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# Class sizes of prime-power order $p'$-elements and normal subgroups

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**Abstract** We prove an extension of the renowned Itô's theorem on groups having two class sizes in three different directions at the same time: normal subgroups, $p'$-elements and prime-power order elements. Let $N$ be a normal subgroup of a finite group $G$ and let $p$ be a fixed prime. Suppose that $|x^G| = 1$ or $m$ for every $q$-element of $N$ and for every prime $q \neq p$. Then, $N$ has nilpotent $p$-complements.

**Keywords** Finite groups · Conjugacy class sizes · Normal subgroups · Prime-power order elements · $p'$-Elements

**Mathematics Subject Classification (2010)** 20E45 · 20D15

## 1 Introduction

A classic problem in Group Theory is the study of the influence of the conjugacy class sizes on the structure of finite groups. However, studying such properties only from partial information, provided by certain class sizes, can be a more complex problem. Several results have recently shown how the class sizes of certain subsets of elements, such as those lying in a normal subgroup, or the $p'$-elements for some prime $p$, or the prime-power order elements, continue to exert a strong control on the normal structure, the $p$-structure or even the whole

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structure of the group. We must note that some results are quite elementary when all class sizes are considered, whereas the corresponding results which are obtained just from the subsets cited above may need deeper results or even the classification of the finite simple groups (CFSG).

Regarding class sizes of elements of a normal subgroup $N$, the main theorem of [2] establishes the nilpotency of $N$ when it has exactly two $G$-class sizes. The nilpotency of $N$ also holds when restricting to prime-power order elements [4]. On the other hand, when only $p'$-elements are considered, the nilpotency of the $p$-complements of a group or a normal subgroup is also preserved (see [1,3,6]). Our research goes further with a generalization in three different directions. Precisely, we only consider the $p'$-part of the class sizes of prime-power order $p'$-elements which are non-central in $N$.

**Theorem** *Let $N$ be a normal subgroup of a finite group $G$ and $p$ a fixed prime. Suppose that $|x^G|_{p'} = m$ for every $q$-element in $N \setminus \mathbf{Z}(N)$ and for every prime $q \neq p$. Then $N$ has nilpotent $p$-complements.*

In particular, the hypotheses of the Theorem imply that if all $G$-class sizes of the $q$-elements ($q \neq p$) of the normal subgroups $N$ are either 1 or a fixed number $m$, then $N$ has normal $p$-complements. From this result, we can recover all the main theorems that we have referenced above as particular cases.

The techniques employed in the mentioned papers are different among them. However, our approach appeals in a novel way to the prime graph of a finite group and its independence numbers in order to analyze certain class size properties of non-abelian simple groups and thus, to achieve the solvability of $N$ within a more general context. On the other hand, we remark that a "dual" problem of our theorem for conjugacy classes in the whole group has been considered by Casolo et al. [7]. They prove that when all non-trivial class sizes have the same $p$-part, then the group is solvable and has normal $p$-complement, and so, obtain a partial analog for conjugacy classes of the well-known Thompson's theorem on character degrees.

All groups are supposed to be finite. If $G$ is a group, then $\pi(G)$ denotes the set of prime divisors of $|G|$, and similarly, if $n$ is an integer, $\pi(n)$ will denote the set of prime divisors of $n$. If $p$ is a prime number, we use the notation $n_p$ for the $p$-part of $n$.

## 2 Preliminaries

Before taking up the problem, we present here some useful results which will be used in the sequel. First, we prove the following lemma.

**Lemma 2.1** *Let $G$ be a finite group and $p, q$ two fixed primes. If $|x^G|$ is a $\{p, q\}$-number for every prime-power order $p'$-element of $G$, then $G$ is not a non-abelian simple group.*

*Proof* Suppose that $G$ is such a group. Let $Q \in \mathrm{Syl}_q(G)$ and $1 \neq x \in \mathbf{Z}(Q)$. Then $q$ does not divide $|x^G|$ and hence $|x^G|$ is a $p$-power, so $G$ cannot be non-abelian simple by Burnside's Theorem (see 15.2 of [8]). □

A repeatedly used result is Thompson's $P \times Q$-Lemma.

