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Characterizations of k-potent elements in rings

Dijana Mosić

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Abstract In this paper, we investigate several characterizations of k-potent elements in rings in purely algebraic terms and considerably simplify proofs of already existing characterizations. A special attention is dedicated to tripotent elements of rings.

Keywords Idempotent · Moore–Penrose inverse · Group inverse · EP element · Ring with involution

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1 Introduction

Let $\mathcal{R}$ be an associative ring with unit 1 and zero 0. The set of all invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{-1}$.

An element $a \in \mathcal{R}$ is called *k-potent*, $k \in \mathbb{N}$, if $a^k = a$. For $k = 2$, an element a satisfying $a^2 = a$ is *idempotent* (or *projector*).

Let $a \in \mathcal{R}$. Then, a is *group invertible* if there is $a^\# \in \mathcal{R}$ such that

$$aa^\# a = a, \quad a^\# aa^\# = a^\#, \quad aa^\# = a^\# a;$$

$a^\#$ is called a *group inverse* of a, and it is uniquely determined by these equations [1]. Recall that $a^\#$ exists if and only if $a \in a^2\mathcal{R} \cap \mathcal{R}a^2$ if and only if $a\mathcal{R} = a^2\mathcal{R}$ and $\mathcal{R}a = \mathcal{R}a^2$ [15]. We use $\mathcal{R}^\#$ to denote the set of all group invertible elements of $\mathcal{R}$.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*.$$

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D. Mosić (✉)
Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia
e-mail: dijana@pmf.ni.ac.rs

An element $a \in \mathcal{R}$ satisfying $aa^* = a^*a$ is called *normal*. An element $a \in \mathcal{R}$ satisfying $a = a^*$ is *Hermitian* (or *self-adjoint*). An element $a \in \mathcal{R}$ is *orthogonal projector* if $a^2 = a = a^*$. If $a \in \mathcal{R}$ satisfies $a^2 = a^*$, then a is *generalized projection*. An element $a \in \mathcal{R}$ satisfying $a = a^* = a^3$ is *extended orthogonal projector*. So, a is an extended orthogonal projector if and only if a is Hermitian and tripotent.

We say that $a^\dagger$ is the *Moore–Penrose inverse* (or *MP-inverse*) of a, if the following hold [17]:

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad (aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a.$$

There is at most one $a^\dagger$ such that above conditions hold. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

Theorem 1.1 [5,11] *For any $a \in \mathcal{R}^\dagger$, the following is satisfied:*

(a) $(a^\dagger)^\dagger = a$;
(b) $(a^*)^\dagger = (a^\dagger)^*$;
(c) $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$;
(d) $(aa^*)^\dagger = (a^\dagger)^*a^\dagger$;
(f) $a^* = a^\dagger aa^* = a^*aa^\dagger$;
(g) $a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger = (a^*a)^\# a^* = a^*(aa^*)^\#$;
(h) $(a^*)^\dagger = a(a^*a)^\dagger = (aa^*)^\dagger a$.

An element a of a ring $\mathcal{R}$ with involution is said to be *EP* if $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ and $a^\# = a^\dagger$. Recall that an element $a \in \mathcal{R}$ is EP if and only if $a \in \mathcal{R}^\dagger$ and $aa^\dagger = a^\dagger a$. The following result is proved in [8].

Theorem 1.2 *An element $a \in \mathcal{R}$ is EP if and only if a is group invertible and $a^\# a$ is self-adjoint.*

An element $a \in \mathcal{R}^\dagger$ satisfying $a^2 = a^\dagger$ is *hypergeneralized projection*. An element $a \in \mathcal{R}$ satisfying $aa^*a = a$ is called *a partial isometry*. We have that $a \in \mathcal{R}$ is a partial isometry if and only if $a \in \mathcal{R}^\dagger$ and $a^* = a^\dagger$.

An element $a \in \mathcal{R}$ is: *left *-cancellable* if $a^*ax = a^*ay$ implies $ax = ay$; it is *right *-cancellable* if $xaa^* = yaa^*$ implies $xa = ya$, and it is **-cancellable* if it is both left and right *-cancellable. We observe that a is left *-cancellable if and only if a^* is right *-cancellable. In C^*-algebras, all elements are *-cancellable. A ring $\mathcal{R}$ is called **-reducing* if every element of $\mathcal{R}$ is *-cancellable. This is equivalent to the implication $a^*a = 0 \Rightarrow a = 0$ for all $a \in \mathcal{R}$.

Using [13, Theorem 2.4] and Remark after [13, Theorem 2.4], we can formulate the following result.

Theorem 1.3 *Let $\mathcal{R}$ be a ring with involution, let $a, b, ab \in \mathcal{R}^\dagger$ and let $(1 - a^\dagger a)b$ be left *-cancellable. Then the following conditions are equivalent:*

(a) $(ab)^\dagger = b^\dagger a^\dagger$;
(b) $a^*abb^\dagger = bb^\dagger a^*a$ *and* $bb^*a^\dagger a = a^\dagger abb^*$.

