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Multi-bump solutions for a class of quasilinear problems involving variable exponents

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Abstract We establish the existence of multi-bump solutions for the following class of quasilinear problems

$$-\Delta_{p(x)}u + \big(\lambda V(x) + Z(x)\big)u^{p(x)-1} = f(x,u) \text{ in } \mathbb{R}^N,\ u \geq 0 \text{ in } \mathbb{R}^N,$$

where the nonlinearity $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function having a subcritical growth and potentials $V, Z\colon \mathbb{R}^N \to \mathbb{R}$ are continuous functions verifying some hypotheses. The main tool used is the variational method.

Keywords Variational Methods · Positive solutions · Asymptotic behavior of solutions · $p(x)$-Laplacian

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1 Introduction

In this paper, we consider the existence and multiplicity of solutions for the following class of problems

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$$(P_\lambda)\ \begin{cases} -\Delta_{p(x)}u + \big(\lambda V(x) + Z(x)\big)u^{p(x)-1} = f(x,u), & \text{in } \mathbb{R}^N,\\ u \geq 0, & \text{in } \mathbb{R}^N,\\ u \in W^{1,p(x)}\big(\mathbb{R}^N\big), \end{cases}$$

where $\Delta_{p(x)}$ is the $p(x)$-Laplacian operator given by

$$\Delta_{p(x)}u = \operatorname{div}\left(\left|\nabla u\right|^{p(x)-2}\nabla u\right).$$

Here, $\lambda > 0$ is a parameter, $p\colon \mathbb{R}^N \to \mathbb{R}$ is a Lipschitz function, $V, Z\colon \mathbb{R}^N \to \mathbb{R}$ are continuous functions with $V \geq 0$, and $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous having a subcritical growth. Furthermore, we take into account the following set of hypotheses:

(H_1) $1 < p_- = \inf_{\mathbb{R}^N} p \leq p_+ = \sup_{\mathbb{R}^N} p < N$.

(H_2) $\Omega = \operatorname{int} V^{-1}(0) \neq \emptyset$ and bounded, $\overline{\Omega} = V^{-1}(0)$ and Ω can be decomposed in k connected components $\Omega_1, \ldots, \Omega_k$ with $\operatorname{dist}\big(\Omega_i, \Omega_j\big) > 0$, $i \neq j$.

(H_3) There exists $M > 0$ such that

$$\lambda V(x) + Z(x) \geq M,\ \forall x \in \mathbb{R}^N, \lambda \geq 1.$$

(H_4) There exists $K > 0$ such that

$$\big|Z(x)\big| \leq K,\ \forall x \in \mathbb{R}^N.$$

(f_1)

$$\limsup_{|t|\to\infty} \frac{|f(x,t)|}{|t|^{q(x)-1}} < \infty,\ \text{uniformly in } x \in \mathbb{R}^N,$$

where $q\colon \mathbb{R}^N \to \mathbb{R}$ is continuous with $p_+ < q_-$ and $q \ll p^* = \frac{Np}{N-p}$. Here, the notation $q \ll p^*$ means that $\inf_{\mathbb{R}^N}(p^* - q) > 0$.

(f_2) $f(x,t) = o\big(|t|^{p_+-1}\big)$, $t \to 0$, uniformly in $x \in \mathbb{R}^N$.

(f_3) There exists $\theta > p_+$ such that

$$0 < \theta F(x,t) \leq f(x,t)t,\ \forall x \in \mathbb{R}^N, t > 0,$$

where $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$.

(f_4) $\dfrac{f(x,t)}{t^{p_+-1}}$ is strictly increasing in $t \in (0,\infty)$, for each $x \in \mathbb{R}^N$.

(f_5) $\forall a, b \in \mathbb{R}$, $a < b$, $\sup_{\substack{x\in\mathbb{R}^N\\ t\in[a,b]}} |f(x,t)| < \infty$.

A typical example of nonlinearity verifying $(f_1) - (f_5)$ is

$$f(x,t) = |t|^{q(x)-2}t,\ \forall\, x \in \mathbb{R}^N \text{ and } \forall t \in \mathbb{R},$$

where $p_+ < q_-$ and $q \ll p^*$.

Partial differential equations involving the $p(x)$-Laplacian arise, for instance, as a mathematical model for problems involving electrorheological fluids and image restorations, see [1,2,11–13,29]. This explains the intense research on this subject in the last decades. A lot of works, mainly treating nonlinearities with subcritical growth, are available (see [4–9,16–18,20–24,28] for interesting works). Nevertheless, to the best of the author's knowledge, this is the first work dealing with multi-bump solutions for this class of problems.

The motivation to investigate problem (P_λ) in the setting of variable exponents has been the papers [3] and [15]. In [15], inspired by del Pino and Felmer [14] and Séré [30], the authors considered (P_λ) for $p = 2$ and $f(u) = u^q$, $q \in \left(1, \frac{N+2}{N-2}\right)$ if $N \geq 3$; $q \in (1, \infty)$ if $N = 1, 2$. The authors showed that (P_λ) has at least $2^k - 1$ solutions u_λ for large values of λ. More precisely, one solution for each non-empty subset Υ of $\{1, \ldots, k\}$. Moreover, fixed $\Upsilon \subset \{1, \ldots, k\}$, it was proved that, for any sequence $\lambda_n \to \infty$, we can extract a subsequence (λ_{n_i}) such that $(u_{\lambda_{n_i}})$ converges strongly in $H^1(\mathbb{R}^N)$ to a function u, which satisfies $u = 0$ outside $\Omega_\Upsilon = \bigcup_{j\in\Upsilon} \Omega_j$ and $u_{|\Omega_j}$, $j \in \Upsilon$ is a least energy solution for

$$\begin{cases} -\Delta u + Z(x)u = u^q, & \text{in } \Omega_j, \\ u \in H_0^1(\Omega_j), \; u > 0, & \text{in } \Omega_j. \end{cases}$$

In [3], employing some different arguments than those used in [15], Alves extended the results described above to the p-Laplacian operator, assuming that in (P_λ) the nonlinearity f possesses a subcritical growth and $2 \leq p < N$. In particular, fixed $\Upsilon \subset \{1, \ldots, k\}$, for any sequence $\lambda_n \to \infty$, we can extract a subsequence (λ_{n_i}) such that $(u_{\lambda_{n_i}})$ converges strongly in $W^{1,p}(\mathbb{R}^N)$ to a function u, which satisfies $u = 0$ outside Ω_Υ and $u_{|\Omega_j}$, $j \in \Upsilon$, is a least energy solution for

$$\begin{cases} -\Delta_p u + Z(x)u = f(u), & \text{in } \Omega_j, \\ u \in W_0^{1,p}(\Omega_j), \; u > 0, & \text{in } \Omega_j. \end{cases}$$

In the present paper, we extend the results found in [3] to the $p(x)$-Laplacian operator. However, we would like to emphasize that in a lot of estimates, we have used different arguments from that found in [3]. The main difference is related to the fact that for equations involving the $p(x)$-Laplacian operator it is not clear that Moser's iteration method is a good tool to get the estimates for the L^∞-norm. Here, we adapt some ideas explored in [18] and [25] to get these estimates. For more details see Sect. 5.

Since we intend to find nonnegative solutions, throughout this paper, we replace f by $f^+ : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$f^+(x,t) = \begin{cases} f(x,t), & \text{if } t > 0 \\ 0, & \text{if } t \leq 0. \end{cases}$$

Nevertheless, for the sake of simplicity, we still write f instead of f^+.

The main theorem in this paper is the following:

Theorem 1.1 *Assume that $(H_1) - (H_4)$ and $(f_1) - (f_5)$ hold. Then, there exist $\lambda_0 > 0$ with the following property: for any non-empty subset Υ of $\{1, 2, ..., k\}$ and $\lambda \geq \lambda_0$, problem (P_λ) has a solution u_λ. Moreover, if we fix the subset Υ, then for any sequence $\lambda_n \to \infty$, we can extract a subsequence (λ_{n_i}) such that $(u_{\lambda_{n_i}})$ converges strongly in $W^{1,p(x)}(\mathbb{R}^N)$ to a function u, which satisfies $u = 0$ outside $\Omega_\Upsilon = \bigcup_{j\in\Upsilon} \Omega_j$ and $u_{|\Omega_j}$, $j \in \Upsilon$, is a least energy solution for*

$$\begin{cases} -\Delta_{p(x)} u + Z(x)u = f(x,u), & \text{in } \Omega_j, \\ u \in W_0^{1,p(x)}(\Omega_j), \; u \geq 0, & \text{in } \Omega_j. \end{cases}$$

Notations: The following notations will be used in the present work:

- C and C_i will denote generic positive constant, which may vary from line to line;
- In all the integrals, we omit the symbol dx.

- If u is a measurable function, we denote u^+ and u^- its positive and negative part, i.e., $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \min\{u(x), 0\}$.
- If u, v are measurable functions, $u_- = \operatorname*{ess\,inf}_{\mathbb{R}^N} u$, $u_+ = \operatorname*{ess\,sup}_{\mathbb{R}^N} u$ and the notation $u \ll v$ means that $\operatorname*{ess\,inf}_{\mathbb{R}^N} (v-u) > 0$. Moreover, we will denote by u^* the function

$$u^*(x) = \begin{cases} \frac{Nu(x)}{N-u(x)}, & \text{if } u(x) < N, \\ \infty, & \text{if } u(x) \geq N. \end{cases}$$

2 Preliminaries on variable exponents Lebesgue and Sobolev spaces

In this section, we recall some results on variable exponents Lebesgue and Sobolev spaces found in [8,19,21] and their references.

Let $h \in L^\infty(\mathbb{R}^N)$ with $h_- = \operatorname*{ess\,inf}_{\mathbb{R}^N} h \geq 1$. The *variable exponent Lebesgue space* $L^{h(x)}(\mathbb{R}^N)$ is defined by

$$L^{h(x)}(\mathbb{R}^N) = \left\{ u\colon \mathbb{R}^N \to \mathbb{R}\,;\; u \text{ is measurable and } \int_{\mathbb{R}^N} |u|^{h(x)} < \infty \right\},$$

endowed with the norm

$$|u|_{h(x)} = \inf\left\{ \lambda > 0\,;\; \int_{\mathbb{R}^N} \left|\frac{u}{\lambda}\right|^{h(x)} \leq 1 \right\}.$$

The *variable exponent Sobolev space* is defined by

$$W^{1,h(x)}(\mathbb{R}^N) = \left\{ u \in L^{h(x)}(\mathbb{R}^N)\,;\; |\nabla u| \in L^{h(x)}(\mathbb{R}^N) \right\},$$

with the norm

$$\|u\|_{1,h(x)} = \inf\left\{ \lambda > 0\,;\; \int_{\mathbb{R}^N} \left(\left|\frac{\nabla u}{\lambda}\right|^{h(x)} + \left|\frac{u}{\lambda}\right|^{h(x)} \right) \leq 1 \right\}.$$

If $h_- > 1$, the spaces $L^{h(x)}(\mathbb{R}^N)$ and $W^{1,h(x)}(\mathbb{R}^N)$ are separable and reflexive with these norms.

We are mainly interested in subspaces of $W^{1,h(x)}(\mathbb{R}^N)$ given by

$$E_W = \left\{ u \in W^{1,h(x)}(\mathbb{R}^N)\,;\; \int_{\mathbb{R}^N} W(x)|u|^{h(x)} < \infty \right\},$$

where $W \in C(\mathbb{R}^N)$ is such that $W_- > 0$. Endowing E_W with the norm

$$\|u\|_W = \inf\left\{ \lambda > 0\,;\; \int_{\mathbb{R}^N} \left(\left|\frac{\nabla u}{\lambda}\right|^{h(x)} + W(x)\left|\frac{u}{\lambda}\right|^{h(x)} \right) \leq 1 \right\},$$

E_W is a Banach space. Moreover, it is easy to see that $E_W \hookrightarrow W^{1,h(x)}(\mathbb{R}^N)$ continuously. In addition, we can show that E_W is reflexive. For the reader's convenience, we recall some basic results.

Proposition 2.1 *The functional* $\varrho\colon E_W \to \mathbb{R}$ *defined by*

$$\varrho(u) = \int_{\mathbb{R}^N} \left(\left|\nabla u\right|^{h(x)} + W(x)\,|u|^{h(x)}\right), \tag{2.1}$$

has the following properties:

(i) *If* $\|u\|_W \geq 1$, *then* $\|u\|_W^{h_-} \leq \varrho(u) \leq \|u\|_W^{h_+}$.
(ii) *If* $\|u\|_W \leq 1$, *then* $\|u\|_W^{h_+} \leq \varrho(u) \leq \|u\|_W^{h_-}$.

In particular, for a sequence (u_n) *in* E_W,

$$\|u_n\|_W \to 0 \iff \varrho(u_n) \to 0, \text{ and,}$$
$$(u_n) \text{ is bounded in } E_W \iff \varrho(u_n) \text{ is bounded in } \mathbb{R}.$$

Remark 2.2 For the functional $\varrho_{h(x)}\colon L^{h(x)}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\varrho_{h(x)}(u) = \int_{\mathbb{R}^N} |u|^{h(x)},$$

an analogous conclusion to that of Proposition 2.1 also holds.