**Lemma 2.2** *Let $P \times Q$ be the direct product of a $p$-group $P$ and a $p'$-group $Q$. Suppose that $P \times Q$ acts on a $p$-group $G$ such that $\mathbf{C}_G(P) \leq \mathbf{C}_G(Q)$. Then $Q$ acts trivially on $G$.*

*Proof* For instance, see 8.2.8 of [10]. □

In order to prove the solvability of $N$ in the main theorem, we will also need the following result which uses coprime action.

**Theorem 2.3** *Let $G$ be a group and $N$ a non-solvable normal subgroup of $G$. If $m$ divides $|x^G|_{p'}$ for every $q$-element $x \in N \setminus \mathbf{Z}(N)$ with $q \neq p$, then $m$ divides the order of $\mathbf{Z}(N)$.*

*Proof* First, we claim that we can assume $\pi(N) \setminus \{p\} = \pi(N/\mathbf{Z}(N)) \setminus \{p\}$. If this does not happen, then $N$ can be factorized as a direct product $N = N_1 \times Q$, with $Q$ a central Sylow $q$-subgroup of $N$, $q \neq p$, and $N_1$ normal in $G$. Then, if $x \in N_1 \setminus \mathbf{Z}(N_1)$, it follows that $x \in N \setminus \mathbf{Z}(N)$ and since $N_1$ is neither solvable, we can apply induction to get that $m$ divides $|\mathbf{Z}(N_1)|$, which clearly divides $|\mathbf{Z}(N)|$, so the theorem is proved.

Now we prove that $\pi(m) \subseteq \pi(N)$. Suppose that $r \in \pi(m) \setminus \pi(N)$. Take $R \in \mathrm{Syl}_r(G)$ and assume that there exists a $q$-element $x \in \mathbf{C}_N(R) \setminus \mathbf{Z}(N)$ with $q \neq p$. Therefore, $r$ is not divisor of $|x^G|$ and thus $|x^G| = 1$. This implies that $x \in \mathbf{Z}(G)$ and in particular $x \in \mathbf{Z}(N)$, a contradiction. So we get $\mathbf{C}_N(R) = P \times L$, where $P$ is a Sylow $p$-subgroup of $\mathbf{C}_N(R)$ and $L \leq \mathbf{Z}(N)$. In particular, $\mathbf{C}_N(R)$ is nilpotent. As $R$ acts coprimely on $N$, by Theorem B of [5], we conclude that $N$ is solvable, a contradiction.

Let $Q \in \mathrm{Syl}_q(G)$ with $q \in \pi(m)$. For every $x \in (Q \cap N) \setminus \mathbf{Z}(N)$, there exists $y \in G$ such that $\mathbf{C}_{Q^y}(x) \in \mathrm{Syl}_q(\mathbf{C}_G(x))$. Moreover, there is $z \in \mathbf{C}_G(x)$ such that $\mathbf{C}_Q(x) \leq (\mathbf{C}_{Q^y}(x))^z = \mathbf{C}_{Q^{yz}}(x)$. Hence, $m_q$ divides $|x^G|_q = |Q^{yz} : \mathbf{C}_{Q^{yz}}(x)|$ which divides $|x^Q|$. Note that $N_q := Q \cap N \in \mathrm{Syl}_q(N)$ and $N_q \trianglelefteq Q$. Therefore, from the class equation in $Q$, we obtain $|N_q| = |N_q \cap \mathbf{Z}(N)| + m_q l$ for some positive integer $l$. As $q \in \pi(N)$, we deduce $q$ divides $|N_q \cap \mathbf{Z}(N)|$. We can reformulate the above equation as

$$\left|\frac{N_q}{N_q \cap \mathbf{Z}(N)}\right| = 1 + \frac{m_q t}{|N_q \cap \mathbf{Z}(N)|}.$$

Since the first member of the equation is a non-trivial $q$-power by the first paragraph, we conclude that $m_q$ divides $|N_q \cap \mathbf{Z}(N)|$, and so, it divides $|\mathbf{Z}(N)|$ for every prime $q \in \pi(m)$. Thus, $m$ divides $|\mathbf{Z}(N)|$. □

## 3 Proof

*Proof of Theorem* We show first that $N$ is solvable and we argue by minimal counterexample. Let $N$ be a counterexample of minimal order. By Burnside's $p^a q^b$ theorem, we may assume that $|\pi(N)| \geq 3$. Let $N/K$ be a chief factor of $G$. If $x \in K \setminus \mathbf{Z}(K)$ is a $q$-element, with $q \neq p$, then $x \in N \setminus \mathbf{Z}(N)$ and $|x^G|_{p'} = m$. By minimality, we have that $K$ is solvable.

**Step 1.** We may assume that $\mathbf{O}_p(N) = 1$.

Otherwise, let $\overline{G} := G/\mathbf{O}_p(N)$ and $\overline{x} \in \overline{N} \setminus \mathbf{Z}(\overline{N})$ a $p'$-element of prime-power order. Then, $x \in N \setminus \mathbf{Z}(N)$ and certainly $x$ can be assumed to be a $q$-element for some prime $q \neq p$. We show that $\mathbf{C}_{\overline{G}}(\overline{x}) = \overline{\mathbf{C}_G(x)}$. Let $Q \in \mathrm{Syl}_q(\mathbf{C}_G(x))$ and let $\overline{y} \in \mathbf{C}_{\overline{G}}(\overline{x})$. Then $[x, y] \in \mathbf{O}_p(N)$, so we can write $x^y = xa \in \mathbf{C}_G(x)\mathbf{O}_p(N)$ with $a \in \mathbf{O}_p(N)$. As $x^y$ is a $q$-element, there exists $t \in \mathbf{O}_p(N)\mathbf{C}_G(x)$ such that $x^{yt} \in Q$. Hence, $[x, yt] = x^{-1}x^{yt} \in Q$. On the other hand, $[x, yt] = [x, t][x, y]^t$. Since $t \in \mathbf{O}_p(N)\mathbf{C}_G(x)$, we may write $t = t'u$ with $t' \in \mathbf{O}_p(N)$ and $u \in \mathbf{C}_G(x)$. Then, $[x, t] = [x, t'u] = [x, t']^u$ and thus, $[x, yt] = [x, t']^u[x, y]^t$. As $[x, t']^u$ and $[x, y]^t$ lie in $\mathbf{O}_p(N)$, we get that $[x, yt] \in \mathbf{O}_p(N)$. Therefore,

$[x, yt] \in \mathbf{O}_p(N) \cap Q = 1$. This means that $yt \in \mathbf{C}_G(x)$, so $yt' \in \mathbf{C}_G(x)$. This implies that $\overline{y} \in \overline{\mathbf{C}_G(x)}$, and we conclude that $\mathbf{C}_{\overline{G}}(\overline{x}) = \overline{\mathbf{C}_G(x)}$, as wanted. Now, as

$$|\overline{x}^{\overline{G}}| = |\overline{G} : \mathbf{C}_{\overline{G}}(\overline{x})| = |G : \mathbf{C}_G(x)\mathbf{O}_p(N)| = |x^G| \frac{|\mathbf{O}_p(N) \cap \mathbf{C}_G(x)|}{|\mathbf{O}_p(G)|}$$

we obtain $|\overline{x}^{\overline{G}}|_{p'} = |x^G|_{p'}$ for every $p'$-element of prime-power order $\overline{x} \in \overline{N} \setminus \mathbf{Z}(\overline{N})$. By minimal counterexample, we deduce that $N/\mathbf{O}_p(N)$ is solvable and so is $N$, a contradiction.

