusion system $\mathcal{H}$, $\mathfrak{rank}(\mathcal{H}) \geq \mathfrak{rank}(\mathcal{H})_C$. Assume $\mathfrak{rank}(\mathcal{H})_C = s$. Let $\sigma : Q_0 < \cdots < Q_s$ be a longest $\mathfrak{m}$-radical chain of $\mathcal{H}$, where $Q_0 = O_p(\mathcal{H})$. Then, $\mathcal{N} = N_{\mathcal{F}}(\sigma_1)$ is a parabolic subsystem of $\mathcal{H}$, as $Q_1 > Q_0$ is $\mathfrak{m}$-radical in $\mathcal{H}$. Since each $\mathfrak{m}$-radical chain of $\mathcal{N}$ can extend to an $\mathfrak{m}$-radical chain of $\mathcal{H}$ by adding Q_0 as an initial subgroup (see the proof of Lemma 4.5), then $\mathfrak{rank}(\mathcal{N})_C = s-1$. By the induction assumption, $\mathfrak{rank}(\mathcal{N}) \geq s - 1$ and hence $\mathfrak{rank}(\mathcal{H}) \geq s$ as required.

Suppose that for any fusion system $\mathcal{H}$ with $\mathfrak{rank}(\mathcal{H}) \leq k$ our assertion holds. If $\mathfrak{rank}(\mathcal{F}) = k + 1$, then there is a parabolic subsystem $\mathcal{N}$ such that $\mathfrak{rank}(\mathcal{N}) = k$. Let $\sigma : Q_0 < \cdots < Q_k$ be one of the longest $\mathfrak{m}$-radical chains of $\mathcal{N}$, where $Q_0 = O_p(\mathcal{N}) > O_p(\mathcal{F})$. Then, $\mathfrak{rank}(\mathcal{F})_C \geq k + 1$ since the chain $O_p(\mathcal{F}) < Q_0 < \cdots < Q_k$ is $\mathfrak{m}$-radical in $\mathcal{F}$. As shown above $\mathfrak{rank}(\mathcal{F}) \geq \mathfrak{rank}(\mathcal{F})_C$, we have $\mathfrak{rank}(\mathcal{F})_C = k + 1$ as required. □

5 Properties of fusion systems

While we define the notion of ranks for all fusion systems, in quite a number of results in this section, we assume that a fusion system $\mathcal{F}$ is saturated. We suppose that S is a p-group for a fixed prime p throughout this section.

5.1 Normal subgroups and products of fusion systems

Lemma 5.1 *Let $\mathcal{F}$ be a fusion system over S and R a subgroup of S. If $T \leq R$ is weakly $\mathcal{F}$-closed, then $T \trianglelefteq N_{\mathcal{F}}(R)$.*

Proof Let $\varphi \in \hom_{N_{\mathcal{F}}(R)}(P, Q)$ where $P, Q \leq N_S(R)$. Then, φ extends to some $\psi \in \mathrm{Hom}_{\mathcal{F}}(RP, RQ)$ such that $\psi(R) = R$. Note that T is weakly $\mathcal{F}$-closed. Take $\overline{\varphi} = \psi|_{TP}$, so $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(TP, TQ)$ is an extension of φ such that $\overline{\varphi}(T) = T$. And $\overline{\varphi}$ is contained in $N_{\mathcal{F}}(R)$, since ψ is the required extension of $\overline{\varphi}$. So $T \trianglelefteq N_{\mathcal{F}}(R)$. □

We consider the products of fusion systems which are defined in the obvious way.

Definition 5.2 For any pair $\mathcal{F}_1$ and $\mathcal{F}_2$ of fusion systems over p-groups S_1 and S_2, respectively, $\mathcal{F}_1 \times \mathcal{F}_2$ is the fusion system over $S_1 \times S_2$ generated by the set of all $(\varphi_1, \varphi_2) \in \mathrm{Hom}(P_1 \times P_2, Q_1 \times Q_2)$ for $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, Q_i)$.

Proposition 5.3 [5] *Assume $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems over p-groups S_1 and S_2, respectively. Then, for all $P, Q \leq S_1 \times S_2$, if P_i and Q_i denote the images of P and Q under the projection to S_i,*

$$\mathrm{Hom}_{\mathcal{F}_1 \times \mathcal{F}_2}(P, Q) = \left\{(\varphi_1, \varphi_2)|_P \mid \varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, Q_i), (\varphi_1, \varphi_2)(P) \leq Q\right\}.$$

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are both saturated, then so is $\mathcal{F}_1 \times \mathcal{F}_2$.