Recall that a ring $\mathcal{R}$ with involution has the *Gelfand–Naimark property* (*GN-property*) if $1 + x^*x \in \mathcal{R}^{-1}$ for all $x \in \mathcal{R}$. It is known that any C^*-algebra has the Gelfand–Naimark property.

Many authors have presented various results about k-potent matrices in recent years [7, 9, 10]. The study on k-potent matrices, particularly idempotents and tripotents, essentially originated from their possible applications in statistics [18, 19].

Baksalary and Trenkler [3] gave a systematic investigation of k-potent complex matrices, with a particular attention paid to tripotent matrices, using the representation of complex matrices provided in [6]. Inspired by [3], in this paper, we use a different approach, exploiting the structure of rings and rings with involution to study k-potent elements. We give some characterizations, and the proofs are based on ring theory only. The paper is organized as follows. In Sect. 2, we give characterizations of tripotent elements and extended orthogonal projectors in rings. In Sect. 3, we study k-potent elements, EP elements, and partial isometries in rings.

2 Characterizations of tripotent elements

In this section, we investigate tripotent elements and extended orthogonal projectors in a ring and a ring with involution. Notice that the set of orthogonal projectors is a proper subset of the set of extended orthogonal projectors.

First, we give the relation between idempotent and tripotent elements in a ring, generalizing [3, Theorem 3.2] for complex matrices without using the rank of matrix.

Theorem 2.1 *Let $a \in \mathcal{R}$ and $2, 3 \in \mathcal{R}^{-1}$. Then a is idempotent if and only if a is tripotent and any if the following conditions is satisfied:*

(i) *$1 - a$ is tripotent;*
(ii) $1 + a \in \mathcal{R}^{-1}$.

Proof The hypothesis $a^2 = a$ gives $a^3 = a$ and $(1 - a)^3 = (1 - a)$, i.e., (i) holds. Also, by $a^2 = a$, we have

$$(1 + a)(2 - a) = 2 - a + 2a - a = 2$$

and $(2 - a)(1 + a) = 2$. Since $2 \in \mathcal{R}^{-1}$, we conclude that $1 + a \in \mathcal{R}^{-1}$. So, part (ii) is satisfied.

Suppose that $a^3 = a$ and $1 - a$ is tripotent. From

$$1 - a = (1 - a)^3 = 1 - 4a + 3a^2,$$

we get $3a = 3a^2$. Hence, $a^2 = a$.

If $a^3 = a$ and $1 + a \in \mathcal{R}^{-1}$, using

$$(1 + a)a^2 = a^2 + a = (1 + a)a,$$

we obtain $a^2 = a$. □

Adding the condition a is EP element in Theorem 2.1, we can prove the next corollary in the same way as [3, Corollary 3.3] for matrices.

Corollary 2.1 *Let $a \in \mathcal{R}$ and $2, 3 \in \mathcal{R}^{-1}$. Then a is an orthogonal projector if and only if a is tripotent, EP and any if the following conditions is satisfied:*

(i) *$1 - a$ is tripotent;*
(ii) $1 + a \in \mathcal{R}^{-1}$.

Proof Observe that every Hermitian element is EP. Also, every idempotent and EP element is an orthogonal projector. Indeed, $aa^\dagger = a^\dagger a$ implies $a = aa^\dagger a = a^\dagger a^2 = a^\dagger a$, i.e., a is an orthogonal projector. This result follows from Theorem 2.1. □

The following result is well known for matrices [2, Theorem 1], and it is equally true in rings with involution:

Lemma 2.1 *Let $a \in \mathcal{R}$. Then a is an orthogonal projector if and only if a is idempotent and either generalized or hypergeneralized projector.*

Proof Obviously, a is an orthogonal projector if and only if a is an idempotent and generalized projector.

If a is an orthogonal projector, then $aa^2a = a^2 = a$ and $a^2aa^2 = a^2$. Also, $aa^2 = a^2a = a^*a$ is self-adjoint, and so, we conclude that $a \in \mathcal{R}^\dagger$ and $a^2 = a^\dagger$.

Let a be an idempotent and hypergeneralized projector. Then, $a^2 = a = a^\dagger$. Since $aa^\dagger$ is self-adjoint and $a = a^2 = aa^\dagger$, we have $a = a^*$. □

If we replace the idempotency condition in Lemma 2.1 with tripotency, we get the next generalization of [3, Theorem 3.5] in rings with involution.

Theorem 2.2 *Let $a \in \mathcal{R}$. Then a is an orthogonal projector if and only if a is tripotent and either generalized or hypergeneralized projector.*

Proof Suppose that a is an orthogonal projector. Now, $a^3 = a^2a = aa = a$. By Lemma 2.1, we deduce that a is generalized and hypergeneralized projector.

Let a be tripotent and generalized projector. Hence, $a^3 = a$ and $a^2 = a^*$ which imply $a = a^2a = a^*a$. The element a^*a is self-adjoint and so $a = a^*$, i.e. a is orthogonal projector.

