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A quasilinear elliptic system with natural growth terms

Lucio Boccardo · Luigi Orsina · Jean-Pierre Puel

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Abstract In this paper, we prove existence of solutions for an elliptic system of the type

$$\begin{cases} -\text{div}(a(x,z)\nabla u) = f, & \text{in } \Omega; \\ -\text{div}(b(x)\nabla z) + h(x,z)|\nabla u|^2 = g, & \text{in } \Omega; \\ u = 0 = z, & \text{on } \partial\Omega, \end{cases}$$

under various assumptions on the functions $a(x,s)$ and $h(x,s)$, and on the data f and g (in Lebesgue spaces).

Keywords Nonlinear elliptic systems · Quasilinear quadratic elliptic equations · Existence and nonexistence of solutions

Mathematics Subject Classification 35J60 · 35J47 · 35J57

1 Introduction

In this paper, we study the existence of solutions of elliptic systems of the type

$$\begin{cases} -\text{div}(a(x,z)\nabla u) = f, & \text{in } \Omega; \\ -\text{div}(b(x)\nabla z) + h(x,z)|\nabla u|^2 = g, & \text{in } \Omega; \\ u = 0 = z, & \text{on } \partial\Omega; \end{cases} \tag{1.1}$$

L. Boccardo · L. Orsina (✉)
Dipartimento di Matematica, "Sapienza" Università di Roma, P.le A. Moro 2, 00185 Rome, Italy
e-mail: orsina@mat.uniroma1.it

L. Boccardo
e-mail: boccardo@mat.uniroma1.it

J.-P. Puel
Laboratoire de Mathématiques de Versailles, Université de Versailles St Quentin,
45 Avenue des Etats Unis, 78035 Versailles Cedex, France
e-mail: jppuel@math.uvsq.fr

where we assume that Ω is a bounded open subset of $\mathbb{R}^N$, with $N > 2$, f and g are functions belonging to suitable Lebesgue spaces, $a(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, and $b : \Omega \to \mathbb{R}$ is a measurable function, such that

$$\alpha \le a(x,s), \quad \alpha \le b(x) \le \beta, \quad h(x,s)\cdot s \ge 0, \tag{1.2}$$

for some $0 < \alpha \le \beta$ in $\mathbb{R}^+$ (see the various sections of this paper for the precise assumptions on a, b and h).

Our interest in studying system (1.1) is motivated by the following example of the calculus of variations. Suppose that f and g are functions in $L^m(\Omega)$, with $m > \frac{N}{2}$, and that $s \mapsto a(x,s)$ is increasing, continuously differentiable, with $a_s(x,s)$ a bounded function with respect to both x and s, and such that $a_s(x,0) \equiv 0$; then, if we define

$$I(v,w) = \frac{1}{2}\int_\Omega a(x,|w|)|\nabla v|^2 + \frac{1}{2}\int_\Omega b(x)|\nabla w|^2 - \int_\Omega f\,v - \int_\Omega g\,w,$$

for every v and w in $H_0^1(\Omega)$ such that

$$\int_\Omega a(x,|w|)|\nabla v|^2 < +\infty,$$

and $I(v,w) = +\infty$ if the above integral is not finite, it is easy to see, using the assumptions on a and b that I is both coercive and weakly lower semicontinuous on $H_0^1(\Omega) \times H_0^1(\Omega)$. This implies that there exists a minimum (u,z) of I. Standard techniques of calculus of variations (together with the assumptions on a, b, f and g) imply that u and z belong to $L^\infty(\Omega)$ and that are weak solutions of the boundary value problem

$$\begin{cases} -\mathrm{div}(a(x,|z|)\nabla u) = f, & \text{in } \Omega; \\ -\mathrm{div}(b(x)\nabla z) + \frac{1}{2}a_s(x,|z|)\mathrm{sgn}(z)|\nabla u|^2 = g, & \text{in } \Omega; \\ u = 0 = z, & \text{in } \partial\Omega, \end{cases} \tag{1.3}$$

i.e., of a problem of type (1.1) with $h(x,s) = \frac{1}{2}a_s(x,|s|)\,\mathrm{sgn}(s)$. Note that the function $s \mapsto a_s(x,|s|)\,\mathrm{sgn}(s)$ is continuous at the origin thanks to the assumption $a_s(x,0) = 0$.

Thus, our interest in the study of the (general and nonvariational) system (1.1) is motivated by (1.3), more specifically, a system with an equation having a lower-order term, which has quadratic dependence with respect to the gradient and satisfies a "sign assumption" [see (1.2), or (2.2) below]. The study of nonvariational systems will allow us to weaken the summability assumptions on the data f and g; we will see that, depending on whether a is bounded from above or not, we will have existence of solutions for $L^1(\Omega)$ data f and g, or for data belonging to smaller Lebesgue spaces.

Even if our system is "natural" and motivated by the study of the very simple functional I, there seems to be no existence result in the literature for this kind of problems. General results on quasilinear elliptic systems, in contexts not applicable in the present case, can be found for example in [2,9–11].

The plan of the paper is as follows: in the next section, we will study existence of solutions for (1.1) under a boundedness assumption on a [see (2.1)], which will allow us to prove results for $L^1(\Omega)$ data f and g. The case of a unbounded from above is dealt with in Sect. 3: after some existence results for bounded solutions z (which will yield that $a(x,z)$ is bounded from above, thus giving results comparable to those of Sect. 2), we will study two particular cases (see Theorems 3.4 and 3.5), where the presence of a suitably growing, with respect to a, lower-order term $h(x,s)$ will help in proving existence of solutions. In Sect. 4, we will prove how the presence of a quadratic (with respect to the gradient) lower-order term in the second equation will "break" the maximum principle: if the datum of the first equation is an actual $L^1(\Omega)$ function f, the solution z of the second equation has to be zero on the set of "explosion" of f, even if the datum g is strictly positive in Ω. The case of measure data, yielding nonexistence of solutions, will be studied in the final Sect. 5.

2 The case of *a* bounded

In this section, we will study system (1.1) if the function $a(x, s)$ which appears in the first equation is bounded: that is, instead of (1.2), we assume that

$$\alpha \le a(x, s) \le \beta, \quad \alpha \le b(x) \le \beta, \tag{2.1}$$

for almost every x in Ω, for every s in $\mathbb{R}$, with $0 < \alpha \le \beta$ in $\mathbb{R}^+$.

Let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $h(x, s)$ is increasing with respect to s and that

$$h(x, s)\, s \ge 0, \quad \sup_{|s|\le t} |h(\cdot, s)| = h_t(\cdot) \in L^\infty(\Omega), \tag{2.2}$$

for almost every x in Ω, for every s and t in $\mathbb{R}$.

Under these assumptions, we will prove an existence result for problem (1.1). We begin with the case of bounded data f and g.