Proposition 2.3 *Let* $m \in L^\infty(\mathbb{R}^N)$ *with* $0 < m_- \leq m(x) \leq h(x)$ *for a.e.* $x \in \mathbb{R}^N$. *If* $u \in L^{h(x)}(\mathbb{R}^N)$, *then* $|u|^{m(x)} \in L^{\frac{h(x)}{m(x)}}(\mathbb{R}^N)$ *and*

$$\left||u|^{m(x)}\right|_{\frac{h(x)}{m(x)}} \leq \max\left\{|u|_{h(x)}^{m_-}, |u|_{h(x)}^{m_+}\right\} \leq |u|_{h(x)}^{m_-} + |u|_{h(x)}^{m_+}.$$

Related to the Lebesgue space $L^{h(x)}(\mathbb{R}^N)$, we have the following generalized Hölder's inequality.

Proposition 2.4 (Hölder's inequality) *If* $h_- > 1$, *let* $h'\colon \mathbb{R}^N \to \mathbb{R}$ *such that*

$$\frac{1}{h(x)} + \frac{1}{h'(x)} = 1 \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Then, for any $u \in L^{h(x)}(\mathbb{R}^N)$ *and* $v \in L^{h'(x)}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |uv|\,dx \leq \left(\frac{1}{h_-} + \frac{1}{h'_-}\right)|u|_{h(x)}|v|_{h'(x)}.$$

We can define *variable exponent Lebesgue spaces with vector values*. We say $u = (u_1, \ldots, u_L)\colon \mathbb{R}^N \to \mathbb{R}^L \in L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)$ if, and only if, $u_i \in L^{h(x)}(\mathbb{R}^N)$, for $i = 1, \ldots, L$. On $L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)$, we consider the norm $|u|_{L^{h(x)}(\mathbb{R}^N,\mathbb{R}^L)} = \sum_{i=1}^{L} |u_i|_{h(x)}$.

We state below lemmas of Brezis–Lieb type. The proof of the two first results follows the same arguments explored at [26], while the proof of the latter can be found at [8].

Proposition 2.5 (Brezis–Lieb lemma, first version) *Let* (u_n) *be a bounded sequence in* $L^{h(x)}\big(\mathbb{R}^N, \mathbb{R}^L\big)$ *such that* $u_n(x) \to u(x)$ *for a.e.* $x \in \mathbb{R}^N$. *Then,* $u \in L^{h(x)}\big(\mathbb{R}^N, \mathbb{R}^L\big)$ *and*

$$\int_{\mathbb{R}^N} \Big| |u_n|^{h(x)} - |u_n - u|^{h(x)} - |u|^{h(x)} \Big| \, dx = o_n(1). \tag{2.2}$$

Proposition 2.6 (Brezis–Lieb lemma, second version) *Let* (u_n) *be a bounded sequence in* $L^{h(x)}\big(\mathbb{R}^N, \mathbb{R}^L\big)$ *with* $h_- > 1$ *and* $u_n(x) \to u(x)$ *for a.e.* $x \in \mathbb{R}^N$. *Then*

$$u_n \rightharpoonup u \quad \text{in } L^{h(x)}\big(\mathbb{R}^N, \mathbb{R}^L\big).$$

Proposition 2.7 (Brezis–Lieb lemma, third version) *Let* (u_n) *be a bounded sequence in* $L^{h(x)}\big(\mathbb{R}^N, \mathbb{R}^L\big)$ *with* $h_- > 1$ *and* $u_n(x) \to u(x)$ *for a.e.* $x \in \mathbb{R}^N$. *Then*

$$\int_{\mathbb{R}^N} \Big| |u_n|^{h(x)-2} u_n - |u_n - u|^{h(x)-2} (u_n - u) - |u|^{h(x)-2} u \Big|^{h'(x)} \, dx = o_n(1), \tag{2.3}$$

To finish this section, we notice that for any open subset $\Omega \subset \mathbb{R}^N$, we can define in the same way the spaces $L^{h(x)}\big(\Omega\big)$ and $W^{1,h(x)}\big(\Omega\big)$. Moreover, all the above propositions have analogous versions for these spaces and, besides, we have the following embedding Theorem of Sobolev's type.

Proposition 2.8 ([21, Theorems 1.1, 1.3]) *Let* $\Omega \subset \mathbb{R}^N$ *an open domain with the cone property,* $h\colon \overline{\Omega} \to \mathbb{R}$ *satisfying* $1 < h_- \le h_+ < N$ *and* $m \in L^{\infty}_{+}\big(\Omega\big)$.

(i) *If* h *is Lipschitz continuous and* $h \le m \le h^*$, *the embedding* $W^{1,h(x)}\big(\Omega\big) \hookrightarrow L^{m(x)}\big(\Omega\big)$ *is continuous;*
(ii) *If* Ω *is bounded,* h *is continuous and* $m \ll h^*$, *the embedding* $W^{1,h(x)}\big(\Omega\big) \hookrightarrow L^{m(x)}\big(\Omega\big)$ *is compact.*

3 An auxiliary problem

In this section, we work with an auxiliary problem adapting the ideas explored in del Pino and Felmer [14] (see also [3]).

We start noting that the energy functional $I_\lambda\colon E_\lambda \to \mathbb{R}$ associated with $\big(P_\lambda\big)$ is given by

$$I_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \Big(\big|\nabla u\big|^{p(x)} + \big(\lambda V(x) + Z(x)\big)|u|^{p(x)} \Big) - \int_{\mathbb{R}^N} F(x,u),$$

where $E_\lambda = \big(E, \|\cdot\|_\lambda\big)$ with

$$E = \left\{ u \in W^{1,p(x)}\big(\mathbb{R}^N\big)\,;\ \int_{\mathbb{R}^N} V(x)|u|^{p(x)} < \infty \right\},$$

and

$$\|u\|_\lambda = \inf\left\{ \sigma > 0\,;\ \varrho_\lambda\left(\frac{u}{\sigma}\right) \le 1 \right\},$$

being

$$\varrho_\lambda(u) = \int\limits_{\mathbb{R}^N} \left(\left|\nabla u\right|^{p(x)} + \left(\lambda V(x) + Z(x)\right)|u|^{p(x)} \right).$$

Thus, $E_\lambda \hookrightarrow W^{1,p(x)}(\mathbb{R}^N)$ continuously for $\lambda \geq 1$ and E_λ is compactly embedded in $L^{h(x)}_{loc}(\mathbb{R}^N)$, for all $1 \leq h \ll p^*$. In addition, we can show that E_λ is a reflexive space. Also, being $\mathcal{O} \subset \mathbb{R}^N$ an open set, from the relation

$$\varrho_{\lambda,\mathcal{O}}(u) = \int\limits_{\mathcal{O}} \left(\left|\nabla u\right|^{p(x)} + \left(\lambda V(x) + Z(x)\right)|u|^{p(x)} \right) \geq M \int\limits_{\mathcal{O}} |u|^{p(x)} = M\varrho_{p(x),\mathcal{O}}(u), \tag{3.1}$$

for all $u \in E_\lambda$ with $\lambda \geq 1$, writing $M = (1-\delta)^{-1}\nu$, for some $0 < \delta < 1$ and $\nu > 0$, we derive

$$\varrho_{\lambda,\mathcal{O}}(u) - \nu\varrho_{p(x),\mathcal{O}}(u) \geq \delta\varrho_{\lambda,\mathcal{O}}(u), \quad \forall u \in E_\lambda,\ \lambda \geq 1. \tag{3.2}$$

Remark 3.1 From the above commentaries, in this work the parameter λ will be always bigger than or equal to 1.

We recall that for any $\epsilon > 0$, the hypotheses (f_1), (f_2) and (f_5) yield

$$f(x,t) \leq \epsilon|t|^{p(x)-1} + C_\epsilon|t|^{q(x)-1}, \quad \forall x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{3.3}$$

and, consequently,

$$F(x,t) \leq \epsilon|t|^{p(x)} + C_\epsilon|t|^{q(x)}, \quad \forall x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{3.4}$$

where C_ϵ depends on ϵ. Moreover, for each $\nu > 0$ fixed, the assumptions (f_2) and (f_3) allow us considering the function $a\colon \mathbb{R}^N \to \mathbb{R}$ given by

$$a(x) = \min\left\{ a > 0;\ \frac{f(x,a)}{a^{p(x)-1}} = \nu \right\}. \tag{3.5}$$

From (f_2), it follows that

$$0 < a_- = \inf_{x\in\mathbb{R}^N} a(x). \tag{3.6}$$

Using the function $a(x)$, we set the function $\tilde{f}\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$\tilde{f}(x,t) = \begin{cases} f(x,t),\ t \leq a(x) \\ \nu t^{p(x)-1},\ t \geq a(x) \end{cases},$$

which fulfills the inequality

$$\tilde{f}(x,t) \leq \nu|t|^{p(x)-1}, \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R}. \tag{3.7}$$

Thus

$$\tilde{f}(x,t)t \leq \nu|t|^{p(x)}, \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R}, \tag{3.8}$$

and

$$\tilde{F}(x,t) \leq \frac{\nu}{p(x)}|t|^{p(x)}, \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R}, \tag{3.9}$$

where $\tilde{F}(x,t) = \int_0^t \tilde{f}(x,s)\,\mathrm{d}s$.

Now, once that $\Omega = \text{int}\, V^{-1}(0)$ is formed by k connected components $\Omega_1, \ldots, \Omega_k$ with $\text{dist}\big(\Omega_i, \Omega_j\big) > 0$, $i \neq j$, then for each $j \in \{1, \ldots, k\}$, we are able to fix a smooth bounded domain Ω'_j such that

$$\overline{\Omega_j} \subset \Omega'_j \quad \text{and} \quad \overline{\Omega'_i} \cap \overline{\Omega'_j} = \emptyset, \quad \text{for } i \neq j. \tag{3.10}$$

From now on, we fix a non-empty subset $\Upsilon \subset \{1, \ldots, k\}$ and

$$\Omega_\Upsilon = \bigcup_{j \in \Upsilon} \Omega_j, \; \Omega'_\Upsilon = \bigcup_{j \in \Upsilon} \Omega'_j, \; \chi_\Upsilon = \begin{cases} 1, & \text{if } x \in \Omega'_\Upsilon \\ 0, & \text{if } x \notin \Omega'_\Upsilon. \end{cases}$$

Using the above notations, we set the functions

$$g(x,t) = \chi_\Upsilon(x) f(x,t) + \big(1 - \chi_\Upsilon(x)\big) \tilde{f}(x,t), \; (x,t) \in \mathbb{R}^N \times \mathbb{R}$$

and

$$G(x,t) = \int_0^t g(x,s)\, \mathrm{d}s, \; (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

and the auxiliary problem

$$\big(A_\lambda\big) \begin{cases} -\Delta_{p(x)} u + \big(\lambda V(x) + Z(x)\big)|u|^{p(x)-2} u = g(x,u), \text{ in } \mathbb{R}^N, \\ u \in W^{1,p(x)}\big(\mathbb{R}^N\big). \end{cases}$$

The problem $\big(A_\lambda\big)$ is related to $\big(P_\lambda\big)$ in the sense that, if u_λ is a solution for $\big(A_\lambda\big)$ verifying

$$u_\lambda(x) \leq a(x), \; \forall x \in \mathbb{R}^N \setminus \Omega'_\Upsilon,$$

then it is a solution for $\big(P_\lambda\big)$.

In comparison with $\big(P_\lambda\big)$, problem $\big(A_\lambda\big)$ has the advantage that the energy functional associated with $\big(A_\lambda\big)$, namely, $\phi_\lambda \colon E_\lambda \to \mathbb{R}$ given by

$$\phi_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \Big(|\nabla u|^{p(x)} + \big(\lambda V(x) + Z(x)\big)|u|^{p(x)}\Big) - \int_{\mathbb{R}^N} G(x,u),$$

satisfies the (PS) condition, whereas I_λ does not necessarily satisfy this condition. In this way, the mountain pass level (see Theorem 3.6) is a critical value for ϕ_λ.