If a is tripotent and hypergeneralized projector, the equalities $a^3 = a$ and $a^2 = a^\dagger$ give $a = a^2a = a^\dagger a$ is self-adjoint. Thus, $a = a^*$ and $a = aa^\dagger a = aa = a^2$. □

Observe that an element a of a ring $\mathcal{R}$ is tripotent if and only if $a \in \mathcal{R}^\#$ and $a^\# = a$. This result was shown in [4, Theorem 8] for matrices and modified in [3, Theorem 3.7]. We extend [3, Theorem 3.7] for complex matrices to elements in rings.

Theorem 2.3 *Let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is tripotent;*
(ii) *$a \in \mathcal{R}^\#$ and a^2 is idempotent;*
(iii) *$a \in \mathcal{R}^\#$ and $aa^\# = a^2$;*
(iv) *$a \in \mathcal{R}^\#$ and $\frac{1}{2}(aa^\# - a)$ is idempotent;*
(v) *$a \in \mathcal{R}^\#$ and $\frac{1}{2}(aa^\# + a)$ is idempotent.*

Proof (i) $\Rightarrow$ (iii): If $a^3 = a$, then we conclude that $a \in \mathcal{R}^\#$ and $a^\# = a$. Hence, $a^2 = aa^\#$.

(iii) $\Rightarrow$ (ii): By $(a^2)^2 = (aa^\#)^2 = aa^\# = a^2$, a^2 is idempotent.

(ii) $\Rightarrow$ (i): Since a^2 is idempotent, we have $a^4 = a^2$. Multiplying this equality by $a^\#$ from the left side, we get $a^3 = a$.

(iv) $\Leftrightarrow$ (iii): Notice that $\frac{1}{2}(aa^\# - a) = \left[\frac{1}{2}(aa^\# - a)\right]^2$ is equivalent to $2aa^\# - 2a = aa^\# - 2a + a^2$, i.e. $aa^\# = a^2$.

(v) $\Leftrightarrow$ (iii): This part follows in the same way as (iv) $\Leftrightarrow$ (iii). □

In a ring with involution, we prove the following result. If we omit the condition $1 + a^*a \in \mathcal{R}^{-1}$ of the next theorem, we can show that [(i) $\Leftrightarrow$ (iii)] $\Rightarrow$ (ii).

Theorem 2.4 *Let $a \in \mathcal{R}$ be tripotent and $1 + a^*a \in \mathcal{R}^{-1}$. Then the following conditions are equivalent:*

(i) *a is Hermitian;*
(ii) *a is normal;*
(iii) *a is EP and a partial isometry.*

Proof (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): If a is normal, by [11, Lemma 1.3], we conclude that a is EP. Now, by Theorem 2.3, we have that $a^2 = aa^\# = aa^\dagger$ is Hermitian. Since $1 + a^*a \in \mathcal{R}^{-1}$ and

$$(1 + a^*a)(a^*a - a^2) = a^*a - a^2 + (a^*)^2a^2 - a^*a = -a^2 + a^4 = 0,$$

we get $a^*a - a^2 = 0$, i.e. $a^*a = a^2$. Thus, $aa^*a = a^3 = a$, that is, a is a partial isometry.

(iii) $\Rightarrow$ (i): Let a be tripotent, EP, and a partial isometry. Then, $a = a^\# = a^\dagger = a^*$. □

Notice that Theorem 2.4 holds in C^*-algebras and rings with involution having the GN-property without the assumption $1+a^*a \in \mathcal{R}^{-1}$ which is then automatically satisfied. Hence, [3, Theorem 3.6] can be obtained as a particular case of Theorem 2.4.

Some equivalent conditions for an element of a ring to be tripotent are presented in the following theorem, generalizing [3, Theorem 3.8] for complex matrices. It is interesting to compare this result with the analogous result for idempotents.

Theorem 2.5 *Let $a \in \mathcal{R}$. Then the following conditions are equivalent:*

(i) *a is tripotent;*
(ii) *$a\mathcal{R} \oplus (1 - a^2)\mathcal{R} = \mathcal{R}$;*
(iii) *$a\mathcal{R} \subseteq (1 - a^2)^\circ$;*
(iv) *$a^2x = x$ for all $x \in a\mathcal{R}$.*

Proof (i) $\Rightarrow$ (ii): Let $a^3 = a$ and $x \in \mathcal{R}$. Then

$$x = a^2x + (1 - a^2)x \in a^2\mathcal{R} + (1 - a^2)\mathcal{R} \subseteq a\mathcal{R} + (1 - a^2)\mathcal{R}.$$

Suppose that $y \in a\mathcal{R} \cap (1-a^2)\mathcal{R}$. There exist $u, v \in \mathcal{R}$ such that $y = au = (1-a^2)v$. Now, $ay = a(1 - a^2)v = 0$ implying $y = au = a^3u = a^2y = 0$. Thus, $a\mathcal{R} \cap (1 - a^2)\mathcal{R} = \{0\}$ and the statement (ii) holds.

(ii) $\Rightarrow$ (iii): If $x \in a\mathcal{R}$, by $a\mathcal{R} \cap (1-a^2)\mathcal{R} = \{0\}$, we have $(1-a^2)x = 0$. So, $x \in (1-a^2)^\circ$.

(iii) $\Rightarrow$ (iv): Obviously.