Proposition 2.1 *Assume that (2.1) and (2.2) hold, and let f and g be in $L^m(\Omega)$, with $m > \frac{N}{2}$. Then, there exists a solution (u, z) in $H_0^1(\Omega) \times H_0^1(\Omega)$ of system (1.1). Furthermore, both u and z belong to $L^\infty(\Omega)$.*

Sketch of the proof We will prove the result using Schauder's fixed-point theorem. To this aim, fix $z_0 \in L^2(\Omega)$ and let u in $H_0^1(\Omega)$ be the unique solution of the first equation; let then z in $H_0^1(\Omega)$ be the unique solution (see [6]) of the second equation:

$$\begin{cases} -\text{div}(a(x, z_0)\nabla u) = f, & \text{in } \Omega; \\ -\text{div}(b(x)\nabla z) + h(x, z)|\nabla u|^2 = g, & \text{in } \Omega; \\ u = 0 = z, & \text{on } \partial\Omega. \end{cases}$$

Therefore, if we define $S(z_0) = z$, then S maps $L^2(\Omega)$ into itself. It is standard to prove that the ball B_R of radius R in $L^2(\Omega)$ is invariant for S, where

$$R = \frac{\|g\|_{L^{2_*}(\Omega)}}{\alpha\, \mathcal{S}},$$

$\mathcal{S}$ is the Sobolev constant, and $2_* = \frac{2N}{N+2}$.

Moreover, by Rellich-Kondrachov theorem, $\overline{S(B_R)}$ is compact in $L^2(\Omega)$.

To prove that S is continuous, we begin with the observation that if z_n converges to z_0 in $L^2(\Omega)$, then, defining u_n as the solution of

$$u_n \in H_0^1(\Omega) : -\text{div}(a(x, z_n)\nabla u_n) = f,$$

we have that u_n is strongly convergent in $H_0^1(\Omega)$ to a function u, which is the unique solution of

$$u \in H_0^1(\Omega) : -\text{div}(a(x, z_0)\nabla u) = f.$$

Therefore, the sequence $\{|\nabla u_n|^2\}$ is strongly convergent in $L^1(\Omega)$ to $|\nabla u|^2$. This fact, and the stability result of [6] (or [7]), proves that the sequence $\tilde{z}_n = S(z_n)$ of solutions of

$$\tilde{z_n} \in H_0^1(\Omega) : -\mathrm{div}(b(x)\nabla \tilde{z}_n) + h(x, \tilde{z}_n)|\nabla u_n|^2 = g,$$

converges in $L^2(\Omega)$ to the unique solution of

$$\tilde{z} \in H_0^1(\Omega) : -\mathrm{div}(b(x)\nabla \tilde{z}) + h(x, \tilde{z})|\nabla u|^2 = g.$$

Thus, $S(z_n)$ converges in $L^2(\Omega)$ to $S(z_0)$, and so S is continuous. By Schauder's theorem, there exists a fixed-point z of S, and so a solution (u, z) of system (1.1). The fact that both u and z belong to $L^\infty(\Omega)$ then follows by standard elliptic results (see [13]). □

We study now to the more general case of $L^1(\Omega)$ data f and g. In what follows, we will use the functions $T_k(s) = \max(-k, \min(s, k))$, and $G_k(s) = s - T_k(s)$, for s in $\mathbb{R}$ and $k \geq 0$.

Theorem 2.2 *Assume that* (2.1) *and* (2.2) *hold, and let f and g be $L^1(\Omega)$ functions. Then, there exists a distributional solution (u, z) of system* (1.1)*, with both u and z belonging to $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Moreover, $h(x, z)|\nabla u|^2 \in L^1(\Omega)$.*

Proof Let $\{f_n\}$ and $\{g_n\}$ be sequences of $L^\infty(\Omega)$ functions, which converge in $L^1(\Omega)$ to f and g, respectively, and are such that

$$|f_n| \leq |f|, \quad |g_n| \leq |g|, \tag{2.3}$$

almost everywhere in Ω [take for example $f_n = T_n(f)$ and $g_n = T_n(g)$]. By the result of Theorem 2.1, there exist (u_n, z_n) weak solutions of

$$\begin{cases} u_n \in H_0^1(\Omega) : -\mathrm{div}(a(x, z_n)\nabla u_n) = f_n, \\ z_n \in H_0^1(\Omega) : -\mathrm{div}(b(x)\nabla z_n) + h(x, z_n)|\nabla u_n|^2 = g_n, \end{cases} \tag{2.4}$$

with both u_n and z_n belonging to $L^\infty(\Omega)$.

We begin with some *a priori* estimates on the sequences $\{u_n\}$, $\{z_n\}$.

Taking $T_k(u_n)$ as test function in the first equation of (2.4), and using (2.1) and the first of (2.3), we easily obtain (see [1])

$$\|T_k(u_n)\|^2_{H_0^1(\Omega)} \leq k\frac{\|f\|_1}{\alpha}. \tag{2.5}$$

Starting from (2.5), and reasoning as in [3], we have that

$$\int_\Omega |\nabla u_n|^q \leq C_q, \quad \forall q < \frac{N}{N-1}, \tag{2.6}$$

where C_q is a positive constant, which tends to infinity as q tends to $\frac{N}{N-1}$. Choosing $T_1(G_k(u_n))$ as test function in the first equation of (2.4), and using again (2.1) and the first of (2.3), we get

$$\alpha \int_{\{k \leq |u_n| < k+1\}} |\nabla u_n|^2 \leq \int_{\{|u_n| \geq k\}} |f|, \tag{2.7}$$

while the choice of $T_k(z_n)$ as test function in the second equation of (2.4) yields [using (2.1) and the second of (2.3)]

$$\|T_k(z_n)\|^2_{H_0^1(\Omega)} \leq k\frac{\|g\|_1}{\alpha}. \tag{2.8}$$

As before, from (2.8), it follows that

$$\int_\Omega |\nabla z_n|^q \le C_q, \quad \forall q < \frac{N}{N-1}, \tag{2.9}$$

while the choice of $T_1(G_k(z_n))$ as test function in the second equation of (2.4) yields (using again (2.1), the second of (2.3) and (2.2))

$$\alpha \int_{\{k \le |z_n| < k+1\}} |\nabla z_n|^2 \le \int_{\{|z_n| \ge k\}} |g|, \tag{2.10}$$

and

$$\int_{\{|z_n| \ge k+1\}} |h(x, z_n)| |\nabla u_n|^2 \le \int_{\{|z_n| \ge k\}} |g|. \tag{2.11}$$

Thanks to these *a priori* estimates, we have that (up to subsequences still denoted by u_n and z_n) u_n and z_n converge to some functions u and z weakly in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$ strongly in $L^1(\Omega)$, and almost everywhere. These convergences (together with the boundedness of $a(x, s)$) are enough to pass to the limit in the first equation, thus proving that u and z are such that

$$\int_\Omega a(x, z) \nabla u \cdot \nabla \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in C_c^1(\Omega). \tag{2.12}$$

Therefore, we only have to pass to the limit in the weak formulation of the second equation, i.e.,

$$\int_\Omega b(x) \nabla z_n \cdot \nabla \varphi + \int_\Omega h(x, z_n) |\nabla u_n|^2 \varphi = \int_\Omega g_n \varphi, \tag{2.13}$$

for every φ in $C_c^1(\Omega)$.