**Step 2.** $\mathbf{F}(N) = \mathbf{Z}(N) = K$.

Suppose that $\mathbf{Z}(N)_r < \mathbf{O}_r(N)$ for some $r \in \pi(N)$ with $r \neq p$. Let $x \in N \setminus \mathbf{Z}(N)$ be an $s$-element and $R \in \mathrm{Syl}_r(\mathbf{C}_G(x))$, where $s \in \pi(N) \setminus \{p, r\}$. Let us consider the action of $R \times \langle x \rangle$ on $\mathbf{O}_r(N)$ and we claim that $\mathbf{C}_{\mathbf{O}_r(N)}(R) \subseteq \mathbf{C}_{\mathbf{O}_r(N)}(x)$. For every $v \in \mathbf{C}_{\mathbf{O}_r(N)}(R)$ we have: if $v \in \mathbf{Z}(N)$, then $v \in \mathbf{C}_{\mathbf{O}_r(N)}(x)$; if $v \notin \mathbf{Z}(N)$, then $\langle R, v \rangle \subseteq \mathbf{C}_G(v)$. Since $|v^G|_{p'} = |x^G|_{p'}$, it follows that $|R| = |\langle R, v \rangle|$ and thus $v \in R$. This shows that $v \in \mathbf{C}_G(x)$ and then $\mathbf{C}_{\mathbf{O}_r(N)}(R) \subseteq \mathbf{C}_{\mathbf{O}_r(N)}(x)$ as claimed. By applying Lemma 2.2, it follows that $x \in \mathbf{C}_N(\mathbf{O}_r(N))$. So we conclude that $|N/\mathbf{C}_N(\mathbf{O}_r(N))|$ is a $\{p, r\}$-number and $N/\mathbf{C}_N(\mathbf{O}_r(N))$ is solvable. Now, let $M := \mathbf{C}_N(\mathbf{O}_r(N))$. For every $t$-element $u \in M \setminus \mathbf{Z}(M)$, with $t \neq p$, we trivially have $u \in N \setminus \mathbf{Z}(N)$, and by hypothesis we also have $|u^G|_{p'} = m$. Moreover, notice that $M < N$, whence $M$ is solvable by minimality of $N$. This forces $N$ to be solvable too, a contradiction.

By using Step 1, we have just proved that $\mathbf{F}(N) = \mathbf{Z}(N)$. On the other hand, it is easy to see that $\mathbf{F}(N) = \mathbf{F}(K)$. Then, we have $K \leq \mathbf{C}_K(\mathbf{F}(K)) \leq \mathbf{F}(K)$, which implies that $K = \mathbf{Z}(N)$.

**Step 3.** $N/\mathbf{Z}(N)$ is simple.

Since $N$ is non-solvable and $N/\mathbf{Z}(N)$ is a chief factor of $G$, we have $N/\mathbf{Z}(N) = L_1/\mathbf{Z}(N) \times \cdots \times L_t/\mathbf{Z}(N)$, where $L_i/\mathbf{Z}(N)$ are isomorphic non-abelian simple groups. We prove that $t = 1$. Otherwise, let $L = L_1$ and observe that since $L/\mathbf{Z}(N)$ is simple, then $L/\mathbf{Z}(N) = L'\mathbf{Z}(N)/\mathbf{Z}(N) \cong L'/\mathbf{Z}(L')$. We consider $\mathbf{N}_G(L')$. For every prime-power order $p'$-element $x \in L' \setminus \mathbf{Z}(L')$, we see that $\mathbf{C}_G(x) \subseteq \mathbf{N}_G(L')$. In fact, if $v \in \mathbf{C}_G(x) \setminus \mathbf{N}_G(L')$, then $x = x^v \in L' \cap L'^v \subseteq L \cap L^v \subseteq \mathbf{Z}(N)$, a contradiction. This yields to

$$|x^G| = |G : \mathbf{C}_G(x)| = |G : \mathbf{N}_G(L')||\mathbf{N}_G(L') : \mathbf{C}_{\mathbf{N}_G(L')}(x)|.$$