Lemma 5.4 *Assume $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems over p-groups S_1 and S_2, respectively. Then, $O_p(\mathcal{F}_i) \trianglelefteq \mathcal{F}_1 \times \mathcal{F}_2$ ($i = 1, 2$), and $O_p(\mathcal{F}_1 \times \mathcal{F}_2) = O_p(\mathcal{F}_1) \times O_p(\mathcal{F}_2)$.*

Proof In fact, this is a direct corollary of [3, (2.5)]. Set $S = S_1 \times S_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and let $\pi_i : S \to S_i$ be the projection of S onto S_i ($i = 1, 2$). Denote $O = O_p(\mathcal{F}_1 \times \mathcal{F}_2)$, $O_i = O_p(\mathcal{F}_i)$, $i = 1, 2$. Note that $\pi_i(O_i) = O_i$, $\pi_{3-i}(O_i) = 1$. By [3, (2.5)], we have $N_{\mathcal{F}}(O_1) = N_{\mathcal{F}}(\pi_1(O_1)\pi_2(O_1)) = N_{\mathcal{F}_1}(O_1) \times N_{\mathcal{F}_2}(1) = \mathcal{F}$. So $O_1 \trianglelefteq \mathcal{F}$ and similarly $O_2 \trianglelefteq \mathcal{F}$, hence $O_1 \times O_2 \leq O$. On the other hand, also by [3, (2.5)], $\mathcal{F} = N_{\mathcal{F}}(O) \leq N_{\mathcal{F}}(\pi_1(O)\pi_2(O)) = N_{\mathcal{F}_1}(\pi_1(O)) \times N_{\mathcal{F}_2}(\pi_2(O)) \leq \mathcal{F}_1 \times \mathcal{F}_2 = \mathcal{F}$. So $N_{\mathcal{F}_i}(\pi_i(O)) = \mathcal{F}_i$, hence $\pi_i(O) \leq O_i$ and $O \leq O_1 \times O_2$. □

Theorem 5.5 *Assume $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems over p-groups S_1 and S_2, respectively. Then, $\mathfrak{rank}(\mathcal{F}_1 \times \mathcal{F}_2) = \mathfrak{rank}(\mathcal{F}_1) + \mathfrak{rank}(\mathcal{F}_2)$.*

Proof Let $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2,\ S = S_1 \times S_2$ and $\pi_i\ (i = 1, 2)$ the natural projection to S_i.

If $S \trianglelefteq \mathcal{F}$, then $S_i \trianglelefteq \mathcal{F}_i$ by Lemma 5.4, hence $\mathfrak{rank}(\mathcal{F}_1 \times \mathcal{F}_2) = 0 = \mathfrak{rank}(\mathcal{F}_1) + \mathfrak{rank}(\mathcal{F}_2)$. By Lemma 4.4, we may take U to be an $\mathfrak{m}$-radical subgroup of $\mathcal{F}$ which properly contains $O_p(\mathcal{F})$. Let $U_i = \pi_i(U), i = 1, 2$. We have $U \le U_1 \times U_2$. Since $U \trianglelefteq S$, then $U_i \trianglelefteq S_i,\ i = 1, 2$. Let $\mathcal{N}_i = N_{\mathcal{F}_i}(U_i)$ and $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$. For any $P, Q \le N_S(U) = S$ and $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}(U)}(P, Q)$, we see that $\varphi = (\varphi_1, \varphi_2)|_P$ for some $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(\pi_i(P), \pi_i(Q)), i = 1, 2$. By definition, φ can extend to $\psi \in \mathrm{Hom}_{\mathcal{F}}(UP, UQ)$. So $\psi = (\psi_1, \psi_2)|_{UP}$, for some $\psi_i \in \mathrm{Hom}_{\mathcal{F}_i}(U_i\pi_i(P), U_i\pi_i(Q)), i = 1, 2$. Since $\psi|_P = \varphi$ and $\psi(U) = U$, we have $\psi_i|_{\pi_i(P)} = \varphi_i$ and $\psi_i(U_i) = U_i$. So φ_i is contained in $\mathcal{N}_i$, and hence that $N_{\mathcal{F}}(U)$ is a subsystem of $\mathcal{N}$. By Lemma 5.4, U_i is normal in $\mathcal{N}$. So by Lemma 4.3, $U_i \le U$ and hence that $U = U_1 \times U_2$. By [3, (2.5)], $N_{\mathcal{F}}(U) = \mathcal{N}$. By Lemma 5.4, U_i is $\mathfrak{m}$-radical in $\mathcal{F}_i$ for $i = 1, 2$. On the other hand, if U_i is $\mathfrak{m}$-radical in $\mathcal{F}_i$ for $i = 1, 2$, then $U_1 \times U_2$ is $\mathfrak{m}$-radical in $\mathcal{F}$ since $N_{\mathcal{F}}(U_1 \times U_2) = N_{\mathcal{F}_1}(U_1) \times N_{\mathcal{F}_2}(U_2)$, and $U_1 \times U_2 = O_p(N_{\mathcal{F}_1}(U_1) \times N_{\mathcal{F}_2}(U_2))$ by Lemma 5.4.

Let $\mathcal{P} = N_{\mathcal{F}}(U)$ be a parabolic subsystem of $\mathcal{F}$, where $U = O_p(\mathcal{P})$, and $\mathcal{P}$ is chosen with $\mathfrak{rank}(\mathcal{P}) = \mathfrak{rank}(\mathcal{F}) - 1$. Then, $U = U_1 \times U_2$ and $\mathcal{P} = N_{\mathcal{F}_1}(U_1) \times N_{\mathcal{F}_2}(U_2)$, where $U_i = \pi_i(U)$. At least one U_i properly contains $O_p(\mathcal{F}_i)$, so we have by induction assumption that $\mathfrak{rank}(\mathcal{P}) = \mathfrak{rank}(N_{\mathcal{F}_1}(U_1)) + \mathfrak{rank}(N_{\mathcal{F}_2}(U_2))$. Thus, $\mathfrak{rank}(\mathcal{P}) \le \mathfrak{rank}(\mathcal{F}_1) + \mathfrak{rank}(\mathcal{F}_2) - 1$, hence $\mathfrak{rank}(\mathcal{F}) \le \mathfrak{rank}(\mathcal{F}_1) + \mathfrak{rank}(\mathcal{F}_2)$.

On the other hand, we may choose a parabolic subsystem $\mathcal{Q}$ of $\mathcal{F}$ such that $\mathcal{Q} = N_{\mathcal{F}_1}(U_1) \times N_{\mathcal{F}_2}(U_2)$, where $U_1 = O_p(\mathcal{F}_1), U_2 = O_p(N_{\mathcal{F}_2}(U_2)) > O_p(\mathcal{F}_2)$, and $\mathfrak{rank}(N_{\mathcal{F}_2}(U_2)) = \mathfrak{rank}(\mathcal{F}_2) - 1$ (where we may assume that $\mathfrak{rank}(\mathcal{F}_2) > 0$). Then, by induction assumption on the rank, we have $\mathfrak{rank}(\mathcal{Q}) = \mathfrak{rank}(N_{\mathcal{F}_1}(U_1)) + \mathfrak{rank}(N_{\mathcal{F}_2}(U_2))$, so $\mathfrak{rank}(\mathcal{Q}) = \mathfrak{rank}(\mathcal{F}_1) + \mathfrak{rank}(\mathcal{F}_2) - 1$, while also $\mathfrak{rank}(\mathcal{Q}) \le \mathfrak{rank}(\mathcal{F}) - 1$. Thus, we have $\mathfrak{rank}(\mathcal{F}) \ge \mathfrak{rank}(\mathcal{F}_1) + \mathfrak{rank}(\mathcal{F}_2)$. □

5.2 Weakly normal subsystems

In this subsection, $\mathcal{F}$ is a saturated fusion system over S.

Lemma 5.6 *Let $\mathcal{E}$ be a subsystem of $\mathcal{F}$ over $T \le S$ and $\mathcal{E} \prec \mathcal{F}$. Suppose that a subgroup R of T is fully normalized in $\mathcal{F}$. Then, $N_{\mathcal{E}}(R) \prec N_{\mathcal{F}}(R)$.*

Proof Since R is fully normalized in $\mathcal{F}$ and $\mathcal{E}$ is a subsystem of $\mathcal{F}$, R is fully normalized in $\mathcal{E}$. So $N_{\mathcal{E}}(R)$ and $N_{\mathcal{F}}(R)$ are both saturated. Since T is strongly $\mathcal{F}$-closed, $N_T(R)$ is strongly $N_{\mathcal{F}}(R)$-closed. For any $P, Q \le N_T(R), \phi \in \mathrm{Hom}_{N_{\mathcal{E}}(R)}(P, Q)$, and $\psi \in \mathrm{Hom}_{N_{\mathcal{F}}(R)}(Q, N_S(R))$, since $\mathcal{E}$ is weakly normal in $\mathcal{F}$, then $\psi\varphi\psi^{-1} \in \mathrm{Hom}_{\mathcal{E}}(\psi(P), \psi(Q))$. We only need to show that $\psi\varphi\psi^{-1}$ is contained in $N_{\mathcal{E}}(R)$. In fact, there are morphisms $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{E}}(RP, RQ)$ and $\overline{\psi} \in \mathrm{Hom}_{\mathcal{F}}(RQ, N_S(R))$ such that $\overline{\varphi}|_P = \varphi$, $\overline{\psi}|_Q = \psi$ and $\overline{\varphi}(R) = \overline{\psi}(R) = R$. So $\theta := \overline{\psi}\overline{\varphi}\overline{\psi}^{-1}$ is contained in $\mathcal{E}$ and $\theta \in \mathrm{Hom}_{\mathcal{E}}(R\psi(P), R\psi(Q))$ such that $\theta(R) = R$ and $\theta|_{\psi(P)} = \psi\varphi\psi^{-1}$. Then, $N_{\mathcal{E}}(R)$ is $N_{\mathcal{F}}(R)$-invariant, hence weakly normal in $N_{\mathcal{F}}(R)$ as required. □

Lemma 5.7 *Let $\mathcal{E}$ be a subsystem of $\mathcal{F}$ over $T \le S$ and $\mathcal{E} \prec \mathcal{F}$. Suppose that a subgroup R of T is $\mathfrak{m}$-radical in $\mathcal{F}$. Then, R is $\mathfrak{m}$-radical in $\mathcal{E}$.*

Proof As $R \trianglelefteq S$, R is normal in T. Then, $N_{\mathcal{E}}(R)$ and $N_{\mathcal{F}}(R)$ are both saturated and $N_{\mathcal{E}}(R) \prec N_{\mathcal{F}}(R)$ by Lemma 5.6. By [6, Proposition 4.1], we have $O_p(N_{\mathcal{E}}(R)) = O_p(N_{\mathcal{F}}(R)) \cap T = R$. □

Theorem 5.8 *Let $\mathcal{E}$ be a fusion system over S and $\mathcal{E} \prec \mathcal{F}$. Then, $\mathcal{F}$ and $\mathcal{E}$ have the same $\mathfrak{m}$-radical subgroups. In particular, $\mathfrak{rank}(\mathcal{F}) = \mathfrak{rank}(\mathcal{E})$.*

Proof By Lemma 5.7, we have that any $\mathfrak{m}$-radical subgroup R of $\mathcal{F}$ is $\mathfrak{m}$-radical in $\mathcal{E}$. Now we suppose that R is an $\mathfrak{m}$-radical subgroup of $\mathcal{E}$. As $\mathcal{E}$ is over S, $R \trianglelefteq S$. Also by Lemma 5.6, $N_{\mathcal{E}}(R)$ and $N_{\mathcal{F}}(R)$ are both saturated and $N_{\mathcal{E}}(R) \prec N_{\mathcal{F}}(R)$. Take $L = O_p(N_{\mathcal{F}}(R))$. By [6, Proposition 4.1], $R = O_p(N_{\mathcal{E}}(R)) = O_p(N_{\mathcal{F}}(R)) \cap S = L \cap S = L$. So R is $\mathfrak{m}$-radical in $\mathcal{F}$. Proceeding by induction on the rank of $\mathcal{F}$, if $R \neq O_p(\mathcal{F})$ is $\mathfrak{m}$-radical then $\mathfrak{rank}(N_{\mathcal{F}}(R)) = \mathfrak{rank}(N_{\mathcal{E}}(R))$, so $\mathfrak{rank}(\mathcal{F}) = \mathfrak{rank}(\mathcal{E})$ by the recursive definition of rank. □

5.3 Solvable systems

There are at least two natural definitions of solvable saturated fusion systems which turn out to be inequivalent. One of them (see [12]) involves taking repeated subsystems as a series which is analogous to the derived series of a group. In this paper, we focus on a weaker one.

Definition 5.9 [6] Let $\mathcal{F}$ be a saturated fusion system over S. $\mathcal{F}$ is said to be *p-solvable* if there exists a chain of strongly $\mathcal{F}$-closed subgroups

$$1 = P_0 \leq P_1 \leq \cdots \leq P_n = S,$$

such that $P_i/P_{i-1} \leq O_p(\mathcal{F}/P_{i-1})$ for all $1 \leq i \leq n$. If $\mathcal{F}$ is p-solvable, then the length n of a smallest such chain above will be called the *p-length*, denoted by $\ell(\mathcal{F})$, of $\mathcal{F}$.

Define $O_p^{(0)}(\mathcal{F}) = 1$, and the ith term by $O_p^{(i)}(\mathcal{F})/O_p^{(i-1)}(\mathcal{F}) = O_p(\mathcal{F}/O_p^{(i-1)}(\mathcal{F}))$.

Lemma 5.10 [6] *Let $\mathcal{F}$ be a saturated fusion system over S, and Q a strongly $\mathcal{F}$-closed subgroup of S.*

1. *If $\mathcal{F}$ is p-solvable, then all saturated subsystems and quotients $\mathcal{F}/Q$ are p-solvable.*
2. *Let $\mathcal{E}$ be a weakly normal subsystem of $\mathcal{F}$, over the subgroup Q. If both $\mathcal{E}$ and $\mathcal{F}/Q$ are p-solvable, then so is $\mathcal{F}$.*
3. *$\mathcal{F}$ is p-solvable if and only if $O_p^{(n)}(\mathcal{F}) = S$ for some n, and the smallest such n is the p-length of $\mathcal{F}$.*

Lemma 5.11 *Let $\mathcal{F}$ be a saturated fusion system over S, and $R \leq S$. Suppose $O = O_p(\mathcal{F}) \leq R$. Denote $\overline{R} = R/O$. If $\overline{R}$ is $\mathfrak{m}$-radical in $\mathcal{F}/O$, then R is $\mathfrak{m}$-radical in $\mathcal{F}$.*

Proof Let $T = O_p(N_{\mathcal{F}}(R))$. Since R/O is $\mathfrak{m}$-radical in $\mathcal{F}/O$, R is normal in S and hence $N_{\mathcal{F}}(R)$ is saturated. By [1,6,16], T/O is strongly $N_{\mathcal{F}}(R)/O$-closed. Suppose $\overline{\varphi} \in \mathrm{Hom}_{N_{\mathcal{F}}(R)/O}(A/O, B/O)$. Let $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}(R)}(A, B)$ which induces $\overline{\varphi}$ in $N_{\mathcal{F}}(R)/O$. Since T is normal in $N_{\mathcal{F}}(R)$, there is a morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(TA, TB)$ such that $\psi|_A = \varphi$ and $\psi(T) = T$. Note that ψ is also in $N_{\mathcal{F}}(R)$. So ψ induces a morphism $\overline{\psi} \in \mathrm{Hom}_{N_{\mathcal{F}}(R)/O}(TA/O, TB/O)$ such that $\overline{\psi}|_{A/O} = \overline{\varphi}$ and $\overline{\psi}(T/O) = T/O$. So T/O is normal in $N_{\mathcal{F}}(R)/O$. Since O is strongly $\mathcal{F}$-closed, by [7, Lemma 3.9], we have $N_{\mathcal{F}}(R)/O = N_{\mathcal{F}/O}(R/O)$ and $N_S(R)/O = N_{S/O}(R/O)$. So T/O is normal in $N_{S/O}(R/O)$, hence $T/O \leq R/O$. Since $T \geq R$, we have $T = R$. □

Theorem 5.12 *Let $\mathcal{F}$ be a saturated fusion system over S. If $\mathcal{F}$ is p-solvable, then $\ell(\mathcal{F}) \leq \mathfrak{rank}(\mathcal{F}) + 1$.*