In order to do that, the convergences proved so far (and the assumption on g_n) allow to pass to the limit in the first and the third integral, while to pass to the limit in the second integral we need to prove that

$$h(x, z_n) |\nabla u_n|^2 \to h(x, z) |\nabla u|^2, \quad \text{strongly in } L^1(\Omega). \tag{2.14}$$

In order to prove (2.14), we begin by following the technique of [12] to prove that

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } H_0^1(\Omega). \tag{2.15}$$

As in [12], we choose in the first equation the test function

$$w = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)),$$

where $h > k > 0$. Thanks to the assumptions on f_n, we have

$$\lim_{h \to +\infty} \lim_{n \to +\infty} \int_\Omega f_n \, T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = 0,$$

so that

$$\lim_{h \to +\infty} \lim_{n \to +\infty} \int_\Omega a(x, z_n) \nabla u_n \cdot \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = 0.$$

Splitting the integral on the sets where $|u_n| \geq k$ and $|u_n| < k$, and using that $a(x, z_n) \geq \alpha > 0$, we obtain (as in [12])

$$\begin{aligned}\int_\Omega a(x, z_n)\nabla u_n \cdot \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \\ \geq \int_\Omega a(x, z_n)\nabla T_k(u_n) \cdot \nabla(T_k(u_n) - T_k(u)) \\ - \int_{\{|u_n|\geq k\}} |a(x, z_n)||\nabla T_{4k+h}(u_n)||\nabla T_k(u)|.\end{aligned}$$

Since the last integral tends to zero as n tends to infinity (here, we use the boundedness of $a(x, s)$ with respect to s), we have adding and subtracting the term

$$\int_\Omega a(x, z_n)\nabla T_k(u) \cdot \nabla(T_k(u_n) - T_k(u)),$$

which tends to zero as n tends to infinity,

$$0 \leq \lim_{n\to+\infty} \int_\Omega a(x, z_n)|\nabla(T_k(u_n) - T_k(u)|^2 \leq 0,$$

which then implies (2.15). Hence, up to subsequences still denoted by u_n,

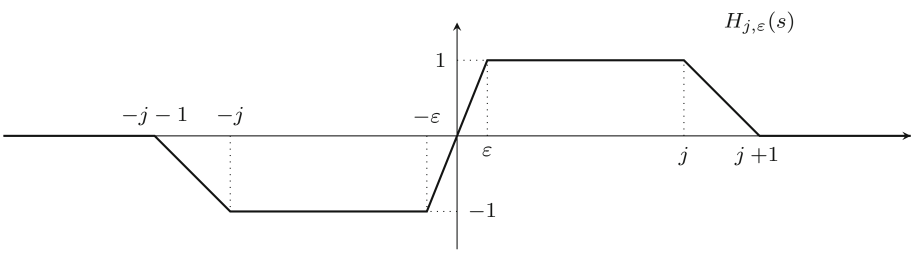
$$\nabla u_n \to \nabla u \quad \text{almost everywhere in } \Omega.$$

Therefore,

$$h(x, z_n)|\nabla u_n|^2 \to h(x, z)|\nabla u|^2, \quad \text{almost everywhere in } \Omega. \tag{2.16}$$

We now define, for $k \geq 0$, $j \geq 0$, and $\varepsilon > 0$, the functions $\psi_k(s) = |T_1(G_k(s))|$ and

$$H_{j,\varepsilon}(s) = \begin{cases} s/\varepsilon & 0 \leq s \leq \varepsilon, \\ 1 & \varepsilon < s \leq j, \\ -s + j + 1 & j < s \leq j+1, \\ 0 & s > j+1, \\ -H_{j,\varepsilon}(-s) & s < 0, \end{cases}$$



and choose $H_{j,\varepsilon}(z_n)\psi_k(u_n)$ as test function in the second equation. We obtain

$$\begin{aligned}\int_\Omega b(x)|\nabla z_n|^2\psi_k(u_n)H'_{j,\varepsilon}(z_n) + \int_\Omega b(x)\nabla z_n\nabla u_n\psi'_k(u_n)H_{j,\varepsilon}(z_n) \\ + \int_\Omega h(x, z_n)|\nabla u_n|^2 H_{j,\varepsilon}(z_n)\psi_k(u_n) = \int_\Omega g_n H_{j,\varepsilon}(z_n)\psi_k(u_n).\end{aligned}$$

Thus, using the definition of $H_{j,\varepsilon}$, and dropping nonnegative terms [thanks to (2.2)],

$$\frac{1}{\varepsilon}\int_{\{|z_n|<\varepsilon\}} b(x)|\nabla z_n|^2\psi_k(u_n) - \int_{\{j\le|z_n|<j+1\}} b(x)|\nabla z_n|^2\psi_k(u_n) + \int_{\Omega} b(x)\nabla z_n\nabla u_n\psi_k'(u_n)H_{j,\varepsilon}(z_n) + \int_{\left\{\substack{\varepsilon\le|z_n|<j\\ |u_n|>k+1}\right\}} |h(x,z_n)||\nabla u_n|^2 \le \int_{\{|u_n|>k\}}|g_n|$$

Dropping the first term, which is positive, and letting ε tend to zero, we obtain [using (2.1) and the second of (2.3)]

$$\int_{\left\{\substack{|z_n|<j\\ |u_n|>k+1}\right\}} |h(x,z_n)||\nabla u_n|^2 \le \int_{\{|u_n|>k\}}|g| + \beta\int_{\{j\le|z_n|<j+1\}}|\nabla z_n|^2 + \beta\int_{\left\{\substack{|z_n|<j+1\\ k<|u_n|\le k+1}\right\}}|\nabla z_n||\nabla u_n|.$$

Now, fix $\delta>0$, and let j^* be such that

$$\int_{\{j^*\le|z_n|<j^*+1\}}|\nabla z_n|^2 \le \int_{\{|z_n|\ge j^*\}}|g| \le \int_{\{|z_n|\ge j^*-1\}}|g| \le \delta. \tag{2.17}$$

Such a choice of j^* is possible thanks to (2.10), to the strong convergence of z_n in $L^1(\Omega)$ and to the absolute continuity of the integral. Thus,

$$\int_{\left\{\substack{|z_n|<j^*\\ |u_n|>k+1}\right\}} |h(x,z_n)||\nabla u_n|^2 \le \int_{\{|u_n|>k\}}|g| + \beta\delta + \beta\int_{\{k<|u_n|\le k+1\}}|\nabla T_{j^*+1}(z_n)||\nabla u_n|,$$

which implies [using (2.8) and (2.7)]

$$\int_{\left\{\substack{|z_n|<j^*\\ |u_n|>k+1}\right\}} |h(x,z_n)||\nabla u_n|^2 \le \int_{\{|u_n|>k\}}|g| + \beta\delta + \beta\left[\frac{\|g\|_1}{\alpha}(j^*+1)\right]^{\frac12}\left[\int_{\{|u_n|>k\}}|f|\right]^{\frac12}. \tag{2.18}$$