Proposition 3.2 *ϕ_λ satisfies the mountain pass geometry.*

Proof From (3.4) and (3.9),

$$\phi_\lambda(u) \geq \frac{1}{p_+} \varrho_\lambda(u) - \epsilon \int_{\mathbb{R}^N} |u|^{p(x)} - C_\epsilon \int_{\mathbb{R}^N} |u|^{q(x)} - \frac{\nu}{p_-} \int_{\mathbb{R}^N} |u|^{p(x)},$$

for $\epsilon > 0$ and $C_\epsilon > 0$ be a constant depending on ϵ. By (3.1), fixing $\epsilon < \frac{M}{p_+}$ and $\nu < p_- M \left(\frac{1}{p_+} - \frac{\epsilon}{M}\right)$ and assuming $\|u\|_\lambda < \min\big\{1, 1/C_q\big\}$, where $|v|_{q(x)} \leq C_q \|v\|_\lambda$, $\forall v \in E_\lambda$, we derive from Proposition 2.1

$$\phi_\lambda(u) \geq \alpha \|u\|_\lambda^{p_+} - C \|u\|_\lambda^{q_-},$$

where $\alpha = \left(\frac{1}{p_+} - \frac{\epsilon}{M}\right) - \frac{\nu}{p_- M} > 0$. Once $p_+ < q_-$, the first part of the mountain pass geometry is satisfied. Now, fixing $v \in C_0^\infty(\Omega_\Upsilon)$, we have for $t \geq 0$

$$\phi_\lambda(tv) = \int_{\mathbb{R}^N} \frac{t^{p(x)}}{p(x)} \left(|\nabla v|^{p(x)} + Z(x))|v|^{p(x)}\right) - \int_{\mathbb{R}^N} F(x, tv).$$

If $t > 1$, by (f_3),

$$\phi_\lambda(tv) \leq \frac{t^{p^+}}{p_-} \int_{\mathbb{R}^N} \left(|\nabla v|^{p(x)} + Z(x))|v|^{p(x)}\right) - C_1 t^\theta \int_{\mathbb{R}^N} |v|^\theta - C_2,$$

and so,

$$\phi_\lambda(tv) \to -\infty \quad \text{as} \quad t \to +\infty.$$

The last limit implies that ϕ_λ verifies the second geometry of the mountain pass. □

Proposition 3.3 *All $(PS)_d$ sequences for ϕ_λ are bounded in E_λ.*

Proof Let (u_n) be a $(PS)_d$ sequence for ϕ_λ. So, there is $n_0 \in \mathbb{N}$ such that

$$\phi_\lambda(u_n) - \frac{1}{\theta}\phi_\lambda'(u_n)u_n \leq d + 1 + \|u_n\|_\lambda, \text{ for } n \geq n_0.$$

On the other hand, by (3.8) and (3.9)

$$\tilde{F}(x, t) - \frac{1}{\theta}\tilde{f}(x, t)t \leq \left(\frac{1}{p(x)} - \frac{1}{\theta}\right) \nu |t|^{p(x)}, \ \forall x \in \mathbb{R}^N, \quad t \in \mathbb{R},$$

which together with (3.2) gives

$$\phi_\lambda(u_n) - \frac{1}{\theta}\phi_\lambda'(u_n)u_n \geq \left(\frac{1}{p_+} - \frac{1}{\theta}\right) \delta \varrho_\lambda(u_n), \ \forall n \in \mathbb{N}.$$

Hence

$$d + 1 + \max\left\{\varrho_\lambda(u_n)^{1/p_-}, \varrho_\lambda(u_n)^{1/p_+}\right\} \geq \left(\frac{1}{p_+} - \frac{1}{\theta}\right) \delta \varrho_\lambda(u_n), \ \forall n \geq n_0,$$

from where it follows that (u_n) is bounded in E_λ. □

Proposition 3.4 *If (u_n) is a $(PS)_d$ sequence for ϕ_λ, then given $\epsilon > 0$, there is $R > 0$ such that*

$$\limsup_n \int_{\mathbb{R}^N \setminus B_R(0)} \left(|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)}\right) < \epsilon. \tag{3.11}$$

Hence, once that g has a subcritical growth, if $u \in E_\lambda$ is the weak limit of (u_n), then

$$\int_{\mathbb{R}^N} g(x, u_n)u_n \, dx \to \int_{\mathbb{R}^N} g(x, u)u \, dx \ \ \textit{and} \ \int_{\mathbb{R}^N} g(x, u_n)v \, dx \to \int_{\mathbb{R}^N} g(x, u)v \, dx, \ \forall v \in E_\lambda.$$

Proof Let (u_n) be a $(PS)_d$ sequence for ϕ_λ, $R > 0$ large such that $\Omega_\Upsilon' \subset B_{\frac{R}{2}}(0)$ and $\eta_R \in C^\infty(\mathbb{R}^N)$ satisfying

$$\eta_R(x) = \begin{cases} 0, & x \in B_{\frac{R}{2}}(0) \\ 1, & x \in \mathbb{R}^N \setminus B_R(0) \end{cases},$$

$0 \leq \eta_R \leq 1$ and $|\nabla \eta_R| \leq \dfrac{C}{R}$, where $C > 0$ does not depend on R. This way,

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + \big(\lambda V(x) + Z(x)\big)|u_n|^{p(x)}\right) \eta_R$$
$$= \phi_\lambda'(u_n)\,(u_n\eta_R) - \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \eta_R + \int_{\mathbb{R}^N \setminus \Omega_\Upsilon'} \tilde{f}(x, u_n) u_n \eta_R.$$

Denoting

$$I = \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + \big(\lambda V(x) + Z(x)\big)|u_n|^{p(x)}\right) \eta_R,$$

it follows from (3.8),

$$I \leq \phi_\lambda'(u_n)\,(u_n\eta_R) + \frac{C}{R} \int_{\mathbb{R}^N} |u_n| |\nabla u_n|^{p(x)-1} + \nu \int_{\mathbb{R}^N} |u_n|^{p(x)} \eta_R.$$

Using Hölder's inequality 2.4 and Proposition 2.3, we derive

$$I \leq \phi_\lambda'(u_n)\,(u_n\eta_R) + \frac{C}{R} |u_n|_{p(x)} \max\left\{ |\nabla u_n|_{p(x)}^{p_- -1}, |\nabla u_n|_{p(x)}^{p_+ -1} \right\} + \frac{\nu}{M} I.$$

Since (u_n) and $\left(|\nabla u_n|\right)$ are bounded in $L^{p(x)}\big(\mathbb{R}^N\big)$ and $\frac{\nu}{M} = 1 - \delta$, we obtain

$$\int_{\mathbb{R}^N \setminus B_R(0)} \left(|\nabla u_n|^{p(x)} + \big(\lambda V(x) + Z(x)\big)|u_n|^{p(x)}\right) \leq o_n(1) + \frac{C}{R}.$$

Therefore

$$\limsup_n \int_{\mathbb{R}^N \setminus B_R(0)} \left(|\nabla u_n|^{p(x)} + \big(\lambda V(x) + Z(x)\big)|u_n|^{p(x)}\right) \leq \frac{C}{R}.$$

So, given $\epsilon > 0$, choosing a $R > 0$ possibly still bigger, we have that $\dfrac{C}{R} < \epsilon$, which proves (3.11). Now, we will show that

$$\int_{\mathbb{R}^N} g(x, u_n) u_n \to \int_{\mathbb{R}^N} g(x, u) u.$$

Using the fact that $g(x, u)u \in L^1(\mathbb{R}^N)$ together with (3.11) and Sobolev embeddings, given $\epsilon > 0$, we can choose $R > 0$ such that

$$\limsup_{n \to +\infty} \int_{\mathbb{R}^N \setminus B_R(0)} |g(x, u_n) u_n| \leq \frac{\epsilon}{4} \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_R(0)} |g(x, u) u| \leq \frac{\epsilon}{4}.$$

On the other hand, since g has a subcritical growth, we have by compact embeddings

$$\int_{B_R(0)} g(x, u_n) u_n \to \int_{B_R(0)} g(x, u) u.$$

Combining the above information, we conclude that

$$\int_{\mathbb{R}^N} g(x, u_n)u_n \to \int_{\mathbb{R}^N} g(x, u)u.$$

The same type of arguments works to prove that

$$\int_{\mathbb{R}^N} g(x, u_n)v \to \int_{\mathbb{R}^N} g(x, u)v \quad \forall v \in E_\lambda.$$

□

Proposition 3.5 *ϕ_λ verifies the (PS) condition.*

Proof Let (u_n) be a $(PS)_d$ sequence for ϕ_λ and $u \in E_\lambda$ such that $u_n \rightharpoonup u$ in E_λ. Thereby, by Proposition 3.4,

$$\int_{\mathbb{R}^N} g(x, u_n)u_n \to \int_{\mathbb{R}^N} g(x, u)u \quad \text{and} \quad \int_{\mathbb{R}^N} g(x, u_n)v \to \int_{\mathbb{R}^N} g(x, u)v, \ \forall v \in E_\lambda.$$

Moreover, the weak limit also gives

$$\int_{\mathbb{R}^N} \left|\nabla u\right|^{p(x)-2} \nabla u \cdot \nabla(u_n - u) \to 0$$

and

$$\int_{\mathbb{R}^N} \big(\lambda V(x) + Z(x)\big)|u|^{p(x)-2}u(u_n - u) \to 0.$$

Now, if

$$P_n^1(x) = \Big(\left|\nabla u_n\right|^{p(x)-2}\nabla u_n - \left|\nabla u\right|^{p(x)-2}\nabla u\Big) \cdot (\nabla u_n - \nabla u)$$

and

$$P_n^2(x) = \Big(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\Big)(u_n - u),$$

we derive

$$\int_{\mathbb{R}^N} \Big(P_n^1(x) + \big(\lambda V(x) + Z(x)\big)P_n^2(x)\Big) = \phi_\lambda'(u_n)u_n + \int_{\mathbb{R}^N} g(x, u_n)u_n - \phi_\lambda'(u_n)u - \int_{\mathbb{R}^N} g(x, u_n)u$$
$$- \int_{\mathbb{R}^N} \Big(\left|\nabla u\right|^{p(x)-2}\nabla u \cdot \nabla(u_n - u) + \big(\lambda V(x) + Z(x)\big)|u|^{p(x)-2}u(u_n - u)\Big).$$

Recalling that $\phi_\lambda'(u_n)u_n = o_n(1)$ and $\phi_\lambda'(u_n)u = o_n(1)$, the above limits lead to

$$\int_{\mathbb{R}^N} \Big(P_n^1(x) + \big(\lambda V(x) + Z(x)\big)P_n^2(x)\Big) \to 0.$$

Now, the conclusion follows as in [8]. □

Theorem 3.6 *The problem (A_λ) has a (nonnegative) solution, for all $\lambda \geq 1$.*

Proof The proof is an immediate consequence of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [10]. □

4 The $(PS)_\infty$ condition

A sequence $(u_n) \subset W^{1,p(x)}(\mathbb{R}^N)$ is called a $(PS)_\infty$ *sequence for the family* $(\phi_\lambda)_{\lambda\geq 1}$, if there is a sequence $(\lambda_n) \subset [1,\infty)$ with $\lambda_n \to \infty$, as $n \to \infty$, verifying

$$\phi_{\lambda_n}(u_n) \to c \quad \text{and} \quad \left\|\phi'_{\lambda_n}(u_n)\right\| \to 0, \text{ as } n \to \infty.$$

Proposition 4.1 *Let $(u_n) \subset W^{1,p(x)}(\mathbb{R}^N)$ be a $(PS)_\infty$ sequence for $(\phi_\lambda)_{\lambda\geq 1}$. Then, up to a subsequence, there exists $u \in W^{1,p(x)}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $W^{1,p(x)}(\mathbb{R}^N)$. Furthermore,*

(i) $\varrho_{\lambda_n}(u_n - u) \to 0$ *and, consequently,* $u_n \to u$ *in* $W^{1,p(x)}(\mathbb{R}^N)$;
(ii) $u = 0$ *in* $\mathbb{R}^N \setminus \Omega_\Upsilon$, $u \geq 0$ *and* $u_{|\Omega_j}$, $j \in \Upsilon$, *is a solution for*

$$(P_j)\ \begin{cases} -\Delta_{p(x)}u + Z(x)|u|^{p(x)-2}u = f(x,u), \text{ in } \Omega_j, \\ u \in W_0^{1,p(x)}(\Omega_j); \end{cases}$$

(iii) $\displaystyle\int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^{p(x)} \to 0$;
(iv) $\displaystyle\varrho_{\lambda_n,\Omega'_j}(u_n) \to \int_{\Omega_j} \left(\left|\nabla u\right|^{p(x)} + Z(x)|u|^{p(x)}\right)$, *for* $j \in \Upsilon$;
(v) $\varrho_{\lambda_n,\mathbb{R}^N\setminus\Omega_\Upsilon}(u_n) \to 0$;
(vi) $\displaystyle\phi_{\lambda_n}(u_n) \to \int_{\Omega_\Upsilon} \frac{1}{p(x)}\left(\left|\nabla u\right|^{p(x)} + Z(x)|u|^{p(x)}\right) - \int_{\Omega_\Upsilon} F(x,u)$.

Proof Using the same reasoning as in the proof of Proposition 3.3, we obtain that $\left(\varrho_{\lambda_n}(u_n)\right)$ is bounded in $\mathbb{R}$. Then $\left(\|u_n\|_{\lambda_n}\right)$ is bounded in $\mathbb{R}$ and (u_n) is bounded in $W^{1,p(x)}(\mathbb{R}^N)$. So, up to a subsequence, there exists $u \in W^{1,p(x)}(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{in } W^{1,p(x)}(\mathbb{R}^N) \quad \text{and} \quad u_n(x) \to u(x) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Now, for each $m \in \mathbb{N}$, we define $C_m = \left\{x \in \mathbb{R}^N\ ;\ V(x) \geq \dfrac{1}{m}\right\}$. Without loss of generality, we can assume $\lambda_n < 2(\lambda_n - 1)$, $\forall n \in \mathbb{N}$. Thus

$$\int_{C_m} |u_n|^{p(x)} \leq \frac{2m}{\lambda_n}\int_{C_m} \left(\lambda_n V(x) + Z(x)\right)|u_n|^{p(x)} \leq \frac{2m}{\lambda_n}\varrho_{\lambda_n}(u_n) \leq \frac{C}{\lambda_n}.$$

By Fatou's lemma, we derive

$$\int_{C_m} |u|^{p(x)} = 0,$$

which implies that $u = 0$ in C_m and, consequently, $u = 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$. From this, we are able to prove $(i) - (vi)$.

(i) Since $u = 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$, repeating the argument explored in Proposition 3.5 we get

$$\int_{\mathbb{R}^N} \left(P_n^1(x) + \left(\lambda_n V(x) + Z(x)\right)P_n^2(x)\right) \to 0,$$

where

$$P_n^1(x) = \left(\left|\nabla u_n\right|^{p(x)-2}\nabla u_n - \left|\nabla u\right|^{p(x)-2}\nabla u\right)\cdot(\nabla u_n - \nabla u)$$

and

$$P_n^2(x) = \left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right)(u_n - u).$$

Therefore, $\varrho_{\lambda_n}(u_n - u) \to 0$, which implies $u_n \to u$ in $W^{1,p(x)}(\mathbb{R}^N)$.