(iv) $\Leftrightarrow$ (i): For $x = a \in a\mathcal{R}$ in (iv), we obtain $a^3 = a$. □

In the following theorem, we study necessary and sufficient conditions for an element of a ring with involution to be tripotent and EP. This result is known for matrices [3, Theorem 3.9], but we present a new proof based on ring theory only.

Theorem 2.6 *Let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is tripotent and EP;*
(ii) *$a \in \mathcal{R}^\dagger$ and $a^\dagger = a$;*
(iii) *a is tripotent and a^2 is Hermitian;*
(iv) *$a \in \mathcal{R}^\dagger$ and $a^2 = aa^\dagger$;*
(v) *$a \in \mathcal{R}^\dagger$ and $a^\dagger = a^2a^\dagger$;*
(vi) *$a \in \mathcal{R}^\dagger$ and $a(a^*)^2 = a$;*
(vii) *$a \in \mathcal{R}^\dagger$ and $(a^*)^2a = a$;*

(viii) *$a \in \mathcal{R}^{\#}$ and a^2 is an orthogonal projector;*
(ix) *$a \in \mathcal{R}^{\#} \cap \mathcal{R}^{\dagger}$ and $a^{\#}aa^{\dagger} = a$;*
(x) *$a \in \mathcal{R}^{\#} \cap \mathcal{R}^{\dagger}$ and $a^{\#} = a^2a^{\dagger}$;*
(xi) *$a \in \mathcal{R}^{\dagger}$ and $a^2 = a^{\dagger}a$.*

Proof (i) $\Rightarrow$ (ii): If $a^3 = a$ and a is EP, then $a \in \mathcal{R}^{\#} \cap \mathcal{R}^{\dagger}$ and $a = a^{\#} = a^{\dagger}$.

(ii) $\Rightarrow$ (iii): From $a^{\dagger} = a$, we get $a = aa^{\dagger}a = a^3$ and $a^2 = aa^{\dagger}$ is Hermitian.

(iii) $\Rightarrow$ (vi) $\wedge$ (vii): Since $a^3 = a$ and $a^2 = (a^2)^* = (a^*)^2$, observe that $a(a^*)^2 = aa^2 = a^3 = a$ and $(a^*)^2a = a^3 = a$. Also, we have $a \in \mathcal{R}^{\dagger}$ and $a^{\dagger} = a$. Thus, the statements (vi) and (vii) are satisfied.

(vii) $\Rightarrow$ (iv): The equality $(a^*)^2a = a$ gives $aa^{\dagger} = (a^*)^2aa^{\dagger} = (a^*)^2$. Hence, $a^2 = [(a^*)^2]^* = (aa^{\dagger})^* = aa^{\dagger}$.

(iv) $\Rightarrow$ (i): By $a^2 = aa^{\dagger}$, we obtain $a^3 = aa^{\dagger}a = a$ and a^2 is Hermitian which imply $a \in \mathcal{R}^{\#}$ and $a^{\#} = a = a^{\dagger}$.

(vi) $\Rightarrow$ (xi) $\Rightarrow$ (i): Similarly as (vii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(v) $\Rightarrow$ (xi): Using $a^{\dagger} = a^2a^{\dagger}$, we get $a^{\dagger}a = a^2a^{\dagger}a = a^2$.

(xi) $\Rightarrow$ (v): Multiplying $a^2 = a^{\dagger}a$ by $a^{\dagger}$ from the right side, we obtain $a^2a^{\dagger} = a^{\dagger}$.

(viii) $\Rightarrow$ (iii): Notice that $a \in \mathcal{R}^{\#}$ and $a^2 = a^4 = (a^2)^*$ implies $a = a^2a^{\#} = a^4a^{\#} = a^3$ and $a^2 = (a^2)^*$.

(iii) $\Rightarrow$ (viii): The hypothesis a is tripotent gives $a \in \mathcal{R}^{\#}$ and $a^2 = a^4$. Because $a^2 = (a^2)^*$, we deduce that (viii) holds.

(ix) $\Rightarrow$ (viii): Let $a \in \mathcal{R}^{\#} \cap \mathcal{R}^{\dagger}$ and $a^{\#}aa^{\dagger} = a$. Then

$$a^2 = aa^{\#}aa^{\dagger} = aa^{\dagger}$$

which yields that a^2 is an orthogonal projector.

(i) $\Rightarrow$ (ix) $\wedge$ (x): Assume that $a \in \mathcal{R}^{\#} \cap \mathcal{R}^{\dagger}$, $a^3 = a$ and $a^{\dagger} = a^{\#}$. Therefore, $a = a^{\#}$, $a = aaa^{\#} = a^{\#}aa^{\dagger}$ and $a = a^2a^{\#} = a^2a^{\dagger}$.

(x) $\Rightarrow$ (i): Applying $a^{\#} = a^2a^{\dagger}$, we obtain $a = aa^{\#}a = aa^2a^{\dagger}a = a^3$. Also, $aa^{\#} = a^3a^{\dagger} = aa^{\dagger}$ is self-adjoint which implies that $a^{\#} = a^{\dagger}$. □

As a consequence of Theorem 2.6, we get the next result.