If $n = |G : \mathbf{N}_G(L')|$, we deduce that $|x^{\mathbf{N}_G(L')}|_{p'} = m/n_{p'}$. This means that every prime-power $p'$-element in $L' \setminus \mathbf{Z}(L')$ satisfies that the $p'$-part of its class size in $\mathbf{N}_G(L')$ is equal to $m/n_{p'}$. Since $L' < N$, by minimal counterexample, we obtain that $L'$ is solvable, a contradiction. Hence, $t = 1$, as desired, that is, $N/\mathbf{Z}(N)$ is a simple group.

**Step 4.** $N$ is solvable.

Notice that $N$ is perfect by minimality, so by Step 3, $N$ is a quasi-simple group. Consequently, $|\mathbf{Z}(N)|$ divides the order of the Schur multiplier of $S := N/\mathbf{Z}(N)$. By Theorem 2.3, we know that $m$ divides $|\mathbf{Z}(N)|$, so $m$ divides the order of the Schur multiplier $M(S)$. The rest of the proof consists in showing that this condition yields to a contradiction for every non-abelian simple group.

By Lemma 2.1, we know that $|M(S)|$ cannot be a prime-power (including 1). Accordingly, $S$ can only belong to the following list (see Section 5.1 of [9]):

(i) $A_1(3^2)$, $A_2(2^2)$, ${}^2A_3(3)$, ${}^2A_5(2)$, ${}^2E_6(2)$, $B_3(3)$, $A_7$, $M_{22}$, $Fi_{22}$ or Suz.
From Tables 2, 3 and 4 of [12], we know that the independence number $t(S) \geq 3$ for every simple group $S$ in the above list. This means that, for each $S$, there are at least

three distinct primes in $\pi(S)$ which are independent (i.e., are not pairwise connected) in the prime graph of $S$. In particular, there exist two different primes $p_1, p_2 \in \pi(S) \setminus \{p\}$ which are not connected between them. Now, let $\overline{x} = x\mathbf{Z}(N)$ be a $p_1$-element of $S$, such that $x \in N$ is a $p_1$-element too. As $S$ has no elements whose order is divisible by $p_1 p_2$, we have that $|S|_{p_2}$ divides $|\overline{x}^S|$. Note that $|\overline{x}^S|$ divides $|x^N|$ and that $|x^N|$ divides $|x^G|$. It follows that $|S|_{p_2}$ divides $m$, so we conclude that $|S|_{p_2}$ divides $|M(S)|$. Now, for each one of the listed groups, we clearly get a contradiction just by computing $|S|$ and $|M(S)|$.

(ii) $A_{n-1}(q)$ for $(n, q) \neq (2, 4), (2, 9), (3, 2), (3, 4)$ or $(4, 2)$.
In all these cases, $|M(S)| = (n, q-1)$. If $n \leq 5$, then $(n, q-1)$ is trivially a prime-power, and we are finished by Lemma 2.1. So, for the remainder of this case we will assume $n \geq 6$. Again by Table 4 of [12], we know that the $r$-independence number $t(r, S)$ is greater than or equal to 3, where $r$ is the characteristic of the underlying finite field, that is, $q = r^t$ for some $t \geq 1$. We recall the reader that $t(r, S)$ is the maximal number of vertices of the independent sets in the prime graph of $S$ containing the prime $r$. Nevertheless, the unequality $t(r, S) \geq 3$ holds except for the cases $(n, q) = (6, 2)$ and $(7, 2)$, but both can be easily ruled out (for instance, by using GAP [11]). Therefore, we can assume that there exist two primes $p_1, p_2 \in \pi(S)$ which are not connected to $r$ in the prime graph of $S$. Since one of them is necessarily distinct from $p$, say $p_1$, we deduce that $|S|_r$ divides the class size of every $p_1$-element $\overline{x}$ of $S$ (and $|S|_{p_2}$ too). As a result, $|S|_r$ divides $|x^G|$. If $r \neq p$, then $|S|_r$ would divide $(n, r^t - 1)$, which is a contradiction. Thus, $r = p$ and $m = |x^G|_{p'}$ must divide $|M(S)| = (n, q-1)$, so in particular, $m$ divides $q - 1$. This is not possible, just take into account that