Let now k^* be such that

$$\left[\frac{\|g\|_1}{\alpha}(j^*+1)\right]^{\frac12}\left[\int_{\{|u_n|>k^*\}}|f|\right]^{\frac12} < \delta, \quad \text{and} \quad \int_{\{|u_n|>k^*\}}|g| < \delta\,;$$

such a choice is possible thanks to the strong convergence of u_n in $L^1(\Omega)$ and to the absolute continuity of the integral. Therefore,

$$\int_{\left\{\substack{|z_n|<j^*\\ |u_n|>k^*+1}\right\}} |h(x,z_n)||\nabla u_n|^2 \le (2\beta+1)\delta. \tag{2.19}$$

Now, we use Vitali Theorem. For every measurable $E \subset \Omega$, we have, using (2.11), (2.17) and (2.19),

$$\begin{aligned}
&\int_E |h(x,z_n)||\nabla u_n|^2 \\
&\quad \le \int_{\{|z_n|>j^*\}} |h(x,z_n)||\nabla u_n|^2 + \int_{E\cap\{|z_n|\le j^*\}} |h(x,z_n)||\nabla u_n|^2 \\
&\quad \le \int_{\{|z_n|>j^*-1\}} |g| + \int_{\left\{\substack{|z_n|\le j^* \\ |u_n|>k^*+1}\right\}} |h(x,z_n)||\nabla u_n|^2 \\
&\qquad + \int_{E\cap\left\{\substack{|z_n|\le j^* \\ |u_n|\le k^*+1}\right\}} |h(x,z_n)||\nabla u_n|^2 \\
&\quad \le (2\beta+2)\delta + \|h_{j^*}\|_{L^\infty(\Omega)} \int_E |\nabla T_{k^*+1}(u_n)|^2.
\end{aligned}$$

Recalling (2.15), it is possible to choose meas(E) small enough so that

$$\|h_{j^*}\|_{L^\infty(\Omega)} \int_E |\nabla T_{k^*+1}(u_n)|^2 \le \delta,$$

uniformly with respect to n. Therefore, we have proved that the sequence $\{h(x,z_n)|\nabla u_n|^2\}$ is equiintegrable. This fact, together with the almost everywhere convergence (2.16), proves (2.14). □

3 The case of *a* unbounded

In this section, we are going to deal with system (1.1) under the assumption that the function $a(x,s)$, which "links" the two equations, may be unbounded with respect to s. In this case, we have to make slight changes in the proof of Theorem 2.1 also if both f and g are such that the solutions are bounded.

Proposition 3.1 *Assume that* (1.2) *and* (2.2) *hold, and let f and g be in $L^m(\Omega)$, with $m > \frac{N}{2}$. Then, there exists a solution (u,z) in $H_0^1(\Omega) \times H_0^1(\Omega)$ of the system* (1.1). *Furthermore, both u and z belong to $L^\infty(\Omega)$.*

Proof Let $k \ge 0$ be an integer; since $a(x, T_k(s))$ satisfies (2.1), by Proposition 2.1, there exists a solution (u_k, z_k) of the system

$$\begin{cases} u_k \in H_0^1(\Omega) : -\text{div}(a(x,T_k(z_k))\nabla u_k) = f, \\ z_k \in H_0^1(\Omega) : -\text{div}(b(x)\nabla z_k) + h(x,z_k)|\nabla u_k|^2 = g. \end{cases}$$

Since g belongs to $L^m(\Omega)$, with $m > \frac{N}{2}$, standard elliptic results, and assumption (2.2), imply that there exists a constant M, independent on k, such that

$$\|z_k\|_{L^\infty(\Omega)} \le M.$$

Thus, taking $\overline{k} > M$, we have that $T_{\overline{k}}(z_{\overline{k}}) = z_{\overline{k}}$ so that $(u,z) = (u_{\overline{k}}, z_{\overline{k}})$ is a solution of (1.1) as desired. □

Now, we turn to the case of less regular data; for example, we would like to take (as in Theorem 2.2) both f and g in $L^1(\Omega)$. However, the fact that a may be unbounded with respect to s, even if it allows to prove (2.5)–(2.7) (where only the ellipticity constant α is used), prevents us to apply the technique of [12] to prove the strong convergence of truncates (2.15), an essential tool to pass to the limit in the "quadratic" lower-order term of the second equation.

Nonetheless, if we make stronger assumptions on the data, we can easily recover existence results, as is in the next theorem.

Theorem 3.2 *Assume that* (1.2) *and* (2.2) *hold, and let* f *be an* $L^1(\Omega)$ *function and let* g *be a function in* $L^m(\Omega)$*, with* $m > \frac{N}{2}$*. Then, there exists a distributional solution* (u, z) *of system* (1.1)*, with* u *belonging to* $W_0^{1,q}(\Omega)$*, for every* $q < \frac{N}{N-1}$*, and* z *belonging to* $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Proof We follow the lines of the proof of Theorem 2.2, approximating f and g with sequences of $L^\infty(\Omega)$ functions such that (2.3) holds, and considering the solutions (u_n, z_n), given by Theorem 3.1, of the system

$$\begin{cases} -\mathrm{div}(a(x, z_n)\nabla u_n) = f_n, & \text{in } \Omega; \\ -\mathrm{div}(b(x)\nabla z_n) + h(x, z_n)|\nabla u_n|^2 = g_n, & \text{in } \Omega; \\ u_n = 0 = z_n, & \text{on } \partial\Omega. \end{cases}$$

Taking $G_k(z_n)$ as test function in the second equation, and dropping the nonnegative second term, we obtain [using (2.1)]

$$\alpha \int_\Omega |\nabla G_k(z_n)|^2 \leq \int_\Omega g_n \, G_k(z_n).$$

Starting from this inequality, one can prove (see [13]) that the sequence $\{z_n\}$ is bounded in $L^\infty(\Omega)$ and, taking $k = 0$, that it is bounded in $H_0^1(\Omega)$. Thus, the term $a(x, z_n)$ in the first equation is bounded in $L^\infty(\Omega)$; this fact allows to prove the strong convergence of $T_k(u_n)$ in $H_0^1(\Omega)$ with the same technique of [12] and to conclude the proof as in Theorem 2.2. □

The result of the previous theorem is however rather unsatisfactory: The unboundedness of a is "invisible" due to the boundedness of z; rather than dealing with the general case, we are going to prove an existence result in a particular case, which will not be variational, to show how the lower-order term in the second equation may be helpful. Before stating and proving the theorem, we need a technical lemma.

Lemma 3.3 *Let* σ_n *be a sequence of nonnegative functions in* $L^\infty(\Omega)$*, almost everywhere convergent to some function* σ*, and let* ρ_n *be a sequence of functions, which is weakly convergent in* $(L^2(\Omega))^N$ *to some function* ρ*. If the sequence* $\sigma_n|\rho_n|^2$ *is bounded in* $L^1(\Omega)$*, then* $\sigma|\rho|^2$ *belongs to* $L^1(\Omega)$ *and*

$$\int_\Omega \sigma|\rho|^2 \leq \liminf_{n\to+\infty} \int_\Omega \sigma_n|\rho_n|^2. \tag{3.1}$$

Proof We begin by proving that, for every $k \geq 0$,

$$\int_\Omega T_k(\sigma)|\rho|^2 \leq \liminf_{n\to+\infty} \int_\Omega T_k(\sigma_n)|\rho_n|^2. \tag{3.2}$$

Indeed, starting from

$$\int_\Omega T_k(\sigma_n)|\rho_n - \rho|^2 \geq 0,$$

we have

$$2\int_\Omega T_k(\sigma_n)\rho_n\cdot\rho - \int_\Omega T_k(\sigma_n)|\rho|^2 \le \int_\Omega T_k(\sigma_n)|\rho_n|^2.$$

Since, by the assumptions on σ_n and ρ, $T_k(\sigma_n)\rho$ converges to $T_k(\sigma)\rho$ strongly in $(L^2(\Omega))^N$ by Lebesgue theorem, and $T_k(\sigma_n)|\rho|^2$ converges to $T_k(\sigma)|\rho|^2$ strongly in $L^1(\Omega)$ (again by Lebesgue theorem), the left-hand side passes to the limit, and (3.2) is proved. But now, since σ_n is nonnegative,

$$\int_\Omega T_k(\sigma)|\rho|^2 \le \liminf_{n\to+\infty}\int_\Omega T_k(\sigma_n)|\rho_n|^2 \le \liminf_{n\to+\infty}\int_\Omega \sigma_n|\rho_n|^2 \le C,$$

and (3.1) then follows letting k tend to infinity and using the monotone convergence theorem.

Theorem 3.4 *Let $p > 0$ be a real number, and let $a : \Omega \to \mathbb{R}$ be a measurable function such that*

$$0 < \alpha \le a(x) \le \beta, \quad \textit{almost everywhere in } \Omega.$$

Let f be a function in $L^{2_*}(\Omega)$, where $2_* = \frac{2N}{N+2}$, and let g be a function in $L^1(\Omega)$. Then, there exists a solution (u, z) of the system say

$$\begin{cases} -\mathrm{div}([a(x)+|z|^p]\nabla u) = f & \textit{in } \Omega, \\ -\mathrm{div}(b(x)\nabla z) + z|z|^{2p-1}\,|\nabla u|^2 = g & \textit{in } \Omega, \\ u = 0 = z & \textit{on } \partial\Omega, \end{cases}$$

with u in $H_0^1(\Omega)$ and z in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$.

Proof As before, we begin by approximating the system: let f_n and g_n be sequences of $L^\infty(\Omega)$ functions converging to f and g in $L^{2_*}(\Omega)$ and $L^1(\Omega)$, respectively, and such that (2.3) holds, and let (u_n, z_n) be a solution of

$$\begin{cases} -\mathrm{div}([a(x)+|z_n|^p]\nabla u_n) = f_n & \text{in } \Omega, \\ -\mathrm{div}(b(x)\nabla z_n) + z_n|z_n|^{2p-1}|\nabla u_n|^2 = g_n & \text{in } \Omega, \\ u_n = 0 = z_n & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

which exists by Theorem 3.2. Choosing u_n as test function in the first equation, we obtain

$$\alpha\int_\Omega |\nabla u_n|^2 \le \int_\Omega [a(x)+|z_n|^p]|\nabla u_n|^2 \le \int_\Omega f_n u_n \le \|f_n\|_{L^{2_*}(\Omega)}\|u_n\|_{L^{2^*}(\Omega)}.$$

Starting from this inequality, and using Sobolev embedding, we have that u_n is bounded in $H_0^1(\Omega)$. Hence, up to subsequences still denoted by u_n, it converges weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and almost everywhere to some function u in $H_0^1(\Omega)$.

Using, as in the proof of Theorem 2.2, the fact that the lower-order term of the second equation has the same sign of z_n, we have that both (2.9) and (2.11) hold true. In particular, (2.11) becomes

$$\int_{\{|z_n|\ge k+1\}} |z_n|^{2p}|\nabla u_n|^2 \le \int_{\{|z_n|\ge k\}} |g|. \tag{3.4}$$

The boundedness of z_n in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, implies that, up to subsequences still denoted by the same name, z_n converges weakly in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, strongly in $L^1(\Omega)$ and almost everywhere in Ω, to some function z which belongs to $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$.

If we take $k = 0$ in (3.4), we obtain

$$\int_{\{|z_n|\geq 1\}} |z_n|^{2p} |\nabla u_n|^2 \leq \int_\Omega |g_n| \leq C.$$

Since u_n is bounded in $H_0^1(\Omega)$, from this inequality we obtain

$$\int_\Omega |z_n|^{2p} |\nabla u_n|^2 \leq \int_{\{|z_n|\leq 1\}} |\nabla u_n|^2 + \int_{\{|z_n|\geq 1\}} |z_n|^{2p} |\nabla u_n|^2 \leq C. \tag{3.5}$$

Applying Lemma 3.3 with $\sigma_n = |z_n|^{2p}$ and $\rho_n = \nabla u_n$, we have, from (3.5),

$$\int_\Omega |z|^{2p} |\nabla u|^2 \leq \liminf_{n\to+\infty} \int_\Omega |z_n|^{2p} |\nabla u_n|^2 \leq C,$$

so that $|z|^{2p}|\nabla u|^2$ belongs to $L^1(\Omega)$.

Now, we prove that $|z_n|^p \nabla u_n$ weakly converges to $|z|^p \nabla u$ in $(L^2(\Omega))^N$. If Ψ is a function in $(L^2(\Omega))^N$, then

$$\begin{aligned}\int_\Omega |z_n|^p \nabla u_n \cdot \Psi &= \int_{\{|z_n|\leq k\}} |z_n|^p \nabla u_n \cdot \Psi + \int_{\{|z_n|\geq k\}} |z_n|^p \nabla u_n \cdot \Psi \\ &= \int_{\{|z|\leq k\}} |z|^p \nabla u \cdot \Psi + \varepsilon_n + \int_{\{|z_n|\geq k\}} |z_n|^p \nabla u_n \cdot \Psi,\end{aligned}$$

where ε_n is a quantity which tends to zero as n tends to infinity. On the other hand,

$$\begin{aligned}\int_\Omega |z|^p \nabla u \cdot \Psi &= \int_{\{|z_n|\leq k\}} |z|^p \nabla u \cdot \Psi + \int_{\{|z_n|\geq k\}} |z|^p \nabla u \cdot \Psi \\ &= \int_{\{|z|\leq k\}} |z|^p \nabla u \cdot \Psi + \varepsilon_n + \int_{\{|z_n|\geq k\}} |z|^p \nabla u \cdot \Psi.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\int_\Omega [|z_n|^p \nabla u_n - |z|^p \nabla u] \cdot \psi\right| \leq &\int_{\{|z_n|\geq k\}} |z_n|^p |\nabla u_n| |\Psi| \\ &+ \int_{\{|z_n|\geq k\}} |z|^p |\nabla u| |\Psi| + 2\varepsilon_n.\end{aligned}$$