Corollary 2.2 *Let $a \in \mathcal{R}$. Then a is an extended orthogonal projector if and only if*

(i) *a is Hermitian and any of the conditions in Theorem 2.6 is satisfied;*
(ii) *a is a partial isometry and any of the conditions in Theorem 2.6 is satisfied;*
(iii) *$a \in \mathcal{R}^{\#}$ and $a = a^{\dagger} = a^* = a^{\#} = a^{\#}aa^{\dagger}$.*

Proof By definition, a is an extended orthogonal projector if and only if $a = a^* = a^3$. The corollary follows from Theorem 2.6. □

Some characterizations of extended orthogonal projectors are given in the next results, extending [3, Theorem 3.11, Theorem 3.12, and Theorem 3.13] to more general settings.

Theorem 2.7 *Let $a \in \mathcal{R}$. Then a is an extended orthogonal projector if and only if a is a partial isometry, EP and either $\frac{1}{2}(aa^{\dagger} - a)$ or $\frac{1}{2}(aa^{\dagger} + a)$ is idempotent.*

Proof If a is EP, then, from $aa^{\dagger} = a^{\dagger}a$, we conclude that $\frac{1}{2}(aa^{\dagger} - a)$ or $\frac{1}{2}(aa^{\dagger} + a)$ is idempotent if and only if $a^2 = aa^{\dagger}$. Now, the result follows from Theorem 2.6(iv) and Corollary 2.2. □

As in a C^*-algebra, for $b, c \in \mathcal{R}$, we define the relation

$$b \le c \quad \Leftrightarrow \quad \exists d \in \mathcal{R} : b - c = dd^*.$$

This relation is antisymmetric in any C^*-algebra, but in a ring with involution this is not true. We say that an element $a \in \mathcal{R}$ is a *contraction* if and only if $0 \leq 1 - aa^*$.

Theorem 2.8 *Let $\mathcal{R}$ be a ring on which the relation "$\leq$" is antisymmetric and let a be an element of $\mathcal{R}$. Then a is an extended orthogonal projector if and only if a is tripotent, EP and a contraction.*

Proof Suppose that $a = a^* = a^3$. So, a is tripotent, and, by Corollary 2.2, a is EP. Since

$$1 - aa^* = 1 - a^2 = (1 - a^2)^2 = (1 - aa^*)^2 = (1 - aa^*)(1 - aa^*)^*,$$

we deduce that $0 \leq 1 - aa^*$, i.e., a is a contraction.

On the other hand, if a is tripotent, EP and a contraction, then $a \in \mathcal{R}^\dagger$, $a = a^3$, $a^\dagger a = aa^\dagger$ and there exists $b \in \mathcal{R}$ such that $1 - aa^* = bb^*$. Using Theorem 1.1,

$$a^\dagger(a^\dagger)^* - a^\dagger a = a^\dagger(a^\dagger)^* - a^*(a^\dagger)^* = a^\dagger(1 - aa^*)(a^\dagger)^* = a^\dagger b(a^\dagger b)^*,$$

which gives $a^\dagger(a^\dagger)^* \geq a^\dagger a$ and

$$aa^\dagger - aa^* = a(a^\dagger(a^\dagger)^* - a^\dagger a)a^* = aa^\dagger b(a^\dagger b)^* a^* = aa^\dagger b(aa^\dagger b)^*,$$

that is, $aa^\dagger \geq aa^*$. By Theorem 2.6, $a = a^\dagger$. Then $a^\dagger(a^\dagger)^* \geq a^\dagger a$ implies $aa^* \geq a^\dagger a$. Since $a^\dagger a = aa^\dagger$ and the relation "$\leq$" is antisymmetric, from $aa^* \geq a^\dagger a$ and $aa^\dagger \geq aa^*$, we have $aa^\dagger = aa^*$. Therefore, $a^* = a^\dagger aa^* = a^\dagger aa^\dagger = a^\dagger = a$, and so a is an extended orthogonal projector. □

Theorem 2.9 *Let $a, b \in \mathcal{R}$ be tripotent. If $ab = ba$, then ab is tripotent. If a and b are Hermitian (i.e. are extended orthogonal projectors), then ab is an extended orthogonal projector if and only if $ab = ba$.*

Proof Assume that $a^3 = a$, $b^3 = b$ and $ab = ba$. Then $(ab)^3 = a^3b^3 = ab$.

Now, let $a^3 = a = a^*$ and $b^3 = b = b^*$. If $(ab)^3 = ab = (ab)^*$, notice that $ab = (ab)^* = b^*a^* = ba$. Conversely, when $ab = ba$, we get $ab = b^*a^* = (ab)^*$, and by the first part of this proof, $(ab)^3 = ab$. □

If a and b are extended orthogonal projectors in a ring with involution, we will now characterize the elements for which $(ab)^\dagger = b^\dagger a^\dagger$. This equality does not hold in general.

Theorem 2.10 *Let $a, b \in \mathcal{R}$ be extended orthogonal projectors, assume $ab \in \mathcal{R}^\dagger$ and suppose $(1 - a^\dagger a)b$ left *-cancellable. Then the following statements are equivalent:*

(i) $(ab)^\dagger = b^\dagger a^\dagger$;
(ii) a^2b^2 *is Hermitian;*
(iii) $(a^2b^2)^2 = b^2a^2$.

Proof Since $a^3 = a = a^*$ and $b^3 = b = b^*$, we get that $a, b \in \mathcal{R}^\dagger$, $a^\dagger = a$ and $b^\dagger = b$.

(i) $\Leftrightarrow$ (ii): By Theorem 1.3,

$$\begin{aligned}(ab)^\dagger = b^\dagger a^\dagger &\Leftrightarrow a^*abb^\dagger = bb^\dagger a^*a \quad \text{and} \quad bb^*a^\dagger a = a^\dagger abb^* \\ &\Leftrightarrow a^2b^2 = b^2a^2 \\ &\Leftrightarrow a^2b^2 = (b^2)^*(a^2)^* \\ &\Leftrightarrow a^2b^2 = (a^2b^2)^* \\ &\Leftrightarrow a^2b^2 \text{ is Hermitian.}\end{aligned}$$

(ii) $\Rightarrow$ (iii): If a^2b^2 is Hermitian, then $a^2b^2 = b^2a^2$, which yields

$$(a^2b^2)^2 = a^2(b^2a^2)b^2 = a^2a^2b^2b^2 = a^4b^4 = a^2b^2 = b^2a^2.$$

(iii) $\Rightarrow$ (ii): Applying the involution to the hypothesis $b^2a^2 = a^2b^2a^2b^2$, we obtain $a^2b^2 = b^2a^2b^2a^2$. Therefore,

$$a^2b^2 = (b^2a^2)b^2a^2 = a^2b^2a^2b^2b^2a^2 = (a^2b^2a^2b^2)a^2$$
$$= b^2a^2a^2 = b^2a^2,$$

that is, a^2b^2 is Hermitian. □

Notice that Theorem 2.10 holds in C^*-algebras and *-reducing rings without the hypothesis that $(1-a^\dagger a)b$ is left *-cancellable, since this hypothesis is then automatically satisfied. So, [3, Theorem 3.14] can be obtained as a special case of our result.

Let $a, b \in \mathcal{R}$. Define

$$a \leq^* b \quad \Leftrightarrow \quad a^*a = a^*b \quad \text{and} \quad aa^* = ba^*.$$

Theorem 2.11 *Let $a, b \in \mathcal{R}$ be extended orthogonal projectors and let $a^2 - ab$ be right *-cancellable. Then the following conditions are equivalent:*

(i) $a \leq^* b$;
(ii) $aba = a$ *and* $bab = a$;
(iii) $aba = a$ *and* $a^2 \leq^* b^2$.

Proof (i) $\Rightarrow$ (ii): Since $a \leq^* b$, we have that $a^2 = ab = ba$, which yields $a = a^3 = aba$ and $bab = ba^2 = a^3 = a$.

(ii) $\Rightarrow$ (i): Using the equalities $aba = a$ and $bab = a$, we obtain $a^2 = a(bab)ab = a^3b = ab$ and $a^2 = (a^2)^* = (ab)^* = ba$. Hence, $a \leq^* b$.

(i) $\Rightarrow$ (iii): From $a^2 = ab = ba$, we get $a^2 = a^4 = abab = a^2b^2 = b^2a^2$ which implies that $a^2 \leq^* b^2$. We can prove that $a = aba$ as in part (i) $\Rightarrow$ (ii).

(iii) $\Rightarrow$ (i): The assumptions $aba = a$ and $a^2 = a^2b^2 = b^2a^2$ give

$$a = a^3 = ab^2 = b^2a$$

and

$$a^2 = ab^2a = a^2ba = aba^2.$$

Now, we have

$$(a^2 - ab)(a^2 - ab)^* = (a^2 - ab)(a^2 - ba)$$
$$= a^4 - a^2ba - aba^2 + ab^2a$$
$$= 0.$$

Because the element $a^2 - ab$ is right *-cancellable, we conclude that $a^2 - ab = 0$, that is $a^2 = ab$. Applying the involution to this equality, it follows $a^2 = ba$. Thus, $a \leq^* b$. □

3 Further results

In the beginning of this section, we characterize the normal and $(k+1)$-potent elements in rings with involution.

Theorem 3.1 *Let $a \in \mathcal{R}$ and let $k \in N$, $k > 2$. Suppose that $x = 1 + aa^* + a^2(a^*)^2 + \cdots + a^{k-1}(a^*)^{k-1} \in \mathcal{R}^{-1}$ and $y = a^2(a^*)^2 + \cdots + a^{k-1}(a^*)^{k-1} \in \mathcal{R}^{-1}$. Then the following conditions are equivalent:*

(i) *a is normal and $(k+1)$-potent;*
(ii) *a is a partial isometry and $(k+1)$-potent;*
(iii) *a is a partial isometry and $a^k = aa^*$;*
(iv) *$a \in \mathcal{R}^{\#}$, a is a partial isometry and $a^{k-1} = a^{\#}$;*
(v) *$a \in \mathcal{R}^{\#}$, $a^* = a^{\#}$ and $a^{k-1} = a^{\#}$;*
(vi) *$a^{k-1} = a^*$.*