$$|A_{n-1}(q)| = |\mathrm{PSL}(n, q)| = \frac{(q^n - 1)\dots(q^n - q^{n-1})}{(q-1)(n, q-1)}$$

is divisible by $(q-1)^2$ whenever $n \geq 4$. In particular, $((q-1)^2)_{p_2}$ divides $|S|_{p_2}$. However, we know by the above comments that $|S|_{p_2}$ divides $m$, and $m$ divides $(q-1)$, which is a contradiction.

(iii) ${}^2A_{n-1}(q)$ for $(n, q) \neq (4, 2), (4, 3)$ or $(6, 2)$.
We can argue similarly as in ii). For these groups, we have $|M(S)| = (n, q+1)$. If $n \leq 5$ then $|M(S)|$ obviously is a prime-power and again we are finished. If $n \geq 6$, we only have to take into account that $t(r, S) = 3$, where $r$ is characteristic of the field, and that $|{}^2A_{n-1}(q)|$ is always divisible by $(q+1)^2$ when $n \geq 4$. Both facts lead to a contradiction as above and this implies that $N$ is solvable.

**Step 5.** $N$ is $p$-nilpotent.

We argue by induction on $|N|$. Let $N/K$ be a chief factor of $G$, and since $N$ is solvable, $N/K$ is a $q$-group for some prime $q$. Let $x \in K \setminus \mathbf{Z}(K)$ be a prime-power order $p'$-element, so $x \in N \setminus \mathbf{Z}(N)$ and $|x^G|_{p'} = m$. By induction, we have that $K$ has nilpotent $p$-complements. Notice that if $q = p$ then the theorem is already proved, so we will assume in the sequel that $q \neq p$.

Let $H$ be a $p$-complement of $N$, so that $H \cap K$ is a nilpotent $p$-complement of $K$. Now, if $Q$ is a Sylow $q$-subgroup of $H$ (and also of $N$), we have $N = KQ$ and, by Dedekind's modular law, $H = (H \cap K)Q$. In particular, $H$ is nilpotent if and only if $Q$ is normal in $H$, or equivalently, if and only if for every $r \in \pi(H)$, $r \neq q$, $Q$ centralizes the (unique) Sylow $r$-subgroup, say $H_r$, of $H$. This is equivalent to prove that $q$ does not divide $|N : \mathbf{C}_N(H_r)|$, and this is what we are proving next.

Let $x \in Q \setminus \mathbf{Z}(N)$ and let $R$ be a Sylow $r$-subgroup of $\mathbf{C}_G(x)$. Since $RN$ is solvable and $R \times \langle x \rangle$ is a $p'$-subgroup of it, there exists a $p$-complement, say $H_1$, of $NR$ which contains $R \times \langle x \rangle$. As $H_1 \cap N$ is a $p$-complement of $N$, for some $t \in N$, we have $H^t = H_1 \cap N \trianglelefteq H_1$ and moreover, since $H_r^t$ is characteristic in $H^t$, it follows that $H_r^t$ is normal in $H_1$. As a consequence, $R \times \langle x \rangle$ acts on $H_r^t$. Now, from the hypotheses, it is easy to see that $\mathbf{C}_{H_r^t}(R) \subseteq \mathbf{C}_{H_r^t}(\langle x \rangle)$. By applying Lemma 2.2, we have $x \in \mathbf{C}_N(H_r^t)$, and hence $Q \subseteq \cup_{t\in N}\mathbf{C}_N(H_r^t)$. From $N = KQ$, we deduce that

$$N = \cup_{t\in N} K\mathbf{C}_N(H_r^t) = \cup_{t\in N}(K\mathbf{C}_N(H_r))^t,$$

which implies that $N = K\mathbf{C}_N(H_r)$. Finally, observe that

$$|N : \mathbf{C}_N(H_r)| = |K : \mathbf{C}_K(H_r)|$$

is a $q'$-number as the nilpotent group $K \cap H$ contains a Sylow $q$-subgroup of $K$, which centralizes its $r$-complement $H_r$. This finishes the proof. □

