Annali di Matematica (2015) 194:1135–1155
DOI 10.1007/s10231-014-0414-6

Spectral multipliers on Heisenberg–Reiter and related groups

Alessio Martini

Received: 19 November 2013 / Accepted: 19 February 2014 / Published online: 7 March 2014

Abstract Let L be a homogeneous sublaplacian on a 2-step stratified Lie group G of topological dimension d and homogeneous dimension Q. By a theorem due to Christ and to Mauceri and Meda, an operator of the form $F(L)$ is bounded on L^p for $1 < p < \infty$ if F satisfies a scale-invariant smoothness condition of order $s > Q/2$. Under suitable assumptions on G and L, here we show that a smoothness condition of order $s > d/2$ is sufficient. This extends to a larger class of 2-step groups the results for the Heisenberg and related groups by Müller and Stein and by Hebisch and for the free group $N_{3,2}$ by Müller and the author.

Keywords Nilpotent Lie groups · