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H-hypersurfaces with three distinct principal curvatures in the Euclidean spaces

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Abstract In this paper, we study hypersurfaces of Euclidean spaces with arbitrary dimension. First, we obtain some results on H-hypersurfaces. Then, we give the complete classification of H-hypersurfaces with three distinct curvatures. We also give some explicit examples.

Keywords Biharmonic submanifolds · Biconservative maps · Null 2-type submanifolds

Mathematics Subject Classification 53C40 · 53C42 · 53C50

1 Introduction

Let M be an n-dimensional submanifold of Euclidean m-space $\mathbb{E}^m$ and $x : M \to \mathbb{E}^m$ an isometric immersion. M is said to be biharmonic if x satisfies $\Delta^2 x = 0$, where Δ is the Laplace operator of M. In [3,5], Bang-Yen Chen conjectured that every biharmonic submanifold of a Euclidean space is minimal. Chen's conjecture is supported by all of the results obtained so far (see for example [8,9,13]).

On the other hand, M is said to be null 2-type if x can be expressed as $x = x_0 + x_1$ for some non-constant vector valued functions x_0 and x_1 satisfying $\Delta x_0 = 0$ and $\Delta x_1 = \lambda x_1$ for a nonzero constant λ, [2,6]. Several works on null 2-type surfaces also have been appeared, [4,10,12].

In particular, there are some recent results on biharmonic and null 2-type hypersurfaces, [7,11,12]. For example, in [7], authors obtained some results on $\delta(2)$-ideal null 2-type hypersurfaces. Most recently, the complete classification of biharmonic hypersurfaces in $\mathbb{E}^5$ with three distinct principle curvatures has been obtained by Fu [11].

Now, suppose that M is a hypersurface in Euclidean space $\mathbb{E}^{n+1}$ and let N be its unit normal vector field. From the definition, one can see that if M is null 2-type or biharmonic,

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then the equation

$$\Delta^2 x = \lambda \Delta x \tag{1.1}$$

is satisfied for a constant λ. In addition, Beltrami's well known formula $\Delta x = -s_1 N$ implies

$$\Delta^2 x = -\left(\Delta s_1 + s_1(s_1^2 - 2s_2)\right) N - \left(S(\nabla s_1) + \frac{s_1}{2}\nabla s_1\right),$$

where S is the shape operator and s_1, s_2 denote the first and second mean curvatures of M, respectively. Therefore, if a hypersurface M in $\mathbb{E}^{n+1}$ is biharmonic or null 2-type, then the system of differential equations

$$S(\nabla s_1) = -\frac{s_1}{2}\nabla s_1, \tag{1.2a}$$

$$\Delta s_1 = -s_1(s_1^2 - 2s_2 - \lambda) \tag{1.2b}$$

is satisfied for a constant λ. A hypersurface with non-constant first mean curvature is said to be an H-hypersurface [13] or biconservative hypersurface [1,14] if it satisfies (1.2a). Classifying H-hypersurfaces, or at least understanding their geometry, may play an important role on the theory of hypersurfaces satisfying (1.1).

In this work, we study hypersurfaces with three distinct principal curvatures in the Euclidean space of arbitrary dimension. In Sect. 2, after we describe our notations, we give a summary of the basic facts and formulas that we will use. In Sect. 3, we obtain some geometrical properties of H-hypersurfaces. In Sect. 4, we give a classification of H-hypersurfaces with three distinct principal curvatures.

2 Preliminieries

Let $\mathbb{E}^m$ denote the Euclidean m-space with the canonical Euclidean metric tensor given by

$$\widetilde{g} = \langle\ ,\ \rangle = \sum_{i=1}^{m} \mathrm{d}x_i^2,$$

where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m$.

Consider an n-dimensional Riemannian submanifold M of the space $\mathbb{E}^m$. We denote Levi-Civita connections of $\mathbb{E}^m$ and M by $\widetilde{\nabla}$ and ∇, respectively. Then, the Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\widetilde{\nabla}_X \rho = -S_\rho(X) + \nabla^\perp_X \rho \tag{2.2}$$

for all tangent vectors fields X, Y and normal vector fields ρ, where h, $\nabla^\perp$ and S are the second fundamental form, the normal connection and the shape operator of M, respectively. Note that for each $\rho \in T^\perp_m M$, the shape operator S_ρ along the normal direction ρ is a symmetric endomorphism of the tangent space $T_m M$ at $m \in M$. The shape operator and the second fundamental form are related by $\langle h(X, Y), \rho\rangle = \left\langle S_\rho X, Y\right\rangle$.

The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X, Y)Z, W\rangle = \langle h(Y, Z), h(X, W)\rangle - \langle h(X, Z), h(Y, W)\rangle, \tag{2.3}$$

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \tag{2.4}$$

where R is the curvature tensor associated with connection ∇ and $\bar{\nabla} h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The mean curvature vector ζ of M is defined by

$$\zeta = \frac{1}{n} \mathrm{tr}\, h.$$

2.1 Hypersurfaces of Euclidean space

Now, let M be an oriented hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, x its position vector and S its shape operator along the unit normal vector field N associated with the orientation of M. We consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n; N\}$ consisting of principal directions of M with corresponding principal curvatures $k_1, k_2, \ldots, k_n$. We denote the dual basis of this frame field by $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Then, the first structural equation of Cartan is

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i = 1, 2, \ldots, n, \tag{2.5}$$

where ω_{ij} denotes the connection forms corresponding to the chosen frame field, i.e., $\omega_{ij}(e_l) = \langle \nabla_{e_l} e_i, e_j \rangle$.

From the Codazzi equation (2.4), we have

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j), \tag{2.6a}$$

$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l) \tag{2.6b}$$

for distinct $i, j, l = 1, 2, \ldots, n$.

We put $s_1 = k_1 + k_2 + \cdots + k_n$ and, by abuse of terminology, we call this function as the (first) mean curvature of M. Note that M is said to be (1-) minimal if $s_1 = 0$. Throughout this work, we assume ∇s_1 does not vanish at any point of M.

3 H-hypersurfaces

In this section, we give some results on H-hypersurfaces of Euclidean spaces by extending the results obtained in [13].

3.1 Connection forms of H-hypersurfaces

Let M be an H-hypersurface of the Euclidean space $\mathbb{E}^{n+1}$. Then, (1.2a) is satisfied and s_1 is not constant. From (1.2a), we have ∇s_1 is a principal direction of M. We consider a frame field $\{e_1, e_2, \ldots, e_n\}$ consisting of principal directions of M with corresponding principal curvatures $k_1, k_2, \ldots, k_n$ such that $e_1 = \nabla s_1 / |\nabla s_1|$ and $k_1 = -s_1/2$. Therefore, we have

$$e_1(k_1) \neq 0, \qquad e_x(k_1) = 0, \quad x = 2, 3, \ldots, n \tag{3.1}$$

and

$$3k_1 + k_2 + k_3 + \cdots + k_n = 0. \tag{3.2}$$

Remark 1 [13] If $k_1 = k_x$ for some $2 \leq x \leq, n$, then Codazzi equation (2.6a) for $i = 1$, $j = x$ implies $e_1(k_1) = e_1(k_x) = \omega_{1x}(e_x)(k_1 - k_x) = 0$ which contradicts with (3.1). Thus, the dimension of the distribution D_0 given by

$$D_0(m) = \{X \in T_m | SX = k_1 X\}$$

is 1. Integral curves of D_0 are planar and geodesics of M. Furthermore, if α and β are integral curves of D_0 passing through m and m', respectively, then α and β are congruent, [13].

By combining (3.1) with Codazzi equation (2.6a) for $i = x,\ j = 1$, we get

$$\omega_{1x}(e_1) = 0, \quad x = 2, 3, \dots, n. \tag{3.3}$$

On the other hand, for a tangent vector field X of M, $\langle X, e_1\rangle = 0$ if and only if $Xk_1 = 0$. Therefore, $[e_x, e_y](k_1) = 0$ implies $\langle [e_x, e_y], e_1\rangle = 0$ from which we have

$$\omega_{1x}(e_y) = \omega_{1y}(e_x), \qquad x, y = 2, 3, \dots, n,\ x \neq y.$$

From this equation and Codazzi equation (2.6b) for $i = 1,\ j = x,\ l = z$, we get

$$\omega_{1x}(e_y) = 0, \quad x, y = 2, 3, \dots, n, x \neq y. \tag{3.4}$$

Therefore, (2.6b) for $i = x, j = y, l = 1$ and (2.6b) for $i = x, j = 1, l = y$ imply

$$\omega_{xy}(e_1) = 0, \quad x, y = 2, 3, \dots, n,\ k_x \neq k_y. \tag{3.5a}$$

In fact, we have

$$\omega_{xy}(e_z) = 0, \quad x, y, z = 1, 2, 3, \dots, n,\ x \neq z,\ k_x = k_z \neq k_y \tag{3.5b}$$

from the Codazzi equation (2.6b) for $i = x,\ j = y,\ l = z$.

Since (3.3) implies $\langle [e_1, e_x], e_1\rangle = 0$, we have $[e_1, e_x](k_1) = 0$ from which and (3.1) we obtain

$$e_x e_1(k_1) = e_x e_1 e_1(k_1) = 0, \quad x = 2, 3, \dots, n. \tag{3.6}$$

3.2 Some lemmas on H-hypersurfaces

In this subsection, we obtain some lemmas that we will use on the rest of the paper.

First, we consider the distribution given by

$$D(m) = \{X \in T_m M | SX = k_2 X\}. \tag{3.7}$$

Remark 2 Obviously, the dimension of D is equal to multiplicity of k_2 as an eigenvalue of the shape operator S of M.

We obtain the following lemma.

Lemma 3.1 *Let M be an H-hypersurface in the Euclidean space $\mathbb{E}^{n+1}$ and k_2 one of its principal curvatures. Then, the distribution D given by (3.7) is involutive.*

Proof If the dimension of D is 1, then it is obviously involutive. Thus, we assume $\dim D = p > 1$ and, by renaming the indices if necessary,

$$k_2 = k_3 = \ldots = k_{p+1}. \tag{3.8}$$

Therefore, (3.5b) implies $\langle \nabla_{e_A} e_B, e_i\rangle = \omega_{Bi}(e_A) = 0$ for all $i = 1, p+2, p+3, \dots, n$ and $A, B = 2, 3, \dots, p+1$ with $A \neq B$. Thus, we have $(\nabla_{e_A} e_B)_m \in D(m)$ from which we see that $X_m, Y_m \in D(m)$ implies $[X_m, Y_m] \in D(m)$. Hence, D is involutive. □

Now, we want to construct integral submanifolds of the distribution D given by (3.7).

Lemma 3.2 *Let M be an H-hypersurface in the Euclidean space $\mathbb{E}^{n+1}$ and k_2 one of its principal curvatures. Assume that the distribution D given by (3.7) has dimension greater than 1. Then, any integral submanifold H of D has parallel mean curvature vector field ζ in $\mathbb{E}^{n+1}$. Moreover, for any normal vector field ρ in H, we have $\hat{S}_\rho = \tau I$ for a function τ, where $\hat{S}$ denotes the shape operator of H in $\mathbb{E}^{n+1}$.*

Proof Let the dimension of D is $p > 1$. Then, by renaming indices if necessary, we assume (3.8). Using (3.5), we obtain

$$\widetilde{\nabla}_{e_A} e_A = -\omega_{1A}(e_A)e_1 + \sum_{C=2}^{p+1} \omega_{AC}(e_A)e_C + \sum_{a=p+2}^{n} \omega_{Aa}(e_A)e_a + k_2 N,$$

$$\widetilde{\nabla}_{e_A} e_B = \sum_{C=2}^{p+1} \omega_{BC}(e_A)e_C \tag{3.9}$$

for all $A, B = 2, 3, \ldots, p+1$ with $A \neq B$. Note that Codazzi equation (2.6a) for $i = 1$, $j = A$ and $i = a$, $j = A$ give $\omega_{1A}(e_A) = \frac{e_1(k_A)}{k_1 - k_A}$ and $\omega_{Aa}(e_A) = \frac{e_a(k_A)}{k_A - k_a}$, respectively. Thus, (3.8) implies

$$\xi = -\omega_{12}(e_2) = -\omega_{13}(e_3) = \cdots = -\omega_{1(p+1)}(e_{p+1}),$$
$$\eta_a = \omega_{a2}(e_2) = \omega_{a3}(e_3) = \cdots = \omega_{a(p+1)}(e_{p+1}) \tag{3.10}$$

for some functions ξ and η_a for $a = p+2, p+3, \ldots, n$.

Now, let H be an integral submanifold of D and consider the local orthonormal frame field

$$\{f_1, f_2, \ldots, f_p; f_{p+1}, f_{p+2}, \ldots f_{n+1}\}$$

on H given by

$$f_{A-1} = e_A|_H\,,\ f_{p+1} = e_1|_H\,, f_a = e_a|_H\,, f_{n+1} = N|_H\,. \tag{3.11}$$

From (3.9) and (3.10), we have

$$\widetilde{\nabla}_{f_i} f_i = \hat{\nabla}_{f_i} f_i + \hat{\xi} f_{p+1} + \sum_{a=p+2}^{n} \hat{\eta}_a f_a + \hat{k}_2 f_{n+1} \tag{3.12a}$$

$$\widetilde{\nabla}_{f_i} f_j = \hat{\nabla}_{f_i} f_j, \quad i, j = 1, 2, \ldots, p,\ i \neq j, \tag{3.12b}$$

where $\hat{\nabla}$ denotes the Levi-Civita connection of H and $\hat{\xi}$, $\hat{\eta}_a$, $\hat{k}_2$ are restrictions of ξ, η_a, k_2 to H, respectively.

Therefore, we have

$$\hat{S}_{p+1} = \hat{\xi} I,\ \hat{S}_a = \hat{\eta}_a I,\ \hat{S}_{n+1} = \hat{k}_2 I \tag{3.13}$$

or, equivalently,

$$\zeta = \hat{h}(f_1, f_1) = \hat{h}(f_2, f_2) = \cdots = \hat{h}(f_p, f_p) = \hat{\xi} f_{p+1} + \sum_{a=p+2}^{n} \hat{\eta}_a f_a + \hat{k}_2 f_{n+1}, \tag{3.14}$$

where $\hat{h}$ stands for the second fundamental form of H in $\mathbb{E}^{n+1}$ and $\hat{S}_\alpha = \hat{S}_{f_\alpha}$.

Furthermore, Codazzi equation (2.4) for $X = Z = f_i$ and $Y = f_j$ for $i \neq j$ gives

$$\hat{\nabla}^\perp_{f_i} \hat{h}(f_i, f_j) - \hat{h}(\hat{\nabla}_{f_i} f_i, f_j) - \hat{h}(f_i, \hat{\nabla}_{f_i} f_j) = \hat{\nabla}^\perp_{f_j} h(f_i, f_i) - 2\hat{h}(\hat{\nabla}_{f_j} f_i, f_i),$$

where $\hat{\nabla}^{\perp}$ is the normal connection of H in $\mathbb{E}^{n+1}$. Using (3.12) in this equation and considering (3.14), we get $\hat{\nabla}^{\perp}_{f_j}\zeta = 0$ for all $j = 1, 2, \ldots, p$. Hence, ζ is parallel. □

Remark 3 Let M be an H-hypersurface in the Euclidean space $\mathbb{E}^{n+1}$ and k_2 one of its principal curvatures. Assume that the distribution D given by (3.7) has dimension greater than 1 and H is an (connected) integral submanifold of D. From the proof of Lemma 3.2, one can see that k_2, $\xi = \omega_{1A}(e_A)$ and $\eta_a = \omega_{Aa}(e_A)$ are constant on H.

By the following proposition, we obtain integral submanfiolds of the distribution D given by (3.7).

Proposition 3.3 *Let M be an H-hypersurface in the Euclidean space $\mathbb{E}^{n+1}$ and k_2 one of its principal curvatures. Assume that the distribution D given by* (3.7) *has dimension $p > 1$ and H is an (connected) integral submanifold of D passing through $m \in M$. If $k_2(m) = 0$ and $(\nabla k_2)_m = 0$ then H is a p-plane of $\mathbb{E}^{n+1}$. Otherwise, H lies on a $(p+1)$-plane of $\mathbb{E}^{n+1}$ and it is congruent to a hypersphere of $\mathbb{E}^{p+1}$.*

Proof First, suppose that k_2 and ∇k_2 vanish at m. Then, we have $\hat{\eta}_a(m) = \hat{\xi}(m) = 0$ for $a = p+2, p+3, \ldots, n$, where $\hat{\eta}_a$ and $\hat{\xi}$ are functions defined in the proof of Lemma 3.2. Remark 3 implies that $\hat{k}_2 \equiv 0$, $\hat{\xi} \equiv 0$ and $\hat{\eta} \equiv 0$ on H. Thus, from (3.14) we have $\hat{h} = 0$, i.e., M is a totally geodesic p-dimensional submanifold of $\mathbb{E}^{n+1}$. Hence, M is a p-plane.

Next, assume $k_2(m) \neq 0$. Define $n - p$ normal vector fields $\zeta_1, \zeta_{p+2}, \ldots, \zeta_n$ by $\zeta_1 = \hat{k}_2 f_p - \hat{\xi} f_{n+1}$ and $\zeta_a = \hat{k}_2 f_a - \hat{\eta}_a f_{n+1}$. Clearly, $\zeta_1, \zeta_{p+2}, \ldots, \zeta_n$ are linearly independent constant vector fields normal to H. Thus, H lies in a $(p+1)$-plane $\Pi \cong \mathbb{E}^{p+1}$ of $\mathbb{E}^{n+1}$. As its mean curvature vector is parallel, and shape operator is proportional to identity operator I, it is a hypersphere of Π.

If $(\nabla k_2)_m \neq 0$, then we have $\hat{\xi}(m) \neq 0$ or $\hat{\eta}_a(m) \neq 0$ for some a because of Codazzi equation (2.6a). Same proof can be done for both cases. □

4 H-hypersurfaces with three distinct principal curvatures

Let M be an H-hypersurfaces in $\mathbb{E}^{n+1}$ with three distinct principal curvatures k_1, k_2 and k_{p+2} and S its shape operator. Because of Remark 1, the multiplicity of k_1 is 1. Therefore, the matrix representation of S is

$$S = \operatorname{diag}(k_1, \underbrace{k_2, k_2, \ldots, k_2}_{p \text{ times}}, \underbrace{k_{p+2}, k_{p+2}, \ldots, k_{p+2}}_{q \text{ times}}), \quad k_2 \neq k_{p+2} \tag{4.1}$$

corresponding to a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ consisting of principal directions of M, where $p + q + 1 = n$. We also assume that the functions $k_1 - k_2$, $k_1 - k_{p+2}$ and $k_2 - k_{p+2}$ do not vanish on M.

First, we consider the distribution $D^{\perp}$ given by

$$D^{\perp}(m) = \{X_m \in T_m M \,|\, \langle X_m, Y\rangle = 0, \text{ for all } Y \in D(m)\}. \tag{4.2}$$

Lemma 4.1 *Let M be an H-hypersurface in the Euclidean space $\mathbb{E}^{n+1}$ with the shape operator given by* (4.1)*. Then, the distribution $D^{\perp}$ given by* (4.2) *is involutive.*

Proof From the definition, we have $D^{\perp}(m) = \operatorname{span}\{(e_1)_m, (e_{p+2})_m, (e_{p+3})_m, \ldots, (e_n)_m\}$. Moreover, from (3.5) and (4.1) we have $\nabla_{e_a} e_b, \nabla_{e_a} e_1, \nabla_{e_1} e_a \in D^{\perp}$ for all $a, b = p+2, p+3, \ldots, n$. Thus, for all $X, Y \in D^{\perp}$, we have $[X, Y] \in D^{\perp}$. Hence $D^{\perp}$ is involutive. □

Remark 4 By combining (3.3) and (3.4) with Cartan's first structural equation (2.5), we obtained $d\theta_1 = 0$, i.e., θ_1 is closed. Thus, Poincarè lemma implies that $d\theta_1$ is exact, i.e., there exists a function s such that $\theta_1 = ds$. Moreover, since the distributions D and $D^{\perp}$ given by (3.7) and (4.2) are involutive, there exists a local coordinate system $t_1, t_2, \ldots, t_n$ on a neighborhood of $m \in M$ such that $t_2, t_3, \ldots, t_{p+1}$ span D and $t_1 = s, t_{p+2}, t_{p+3}, \ldots, t_n$ span $D^{\perp}$ because of the local Frobenius theorem. Thus, by redefining the vector fields e_i, $i = 2, 3, \ldots, n$ properly, we can assume

$$e_1 = \partial_s, \quad e_i = F_i \partial_{t_i}, \ i = 2, 3, \ldots, n \tag{4.3}$$

for some smooth non-vanishing functions $F_i = F_i(s, t_2, t_3, \ldots, t_n)$.

Since the study on $\mathbb{E}^4$ is completed in [13], we focus on the case $n > 3$. Therefore, without loss of generality, we may assume $p > 1$. Thus, we have (3.8) and Codazzi equation (2.6a) implies

$$e_A(k_2) = 0, \quad A = 2, 3, \ldots, p+1. \tag{4.4}$$

From (3.1), (3.2), (4.1) and (4.4), we also get

$$e_A(k_{p+2}) = 0, \quad a = p+2, p+3, \ldots, n \tag{4.5}$$

from which and Codazzi equation (2.6a) for $i = A$, $j = a$, we obtain

$$\omega_{Aa}(e_a) = 0. \tag{4.6}$$

4.1 Case $p > 1$ and $q > 1$

In this case, we have

$$e_a(k_{p+2}) = 0, \tag{4.7a}$$
$$e_a(k_2) = 0, \tag{4.7b}$$
$$\omega_{Aa}(e_A) = 0, \quad A = 2, 3, \ldots, p+1, \ a = p+2, p+3, \ldots, n. \tag{4.7c}$$

By combining (3.3), (3.4), (3.5), (4.6) and (4.7c), we get

$$\widetilde{\nabla}_{e_1} e_A = \sum_{C=2}^{p+1} \omega_{AC}(e_1) e_C, \tag{4.8a}$$
$$\widetilde{\nabla}_{e_A} e_1 = \omega_{12}(e_2) e_A, \tag{4.8b}$$
$$\widetilde{\nabla}_{e_a} e_1 = \omega_{1(p+2)}(e_{p+2}) e_a. \tag{4.8c}$$

On the other hand, since $[e_1, e_A](k_2) = e_A e_1(k_2)$, we have

$$e_A e_1(k_2) = \left(\sum_{C=2}^{p+1} \omega_{AC}(e_1) e_C - \omega_{1A}(e_A) e_A \right)(k_2)$$

from (4.8a) and (4.8b). The right- hand side of this equation is zero because of (4.4). Thus, we obtain

$$e_A e_1(k_2) = 0. \tag{4.9}$$

Furthermore, from (2.6a) for $i = 1$, $j = 2$ we have

$$e_A(\omega_{12}(e_2)) = e_A \left(\frac{e_1(k_2)}{k_1 - k_2} \right).$$

By combining this equation with (3.1), (4.4) and (4.9) we get

$$e_A(\omega_{12}(e_2)) = 0. \tag{4.10a}$$

By a similar way, we obtain

$$e_a(\omega_{12}(e_2)) = 0, \tag{4.10b}$$

$$e_A(\omega_{1(p+2)}(e_{p+2})) = e_a(\omega_{1(p+2)}(e_{p+2})) = 0. \tag{4.10c}$$

By combining the equations in (4.10), we get

$$\omega_{12}(e_2) = \xi(s), \qquad \omega_{1(p+2)}(e_{p+2}) = \eta(s) \tag{4.11}$$

for some functions ξ, η, where s is the local coordinate given in Remark 4. Now, we are ready to prove

Theorem 1 *Let M be a hypersurface in $\mathbb{E}^{n+1}$ with the shape operator given by (4.1), $k_2 \neq k_{p+2}$ and $p > 1, q > 1$. Then, M is an H-hypersurface if and only if it is congruent to one of the following hypersurfaces.*

(i) *A generalized rotational hypersurface given by*

$$\begin{aligned} x(s, t_2, \dots, t_n) = &\big(\psi(s)\cos t_2, \psi(s)\sin t_2 \cos t_3, \dots, \psi(s)\sin t_2 \dots \sin t_p \cos t_{p+1}, \\ &\psi(s)\sin t_2 \dots \sin t_p \sin t_{p+1}, \phi(s)\cos t_{p+2}, \phi(s)\sin t_{p+2}\cos t_{p+3}, \dots, \\ &\phi(s)\sin t_{p+2}\dots \sin t_{n-1}\cos t_n, \phi(s)\sin t_{p+2}\dots\sin t_{n-1}\sin t_n\big) \end{aligned} \tag{4.12}$$

with the profile curve (ψ, ϕ) satisfying $\psi'^2 + \phi'^2 = 1$ and

$$\phi'\psi'' - \phi''\psi' = \frac{1}{3}\left(p\frac{\phi'}{\psi} - q\frac{\psi'}{\phi}\right). \tag{4.13}$$

(ii) *A generalized cylinder over a rotational hypersurface given by*

$$\begin{aligned} x(s, t_1, \dots, t_n) = &\big(\psi(s)\cos t_2, \psi(s)\sin t_2 \cos t_3, \dots, \psi(s)\sin t_2 \dots \sin t_p \cos t_{p+1}, \\ &\psi(s)\sin t_2 \dots \sin t_p \sin t_{p+1}, \phi(s), t_{p+2}, t_{p+3}, \dots, t_n\big) \end{aligned} \tag{4.14}$$

with the profile curve (ψ, ϕ) satisfying $\psi'^2 + \phi'^2 = 1$ and

$$\phi'\psi'' - \phi''\psi' = \frac{p}{3}\frac{\phi'}{\psi}. \tag{4.15}$$

Proof We assume that M is an H-hypersurface. Then, (3.2) is satisfied. Let $s, t_2, t_3, \dots, t_n$ be the local coordinate system given in Remark 4. From (4.8b) and (4.8c) we have

$$x_{st_A} = \omega_{12}(e_2)x_{t_A}, \quad A = 2, 3, \dots, p+1 \tag{4.16a}$$

$$x_{st_a} = \omega_{1(p+2)}(e_{p+2})x_{t_a}, \quad a = p+2, p+3, \dots, n. \tag{4.16b}$$

By taking into account the (4.11), we integrate (4.16) to obtain

$$x_s = \xi(s)x + \tilde{\Theta}_2(s, t_{p+2}, t_{p+3}, \dots, t_n) = \eta(s)x + \tilde{\Theta}_1(s, t_2, t_3, \dots, t_{p+1})$$

for some vector valued functions $\tilde{\Theta}_1$, $\tilde{\Theta}_2$. Therefore, we have

$$x(s, t_2, t_3, \dots, t_n) = \hat{\Theta}_1(s, t_2, t_3, \dots, t_{p+1}) + \hat{\Theta}_2(s, t_{p+2}, t_{p+3}, \dots, t_n) \tag{4.17}$$

for some vector valued functions $\hat{\Theta}_1$ and $\hat{\Theta}_2$.

Next, we put (4.17) in (4.16) to get

$$\hat{\Theta}_{1,st_A} = \xi(s)\hat{\Theta}_{1,t_A}, \quad \hat{\Theta}_{2,st_a} = \eta(s)\hat{\Theta}_{2,t_a}.$$

By integrating these equations, we obtain

$$x(s, t_2, t_3, \ldots, t_n) = \psi(s)\Theta_1(t_2, t_3, \ldots, t_{p+1}) + \phi(s)\Theta_2(t_{p+2}, t_{p+3}, \ldots, t_n) + \varphi(s)$$

for some functions ϕ, ψ and vector valued functions $\Theta_1, \Theta_2, \varphi$. By taking into account Remark 1, we see that φ is a constant vector. Thus, we may assume

$$x(s, t_2, t_3, \ldots, t_n) = \psi(s)\Theta_1(t_2, t_3, \ldots, t_{p+1}) + \phi(s)\Theta_2(t_{p+2}, t_{p+3}, \ldots, t_n). \tag{4.18}$$

Because of (4.3), we have

$$\left\langle \Theta_{1,t_A}, \Theta_{2,t_a} \right\rangle = 0, \tag{4.19a}$$

$$\langle x_s, x_s \rangle = 1. \tag{4.19b}$$

Since $k_2 \neq k_{p+2}$, without loss of generality, we may assume $k_2 \neq 0$. Now, we consider the slice H of M given by

$$y_1(t_2, t_3, \ldots, t_{p+1}) = x(\bar{s}, t_2, t_3, \ldots, t_{p+1}, \bar{t}_{p+2}, \bar{t}_{p+3}, \bar{t}_n)$$

passing through the point $m = x(\bar{s}, \bar{t}_2, \bar{t}_3, \ldots, \bar{t}_n) \in M$. From (4.18) we have

$$y_1(t_2, t_3, \ldots, t_{p+1}) = c_0\Theta_1(t_2, t_3, \ldots, t_{p+1}) + v_0, \tag{4.20}$$

where $c_0 = \phi(\bar{s})$ is a constant and $v_0 = \psi(\bar{s})\Theta_2(\bar{t}_{p+2}, \bar{t}_{p+3}, \ldots, \bar{t}_n)$ is a constant vector.

Since H is an integral submanifold of the distribution D given by (3.7), and $k_2 \neq 0$, it is congruent to hypersphere of $\mathbb{E}^{p+1}$ because of Proposition 3.3. Thus, by choosing suitable coordinates and redefining ψ, we may assume

$$\begin{aligned}\Theta_1(t_2, \ldots, t_{p+2}) = \big(&\cos t_2, \sin t_2 \cos t_3, \ldots, \sin t_2 \ldots \sin t_p \cos t_{p+1},\\ &\sin t_2 \ldots \sin t_p \sin t_{p+1}, 0, 0, \ldots, 0\big).\end{aligned} \tag{4.21}$$

Now, consider the submanifold H' given by

$$y_2(t_{p+2}, t_{p+3}, \ldots, t_n) = x(\bar{s}, \bar{t}_2, \bar{t}_3, \ldots, \bar{t}_{p+1}, t_{p+2}, t_{p+3}, t_n)$$

which is an integral submanifold of the distribution D' given by $D'(m') = \{X \in T_{m'}M | SX = k_{p+2}X\}$ passing through the point m. Now, we have two cases: $k_{p+2} = 0$ and $k_{p+2} \neq 0$.

Case 1. $k_{p+2} = 0$. In this case, H' is a q-plane because of Proposition 3.3. Thus, Θ_2 is the position vector of a q-plane. Because of (4.19a) without loss of generality, we may assume

$$\Theta_2(t_{p+2}, t_{p+3}, \ldots, t_n) = \big(0, 0, \ldots, 1, t_{p+2}, t_{p+3}, \ldots, t_n\big).$$

By redefining $t_{p+2}, \ldots, t_n$, we obtain (4.14). Because of (4.19b), we have $\psi'^2 + \phi'^2 = 1$.

Moreover, the shape operator of this hypersurface is

$$S = \mathrm{diag}(k_1, \underbrace{\frac{\phi'}{\psi}, \frac{\phi'}{\psi}, \ldots, \frac{\phi'}{\psi}}_{p \text{ times}}, \underbrace{0, 0, \ldots, 0}_{q \text{ times}}). \tag{4.22}$$

From (3.2) and (4.22), we get (4.15). Hence, we have the case (ii) of theorem.

Case 2. $k_{p+2} \neq 0$. In this case, H' is congruent to a hypersphere of $\mathbb{E}^{q+1}$ because of Proposition 3.3. Because of (4.19a), without loss of generality, we choose

$$\Theta_2(t_{p+2}, t_{p+3}, \dots, t_n) = \big(\underbrace{0, 0, \dots, 0}_{p+1 \text{ times}}, \cos t_{p+2}, \sin t_{p+2} \cos t_{p+3}, \dots, \\ \sin t_{p+2} \dots \sin t_{n-1} \cos t_n, \psi \sin t_{p+2} \dots \sin t_n\big).$$

Therefore, we obtain (4.12). Because of (4.19b), we have $\psi'^2 + \phi'^2 = 1$.

Moreover, the shape operator of this hypersurface is

$$S = \mathrm{diag}(k_1, \underbrace{\frac{\phi'}{\psi}, \frac{\phi'}{\psi}, \dots, \frac{\phi'}{\psi}}_{p \text{ times}}, \underbrace{-\frac{\psi'}{\phi}, -\frac{\psi'}{\phi}, \dots, -\frac{\psi'}{\phi}}_{q \text{ times}}). \tag{4.23}$$

From (3.2) and (4.23) we get (4.13). Hence, we have the case (i) of theorem. □

Remark 5 In [14], Montaldo et al. proved that a curve satisfying (4.13) is of catenary type. The authors also proved that none of these type of hypersurfaces are biharmonic. Recently, in [11], Yu Fu remarked that he extended this result by proving that there is no non-minimal biharmonic hypersurface in $\mathbb{E}^{n+1}$ with three distinct principal curvature. However, classifying null 2-type hypersurfaces with three distinct principal curvature is an open problem.

4.2 Case $p > 1$ and $q = 1$

In the remaining part, we consider the case $p > 1$ and $q = 1$ to obtain a necessary condition for null 2-type hypersurfaces with 3 principal curvatures. In this case (4.1) becomes

$$S = \mathrm{diag}(k_1, \underbrace{k_2, k_2, \dots, k_2}_{n-2 \text{ times}}, k_n). \tag{4.24}$$

Since $p = n - 2 > 1$, the equations (4.4)-(4.6) are still satisfied. Moreover, the distribution D given in (3.7) is involutive and its integral submanifold are congruent to hyperspheres or hyperplanes of $\mathbb{E}^{n-1}$ because of Lemma 3.1 and Lemma 3.2. From [13, Lemma 2.2], we also know that integral curves of $e_1 = \partial_s$ are some planar curves and congruent to each other. Therefore, we first want to focus on the remaining part, integral curves of e_n.

Let M be a hypersurface with the shape operator given in (4.24). We also suppose that the functions $k_1 - k_2$, $k_1 - k_n$ and $k_2 - k_n$ do not vanish on M. Now, assume that M is a null 2-type hypersurface. Then, M is an H-surface satisfying (1.2b). Moreover, from (3.2) and (4.24), we have

$$3k_1 + (n-2)k_2 + k_n = 0 \tag{4.25}$$

because M is an H-hypersurface.

By combining (4.4) and (4.5) with Codazzi equation (2.6a), we have $\omega_{An}(e_n) = 0$. Therefore, we have

$$\widetilde{\nabla}_{e_n} e_1 = \omega_{1n}(e_n) e_n, \quad \widetilde{\nabla}_{e_A} e_1 = \omega_{1A}(e_A) e_A \tag{4.26a}$$

$$\widetilde{\nabla}_{e_A} e_n = -\omega_{An}(e_A) e_A, \quad \widetilde{\nabla}_{e_n} e_A = \omega_{AB}(e_n) e_B \tag{4.26b}$$

Now, we want to show $e_n(k_2) = 0$ using a method similar with [11].

Since $e_A(k_2) = 0$ and $e_A(k_n) = 0$, we have $[e_A, e_1](k_2) = e_A e_1(k_2)$ and $[e_A, e_1](k_n) = e_A e_1(k_n)$. By computing the left-hand side of each of these equations using (4.26b), we get

$$e_A e_1(k_2) = e_A e_1(k_n) = 0, \quad A = 2, 3, \dots, n-1. \tag{4.27}$$

Furthermore, from the Gauss equation (2.3) for $X = e_A$, $Y = e_n$, $Z = e_1$, and $W = e_A$ we obtained

$$e_n(\omega_{12}(e_2)) = \frac{e_n(k_2)}{k_2 - k_n}(\omega_{12}(e_2) - \omega_{1n}(e_n)). \tag{4.28}$$

By a direct calculation using Codazzi equation (2.6a), (3.1), (4.25) and (4.28), we also obtain

$$e_n(\omega_{1n}(e_n)) = (n-2)\frac{(2k_2 - k_1 - k_n)e_n(k_2)}{(k_1 - k_n)(k_2 - k_n)}(\omega_{12}(e_2) - \omega_{1n}(e_n)). \tag{4.29}$$

On the other hand, from (4.24) and (1.2b) we have

$$e_1e_1(k_1)+(n-2)\omega_{12}(e_2)e_1(k_1)+\omega_{1n}(e_n)e_1(k_1)=k_1(k_1^2 + (n-2)k_2^2 + k_n^2 - \lambda). \tag{4.30}$$

By applying e_n to both hand side of this equation and using (3.6), (4.28) and (4.29) we obtain

$$e_n(k_n)\Big(e_1(k_1)(\omega_{12}(e_2) - \omega_{1n}(e_n)) - k_1(k_2 - k_n)(k_1 - k_n)\Big) = 0. \tag{4.31}$$

From the assumptions, we have the functions $\omega_{12}(e_2) - \omega_{1n}(e_n)$ and k_1 do not vanish. Thus, if $e_n(k_n) \neq 0$, then we have

$$\frac{e_1(k_1)}{k_1} = \frac{(k_2 - k_n)(k_1 - k_n)}{\omega_{12}(e_2) - \omega_{1n}(e_n)}$$

because of (4.31). By applying e_n to this equation we obtain

$$e_n\left(\frac{(k_2 - k_n)(k_1 - k_n)}{\omega_{12}(e_2) - \omega_{1n}(e_n)}\right) = 0.$$

Next, we compute the left-hand side of this equation using (3.1), (4.25), (4.28) and (4.29) to get $k_n = a_0 k_2$ for a constant a_0. However, this equation, (3.1) and (4.25) give us $e_n(k_2) = 0$ which is a contradiction. Therefore, we have

$$e_n(k_2) = 0 \tag{4.32}$$

and (3.1), (4.25) imply

$$e_n(k_n) = 0. \tag{4.33}$$

Moreover, from Codazzi equation (2.6a) and (4.32) we have

$$\omega_{An}(e_A) = 0. \tag{4.34}$$

On the other hand, from (4.28), (4.29) and (4.32) we get

$$e_n(\omega_{1n}(e_n)) = e_n(\omega_{1A}(e_A)) = 0 \tag{4.35a}$$

and by taking into account (4.26) and using Gauss equation (2.3) for $X = e_A$, $Y = e_n$, $Z = e_1$, $W = e_n$ we obtain

$$e_A(\omega_{1n}(e_n)) = 0. \tag{4.35b}$$

From Codazzi equation (2.6a) for $i = 1$, $j = A$ we have $\omega_{1A}(e_A) = e_1(k_A)/(k_1 - k_A)$. Thus, we have

$$e_A(\omega_{1A}(e_A)) = 0. \tag{4.35c}$$

Next, we want to give a geometric interpretation of these results.

Proposition 4.2 *Let M be a null 2-type hypersurface in $\mathbb{E}^{n+1}$ with the shape operator given by (4.24) and non-constant first mean curvature. Then, an integral curve of e_n is either a circle or line.*

Proof Using (4.34), we get

$$\widetilde{\nabla}_{e_n} e_n = -\omega_{1n}(e_n)e_1 + k_n N, \quad \widetilde{\nabla}_{e_n} e_1 = \omega_{1n}(e_n)e_n, \quad \widetilde{\nabla}_{e_n} N = -k_n e_n, \tag{4.36}$$

Moreover, (4.35a) and (4.33) imply that $\omega_{1n}(e_n)$ and k_n are constant on any (connected) integral curve α of e_n. Let t, n be tangent and normal vector fields of α. Note that we have $t = e_n|_\alpha$. If

$$\|\hat{\nabla}_t t\| = a = 0$$

then α is a line and proof is completed, where $\hat{\nabla}$ is the Levi-Civita connection of α and a is the constant given by $\left(\omega_{1n}(e_n)^2 + k_n^2\right)^{1/2}\Big|_\alpha$.

We assume $\hat{\nabla}_t t \neq 0$. Then, we have $n = \hat{\nabla}_t t/\|\hat{\nabla}_t t\|$. From (4.36) we have $\hat{\nabla}_t t = an, \quad \hat{\nabla}_t n = -an$. Thus, α is planar and its curvature $a > 0$. □

By summing up (4.35), we see that (4.11) is satisfied for $q = 1$. Thus, by taking into account Proposition 4.2, we obtain a *necessary condition* for being null 2-type of hypersurfaces that we are considering. The following proposition can be proved like Theorem 1.

Proposition 4.3 *Let M be an hypersurface with non-constant first mean curvature and the shape operator given by* (4.24). *If M is a null 2-type hypersurface then it must be congruent to one of the following hypersurfaces.*

(i) *A generalized rotational hypersurfaces given by* (4.12) *with $p = n - 2$, $q = 1$ for some functions satisfying $\psi'^2 + \phi'^2 = 1$ and* (4.13),
(ii) *A generalized cylinder over a rotational hypersurface, given by* (4.14) *with $p = n - 2$, $q = 1$ for some functions satisfying $\psi'^2 + \phi'^2 = 1$ and* (4.15),
(iii) *A generalized cylinder over a rotational surface, given by* (4.14) *with $p = 1$, $q = n - 2$, for some functions satisfying $\psi'^2 + \phi'^2 = 1$ and* (4.15).

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