(ii) Since $u \in W^{1,p(x)}(\mathbb{R}^N)$ and $u = 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$, we have $u \in W_0^{1,p(x)}(\Omega)$ or, equivalently, $u_{|\Omega_j} \in W_0^{1,p(x)}(\Omega_j)$, for $j = 1, \ldots, k$. Moreover, the limit $u_n \to u$ in $W^{1,p(x)}(\mathbb{R}^N)$ combined with $\phi'_{\lambda_n}(u_n)\varphi \to 0$ for $\varphi \in C_0^\infty(\Omega_j)$ implies that

$$\int_{\Omega_j} \left(\left|\nabla u\right|^{p(x)-2}\nabla u \cdot \nabla\varphi + Z(x)|u|^{p(x)-2}u\varphi\right) - \int_{\Omega_j} g(x,u)\varphi = 0, \tag{4.1}$$

showing that $u_{|\Omega_j}$ is a solution for

$$\begin{cases} -\Delta_{p(x)}u + Z(x)|u|^{p(x)-2}u = g(x,u), \text{ in } \Omega_j, \\ u \in W_0^{1,p(x)}(\Omega_j). \end{cases}$$

This way, if $j \in \Upsilon$, then $u_{|\Omega_j}$ satisfies (P_j). On the other hand, if $j \notin \Upsilon$, we must have

$$\int_{\Omega_j} \left(\left|\nabla u\right|^{p(x)} + Z(x)|u|^{p(x)}\right) - \int_{\Omega_j} \tilde{f}(x,u)u = 0.$$

The above equality combined with (3.8) and (3.2) gives

$$0 \ge \varrho_{\lambda,\Omega_j}(u) - \nu\varrho_{p(x),\Omega_j}(u) \ge \delta\varrho_{\lambda,\Omega_j}(u) \ge 0,$$

from where it follows $u_{|\Omega_j} = 0$. This proves $u = 0$ outside Ω_Υ and $u \ge 0$ in $\mathbb{R}^N$.

(iii) It follows from (i), since

$$\int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^{p(x)} = \int_{\mathbb{R}^N} \lambda_n V(x)|u_n - u|^{p(x)} \le 2\varrho_{\lambda_n}(u_n - u).$$

(iv) Let $j \in \Upsilon$. From (i),

$$\varrho_{p(x),\Omega'_j}(u_n - u), \varrho_{p(x),\Omega'_j}\left(\nabla u_n - \nabla u\right) \to 0.$$

Then by Proposition 2.5,

$$\int_{\Omega'_j} \left(\left|\nabla u_n\right|^{p(x)} - \left|\nabla u\right|^{p(x)}\right) \to 0 \quad \text{and} \quad \int_{\Omega'_j} Z(x)\left(|u_n|^{p(x)} - |u|^{p(x)}\right) \to 0.$$

From (iii),

$$\int_{\Omega'_j} \lambda_n V(x)\left(|u_n|^{p(x)} - |u|^{p(x)}\right) = \int_{\Omega'_j \setminus \overline{\Omega_j}} \lambda_n V(x)|u_n|^{p(x)} \to 0.$$

This way

$$\varrho_{\lambda_n,\Omega'_j}(u_n) - \varrho_{\lambda_n,\Omega'_j}(u) \to 0.$$

Once $u = 0$ in $\Omega'_j \setminus \Omega_j$, we get

$$\varrho_{\lambda_n,\Omega'_j}(u_n) \to \int_{\Omega_j} \left(|\nabla u|^{p(x)} + Z(x)|u|^{p(x)}\right).$$

(v) By (i), $\varrho_{\lambda_n}(u_n - u) \to 0$, and so,

$$\varrho_{\lambda_n,\mathbb{R}^N\setminus\Omega_\Upsilon}(u_n) \to 0.$$

(vi) We can write the functional ϕ_{λ_n} in the following way

$$\phi_{\lambda_n}(u_n) = \sum_{j\in\Upsilon}\int_{\Omega'_j} \frac{1}{p(x)}\left(\left|\nabla u_n\right|^{p(x)} + \left(\lambda_n V(x) + Z(x)\right)|u_n|^{p(x)}\right)$$
$$+ \int_{\mathbb{R}^N\setminus\Omega'_\Upsilon} \frac{1}{p(x)}\left(\left|\nabla u_n\right|^{p(x)} + \left(\lambda_n V(x) + Z(x)\right)|u_n|^{p(x)}\right) - \int_{\mathbb{R}^N} G(x, u_n).$$

From $(i) - (v)$,

$$\int_{\Omega'_j} \frac{1}{p(x)}\left(\left|\nabla u_n\right|^{p(x)} + \left(\lambda_n V(x) + Z(x)\right)|u_n|^{p(x)}\right)$$
$$\to \int_{\Omega_j} \frac{1}{p(x)}\left(\left|\nabla u\right|^{p(x)} + Z(x)|u|^{p(x)}\right),$$
$$\int_{\mathbb{R}^N\setminus\Omega'_\Upsilon} \frac{1}{p(x)}\left(\left|\nabla u_n\right|^{p(x)} + \left(\lambda_n V(x) + Z(x)\right)|u_n|^{p(x)}\right) \to 0.$$

and

$$\int_{\mathbb{R}^N} G(x, u_n) \to \int_{\Omega_\Upsilon} F(x, u).$$

Therefore

$$\phi_{\lambda_n}(u_n) \to \int_{\Omega_\Upsilon} \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + Z(x)|u|^{p(x)}\right) - \int_{\Omega_\Upsilon} F(x, u).$$

□

5 The boundedness of the (A_λ) solutions

In this section, we study the boundedness outside Ω'_Υ for some solutions of (A_λ). To this end, we adapt for our problem arguments found in [18] and [25].

Proposition 5.1 *Let (u_λ) be a family of solutions for (A_λ) such that $u_\lambda \to 0$ in $W^{1,p(x)}(\mathbb{R}^N \setminus \Omega_\Upsilon)$, as $\lambda \to \infty$. Then, there exists $\lambda^* > 0$ with the following property:*

$$|u_\lambda|_{\infty,\mathbb{R}^N\setminus\Omega'_\Upsilon} \le a_-,\ \forall\lambda \ge \lambda^*.$$

Hence, u_λ is a solution for (P_λ) for $\lambda \ge \lambda^*$.

Before to prove the above proposition, we need to show some technical lemmas.

Lemma 5.2 *There exist* $x_1, \dots, x_l \in \partial\Omega'_\Upsilon$ *and corresponding* $\delta_{x_1}, \dots, \delta_{x_l} > 0$ *such that*

$$\partial\Omega'_\Upsilon \subset \mathcal{N}\left(\partial\Omega'_\Upsilon\right) := \bigcup_{i=1}^{l} B_{\frac{\delta_{x_i}}{2}}(x_i).$$

Moreover,

$$q_+^{x_i} \le \left(p_-^{x_i}\right)^*, \tag{5.1}$$

where

$$q_+^{x_i} = \sup_{B_{\delta_{x_i}}(x_i)} q, \; p_-^{x_i} = \inf_{B_{\delta_{x_i}}(x_i)} p \text{ and } \left(p_-^{x_i}\right)^* = \frac{Np_-^{x_i}}{N - p_-^{x_i}}.$$

Proof From (3.10), $\overline{\Omega_\Upsilon} \subset \Omega'_\Upsilon$. So, there is $\delta > 0$ such that

$$\overline{B_\delta(x)} \subset \mathbb{R}^N \setminus \overline{\Omega_\Upsilon}, \; \forall x \in \partial\Omega'_\Upsilon.$$

Once $q \ll p^*$, there exists $\epsilon > 0$ such that $\epsilon \le p^*(y) - q(y)$, for all $y \in \mathbb{R}^N$. Then, by continuity, for each $x \in \partial\Omega'_\Upsilon$, we can choose a sufficiently small $0 < \delta_x \le \delta$ such that

$$q_+^x \le \left(p_-^x\right)^*,$$

where

$$q_+^x = \sup_{B_{\delta_x}(x)} q, \; p_-^x = \inf_{B_{\delta_x}(x)} p \text{ and } \left(p_-^x\right)^* = \frac{Np_-^x}{N - p_-^x}.$$

Covering $\partial\Omega'_\Upsilon$ by the balls $B_{\frac{\delta_x}{2}}(x)$, $x \in \partial\Omega'_\Upsilon$, and using its compactness, there are $x_1, \dots, x_l \in \partial\Omega'_\Upsilon$ such that

$$\partial\Omega'_\Upsilon \subset \bigcup_{i=1}^{l} B_{\frac{\delta_{x_i}}{2}}(x_i).$$

□

Lemma 5.3 *If* u_λ *is a solution for* $\left(A_\lambda\right)$, *in each* $B_{\delta_{x_i}}(x_i)$, $i = 1, \dots, l$, *given by Lemma 5.2, it is fulfilled*

$$\int_{A_{k,\overline{\delta},x_i}} \left|\nabla u_\lambda\right|^{p_-^{x_i}} \le C\left(\left(k^{q_+} + 2\right)\left|A_{k,\widetilde{\delta},x_i}\right| + \left(\widetilde{\delta} - \overline{\delta}\right)^{-\left(p_-^{x_i}\right)^*} \int_{A_{k,\widetilde{\delta},x_i}} (u_\lambda - k)^{\left(p_-^{x_i}\right)^*}\right),$$

where $0 < \overline{\delta} < \widetilde{\delta} < \delta_{x_i}, k \ge \dfrac{a_-}{4}, C = C\left(p_-, p_+, q_-, q_+, \nu, \delta_{x_i}\right) > 0$ *is a constant independent of* k, *and for any* $R > 0$, *we denote by* A_{k,R,x_i} *the set*

$$A_{k,R,x_i} = B_R(x_i) \cap \left\{x \in \mathbb{R}^N \,;\, u_\lambda(x) > k\right\}.$$

Proof We choose arbitrarily $0 < \overline{\delta} < \widetilde{\delta} < \delta_{x_i}$ and $\xi \in C^\infty\left(\mathbb{R}^N\right)$ with

$$0 \le \xi \le 1, \; \operatorname{supp} \xi \subset B_{\widetilde{\delta}}(x_i), \; \xi = 1 \text{ in } B_{\overline{\delta}}(x_i) \quad \text{and} \quad \left|\nabla\xi\right| \le \frac{2}{\widetilde{\delta} - \overline{\delta}}.$$

For $k \ge \dfrac{a_-}{4}$, we define $\eta = \xi^{p_+}(u_\lambda - k)^+$. We notice that

$$\nabla\eta = p_+\xi^{p_+-1}(u_\lambda - k)\nabla\xi + \xi^{p_+}\nabla u_\lambda$$

on the set $\{u_\lambda > k\}$. Then, writing $u_\lambda = u$ and taking η as a test function, we obtain

$$p_+ \int_{A_{k,\widetilde{\delta},x_i}} \xi^{p_+-1}(u-k)\big|\nabla u\big|^{p(x)-2}\nabla u\cdot\nabla\xi + \int_{A_{k,\widetilde{\delta},x_i}} \xi^{p_+}\big|\nabla u\big|^{p(x)}$$
$$+ \int_{A_{k,\widetilde{\delta},x_i}} \big(\lambda V(x)+Z(x)\big)u^{p(x)-1}\xi^{p_+}(u-k) = \int_{A_{k,\widetilde{\delta},x_i}} g(x,u)\xi^{p_+}(u-k).$$

If we set

$$J = \int_{A_{k,\widetilde{\delta},x_i}} \xi^{p_+}\big|\nabla u\big|^{p(x)},$$

using that $\nu \le \lambda V(x) + Z(x),\ \forall x \in \mathbb{R}^N$, we get

$$J \le p_+ \int_{A_{k,\widetilde{\delta},x_i}} \xi^{p_+-1}(u-k)\big|\nabla u\big|^{p(x)-1}\big|\nabla\xi\big|$$
$$- \int_{A_{k,\widetilde{\delta},x_i}} \nu u^{p(x)-1}\xi^{p_+}(u-k) + \int_{A_{k,\widetilde{\delta},x_i}} g(x,u)\xi^{p_+}(u-k). \tag{5.2}$$

From (5.2), (3.3) and (3.7),

$$J \le p_+ \int_{A_{k,\widetilde{\delta},x_i}} \xi^{p_+-1}(u-k)\big|\nabla u\big|^{p(x)-1}\big|\nabla\xi\big| - \int_{A_{k,\widetilde{\delta},x_i}} \nu u^{p(x)-1}\xi^{p_+}(u-k)$$
$$+ \int_{A_{k,\widetilde{\delta},x_i}} \big(\nu u^{p(x)-1} + C_\nu u^{q(x)-1}\big)\xi^{p_+}(u-k),$$

from where it follows

$$J \le p_+ \int_{A_{k,\widetilde{\delta},x_i}} \xi^{p_+-1}(u-k)\big|\nabla u\big|^{p(x)-1}\big|\nabla\xi\big| + C_\nu \int_{A_{k,\widetilde{\delta},x_i}} u^{q(x)-1}(u-k).$$

Using Young's inequality, we obtain, for $\chi \in (0, 1)$,

$$J \le \frac{p_+(p_+-1)}{p_-}\chi^{\frac{p_-}{p_+-1}} J + \frac{2^{p_+}p_+}{p_-}\chi^{-p_+} \int_{A_{k,\widetilde{\delta},x_i}} \left(\frac{u-k}{\widetilde{\delta}-\overline{\delta}}\right)^{p(x)}$$
$$+\frac{C_\nu(q_+-1)}{q_-} \int_{A_{k,\widetilde{\delta},x_i}} u^{q(x)} + \frac{C_\nu\left(1+\delta_{x_i}^{q_+}\right)}{q_-} \int_{A_{k,\widetilde{\delta},x_i}} \left(\frac{u-k}{\widetilde{\delta}-\overline{\delta}}\right)^{q(x)}.$$