Proof (i) $\Rightarrow$ (ii): If a is normal and $(k+1)$-potent, then $a \in \mathcal{R}^{\#} \cap \mathcal{R}^{\dagger}$ and $a^{\dagger} = a^{\#} = a^{k-1}$. Hence, $a^k = aa^{\dagger}$ is self-adjoint. From

$$\begin{aligned} x(a - a^2a^*) &= a - a^2a^* + a^2a^* - a^3(a^*)^2 + \cdots \\ &\quad + a^{k-1}(a^*)^{k-2} - (a^*)^{k-1} + (a^*)^{k-1} - a \\ &= 0 \end{aligned}$$

and $x \in \mathcal{R}^{-1}$, we deduce that $a - a^2a^* = 0$. This implies that $a = aa^*a$.

(ii) $\Rightarrow$ (iii) $\wedge$ (v) $\wedge$ (vi): Assume that a is a partial isometry and $(k+1)$-potent. The second condition imply $a \in \mathcal{R}^{\#}$, $a^{k-1} = a^{\#}$, $(a^*)^{k+1} = a^*$ and

$$\begin{aligned} y((a^*)^k a^k - a^k) &= a^2(a^*)^2a^k - a^2(a^*)^2a^k + \cdots \\ &\quad + a^{k-1}(a^*)^{k-1}a^k - a^{k-1}(a^*)^{k-1}a^k \\ &= 0. \end{aligned}$$

Since $y \in \mathcal{R}^{-1}$, we obtain $(a^*)^k a^k - a^k = 0$ which gives that $a^k = (a^*)^k a^k$ is self-adjoint. Therefore, $aa^{\#} = aa^{k-1} = a^k$ is self-adjoint and, by Theorem 1.2, a is EP. Now, because a is a partial isometry, $a \in \mathcal{R}^{\dagger}$, $a^{k-1} = a^{\#} = a^{\dagger} = a^*$ and $a^k = aa^*$.

(iii) $\Rightarrow$ (ii): Using that a is a partial isometry and $a^k = aa^*$, we have $a^{k+1} = aa^*a = a$.

(ii) $\Leftrightarrow$ (iv): This is clear.

(vi) $\Rightarrow$ (iii) $\wedge$ (i): The condition $a^{k-1} = a^*$ gives $a^*a = a^k = aa^*$. Thus, a^k is self-adjoint and a is normal, which imply that a is EP. From

$$\begin{aligned} y(a - a^2a^*) &= a^2(a^*)^2a^k - a^2(a^*)^2a^k + \cdots \\ &\quad + a^{k-1}(a^*)^{k-1}a^k - a^{k-1}(a^*)^{k-1}a^k \\ &= 0, \end{aligned}$$

we get $a = aa^*a$ and $a^{k+1} = aa^*a = a$.

(v) $\Rightarrow$ (iii): Let $a \in \mathcal{R}^{\#}$, $a^* = a^{\#}$ and $a^{k-1} = a^{\#}$. Then $aa^*a = aa^{\#}a = a$ and $a^k = aa^{\#} = aa^*$. □

Observe that, without the assumptions $x, y \in \mathcal{R}^{-1}$, Theorem 3.1 is true for complex matrices [3, Theorem 4.2] and for Hilbert space operators. This can be proved using the corresponding decomposition of the operator. Also, we can see that the condition $x \in \mathcal{R}^{-1}$ is satisfied in both these cases using the mentioned decompositions.

We prove the next characterization of EP and $(k+1)$-potent elements of rings with involution, generalizing [3, Theorem 4.3], where the result was verified using the corresponding matrix representation.

Theorem 3.2 *Let $a \in \mathcal{R}$ and let $k \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) *a is EP and $(k+1)$-potent;*
(ii) *$a \in \mathcal{R}^{\dagger}$ and $a^k = aa^{\dagger}$.*

Proof (i) $\Rightarrow$ (ii): Suppose that $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$, $a^\dagger = a^\#$ and $a^{k+1} = a$. Then $a^k = a^{k+1}a^\# = aa^\dagger$.

(ii) $\Rightarrow$ (i): If $a \in \mathcal{R}^\dagger$ and $a^k = aa^\dagger$, then

$$a^{k+1} = a^k a = aa^\dagger a = a.$$

Note that, for $k > 1$, $a^{k+1} = a$ is equivalent to $a \in \mathcal{R}^\#$ and $a^\# = a^{k-1}$. Therefore, $a^\# a = a^k = aa^\dagger$ is self-adjoint. By Theorem 1.2, we deduce that a is EP.

If $k = 1$, then $a = aa^\dagger$ implies $a^2 = a$. So, $a \in \mathcal{R}^\#$ and $a^\# = a$. Since $aa^\# = a^2 = a = aa^\dagger$ is self-adjoint, a is EP. □

In the following theorem, we give two characterizations of the elements in a ring with involution which are EP and partial isometries and we recover [3, Theorem 4.4].