Using that $|z_n|^p \nabla u_n$ is bounded in $(L^2(\Omega))^N$, and that $|z|^p \nabla u$ belongs to $(L^2(\Omega))^N$, we have

$$\begin{aligned}\int_{\{|z_n|\geq k\}} |z_n|^p |\nabla u_n| |\Psi| &\leq \left(\int_\Omega |z_n|^{2p} |\nabla u_n|^2\right)^{\frac{1}{2}} \left(\int_{\{|z_n|\geq k\}} |\Psi|^2\right)^{\frac{1}{2}} \\ &\leq C \left(\int_{\{|z_n|\geq k\}} |\Psi|^2\right)^{\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}\int_{\{|z_n|\geq k\}} |z|^p |\nabla u| |\Psi| &\leq \left(\int_\Omega |z|^{2p} |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{\{|z_n|\geq k\}} |\Psi|^2\right)^{\frac{1}{2}} \\ &\leq C \left(\int_{\{|z_n|\geq k\}} |\Psi|^2\right)^{\frac{1}{2}},\end{aligned}$$

so that since z_n is strongly convergent in $L^1(\Omega)$, for every $\delta > 0$, there exists $k_\delta \geq 0$ such that

$$\left|\int_{\{|z_n|\geq k\}} |z_n|^p \nabla u_n \cdot \Psi\right| \leq \delta, \quad \left|\int_{\{|z_n|\geq k\}} |z|^p \nabla u \cdot \Psi\right| \leq \delta,$$

for every n in $\mathbb{N}$. Thus, choosing $n \geq n_\delta$, where n_δ is such that $\varepsilon_n \leq \delta$, we have

$$\left|\int_\Omega [|z_n|^p \nabla u_n - |z|^p \nabla u] \cdot \psi\right| \leq 4\delta,$$

for every $n \geq n_\delta$, which implies the desired weak convergence.

Choosing v in $H_0^1(\Omega)$ as test function in the first equation, we obtain

$$\int_\Omega [a(x) + |z_n|^p] \nabla u_n \cdot \nabla v = \int_\Omega f_n\, v,$$

and, passing to the limit as n tends to infinity, using the weak convergence just proved,

$$\int_\Omega [a(x) + |z|^p] \nabla u \cdot \nabla v = \int_\Omega f\, v,$$

so that u is a solution of the first equation of the system. If we take $v = u$, we then have

$$\int_\Omega [a(x) + |z|^p] |\nabla u|^2 = \int_\Omega f\, u. \tag{3.6}$$

On the other hand, choosing u_n as test function in the first approximate equation, we obtain

$$\int_\Omega [a(x) + |z_n|^p] |\nabla u_n|^2 = \int_\Omega f_n\, u_n.$$

Passing to the limit as n tends to infinity, and using (3.6), and the weak convergence of u_n to u in $H_0^1(\Omega)$, we get

$$\begin{aligned}\lim_{n\to+\infty} \int_\Omega [a(x) + |z_n|^p] |\nabla u_n|^2 &= \lim_{n\to+\infty} \int_\Omega f_n\, u_n \\ &= \int_\Omega f\, u = \int_\Omega [a(x) + |z|^p] |\nabla u|^2.\end{aligned}$$

Therefore,

$$\lim_{n\to+\infty} \int_\Omega [a(x) + |z_n|^p] |\nabla u_n|^2 = \int_\Omega [a(x) + |z|^p] |\nabla u|^2. \tag{3.7}$$

Choosing $\sigma_n = a(x)$ and $\rho_n = \nabla u_n$ in Lemma 3.3, or $\sigma_n = |z_n|^p$ and $\rho_n = \nabla u_n$ in the same lemma, we have

$$\int_\Omega a(x) |\nabla u|^2 \leq \liminf_{n\to+\infty} \int_\Omega a(x) |\nabla u_n|^2,$$

and

$$\int_\Omega |z|^p |\nabla u|^2 \leq \liminf_{n\to+\infty} \int_\Omega |z_n|^p |\nabla u_n|^2 \leq C,$$

with the latter inequality being due to the fact that

$$\int_\Omega |z_n|^p |\nabla u_n|^2 \leq \int_{\{|z_n|\leq 1\}} |\nabla u_n|^2 + \int_{\{|z_n|\geq 1\}} |z_n|^{2p} |\nabla u_n|^2 \leq C.$$

Define now

$$A_n = \int_\Omega a(x)|\nabla u_n|^2, \quad A = \int_\Omega a(x)|\nabla u|^2,$$

and

$$B_n = \int_\Omega |z_n|^p|\nabla u_n|^2, \quad B = \int_\Omega |z|^p|\nabla u|^2$$

From (3.7), it follows that

$$\lim_{n\to+\infty}(A_n + B_n) = A + B,$$

with

$$A \le \liminf_{n\to+\infty} A_n, \quad \text{and} \quad B \le \liminf_{n\to+\infty} B_n.$$

But then, since $A_n = (A_n + B_n) - B_n$, we have

$$\liminf_{n\to+\infty} A_n = A + B - \limsup_{n\to+\infty} B_n,$$

and so

$$A + B = \liminf_{n\to+\infty} A_n + \limsup_{n\to+\infty} B_n \ge A + \limsup_{n\to+\infty} B_n,$$

which then implies

$$\limsup_{n\to+\infty} B_n \le B.$$

Therefore, B_n converges to B, and so A_n converges to A. Thus,

$$\lim_{n\to+\infty}\int_\Omega a(x)|\nabla u_n|^2 = \int_\Omega a(x)|\nabla u|^2,$$

which then implies that u_n strongly converges to u in $H_0^1(\Omega)$. In particular, and always up to subsequences, ∇u_n almost everywhere converges to ∇u.

We can now pass to the limit in the second equation; as in the proof of Theorem 2.2, the only difficult part is the proof of the strong convergence in $L^1(\Omega)$ of the lower-order term. Due to the almost everywhere convergence of z_n to z and of ∇u_n to ∇u, we have

$$z_n|z_n|^{2p-1}|\nabla u_n|^2 \to z|z|^{2p-1}|\nabla u|^2, \quad \text{almost everywhere in } \Omega,$$

so that we only have to prove the equiintegrability in order to apply Vitali theorem. If E is a measurable subset of Ω, we have, recalling (2.11),

$$\begin{aligned}\int_E |z_n|^{2p}|\nabla u_n|^2 &\le k^{2p}\int_{\{|z_n|\le k\}\cap E}|\nabla u_n|^2 + \int_{\{|z_n|\ge k\}}|z_n|^{2p}|\nabla u_n|^2 \\ &\le k^{2p}\int_E|\nabla u_n|^2 + \int_{\{|z_n|\ge k-1\}}|g_n|.\end{aligned}$$

We now first choose k such that the second term is small (which can be done since g_n is strongly convergent in $L^1(\Omega)$, as is z_n) and then choose meas(E) small enough so that the first term is small [and this can be done since $|\nabla u_n|$ is strongly convergent in $L^1(\Omega)$]. Thus, the sequence $|z_n|^{2p}|\nabla u_n|^2$ is equiintegrable, and so, the lower-order term is strongly convergent in $L^1(\Omega)$, as desired. □

If we want to put ourselves in a "variational" setting, we have to make more restrictive assumptions on the data and on p.