Writing

$$Q = \int_{A_{k,\widetilde{\delta},x_i}} \left(\frac{u-k}{\widetilde{\delta}-\overline{\delta}}\right)^{\left(p_-^{x_i}\right)^*},$$

for $\chi \approx 0^+$ fixed, due to (5.1), we deduce

$$J \le \frac{1}{2}J + \frac{2^{p_+}p_+}{p_-}\chi^{-p_+}\left(\left|A_{k,\widetilde{\delta},x_i}\right| + Q\right) + \frac{C_\nu 2^{q_+}(q_+-1)\left(1+\delta_{x_i}^{q_+}\right)}{q_-}\left(\left|A_{k,\widetilde{\delta},x_i}\right| + Q\right)$$
$$+\frac{C_\nu 2^{q_+}(q_+-1)\left(1+k^{q_+}\right)}{q_-}\left|A_{k,\widetilde{\delta},x_i}\right| + \frac{C_\nu\left(1+\delta_{x_i}^{q_+}\right)}{q_-}\left(\left|A_{k,\widetilde{\delta},x_i}\right| + Q\right).$$

Therefore

$$\int_{A_{k,\overline{\delta},x_i}} \left|\nabla u\right|^{p(x)} \le J \le C\left[\left(k^{q_+}+1\right)\left|A_{k,\widetilde{\delta},x_i}\right| + Q\right],$$

for a positive constant $C = C\left(p_-, p_+, q_-, q_+, \nu, \delta_{x_i}\right)$ which does not depend on k. Since

$$\left|\nabla u\right|^{p_-^{x_i}} - 1 \le \left|\nabla u\right|^{p(x)}, \ \forall x \in B_{\delta_{x_i}}(x_i),$$

we obtain

$$\int_{A_{k,\overline{\delta},x_i}} \left|\nabla u\right|^{p_-^{x_i}} \le C\left[\left(k^{q_+}+1\right)\left|A_{k,\widetilde{\delta},x_i}\right| + Q\right] + \left|A_{k,\widetilde{\delta},x_i}\right|$$
$$\le C\left(\left(k^{q_+}+2\right)\left|A_{k,\widetilde{\delta},x_i}\right| + \left(\widetilde{\delta}-\overline{\delta}\right)^{-\left(p_-^{x_i}\right)^*}\int_{A_{k,\widetilde{\delta},x_i}} (u-k)^{\left(p_-^{x_i}\right)^*}\right),$$

for a positive constant $C = C\left(p_-, p_+, q_-, q_+, \nu, \delta_{x_i}\right)$ which does not depend on k. □

The next lemma can be found at ([27, Lemma 4.7]).

Lemma 5.4 *Let* (J_n) *be a sequence of nonnegative numbers satisfying*

$$J_{n+1} \le CB^n J_n^{1+\eta}, \ n = 0, 1, 2, \dots,$$

where $C, \eta > 0$ *and* $B > 1$. *If*

$$J_0 \le C^{-\frac{1}{\eta}} B^{-\frac{1}{\eta^2}},$$

then $J_n \to 0$, *as* $n \to \infty$.

Lemma 5.5 *Let* $\left(u_\lambda\right)$ *be a family of solutions for* $\left(A_\lambda\right)$ *such that* $u_\lambda \to 0$ *in* $W^{1,p(x)}\left(\mathbb{R}^N \setminus \Omega_\Upsilon\right)$, *as* $\lambda \to \infty$. *Then, there exists* $\lambda^* > 0$ *with the following property:*

$$|u_\lambda|_{\infty,\mathcal{N}(\partial\Omega'_\Upsilon)} \le a_-, \ \forall \lambda \ge \lambda^*.$$

Proof It is enough to prove the inequality in each ball $B_{\frac{\delta_{x_i}}{2}}(x_i)$, $i = 1, \dots, l$, given by Lemma 5.2. We set

$$\widetilde{\delta}_n = \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2^{n+1}}, \ \overline{\delta}_n = \frac{\widetilde{\delta}_n + \widetilde{\delta}_{n+1}}{2}, \ k_n = \frac{a_-}{2}\left(1 - \frac{1}{2^{n+1}}\right), \ \forall n = 0, 1, 2, \dots.$$

Then

$$\widetilde{\delta}_n \downarrow \frac{\delta_{x_i}}{2}, \quad \widetilde{\delta}_{n+1} < \overline{\delta}_n < \widetilde{\delta}_n, \quad k_n \uparrow \frac{a_-}{2}.$$

From now on, we fix

$$J_n(\lambda) = J_n = \int\limits_{A_{k_n,\widetilde{\delta}_n,x_i}} \left(u_\lambda(x) - k_n\right)^{\left(p_-^{x_i}\right)^*},\ n = 0, 1, 2, \ldots.$$

and $\xi \in C^1(\mathbb{R})$ such that

$$0 \leq \xi \leq 1, \quad \xi(t) = 1,\ \text{for } t \leq \frac{1}{2}, \quad \text{and} \quad \xi(t) = 0,\ \text{for } t \geq \frac{3}{4}.$$

Setting

$$\xi_n(x) = \xi\left(\frac{2^{n+1}}{\delta_{x_i}}\left(\left|x - x_i\right| - \frac{\delta_{x_i}}{2}\right)\right), \quad x \in \mathbb{R}^N, \quad n = 0, 1, 2, \ldots,$$

we have $\xi_n = 1$ in $B_{\widetilde{\delta}_{n+1}}(x_i)$ and $\xi_n = 0$ outside $B_{\overline{\delta}_n}(x_i)$. Writing $u_\lambda = u$, we get

$$\begin{aligned}
J_{n+1} &\leq \int\limits_{A_{k_{n+1},\overline{\delta}_n,x_i}} \left((u(x) - k_{n+1})\xi_n(x)\right)^{\left(p_-^{x_i}\right)^*} \\
&= \int\limits_{B_{\delta_{x_i}}(x_i)} \left((u - k_{n+1})^+(x)\xi_n(x)\right)^{\left(p_-^{x_i}\right)^*} \\
&\leq C\left(N, p_-^{x_i}\right)\left(\int\limits_{B_{\delta_{x_i}}(x_i)} \left|\nabla\left((u - k_{n+1})^+\xi_n\right)(x)\right|^{p_-^{x_i}}\right)^{\frac{\left(p_-^{x_i}\right)^*}{p_-^{x_i}}} \\
&\leq C\left(N, p_-^{x_i}\right)\left(\int\limits_{A_{k_{n+1},\overline{\delta}_n,x_i}} \left|\nabla u\right|^{p_-^{x_i}} + \int\limits_{A_{k_{n+1},\overline{\delta}_n,x_i}} (u - k_{n+1})^{p_-^{x_i}}\left|\nabla\xi_n\right|^{p_-^{x_i}}\right)^{\frac{\left(p_-^{x_i}\right)^*}{p_-^{x_i}}}.
\end{aligned}$$

Since

$$\left|\nabla\xi_n(x)\right| \leq C\left(\delta_{x_i}\right)2^{n+1},\ \forall x \in \mathbb{R}^N,$$

writing $J_{n+1}^{\frac{p_-^{x_i}}{\left(p_-^{x_i}\right)^*}} = \widetilde{J}_{n+1}$, we obtain

$$\widetilde{J}_{n+1} \leq C\left(N, p_-^{x_i}, \delta_{x_i}\right)\left(\int\limits_{A_{k_{n+1},\overline{\delta}_n,x_i}} \left|\nabla u\right|^{p_-^{x_i}} + 2^{np_-^{x_i}}\int\limits_{A_{k_{n+1},\overline{\delta}_n,x_i}} (u - k_{n+1})^{p_-^{x_i}}\right).$$

Using Lemma 5.3,

$$\begin{aligned}\widetilde{J}_{n+1} &\le C\Big(N, p_-^{x_i}, \delta_{x_i}\Big)\Bigg((k_{n+1}^{q+}+2)\left|A_{k_{n+1},\widetilde{\delta}_n,x_i}\right| \\ &\quad + \left(\frac{2^{n+3}}{\delta_{x_i}}\right)^{\left(p_-^{x_i}\right)^*} \int\limits_{A_{k_{n+1},\widetilde{\delta}_n,x_i}} (u-k_{n+1})^{\left(p_-^{x_i}\right)^*} + 2^{np_-^{x_i}} \int\limits_{A_{k_{n+1},\widetilde{\delta}_n,x_i}} (u-k_{n+1})^{p_-^{x_i}}\Bigg) \\ &\le C\Big(N, p_-^{x_i}, \delta_{x_i}\Big)\Bigg((k_{n+1}^{q+}+2)\left|A_{k_{n+1},\widetilde{\delta}_n,x_i}\right| \\ &\quad + 2^{n\left(p_-^{x_i}\right)^*} \int\limits_{A_{k_{n+1},\widetilde{\delta}_n,x_i}} (u-k_{n+1})^{\left(p_-^{x_i}\right)^*} + 2^{np_-^{x_i}} \int\limits_{A_{k_{n+1},\widetilde{\delta}_n,x_i}} (u-k_{n+1})^{p_-^{x_i}}\Bigg).\end{aligned}$$

From Young's inequality

$$\int\limits_{A_{k_{n+1},\widetilde{\delta}_n,x_i}} (u-k_{n+1})^{p_-^{x_i}} \le C\Big(p_-^{x_i}\Big)\left(\left|A_{k_{n+1},\widetilde{\delta}_n,x_i}\right| + \int\limits_{A_{k_{n+1},\widetilde{\delta}_n,x_i}} (u-k_{n+1})^{\left(p_-^{x_i}\right)^*}\right).$$

Thus

$$\widetilde{J}_{n+1} \le C\Big(N, p_-^{x_i}, \delta_{x_i}\Big)\Bigg(\bigg(\Big(\frac{a_-}{2}\Big)^{q_+} + 2 + 2^{np_-^{x_i}}\bigg)\left|A_{k_{n+1},\widetilde{\delta}_n,x_i}\right| + 2^{n\left(p_-^{x_i}\right)^*} J_n + 2^{np_-^{x_i}} J_n\Bigg).$$

Now, since

$$J_n \ge \int\limits_{A_{k_{n+1},\widetilde{\delta}_n,x_i}} (u-k_n)^{\left(p_-^{x_i}\right)^*} \ge (k_{n+1}-k_n)^{\left(p_-^{x_i}\right)^*}\left|A_{k_{n+1},\widetilde{\delta}_n,x_i}\right|$$

it follows that

$$\left|A_{k_{n+1},\widetilde{\delta}_n,x_i}\right| \le \left(\frac{2^{n+3}}{a_-}\right)^{\left(p_-^{x_i}\right)^*} J_n,$$

and so,

$$\widetilde{J}_{n+1} \le C\Big(N, p_-^{x_i}, \delta_{x_i}, a_-, q_+\Big)\left(2^{n\left(p_-^{x_i}\right)^*} J_n + 2^{n\left(p_-^{x_i}+\left(p_-^{x_i}\right)^*\right)} J_n + 2^{n\left(p_-^{x_i}\right)^*} J_n + 2^{np_-^{x_i}} J_n\right).$$

Fixing $\alpha = \left(p_-^{x_i} + \left(p_-^{x_i}\right)^*\right)$, it follows that

$$J_{n+1} \le C\Big(N, p_-^{x_i}, \delta_{x_i}, a_-, q_+\Big)\left(2^{\alpha\frac{\left(p_-^{x_i}\right)^*}{p_-^{x_i}}}\right)^n J_n^{\frac{\left(p_-^{x_i}\right)^*}{p_-^{x_i}}},$$

and consequently

$$J_{n+1} \le C B^n J_n^{1+\eta},$$

where $C = C\left(N, p_-^{x_i}, \delta_{x_i}, a_-, q_+\right)$, $B = 2^{\alpha \frac{\left(p_-^{x_i}\right)^*}{p_-^{x_i}}}$ and $\eta = \frac{\left(p_-^{x_i}\right)^*}{p_-^{x_i}} - 1$. Now, once that $u_\lambda \to 0$ in $W^{1,p(x)}\left(\mathbb{R}^N \setminus \Omega_\Upsilon\right)$, as $\lambda \to \infty$, there exists $\lambda_i > 0$ such that

$$\int_{A_{\frac{a_-}{4},\delta_{x_i},x_i}} \left(u_\lambda - \frac{a_-}{4}\right)^{\left(p_-^{x_i}\right)^*} = J_0(\lambda) \le C^{-\frac{1}{\eta}} B^{-\frac{1}{\eta^2}}, \quad \lambda \ge \lambda_i.$$

From Lemma 5.4, $J_n(\lambda) \to 0$, $n \to \infty$, for all $\lambda \ge \lambda_i$, and so,

$$u_\lambda \le \frac{a_-}{2} < a_-, \text{ in } B_{\frac{\delta_{x_i}}{2}}, \text{ for all } \lambda \ge \lambda_i.$$

Now, taking $\lambda^* = \max\{\lambda_1, \ldots, \lambda_l\}$, we conclude that

$$|u_\lambda|_{\infty,\mathcal{N}(\partial\Omega'_\Upsilon)} < a_-, \ \forall \lambda \ge \lambda^*.$$