Proof If S is trivial, there is nothing to prove. Suppose $S \neq 1$. Note that $\ell(\mathcal{F}) = 1$ if and only if $O_p(\mathcal{F}) = S$ and hence if and only if $\mathfrak{rank}(\mathcal{F}) = 0$, by the definitions. Suppose that $\mathfrak{rank}(\mathcal{F}) \geq 1$. We have that $1 < O_p(\mathcal{F}) < S$ since $\mathcal{F}$ is p-solvable. Let $O = O_p(\mathcal{F})$ and $\overline{S} = S/O$. Take $\overline{\sigma} : \overline{Q}_0 < \cdots < \overline{Q}_t$ to be a longest $\mathfrak{m}$-radical chain of $\mathcal{F}/O$, where Q_i is the preimage of $\overline{Q}_i$ in S. By Lemma 5.11 and [7], $\sigma : Q_0 < \cdots < Q_t$ is an $\mathfrak{m}$-radical chain of $\mathcal{F}$. Note that $Q_0 = O_p^{(2)}(\mathcal{F}) > O$ since $\ell(\mathcal{F}) > 1$. So $\varsigma : O < Q_0 < \cdots < Q_t$ is an $\mathfrak{m}$-radical chain of $\mathcal{F}$, hence $\mathfrak{rank}(\mathcal{F}) > \mathfrak{rank}(\mathcal{F}/O)$. By the induction assumption on the p-length, we have $\ell(\mathcal{F}) - 1 \leq \mathfrak{rank}(\mathcal{F}/O) + 1 < \mathfrak{rank}(\mathcal{F}) + 1$, hence $\ell(\mathcal{F}) \leq \mathfrak{rank}(\mathcal{F}) + 1$ as required. □

Following [2], in a saturated fusion system $\mathcal{F}$ over a finite p-group S, a fully normalized non-trivial subgroup U is called an *$\mathcal{F}$TI-subgroup* of S if whenever $P \leq N_S(U)$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ with $1 \neq \varphi(U \cap P) \cap U$, then φ extends to $\psi \in \mathrm{Hom}_{\mathcal{F}}(UP, S)$ with $\psi(U) = U$. In fact, in [2], $\mathcal{F}$TI-subgroups are assumed to be abelian. Here, we eliminate the constraint "abelian" and consider general cases.

Proposition 5.13 *Let $\mathcal{F}$ be a saturated fusion system over S which is non-trivial. Then, the following conditions are equivalent.*

(1) $\mathfrak{rank}(\mathcal{F}) = 0$;
(2) *$\mathcal{F}$ is p-solvable and $\ell(\mathcal{F}) = 1$;*
(3) $S \trianglelefteq \mathcal{F}$;
(4) *S is a $\mathcal{F}$ TI-subgroup.*

Proof By the discussion in the proof of Theorem 5.12, (1) and (2) are equivalent, while (1) and (3) are equivalent by the definition of rank. Thus, we only need to show (3) $\Leftrightarrow$ (4).

(3) $\Rightarrow$ (4). Since S is normal in $\mathcal{F}$, for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, φ extends to $\psi \in \mathrm{Hom}_{\mathcal{F}}(PS, QS)$. So S is an $\mathcal{F}$TI-subgroup.

(4) $\Rightarrow$ (3). Let $P, Q \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. We must show that φ extends to S. If $P = 1$, there is nothing to prove. Suppose that $P \neq 1$. So $1 \neq \varphi(S \cap P) \cap S$, since φ is an injection. By definition, φ extends to $\psi \in \mathrm{Hom}_{\mathcal{F}}(PS, S)$ with $\psi(S) = S$. So S is normal in $\mathcal{F}$. □

5.4 Miscellany

Proposition 5.14 *Let $\mathcal{F}$ be a fusion system over S with $\mathfrak{rank}(\mathcal{F}) > 0$ and $O = O_p(\mathcal{F})$. Then, $\mathfrak{rank}(\mathcal{F}) = 1$ if and only if for any $O < U \trianglelefteq S$, $N_{\mathcal{F}}(U) \subseteq N_{\mathcal{F}}(S)$.*

Proof Since $\mathfrak{rank}(\mathcal{F}) > 0$, O is a proper subgroup of S.