Theorem 3.5 *Let $p > 1$ be a real number, and let $a : \Omega \to \mathbb{R}$ be a measurable function such that*

$$0 < \alpha \le a(x) \le \beta, \quad \textit{almost everywhere in } \Omega.$$

Let f be a function in $L^{2_*}(\Omega)$, where $2_* = \frac{2N}{N+2}$, let g be a function in $L^r(\Omega)$, with

$$r \ge \frac{N(p+2)}{N+2+2p}, \tag{3.8}$$

and let $B > 0$. Then, there exists a solution (u, z) of the system

$$\begin{cases} -\mathrm{div}([a(x) + |z|^p]\nabla u) = f & \textit{in } \Omega, \\ -\mathrm{div}(b(x)\nabla z) + B\, z|z|^{p-2}\,|\nabla u|^2 = g & \textit{in } \Omega, \\ u = 0 = z & \textit{on } \partial\Omega, \end{cases}$$

with u and z in $H_0^1(\Omega)$. Moreover, $z^p|\nabla u|^2$ belongs to $L^1(\Omega)$.

Remark 3.6 We remark that for every choice of $p > 1$, we have

$$\frac{N(p+2)}{N+2+2p} > 2_*,$$

so that the datum g always belongs to the dual space of $H_0^1(\Omega)$. Furthermore, if p tends to infinity, then r has to be larger than $\frac{N}{2}$ (which is the assumption on g in order to have bounded solutions z).

Remark 3.7 If we choose $B = \frac{p}{2}$, the system becomes variational, and the solutions can be found as minima of the functional

$$J(v, w) = \frac{1}{2}\int_\Omega [a(x) + |w|^p]|\nabla v|^2 + \frac{1}{2}\int_\Omega b(x)|\nabla w|^2 - \int_\Omega f\, v - \int_\Omega g\, w.$$

Proof The proof of the result follows the lines of the argument in Theorem 3.4. We begin by approximating the system, taking f_n and g_n sequences of $L^\infty(\Omega)$ functions converging to f and g in $L^{2_*}(\Omega)$ and $L^r(\Omega)$, respectively, and such that (2.3) holds. We then consider a solution (u_n, z_n), which exists by Theorem 3.2, of

$$\begin{cases} -\mathrm{div}([a(x) + |z_n|^p]\nabla u_n) = f_n & \text{in } \Omega, \\ -\mathrm{div}(b(x)\nabla z_n) + B\, z_n|z_n|^{p-2}|\nabla u_n|^2 = g_n & \text{in } \Omega, \\ u_n = 0 = z_n & \text{on } \partial\Omega. \end{cases}$$

Since the lower-order term of the second equation has the same sign as the solution, standard elliptic techniques (see for example [4]) imply that z_n is bounded in $H_0^1(\Omega)$ (see Remark 3.6) and in $L^{r^{**}}(\Omega)$. Since the assumption on r implies

$$\frac{1}{r} + \frac{p+1}{r^{**}} \le 1,$$

choosing $|z_n|^{p+1}T_1(G_k(z_n))$ as test function in the second equation, we obtain, dropping two nonnegative terms,

$$B\int_{\{|z_n|\geq k+1\}}|z_n|^{2p}|\nabla u_n|^2 \leq \int_{\{|z_n|\geq k\}}|g_n||z_n|^{p+1}$$
$$\leq \left(\int_{\{|z_n|\geq k\}}|g_n|^r\right)^{\frac{1}{r}}\|z_n\|^{p+1}_{L^{r^{**}}(\Omega)}$$
$$\leq C\left(\int_{\{|z_n|\geq k\}}|g_n|^r\right)^{\frac{1}{r}},$$

which is exactly (2.11). From now on, the proof is identical to that of Theorem 3.4, and therefore, we omit it. □

4 Breaking of the maximum principle

When considering system (1.1), both with a bounded or unbounded, one of the main properties of the solutions is that (no matter which assumptions we make on the data) $|h(x,z)||\nabla u|^2$ belongs to $L^1(\Omega)$, as a consequence of the sign assumption on h. However, if the datum f is in $L^1(\Omega)$ [or, in general, if it does not belong to $H^{-1}(\Omega)$], u does not belong to $H_0^1(\Omega)$, so that the function $|\nabla u|^2$ may not belong to $L^1(\Omega)$. This fact, as we will prove in the next theorem, implies that the solution z has to be zero inside Ω even if the datum g is nonnegative. In other words, the maximum principle fails for the second equation of system (1.1).

Theorem 4.1 *Let $f\geq 0$ be a function in $L^1(\Omega)$, and let $g\geq 0$ belong to $L^1(\Omega)$. Let $a(x,s)$, $b(x)$ and $h(x,s)$ be such that* (2.1) *and* (2.2) *hold, and let (u,z) be a solution of* (1.1) *given by Theorem* 2.2. *If, for every $\delta>0$,*

$$\inf_{\Omega\times[\delta,+\infty)} h(x,s)=\Gamma_\delta>0,$$

and if there exists x_0 in Ω such that f belongs to $L^\infty(\Omega\backslash B_r(x_0))$, and f does not belong to $H^{-1}(B_r(x_0))$ for every $r>0$, then

$$\operatorname{ess\,inf}_{B_r(x_0)} z=0,\quad \forall r>0. \tag{4.1}$$

Proof First of all, observe that both u and z are nonnegative functions. If (4.1) is false, then there exist $R>0$ and $\delta>0$ such that

$$\operatorname{ess\,inf}_{B_R(x_0)} z=\delta.$$

Taking $T_\delta(z_n)$ as test function in the second equation of (2.4), we obtain, dropping a nonnegative term,

$$\int_\Omega h(x,z_n)\,T_\delta(z_n)|\nabla u_n|^2\leq\delta\int_\Omega|g|.$$

Recalling (2.14), and the fact that $z\geq 0$, we then have

$$\delta\int_{\{z\geq\delta\}}h(x,z)|\nabla u|^2\leq\int_\Omega h(x,z)\,T_\delta(z)|\nabla u|^2\leq\delta\int_\Omega|g|,$$

so that

$$\int_{B_R(x_0)}h(x,z)|\nabla u|^2=\int_{B_R(x_0)\cap\{z\geq\delta\}}h(x,z)|\nabla u|^2\leq\int_{\{z\geq\delta\}}h(x,z)|\nabla u|^2\leq\int_\Omega|g|.$$

Therefore, recalling the assumption on h, we have

$$\Gamma_\delta \int_{B_R(x_0)} |\nabla u|^2 \le \int_{B_R(x_0)} h(x,z)|\nabla u|^2 \le \int_\Omega |g|.$$

By the result of [5], we also have

$$\int_{\Omega \setminus B_R(x_0)} |\nabla u|^2 \le C_0 \|f\|_{L^\infty(\Omega\setminus B_R(x_0))},$$

so that u belongs to $H_0^1(\Omega)$. This, and the boundedness of $a(x,s)$, implies that f belongs to $H^{-1}(\Omega)$, which is false. □

Note that, as a consequence of this result, if f is an $L^1(\Omega) \setminus H^{-1}(\Omega)$ function which has an isolated singularity in x_0, then the coupling between the equations of the system forces z to be zero at x_0.