□

Proof of Proposition 5.1 Fix $\lambda \ge \lambda^*$, where λ^* is given at Lemma 5.5, and define $\widetilde{u}_\lambda : \mathbb{R}^N \setminus \Omega'_\Upsilon \to \mathbb{R}$ given by

$$\widetilde{u}_\lambda(x) = (u_\lambda - a_-)^+(x).$$

From Lemma 5.5, $\widetilde{u}_\lambda \in W_0^{1,p(x)}\left(\mathbb{R}^N \setminus \Omega'_\Upsilon\right)$. Our goal is showing that $\widetilde{u}_\lambda = 0$ in $\mathbb{R}^N \setminus \Omega'_\Upsilon$. This implies

$$|u_\lambda|_{\infty,\mathbb{R}^N\setminus\Omega'_\Upsilon} \le a_-.$$

In fact, extending $\widetilde{u}_\lambda = 0$ in Ω'_Υ and taking $\widetilde{u}_\lambda$ as a test function, we obtain

$$\int_{\mathbb{R}^N\setminus\Omega'_\Upsilon} \left|\nabla u_\lambda\right|^{p(x)-2}\nabla u_\lambda \cdot \nabla\widetilde{u}_\lambda + \int_{\mathbb{R}^N\setminus\Omega'_\Upsilon} \left(\lambda V(x) + Z(x)\right)u_\lambda^{p(x)-2}u_\lambda\widetilde{u}_\lambda = \int_{\mathbb{R}^N\setminus\Omega'_\Upsilon} g\left(x, u_\lambda\right)\widetilde{u}_\lambda.$$

Since

$$\int_{\mathbb{R}^N\setminus\Omega'_\Upsilon} \left|\nabla u_\lambda\right|^{p(x)-2}\nabla u_\lambda \cdot \nabla\widetilde{u}_\lambda = \int_{\mathbb{R}^N\setminus\Omega'_\Upsilon} \left|\nabla\widetilde{u}_\lambda\right|^{p(x)},$$

$$\int_{\mathbb{R}^N\setminus\Omega'_\Upsilon} \left(\lambda V(x) + Z(x)\right)u_\lambda^{p(x)-2}u_\lambda\widetilde{u}_\lambda = \int_{(\mathbb{R}^N\setminus\Omega'_\Upsilon)_+} \left(\lambda V(x) + Z(x)\right)u_\lambda^{p(x)-2}\left(\widetilde{u}_\lambda + a_-\right)\widetilde{u}_\lambda$$

and

$$\int_{\mathbb{R}^N\setminus\Omega'_\Upsilon} g\left(x, u_\lambda\right)\widetilde{u}_\lambda = \int_{(\mathbb{R}^N\setminus\Omega'_\Upsilon)_+} \frac{g\left(x, u_\lambda\right)}{u_\lambda}\left(\widetilde{u}_\lambda + a_-\right)\widetilde{u}_\lambda,$$

where

$$\left(\mathbb{R}^N \setminus \Omega'_\Upsilon\right)_+ = \left\{x \in \mathbb{R}^N \setminus \Omega'_\Upsilon \,;\, u_\lambda(x) > a_-\right\},$$

we derive

$$\int_{\mathbb{R}^N\setminus\Omega'_\Upsilon} \left|\nabla\widetilde{u}_\lambda\right|^{p(x)} + \int_{(\mathbb{R}^N\setminus\Omega'_\Upsilon)_+} \left(\left(\lambda V(x) + Z(x)\right)u_\lambda^{p(x)-2} - \frac{g\left(x, u_\lambda\right)}{u_\lambda}\right)\left(\widetilde{u}_\lambda + a_-\right)\widetilde{u}_\lambda = 0,$$

Now, by (3.7),

$$\big(\lambda V(x)+Z(x)\big)u_\lambda^{p(x)-2}-\frac{g\,(x,u_\lambda)}{u_\lambda}>\nu u_\lambda^{p(x)-2}-\frac{\tilde f\,(x,u_\lambda)}{u_\lambda}\geq 0 \quad \text{in} \quad \Big(\mathbb{R}^N\setminus\Omega'_\Upsilon\Big)_+.$$

This form, $\widetilde{u}_\lambda=0$ in $\big(\mathbb{R}^N\setminus\Omega'_\Upsilon\big)_+$. Obviously, $\widetilde{u}_\lambda=0$ at the points where $u_\lambda\leq a_-$, consequently, $\widetilde{u}_\lambda=0$ in $\mathbb{R}^N\setminus\Omega'_\Upsilon$.

6 A special critical value for ϕ_λ

For each $j=1,\ldots,k$, consider

$$I_j(u)=\int\limits_{\Omega_j}\frac{1}{p(x)}\Big(\big|\nabla u\big|^{p(x)}+Z(x)|u|^{p(x)}\Big)-\int\limits_{\Omega_j}F(x,u),\ u\in W_0^{1,p(x)}\big(\Omega_j\big),$$

the energy functional associated to (P_j), and

$$\phi_{\lambda,j}(u)=\int\limits_{\Omega'_j}\frac{1}{p(x)}\Big(\big|\nabla u\big|^{p(x)}+\big(\lambda V(x)+Z(x)\big)|u|^{p(x)}\Big)-\int\limits_{\Omega'_j}F(x,u),\ u\in W^{1,p(x)}\big(\Omega'_j\big),$$

the energy functional associated to

$$\begin{cases}-\Delta_{p(x)}u+\big(\lambda V(x)+Z(x)\big)|u|^{p(x)-2}u=f(x,u), & \text{in } \Omega'_j,\\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \frac{\partial u}{\partial\eta}=0, & \text{on } \partial\Omega'_j.\end{cases}$$

It is fulfilled that I_j and $\phi_{\lambda,j}$ satisfy the mountain pass geometry and let

$$c_j=\inf_{\gamma\in\Gamma_j}\max_{t\in[0,1]}I_j\big(\gamma(t)\big)\ \text{ and }\ c_{\lambda,j}=\inf_{\gamma\in\Gamma_{\lambda,j}}\max_{t\in[0,1]}\phi_{\lambda,j}\big(\gamma(t)\big),$$

their respective mountain pass levels, where

$$\Gamma_j=\Big\{\gamma\in C\Big([0,1],W_0^{1,p(x)}\big(\Omega_j\big)\Big);\ \gamma(0)=0 \text{ and } I_j\big(\gamma(1)\big)<0\Big\}$$

and

$$\Gamma_{\lambda,j}=\Big\{\gamma\in C\Big([0,1],W^{1,p(x)}\big(\Omega'_j\big)\Big);\ \gamma(0)=0 \text{ and } \phi_{\lambda,j}\big(\gamma(1)\big)<0\Big\}.$$

Invoking the (PS) condition on I_j and $\phi_{\lambda,j}$, we ensure that there exist $w_j\in W_0^{1,p(x)}\big(\Omega_j\big)$ and $w_{\lambda,j}\in W^{1,p(x)}\big(\Omega'_j\big)$ such that

$$I_j\big(w_j\big)=c_j\ \text{ and }\ I'_j\big(w_j\big)=0$$

and

$$\phi_{\lambda,j}\big(w_{\lambda,j}\big)=c_{\lambda,j}\ \text{ and }\ \phi'_{\lambda,j}\big(w_{\lambda,j}\big)=0.$$

Lemma 6.1 *There holds that*

(i) $0<c_{\lambda,j}\leq c_j,\ \forall\lambda\geq 1,\ \forall j\in\{1,\ldots,k\}$;
(ii) $c_{\lambda,j}\to c_j$, *as* $\lambda\to\infty$, $\forall j\in\{1,\ldots,k\}$.

Proof (i) Once $W_0^{1,p(x)}(\Omega_j) \subset W^{1,p(x)}(\Omega'_j)$ and $\phi_{\lambda,j}(\gamma(1)) = I_j(\gamma(1))$ for $\gamma \in \Gamma_j$, we have $\Gamma_j \subset \Gamma_{\lambda,j}$. This way

$$c_{\lambda,j} = \inf_{\gamma\in\Gamma_{\lambda,j}} \max_{t\in[0,1]} \phi_{\lambda,j}(\gamma(t)) \le \inf_{\gamma\in\Gamma_j} \max_{t\in[0,1]} \phi_{\lambda,j}(\gamma(t)) = \inf_{\gamma\in\Gamma_j} \max_{t\in[0,1]} I_j(\gamma(t)) = c_j.$$

(ii) It suffices to show that $c_{\lambda_n,j} \to c_j$, as $n \to \infty$, for all sequences (λ_n) in $[1,\infty)$ with $\lambda_n \to \infty$, as $n \to \infty$. Let (λ_n) be such a sequence and consider an arbitrary subsequence of $(c_{\lambda_n,j})$ (not relabeled) . Let $w_n \in W^{1,p(x)}(\Omega'_j)$ with

$$\phi_{\lambda_n,j}(w_n) = c_{\lambda_n,j} \text{ and } \phi'_{\lambda_n,j}(w_n) = 0.$$

By the previous item, $(c_{\lambda_n,j})$ is bounded. Then, there exists (w_{n_k}) subsequence of (w_n) such that $\phi_{\lambda_{n_k},j}(w_{n_k})$ converges and $\phi'_{\lambda_{n_k},j}(w_{n_k}) = 0$. Now, repeating the same type of arguments explored in the proof of Proposition 4.1, there is $w \in W_0^{1,p(x)}(\Omega_j) \setminus \{0\} \subset W^{1,p(x)}(\Omega'_j)$ such that

$$w_{n_k} \to w \text{ in } W^{1,p(x)}(\Omega'_j), \text{ as } k \to \infty.$$

Furthermore, we also can prove that

$$c_{\lambda_{n_k},j} = \phi_{\lambda_{n_k},j}(w_{n_k}) \to I_j(w)$$

and

$$0 = \phi'_{\lambda_{n_k},j}(w_{n_k}) \to I'_j(w).$$

Then, by (f_4),

$$\lim_k c_{\lambda_{n_k},j} \ge c_j.$$

The last inequality together with item (i) implies

$$c_{\lambda_{n_k},j} \to c_j, \text{ as } k \to \infty.$$

This establishes the asserted result.

□

In the sequel, let $R > 1$ verifying

$$0 < I_j\left(\frac{1}{R} w_j\right), I_j(Rw_j) < c_j, \text{ for } j = 1, \dots, k. \tag{6.1}$$

There holds that

$$c_j = \max_{t\in[1/R^2,1]} I_j(tRw_j), \text{ for } j = 1, \dots, k.$$

Moreover, to simplify the notation, we rename the components Ω_j of Ω in way such that $\Upsilon = \{1, 2, \dots, l\}$ for some $1 \le l \le k$. Then, we define:

$$\gamma_0(t_1, \dots, t_l)(x) = \sum_{j=1}^{l} t_j R w_j(x), \ \forall (t_1, \dots, t_l) \in [1/R^2, 1]^l,$$

$$\Gamma_* = \left\{\gamma \in C\left([1/R^2, 1]^l, E_\lambda \setminus \{0\}\right); \ \gamma = \gamma_0 \text{ on } \partial[1/R^2, 1]^l\right\}$$

and

$$b_{\lambda,\Upsilon} = \inf_{\gamma\in\Gamma_*} \max_{(t_1,\dots,t_l)\in[1/R^2,1]^l} \phi_\lambda\big(\gamma(t_1,\dots,t_l)\big).$$

Next, our intention is proving that $b_{\lambda,\Upsilon}$ is a critical value for ϕ_λ. However, to do this, we need to some technical lemmas. The arguments used are the same found in [3]; however, for reader's convenience, we will repeat their proofs

Lemma 6.2 *For all $\gamma \in \Gamma_*$, there exists $(s_1,\dots,s_l) \in [1/R^2,1]^l$ such that*

$$\phi'_{\lambda,j}\big(\gamma(s_1,\dots,s_l)\big)\big(\gamma(s_1,\dots,s_l)\big) = 0,\ \forall j \in \Upsilon.$$

Proof Given $\gamma \in \Gamma_*$, consider $\widetilde{\gamma}\colon [1/R^2,1]^l \to \mathbb{R}^l$ such that

$$\widetilde{\gamma}(\mathbf{t}) = \Big(\phi'_{\lambda,1}\big(\gamma(\mathbf{t})\big)\gamma(\mathbf{t}),\dots,\phi'_{\lambda,l}\big(\gamma(\mathbf{t})\big)\gamma(\mathbf{t})\Big),\ \text{ where } \mathbf{t} = (t_1,\dots,t_l).$$

For $\mathbf{t} \in \partial[1/R^2,1]^l$, it holds $\widetilde{\gamma}(\mathbf{t}) = \widetilde{\gamma_0}(\mathbf{t})$. From this, we observe that there is no $\mathbf{t} \in \partial[1/R^2,1]^l$ with $\widetilde{\gamma}(\mathbf{t}) = 0$. Indeed, for any $j \in \Upsilon$,

$$\phi'_{\lambda,j}\big(\gamma_0(\mathbf{t})\big)\gamma_0(\mathbf{t}) = I'_j(t_jRw_j)(t_jRw_j).$$

This form, if $\mathbf{t} \in \partial[1/R^2,1]^l$, then $t_{j_0} = 1$ or $t_{j_0} = \frac{1}{R^2}$, for some $j_0 \in \Upsilon$. Consequently,

$$\phi'_{\lambda,j_0}\big(\gamma_0(\mathbf{t})\big)\gamma_0(\mathbf{t}) = I'_{j_0}(Rw_{j_0})(Rw_{j_0}) \text{ or } \phi'_{\lambda,j_0}\big(\gamma_0(\mathbf{t})\big)\gamma_0(\mathbf{t}) = I'_{j_0}\left(\frac{1}{R}w_{j_0}\right)\left(\frac{1}{R}w_{j_0}\right).$$

Therefore, if $\phi'_{\lambda,j_0}\big(\gamma_0(\mathbf{t})\big)\gamma_0(\mathbf{t}) = 0$, we get $I_{j_0}(Rw_{j_0}) \ge c_{j_0}$ or $I_{j_0}\left(\frac{1}{R}w_{j_0}\right) \ge c_{j_0}$, which is a contradiction with (6.1).