Firstly, suppose that for any $O < U \trianglelefteq S$, $N_{\mathcal{F}}(U) \subseteq N_{\mathcal{F}}(S)$. Let $O \neq R$ be an $\mathfrak{m}$-radical subgroup of $\mathcal{F}$. Then, R properly contains O and $R \trianglelefteq S$; hence, $N_{\mathcal{F}}(R) \subseteq N_{\mathcal{F}}(S)$. By Lemma 4.3, $R = S$. So $\mathfrak{rank}(\mathcal{F}) = 1$, since the length of an $\mathfrak{m}$-radical p-chain of $\mathcal{F}$ is at most 1.

Now suppose that $\mathfrak{rank}(\mathcal{F}) = 1$. Take T to be a normal subgroup of S with $T > O$. We will prove our assertion by induction on $|S : T|$. If $T = S$, there is nothing to prove. Suppose that T is a proper normal subgroup of S. Let $P = O_p(N_{\mathcal{F}}(T))$. Since $\mathfrak{rank}(\mathcal{F}) = 1$, there are only two $\mathfrak{m}$-radical p-subgroups, i.e., O and S, in $\mathcal{F}$. So T is not $\mathfrak{m}$-radical in $\mathcal{F}$ and $P > T$. Note that P is strongly $N_{\mathcal{F}}(T)$-closed. Then, P is normal in S, since $N_{\mathcal{F}}(T)$ is over S. Note that $N_{\mathcal{F}}(T) \subseteq N_{\mathcal{F}}(P)$. By induction assumption, we have $N_{\mathcal{F}}(T) \subseteq N_{\mathcal{F}}(P) \subseteq N_{\mathcal{F}}(S)$ as required. □

A saturated fusion system $\mathcal{F}$ is called *realizable* if there is a finite group G with $S \in Syl_p(G)$ such that $\mathcal{F} = \mathcal{F}_S(G)$. In [15], p-local rank of a finite group G is defined recursively, where p is a prime divisor of the order of G. For a realizable fusion system, we have the followings.

Proposition 5.15 *Let $\mathcal{F} = \mathcal{F}_S(G)$, where G is a finite group and $S \in Syl_p(G)$. Then, $p\mathrm{lr}(G) \geq \mathfrak{rank}(\mathcal{F})$, where $p\mathrm{lr}(H)$ denotes the p-local rank of a finite group H.*

Proof We prove our assertion by induction on $p\mathrm{lr}(G)$. If $p\mathrm{lr}(G) = 0$, then $S \trianglelefteq G$, hence $S \trianglelefteq \mathcal{F}$. So $p\mathrm{lr}(G) = \mathfrak{rank}(\mathcal{F}) = 0$. Let $O_p(\mathcal{F}) < R \leq S$ be an $\mathfrak{m}$-radical subgroup of $\mathcal{F}$. Then, $R \trianglelefteq S$ and $S \leq N = N_G(R)$ is a Sylow p-subgroup of N. Since R is fully normalized, $N_{\mathcal{F}}(R) = \mathcal{F}_S(N)$ (see [4, I.5.4]). Note that $O_p(G) \leq O_p(\mathcal{F})$, since $O_p(G)$ is normal in G. So $R > O_p(G)$ and there is a radical p-subgroup T such that $R \leq T \leq S$ and $N_G(T) \geq N$. Note that $p\mathrm{lr}(G) > p\mathrm{lr}(N_G(T))$ since T is radical in G. By induction assumption, we have $p\mathrm{lr}(N_G(T)) \geq p\mathrm{lr}(N) \geq \mathfrak{rank}(\mathcal{F}_S(N))$. By definition, we have

$$\begin{aligned} p\mathrm{lr}(G) &\geq 1 + \max\left\{p\mathrm{lr}(N_G(R)) \mid O_p(\mathcal{F}) < R \text{ is an } \mathfrak{m}\text{-radical subgroup of } \mathcal{F}\right\} \\ &\geq 1 + \max\left\{\mathfrak{rank}(\mathcal{F}_S(N_G(R))) \mid O_p(\mathcal{F}) < R \text{ is an } \mathfrak{m}\text{-radical subgroup of } \mathcal{F}\right\} \\ &= 1 + \max\{\mathfrak{rank}(\mathcal{P}) \mid \mathcal{P} \text{ is a parabolic subsystem of } \mathcal{F}\} \\ &= \mathfrak{rank}(\mathcal{F}) \end{aligned}$$

as required. □

Acknowledgments The authors cordially thank the referees for their careful reading and helpful comments.

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