5 Measure data

Now, we turn to the case of measure data for system (1.1). In this case, as we will see, we can prove a negative (i.e., nonexistence) result also in the "simple" case of $a(x,s)$ bounded.

We recall that the main tool in the proof of the existence result of Theorem 2.2 is the fact that if u_n and z_n are sequences of approximating solutions, we can choose k and j large enough so that

$$\int_{\{|z_n|\ge j\}} |g| \le \delta, \quad \int_{\{|u_n|\ge k\}} |f| \le \delta, \quad \int_{\{|u_n|\ge k\}} |g| \le \delta,$$

uniformly with respect to n. To achieve that, it is of fundamental importance the fact that the sequences f_n and g_n strongly converge in $L^1(\Omega)$, and not (for example) in the weak$*$ topology of measures. And indeed, if f_n converges to (for example) a Dirac mass, and if g_n converges to another (different) Dirac mass, there may not be existence of solutions for the system (1.1), as the following example shows.

Example 5.1 Let $\Omega = B_1(0)$ be the unit ball of $\mathbb{R}^N$, and let us consider the sequences u_n and z_n of solutions of the system

$$\begin{cases} -\Delta u_n = f_n & \text{in } \Omega, \\ -\Delta z_n + z_n^\gamma |\nabla u_n|^2 = g_n & \text{in } \Omega, \\ u_n = 0 = z_n & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where f_n is the sequence of nonnegative $L^\infty(\Omega)$ functions equal to $D_N\, n^N$ on $B_{\frac{1}{n}}(0)$, and zero elsewhere, with D_N such that f_n converges to δ_0, the Dirac mass at the origin, and g_n is a sequence of nonnegative $L^\infty(\Omega)$ functions with support contained in $B_{\frac{1}{n}}(x_0)$ and convergent to δ_{x_0}, the Dirac mass at $x_0 \ne 0$; on γ, we suppose that $\gamma > \frac{N}{N-2}$.

It is well known that u_n converges to $u(x) = C_N(|x|^{2-N} - 1)$, the Green function of the Laplacian relative to the origin; calculating explicitly u_n, one can see that $u_n \equiv u$ outside of $B_{\frac{1}{n}}(0)$, so that

$$|\nabla u_n|^2 = |\nabla u|^2 = C_N'|x|^{2-2N} \ge C_N', \quad \text{for} \quad \frac{1}{n} \le |x| \le 1. \tag{5.2}$$

Since from the second equation we have that

$$\int_{\Omega} z_n^{\gamma} |\nabla u_n|^2 \leq \int_{\Omega} g_n \leq C,$$

from (5.2), we obtain

$$C'_N k^{\gamma} \operatorname{meas}(\{z_n \geq k\} \cap (\Omega \setminus B_{\frac{1}{n}}(0))) \leq \int_{\Omega \setminus B_{\frac{1}{n}}(0)} z_n^{\gamma} |\nabla u_n|^2 \leq C,$$

and so

$$\operatorname{meas}(\{z_n \geq k\} \cap \left(\Omega \setminus B_{\frac{1}{n}}(0)\right)) \leq \frac{C}{k^{\gamma}}. \tag{5.3}$$

On the other hand, since the support of g_n is disjoint from the ball $B_{\frac{1}{n}}(0)$ if $n \geq n_0$, with n_0 large enough, the result of [5] implies that z_n is bounded in $L^{\infty}(B_{\frac{1}{n_0}})$ by a constant $\overline{k}$. Therefore, if $k \geq \overline{k}$, we have

$$\operatorname{meas}(\{z_n \geq k\}) = \operatorname{meas}(\{z_n \geq k\} \cap \left(\Omega \setminus B_{\frac{1}{n_0}}(0)\right) \leq \frac{C}{k^{\gamma}}.$$

On the other hand, choosing $T_k(z_n)$ as test function in the second equation, and dropping the positive second term, we have

$$\int_{\Omega} |\nabla T_k(z_n)|^2 \leq \int_{\Omega} g_n \, T_k(z_n) \leq C\, k.$$

Reasoning as in [1], the latter two inequalities imply

$$\operatorname{meas}(\{|\nabla z_n| \geq \lambda\}) \leq \frac{C}{\lambda^{\frac{2\gamma}{\gamma+1}}},$$

so that z_n is bounded in $W_0^{1,q}(\Omega)$ for every $q < \frac{2\gamma}{\gamma+1}$. Since $\gamma > \frac{N}{N-2}$, then $\frac{2\gamma}{\gamma+1} > \frac{N}{N-1}$, which implies that z_n is bounded in $W_0^{1,p}(\Omega)$ for some $p > \frac{N}{N-1}$.

Since the set $\{x_0\}$ has zero p'-capacity (as every point in $\mathbb{R}^N$), for every $\delta > 0$ there exists a function ψ_δ in $C_0^{\infty}(\Omega)$ (see [8]) such that

$$0 \leq \psi_\delta \leq 1, \quad \int_{\Omega} |\nabla \psi_\delta|^{p'} \leq \delta, \quad \int_{\Omega} g_n \, (1 - \psi_\delta) = 0,$$

the latter being true for every n large enough.

We choose now $T_k(z_n)\,(1 - \psi_\delta)$ as test function in the second equation, to obtain:

$$\int_{\Omega} |\nabla T_k(z_n)|^2 (1 - \psi_\delta) + \int_{\Omega} z_n^{\gamma} \, T_k(z_n)\,(1 - \psi_\delta) |\nabla u_n|^2$$
$$= \int_{\Omega} \nabla z_n \cdot \nabla \psi_\delta \, T_k(z_n) + \int_{\Omega} g_n \, T_k(z_n)\,(1 - \psi_\delta).$$

Dropping the nonnegative second term, observing that the last term is zero for n large enough, and passing to the limit as n tends to infinity, we obtain (by weak lower semicontinuity)

$$\int_{\Omega} |\nabla T_k(z)|^2 (1 - \psi_\delta) \leq \int_{\Omega} \nabla z \cdot \nabla \psi_\delta \, T_k(z).$$

Letting δ tends to zero (and recalling that $|\nabla\psi_\delta|$ tends to zero in $L^{p'}(\Omega)$, while $|\nabla z|$ belongs to $L^p(\Omega)$), we obtain

$$0 \leq \int_\Omega |\nabla T_k(z)|^2 \leq 0,$$

which implies that $z \equiv 0$. However, $z \equiv 0$ is not a solution of the second equation of (1.1).

Remark 5.2 Remark that the conclusion of the previous example remains true every time that g_n converges to a Dirac mass concentrated in a point $x_0 \neq 0$, and the sequence u_n of solutions of the first equation is such that $|\nabla u_n| \geq C > 0$ outside a neighborhood of 0.

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