*Remark* The hypotheses of the Theorem do not imply that the $p$-complements of $N$ need to be the direct product of a $q$-group times an abelian group, even the $p$-complements of $N$ may have all its Sylow subgroups non-abelian. For instance, let

$$L = \left\langle x, y | x^3 = y^3 = 1, [x, y]^3 = 1, [x, [x, y]] = [y, [x, y]] = 1 \right\rangle$$

be the extra special group of order $3^3$ and exponent 3. If $z = [x, y]$, then $\mathbf{Z}(L) = \langle z \rangle$. Let $\langle a \rangle$ be the automorphism of $L$ defined by $x^a = x^2$ and $y^a = y^2$. The set of fixed points of $a$ on $L$ is exactly $\mathbf{Z}(L)$. On the other hand, let us consider an automorphism $\alpha$ of order 3 acting non-trivially on the quaternion group $Q$ of order 8. Observe that $\alpha$ exactly fixes the elements in $\mathbf{Z}(Q)$. We form the group $G := Q\langle \alpha \rangle \times L\langle a \rangle$ and take the normal subgroup $N = Q \times L$. For any choice of $p \in \{2, 3\}$, the $G$-class size of every $p'$-element of prime-power order of $N \setminus \mathbf{Z}(N)$ is exactly 1 or 6, while no Sylow subgroup of $N$ is abelian. This example also shows that $m$ in the Theorem need not be a prime-power as it happens in Itô's theorem (see 33.6 of [8]).

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## References

1. Akhlaghi, Z., Beltrán, A., Felipe, M.J.: The influence of $p$-regular class sizes on normal subgroups. J. Group Theory. **16**, 585–593 (2013)
2. Alemany, E., Beltrán, A., Felipe, M.J.: Nilpotency of normal subgroups having two $G$-class sizes. Proc. Am. Math. Soc. **139**, 2663–2669 (2011)
3. Alemany, E., Beltrán, A., Felipe, M.J.: Finite groups with two $p$-regular conjugacy class lengths II. Bull. Aust. Math. Soc. **797**, 419–425 (2009)
4. Beltrán, A., Felipe, M.J.: Normal subgroups and class sizes elements of prime-power order. Proc. Am. Math. Soc. **140**, 4105–4109 (2012)
5. Beltrán, A.: Action with nilpotent fixed point subgroup. Arch. Math. (Basel) **69**, 177–184 (1997)
6. Camina, A.R.: Finite groups of conjugate rank 2. Nagoya Math. J. **53**, 47–57 (1974)

7. Casolo, C., Dolfi, S., Jabara, E.: Finite groups whose noncentral class sizes have the same $p$-part for some prime $p$. Isr. J. Math. **192**, 197–219 (2012)
8. Huppert, B.: Character Theory of Finite groups, vol. 25. De Gruyter Expositions in Mathemathics, Berlin, New York (1998)
9. Kleidman, P., Liebeck, M.: The Subgroup Structure of The Finite Classical Groups. London Mathematical Society Lecture Note Series, 129. Cambridge University Press, Cambridge (1990)
10. Kurzweil, K., Stellmacher, B.: The Theory of Finite Groups. An Introduction. Springer, New York (2004)
11. The GAP Group, GAP—Groups, Algorithms and Programming, Vers. 4.4.12 (2008). http://www.gap-system.org
12. Vasiliev, A.V., Vdovin, E.P.: An adjacency criterion for the prime graph of a finite simple group. Algebra Logic **44**(6), 381–406 (2005)