Now, we compute the degree $\deg\big(\widetilde{\gamma},(1/R^2,1)^l,(0,\dots,0)\big)$. Since

$$\deg\big(\widetilde{\gamma},(1/R^2,1)^l,(0,\dots,0)\big) = \deg\big(\widetilde{\gamma_0},(1/R^2,1)^l,(0,\dots,0)\big),$$

and, for $\mathbf{t} \in (1/R^2,1)^l$,

$$\widetilde{\gamma_0}(\mathbf{t}) = 0 \iff \mathbf{t} = \left(\frac{1}{R},\dots,\frac{1}{R}\right),$$

we derive

$$\deg\big(\widetilde{\gamma},(1/R^2,1)^l,(0,\dots,0)\big) \ne 0.$$

This shows what was stated. □

Proposition 6.3 *If $c_{\lambda,\Upsilon} = \sum_{j=1}^{l} c_{\lambda,j}$ and $c_\Upsilon = \sum_{j=1}^{l} c_j$, then*

(i) $c_{\lambda,\Upsilon} \le b_{\lambda,\Upsilon} \le c_\Upsilon$, $\forall\lambda \ge 1$;
(ii) $b_{\lambda,\Upsilon} \to c_\Upsilon$, *as* $\lambda \to \infty$;
(iii) $\phi_\lambda\big(\gamma(\mathbf{t})\big) < c_\Upsilon$, $\forall\lambda \ge 1, \gamma \in \Gamma_*$ *and* $\mathbf{t} = (t_1,\dots,t_l) \in \partial[1/R^2,1]^l$.

Proof (i) Once $\gamma_0 \in \Gamma_*$,

$$b_{\lambda,\Upsilon} \le \max_{(t_1,\dots,t_l)\in[1/R^2,1]^l} \phi_\lambda\big(\gamma_0(t_1,\dots,t_l)\big) = \max_{(t_1,\dots,t_l)\in[1/R^2,1]^l} \sum_{j=1}^{l} I_j(t_jRw_j) = c_\Upsilon.$$

Now, fixing $\mathbf{s} = (s_1, \ldots, s_l) \in [1/R^2, 1]^l$ given in Lemma 6.2 and recalling that

$$c_{\lambda,j} = \inf \left\{ \phi_{\lambda,j}(u)\,;\ u \in W^{1,p(x)}(\Omega'_j) \setminus \{0\} \text{ and } \phi'_{\lambda,j}(u)u = 0 \right\},$$

it follows that

$$\phi_{\lambda,j}\big(\gamma(\mathbf{s})\big) \geq c_{\lambda,j},\ \forall j \in \Upsilon.$$

From (3.9),

$$\phi_{\lambda,\mathbb{R}^N \setminus \Omega'_\Upsilon}(u) \geq 0,\ \forall u \in W^{1,p(x)}\big(\mathbb{R}^N \setminus \Omega'_\Upsilon\big),$$

which leads to

$$\phi_\lambda\big(\gamma(\mathbf{t})\big) \geq \sum_{j=1}^{l} \phi_{\lambda,j}\big(\gamma(\mathbf{t})\big),\ \forall \mathbf{t} = (t_1, \ldots, t_l) \in [1/R^2, 1]^l.$$

Thus

$$\max_{(t_1,\ldots,t_l)\in[1/R^2,1]^l} \phi_\lambda\big(\gamma(t_1, \ldots, t_l)\big) \geq \phi_\lambda\big(\gamma(\mathbf{s})\big) \geq c_{\lambda,\Upsilon},$$

showing that

$$b_{\lambda,\Upsilon} \geq c_{\lambda,\Upsilon};$$

(ii) This limit is clear by the previous item, since we already know $c_{\lambda,j} \to c_j$, as $\lambda \to \infty$;

(iii) For $\mathbf{t} = (t_1, \ldots, t_l) \in \partial[1/R^2, 1]^l$, it holds $\gamma(\mathbf{t}) = \gamma_0(\mathbf{t})$. From this,

$$\phi_\lambda\big(\gamma(\mathbf{t})\big) = \sum_{j=1}^{l} I_j(t_j R w_j).$$

Writing

$$\phi_\lambda\big(\gamma(\mathbf{t})\big) = \sum_{\substack{j=1\\ j\neq j_0}}^{l} I_j(t_j R w_j) + I_{j_0}(t_{j_0} R w_{j_0}),$$

where $t_{j_0} \in \left\{\frac{1}{R^2}, 1\right\}$, from (6.1) we derive

$$\phi_\lambda\big(\gamma(\mathbf{t})\big) \leq c_\Upsilon - \epsilon,$$

for some $\epsilon > 0$, so (iii).

□

Corollary 6.4 *$b_{\lambda,\Upsilon}$ is a critical value of ϕ_λ, for λ sufficiently large.*

Proof Assume $b_{\widetilde{\lambda},\Upsilon}$ is not a critical value of $\phi_{\widetilde{\lambda}}$ for some $\widetilde{\lambda}$. We will prove that exists λ_1 such that $\widetilde{\lambda} < \lambda_1$. Indeed, by item (iii) of Proposition 6.3, we have seen that

$$\phi_\lambda\big(\gamma_0(\mathbf{t})\big) < c_\Upsilon,\ \forall \lambda \geq 1,\ \mathbf{t} \in \partial[1/R^2, 1]^l.$$

This way

$$\mathcal{M} = \max_{\mathbf{t}\in\partial[1/R^2,1]^l} \phi_{\widetilde{\lambda}}\big(\gamma_0(\mathbf{t})\big) < c_\Upsilon.$$

Since $b_{\lambda,\Upsilon} \to c_\Upsilon$ (item (ii) of Proposition 6.3), there exists $\lambda_1 > 1$ such that if $\lambda \geq \lambda_1$, then

$$\mathcal{M} < b_{\lambda,\Upsilon}.$$

So, if $\widetilde{\lambda} \geq \lambda_1$, we can find $\tau = \tau(\widetilde{\lambda}) > 0$ small enough, with the ensuing property

$$\mathcal{M} < b_{\widetilde{\lambda},\Upsilon} - 2\tau. \tag{6.2}$$

From the deformation's lemma [31, Page 38], there is $\eta \colon E_\lambda \to E_\lambda$ such that

$$\eta\left(\phi_{\widetilde{\lambda}}^{b_{\widetilde{\lambda},\Upsilon}+\tau}\right) \subset \phi_{\widetilde{\lambda}}^{b_{\widetilde{\lambda},\Upsilon}-\tau} \text{ and } \eta(u) = u, \text{ for } u \notin \phi_{\widetilde{\lambda}}^{-1}\big([b_{\widetilde{\lambda},\Upsilon} - 2\tau, b_{\widetilde{\lambda},\Upsilon} + 2\tau]\big).$$

Then, by (6.2),

$$\eta\big(\gamma_0(\mathbf{t})\big) = \gamma_0(\mathbf{t}), \ \forall \mathbf{t} \in \partial[1/R^2, 1]^l.$$

Now, using the definition of $b_{\widetilde{\lambda},\Upsilon}$, there exists $\gamma_* \in \Gamma_*$ satisfying

$$\max_{\mathbf{t}\in[1/R^2,1]^l} \phi_{\widetilde{\lambda}}\big(\gamma_*(\mathbf{t})\big) < b_{\widetilde{\lambda},\Upsilon} + \tau. \tag{6.3}$$

Defining

$$\widetilde{\gamma}(\mathbf{t}) = \eta\big(\gamma_*(\mathbf{t})\big), \ \mathbf{t} \in [1/R^2, 1]^l,$$

due to (6.3), we obtain

$$\phi_{\widetilde{\lambda}}\big(\widetilde{\gamma}(\mathbf{t})\big) \leq b_{\widetilde{\lambda},\Upsilon} - \tau, \ \forall \mathbf{t} \in [1/R^2, 1]^l.$$

But since $\widetilde{\gamma} \in \Gamma_*$, we deduce

$$b_{\widetilde{\lambda},\Upsilon} \leq \max_{\mathbf{t}\in[1/R^2,1]^l} \phi_{\widetilde{\lambda}}\big(\widetilde{\gamma}(\mathbf{t})\big) \leq b_{\widetilde{\lambda},\Upsilon} - \tau,$$

a contradiction. So, $\widetilde{\lambda} < \lambda_1$. □

7 The proof of the main theorem

To prove Theorem 1.1, we need to find nonnegative solutions u_λ for large values of λ, which converges to a least energy solution in each Ω_j $(j \in \Upsilon)$ and to 0 in Ω_Υ^c as $\lambda \to \infty$. To this end, we will show two propositions which together with the Propositions 4.1 and 5.1 will imply that Theorem 1.1 holds.

Henceforth, we denote by

$$r = R^{p_+} \sum_{j=1}^{l} \left(\frac{1}{p_+} - \frac{1}{\theta}\right)^{-1} c_j, \quad \mathcal{B}_r^\lambda = \big\{u \in E_\lambda \,;\, \varrho_\lambda(u) \leq r\big\}$$

and

$$\phi_\lambda^{c_\Upsilon} = \big\{u \in E_\lambda \,;\, \phi_\lambda(u) \leq c_\Upsilon\big\}.$$

Moreover, for small values of μ,

$$\mathcal{A}_\mu^\lambda = \big\{u \in \mathcal{B}_r^\lambda \,;\, \varrho_{\lambda,\mathbb{R}^N\setminus\Omega_\Upsilon}(u) \leq \mu, \ \big|\phi_{\lambda,j}(u) - c_j\big| \leq \mu, \ \forall j \in \Upsilon\big\}.$$

We observe that

$$w = \sum_{j=1}^{l} w_j \in \mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon},$$

showing that $\mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon} \neq \emptyset$. Fixing

$$0 < \mu < \frac{1}{4} \min_{j \in \Gamma} c_j, \tag{7.1}$$

we have the following uniform estimate of $\|\phi_\lambda'(u)\|$ on the region $\left(\mathcal{A}_{2\mu}^\lambda \setminus \mathcal{A}_\mu^\lambda\right) \cap \phi_\lambda^{c_\Upsilon}$.

Proposition 7.1 *Let $\mu > 0$ satisfying (7.1). Then, there exist $\Lambda_* \geq 1$ and $\sigma_0 > 0$ independent of λ such that*

$$\|\phi_\lambda'(u)\| \geq \sigma_0, \text{ for } \lambda \geq \Lambda_* \quad \text{and all} \quad u \in \left(\mathcal{A}_{2\mu}^\lambda \setminus \mathcal{A}_\mu^\lambda\right) \cap \phi_\lambda^{c_\Upsilon}. \tag{7.2}$$

Proof We assume that there exist $\lambda_n \to \infty$ and $u_n \in \left(\mathcal{A}_{2\mu}^{\lambda_n} \setminus \mathcal{A}_\mu^{\lambda_n}\right) \cap \phi_{\lambda_n}^{c_\Upsilon}$ such that

$$\|\phi_{\lambda_n}'(u_n)\| \to 0.$$

Since $u_n \in \mathcal{A}_{2\mu}^{\lambda_n}$, this implies $\left(\varrho_{\lambda_n}(u_n)\right)$ is a bounded sequence and, consequently, it follows that $\left(\phi_{\lambda_n}(u_n)\right)$ is also bounded. Thus, passing a subsequence if necessary, we can assume $\phi_{\lambda_n}(u_n)$ converges. Thus, from Proposition 4.1, there exists $0 \leq u \in W_0^{1,p(x)}\left(\Omega_\Upsilon\right)$ such that $u_{|\Omega_j}$, $j \in \Upsilon$, is a solution for (P_j),

$$\varrho_{\lambda_n, \mathbb{R}^N \setminus \Omega_\Upsilon}(u_n) \to 0 \quad \text{and} \quad \phi_{\lambda_n, j}(u_n) \to I_j(u).$$

We know that c_j is the least energy level for I_j. So, if $u_{|\Omega_j} \neq 0$, then $I_j(u) \geq c_j$. But since $\phi_{\lambda_n}(u_n) \leq c_\Upsilon$, we must analyze the following possibilities:

(i) $I_j(u) = c_j$, $\forall j \in \Upsilon$;
(ii) $I_{j_0}(u) = 0$, for some $j_o \in \Upsilon$.