Theorem 3.3 *Let $a \in \mathcal{R}$. Then the following conditions are equivalent:*

(i) *a is EP and a partial isometry;*
(ii) *$a \in \mathcal{R}^\#$ and $a = a^* a^2$;*
(iii) *$a \in \mathcal{R}^\#$ and $a = a^2 a^*$.*

Proof (i) $\Rightarrow$ (ii) $\wedge$ (iii): Let a be EP and a partial isometry. By [12, Theorem 2.3], $a \in \mathcal{R}^\#$ and $a^* = a^\#$. Hence, $a = a^2 a^\# = a^2 a^*$ and $a = a^\# a^2 = a^* a^2$.

(ii) $\Rightarrow$ (i): If $a \in \mathcal{R}^\#$ and $a = a^* a^2$, then $aa^\# = a^* a^2 a^\# = a^* a$ is self-adjoint. Using Theorem 1.2, we conclude that a is EP. From $a = a(a^\# a) = aa^* a$, we have that a is a partial isometry.

(iii) $\Rightarrow$ (i): Similar to (ii) $\Rightarrow$ (i). □

The next result concerns k-potent elements of a ring and extends [3, Theorem 4.5].

Theorem 3.4 *Let $a \in \mathcal{R}$ and let $k \in \mathbb{N}$, $k > 1$. Then the following conditions are equivalent:*

(i) *a is k-potent;*
(ii) *$a \in \mathcal{R}^\#$ and $a^{k+1} = a^2$.*

Proof (i) $\Rightarrow$ (ii): The equality $a^k = a$ gives $a^{k+1} = aa^k = a^2$,

$$a\mathcal{R} = a^k\mathcal{R} \subseteq a^2\mathcal{R} \subseteq a\mathcal{R}$$

and

$$\mathcal{R}a = \mathcal{R}a^k \subseteq \mathcal{R}a^2 \subseteq \mathcal{R}a.$$

Therefore, $a\mathcal{R} = a^2\mathcal{R}$ and $\mathcal{R}a = \mathcal{R}a^2$, which implies that $a \in \mathcal{R}^\#$.

(ii) $\Rightarrow$ (i): Assume that $a \in \mathcal{R}^\#$ and $a^{k+1} = a^2$. Then

$$a^k = a^{k+1}a^\# = a^2 a^\# = a.$$

□

EP elements of a ring with involution are characterized in the following theorem without using the corresponding matrix decomposition as in the proof of [3, Theorem 4.7].

Theorem 3.5 *Let $a \in \mathcal{R}$. Then the following conditions are equivalent:*

(i) *a is EP;*
(ii) *$a \in \mathcal{R}^\#$ and $a^2\mathcal{R} = a^*\mathcal{R}$;*

(iii) $a \in \mathcal{R}^{\#}$ *and* a^2 *is EP.*

Proof (i) $\Rightarrow$ (ii): By [16, Corollary 3] or [14, Theorem 3.1], a is EP if and only if $a \in \mathcal{R}^{\#}$ and $a\mathcal{R} = a^*\mathcal{R}$. Since a is EP, we also have that $a \in \mathcal{R}^{\#}$, which yields $a\mathcal{R} = a^2\mathcal{R}$. So, $a^2\mathcal{R} = a\mathcal{R} = a^*\mathcal{R}$.

(ii) $\Rightarrow$ (iii): If $a \in \mathcal{R}^{\#}$ and $a^2\mathcal{R} = a^*\mathcal{R}$, then $a\mathcal{R} = a^2\mathcal{R} = a^*\mathcal{R}$. Now, from

$$(a^*)^2\mathcal{R} \subseteq a^*\mathcal{R} = a^2\mathcal{R}$$

and

$$a^2\mathcal{R} = a^*\mathcal{R} = (a^{\#}a^2)^*\mathcal{R} = (a^2)^*(a^{\#})^*\mathcal{R} \subseteq (a^*)^2\mathcal{R},$$

it follows $(a^*)^2\mathcal{R} = a^2\mathcal{R}$. Because

$$a^2\mathcal{R} = a^4(a^{\#})^2\mathcal{R} \subseteq a^4\mathcal{R} \subseteq a^2\mathcal{R}$$

and

$$\mathcal{R}a^2 = \mathcal{R}(a^{\#})^2a^4 \subseteq \mathcal{R}a^4 \subseteq \mathcal{R}a^2,$$

we get $a^2\mathcal{R} = a^4\mathcal{R}$ and $\mathcal{R}a^2 = \mathcal{R}a^4$. Thus $a^2 \in \mathcal{R}^{\#}$. Applying [16, Corollary 3] or [14, Theorem 3.1] again, we deduce that a^2 is EP.

(iii) $\Rightarrow$ (i): Let $a \in \mathcal{R}^{\#}$ and a^2 is EP. Then

$$a\mathcal{R} = a^2\mathcal{R} = (a^*)^2\mathcal{R} \subseteq a^*\mathcal{R}$$

and

$$a^*\mathcal{R} = (a^{\#}a^2)^*\mathcal{R} \subseteq (a^*)^2\mathcal{R} = a^2\mathcal{R} = a\mathcal{R}$$

imply $a\mathcal{R} = a^*\mathcal{R}$. Hence, we conclude that a is EP. □

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