If (i) occurs, then for n large, it holds

$$\varrho_{\lambda_n, \mathbb{R}^N \setminus \Omega_\Upsilon}(u_n) \leq \mu \ \text{ and } \ \left|\phi_{\lambda_n, j}(u_n) - c_j\right| \leq \mu, \ \forall j \in \Upsilon.$$

So $u_n \in \mathcal{A}_\mu^{\lambda_n}$, a contradiction.
If (ii) occurs, then

$$\left|\phi_{\lambda_n, j_0}(u_n) - c_{j_0}\right| \to c_{j_0} > 4\mu,$$

which is a contradiction with the fact that $u_n \in \mathcal{A}_{2\mu}^{\lambda_n}$. Thus, we have completed the proof. □

Proposition 7.2 *Let $\mu > 0$ satisfying (7.1) and $\Lambda_* \geq 1$ given in the previous proposition. Then, for $\lambda \geq \Lambda_*$, there exists a solution u_λ of (A_λ) such that $u_\lambda \in \mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon}$.*

Proof Let $\lambda \geq \Lambda_*$. Assume that there are no critical points of ϕ_λ in $\mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon}$. Since ϕ_λ is a (PS) functional, there exists a constant $d_\lambda > 0$ such that

$$\|\phi_\lambda'(u)\| \geq d_\lambda, \ \text{ for all } u \in \mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon}.$$

From Proposition 7.1, we have

$$\left\|\phi_\lambda'(u)\right\| \geq \sigma_0, \text{ for all } u \in \left(\mathcal{A}_{2\mu}^\lambda \setminus \mathcal{A}_\mu^\lambda\right) \cap \phi_\lambda^{c_\Upsilon},$$

where $\sigma_0 > 0$ does not depend on λ. In what follows, $\Psi\colon E_\lambda \to \mathbb{R}$ is a continuous functional verifying

$$\Psi(u) = 1, \text{ for } u \in \mathcal{A}_{\frac{3}{2}\mu}^\lambda, \ \Psi(u) = 0, \text{ for } u \notin \mathcal{A}_{2\mu}^\lambda \text{ and } 0 \leq \Psi(u) \leq 1, \ \forall u \in E_\lambda.$$

We also consider $H\colon \phi_\lambda^{c_\Upsilon} \to E_\lambda$ given by

$$H(u) = \begin{cases} -\Psi(u)\left\|Y(u)\right\|^{-1} Y(u), & \text{for } u \in \mathcal{A}_{2\mu}^\lambda, \\ 0, & \text{for } u \notin \mathcal{A}_{2\mu}^\lambda, \end{cases}$$

where Y is a pseudo-gradient vector field for Φ_λ on $\mathcal{K} = \left\{u \in E_\lambda\,;\, \phi_\lambda'(u) \neq 0\right\}$. Observe that H is well defined, once $\phi_\lambda'(u) \neq 0$, for $u \in \mathcal{A}_{2\mu}^\lambda \cap \phi_\lambda^{c_\Upsilon}$. The inequality

$$\left\|H(u)\right\| \leq 1, \ \forall \lambda \geq \Lambda_* \quad \text{and} \quad u \in \phi_\lambda^{c_\Upsilon},$$

guarantees that the deformation flow $\eta\colon [0, \infty) \times \phi_\lambda^{c_\Upsilon} \to \phi_\lambda^{c_\Upsilon}$ defined by

$$\frac{d\eta}{dt} = H(\eta), \ \eta(0, u) = u \in \phi_\lambda^{c_\Upsilon}$$

verifies

$$\frac{d}{dt}\phi_\lambda\big(\eta(t,u)\big) \leq -\frac{1}{2}\Psi\big(\eta(t,u)\big)\left\|\phi_\lambda'\big(\eta(t,u)\big)\right\| \leq 0, \tag{7.3}$$

$$\left\|\frac{d\eta}{dt}\right\|_\lambda = \left\|H(\eta)\right\|_\lambda \leq 1 \tag{7.4}$$

and

$$\eta(t,u) = u \text{ for all } t \geq 0 \text{ and } u \in \phi_\lambda^{c_\Upsilon} \setminus \mathcal{A}_{2\mu}^\lambda. \tag{7.5}$$

We study now two paths, which are relevant for what follows:

- The path $\mathbf{t} \mapsto \eta\big(t, \gamma_0(\mathbf{t})\big)$, where $\mathbf{t} = (t_1, \ldots, t_l) \in [1/R^2, 1]^l$.
  The definition of γ_0 combined with the condition on μ gives

$$\gamma_0(\mathbf{t}) \notin \mathcal{A}_{2\mu}^\lambda, \ \forall \mathbf{t} \in \partial[1/R^2, 1]^l.$$

Since

$$\phi_\lambda\big(\gamma_0(\mathbf{t})\big) < c_\Upsilon, \ \forall \mathbf{t} \in \partial[1/R^2, 1]^l,$$

from (7.5), it follows that

$$\eta\big(t, \gamma_0(\mathbf{t})\big) = \gamma_0(\mathbf{t}), \ \forall \mathbf{t} \in \partial[1/R^2, 1]^l.$$

So, $\eta\big(t, \gamma_0(\mathbf{t})\big) \in \Gamma_*$, for each $t \geq 0$.

- The path $\mathbf{t} \mapsto \gamma_0(\mathbf{t})$, where $\mathbf{t} = (t_1, \ldots, t_l) \in [1/R^2, 1]^l$.
  We observe that

$$\operatorname{supp}\big(\gamma_0(\mathbf{t})\big) \subset \overline{\Omega_\Upsilon}$$

and

$$\phi_\lambda\big(\gamma_0(\mathbf{t})\big) \text{ does not depend on } \lambda \geq 1,$$

forall $\mathbf{t} \in [1/R^2, 1]^l$. Moreover,

$$\phi_\lambda\big(\gamma_0(\mathbf{t})\big) \le c_\Upsilon, \ \forall \mathbf{t} \in [1/R^2, 1]^l$$

and

$$\phi_\lambda\big(\gamma_0(\mathbf{t})\big) = c_\Upsilon \text{if,} \quad \text{and} \quad \text{only if,} \ \ t_j = \frac{1}{R}, \ \forall j \in \Upsilon.$$

Therefore

$$m_0 = \sup\Big\{\phi_\lambda(u)\,;\ u \in \gamma_0\big([1/R^2, 1]^l\big) \setminus A^\lambda_\mu\Big\}$$

is independent of λ and $m_0 < c_\Upsilon$. Now, observing that there exists $K_* > 0$ such that

$$\big|\phi_{\lambda,j}(u) - \phi_{\lambda,j}(v)\big| \le K_*\|u - v\|_{\lambda,\Omega'_j}, \ \forall u, v \in \mathcal{B}^\lambda_r \text{ and } \forall j \in \Upsilon,$$

we derive

$$\max_{\mathbf{t}\in[1/R^2,1]^l} \phi_\lambda\Big(\eta\big(T, \gamma_0(\mathbf{t})\big)\Big) \le \max\left\{m_0, c_\Upsilon - \frac{1}{2K_*}\sigma_0\mu\right\}, \tag{7.6}$$

for $T > 0$ large.

In fact, writing $u = \gamma_0(\mathbf{t})$, $\mathbf{t} \in [1/R^2, 1]^l$, if $u \notin A^\lambda_\mu$, from (7.3),

$$\phi_\lambda\big(\eta(t, u)\big) \le \phi_\lambda(u) \le m_0, \ \forall t \ge 0,$$

and we have nothing more to do. We assume then $u \in A^\lambda_\mu$ and set

$$\widetilde{\eta}(t) = \eta(t, u), \ \widetilde{d_\lambda} = \min\{d_\lambda, \sigma_0\} \ \text{ and } T = \frac{\sigma_0\mu}{K_*\widetilde{d_\lambda}}.$$

Now, we will analyze the ensuing cases:

Case 1: $\widetilde{\eta}(t) \in \mathcal{A}^\lambda_{\frac{3}{2}\mu}$, $\forall t \in [0, T]$.
Case 2: $\widetilde{\eta}(t_0) \in \partial\mathcal{A}^\lambda_{\frac{3}{2}\mu}$, for some $t_0 \in [0, T]$.

Analysis of Case 1

In this case, we have $\Psi\big(\widetilde{\eta}(t)\big) = 1$ and $\big\|\phi'_\lambda\big(\widetilde{\eta}(t)\big)\big\| \ge \widetilde{d_\lambda}$ for all $t \in [0, T]$. Hence, from (7.3),

$$\phi_\lambda\big(\widetilde{\eta}(T)\big) = \phi_\lambda(u) + \int_0^T \frac{\mathrm{d}}{\mathrm{d}s}\phi_\lambda\big(\widetilde{\eta}(s)\big)\,\mathrm{d}s \le c_\Upsilon - \frac{1}{2}\int_0^T \widetilde{d_\lambda}\,\mathrm{d}s,$$

that is,

$$\phi_\lambda\big(\widetilde{\eta}(T)\big) \le c_\Upsilon - \frac{1}{2}\widetilde{d_\lambda}T = c_\Upsilon - \frac{1}{2K_*}\sigma_0\mu,$$

showing (7.6).

Analysis of Case 2

In this case, there exist $0 \le t_1 \le t_2 \le T$ satisfying

$$\widetilde{\eta}(t_1) \in \partial\mathcal{A}^\lambda_\mu,$$
$$\widetilde{\eta}(t_2) \in \partial\mathcal{A}^\lambda_{\frac{3}{2}\mu},$$

and

$$\widetilde{\eta}(t) \in \mathcal{A}^{\lambda}_{\frac{3}{2}\mu} \setminus \mathcal{A}^{\lambda}_{\mu}, \ \forall t \in (t_1, t_2].$$

We claim that

$$\left\| \widetilde{\eta}(t_2) - \widetilde{\eta}(t_1) \right\| \geq \frac{1}{2K_*}\mu.$$

Setting $w_1 = \widetilde{\eta}(t_1)$ and $w_2 = \widetilde{\eta}(t_2)$, we get

$$\varrho_{\lambda, \mathbb{R}^N \setminus \Omega_\Upsilon}(w_2) = \frac{3}{2}\mu \ \text{ or } \left| \phi_{\lambda, j_0}(w_2) - c_{j_0} \right| = \frac{3}{2}\mu,$$

for some $j_0 \in \Upsilon$. We analyze the latter situation, once that the other one follows the same reasoning. From the definition of $\mathcal{A}^{\lambda}_{\mu}$,

$$\left| \phi_{\lambda, j_0}(w_1) - c_{j_0} \right| \leq \mu,$$

consequently,

$$\| w_2 - w_1 \| \geq \frac{1}{K_*} \left| \phi_{\lambda, j_0}(w_2) - \phi_{\lambda, j_0}(w_1) \right| \geq \frac{1}{2K_*}\mu.$$

Then, by mean value theorem, $t_2 - t_1 \geq \frac{1}{2K_*}\mu$ and, this form,

$$\phi_\lambda\left(\widetilde{\eta}(T)\right) \leq \phi_\lambda(u) - \int_0^T \Psi\left(\widetilde{\eta}(s)\right) \left\| \phi'_\lambda\left(\widetilde{\eta}(s)\right) \right\| \mathrm{d}s$$

implying

$$\phi_\lambda\left(\widetilde{\eta}(T)\right) \leq c_\Upsilon - \int_{t_1}^{t_2} \sigma_0 \, \mathrm{d}s = c_\Upsilon - \sigma_0(t_2 - t_1) \leq c_\Upsilon - \frac{1}{2K_*}\sigma_0\mu,$$

which proves 7.6. Fixing $\widehat{\eta}(t_1, \ldots, t_l) = \eta\left(T, \gamma_0(t_1, \ldots, t_l)\right)$, we have that $\widehat{\eta} \in \Gamma_*$ and, hence,

$$b_{\lambda,\Gamma} \leq \max_{(t_1,\ldots,t_l) \in [1/R^2, 1]} \phi_\lambda\left(\widehat{\eta}(t_1, \ldots, t_l)\right) \leq \max\left\{ m_0, c_\Upsilon - \frac{1}{2K_*}\sigma_0\mu \right\} < c_\Upsilon,$$

which contradicts the fact that $b_{\lambda,\Upsilon} \to c_\Upsilon$. □

Proof of Theorem 1.1 According Proposition 7.2, for μ satisfying (7.1) and $\Lambda_* \geq 1$, there exists a solution u_λ for (A_λ) such that $u_\lambda \in \mathcal{A}^{\lambda}_{\mu} \cap \phi_\lambda^{c_\Upsilon}$, for all $\lambda \geq \Lambda_*$.

Claim: There are $\lambda_0 \geq \Lambda_*$ and $\mu_0 > 0$ small enough, such that u_λ is a solution for $\left(P_\lambda\right)$ for $\lambda \geq \Lambda_0$ and $\mu \in (0, \mu_0)$.

Indeed, assume by contradiction that there are $\lambda_n \to \infty$ and $\mu_n \to 0$, such that (u_{λ_n}) is not a solution for (P_{λ_n}). From Proposition 7.2, the sequence (u_{λ_n}) verifies:

(a) $\phi'_{\lambda_n}(u_{\lambda_n}) = 0, \ \forall n \in \mathbb{N}$;
(b) $\varrho_{\lambda_n, \mathbb{R}^N \setminus \Omega_\Upsilon}(u_{\lambda_n}) \to 0$;
(c) $\phi_{\lambda_n, j}(u_{\lambda_n}) \to c_j, \ \forall j \in \Upsilon$.

The item (b) ensures we can use Proposition 5.1 to deduce u_{λ_n} is a solution for $\left(P_{\lambda_n}\right)$, for large values of n, which is a contradiction, showing this way the claim.

Now, our goal is to prove the second part of the theorem. To this end, let (u_{λ_n}) be a sequence verifying the above limits. Since $\phi_{\lambda_n}(u_{\lambda_n})$ is bounded, passing a subsequence, we obtain that $\phi_{\lambda_n}(u_{\lambda_n}) \to c$. This way, using Proposition 4.1 combined with item (c), we derive u_{λ_n} converges in $W^{1,p(x)}(\mathbb{R}^N)$ to a function $u \in W^{1,p(x)}(\mathbb{R}^N)$, which satisfies $u = 0$ outside Ω_Υ and $u_{|\Omega_j}$, $j \in \Upsilon$, is a least energy solution for

$$\begin{cases} -\Delta_{p(x)} u + Z(x)u = f(u), & \text{in } \Omega_j, \\ u \in W_0^{1,p(x)}(\Omega_j), \ u \geq 0, & \text{in } \Omega_j. \end{cases}$$

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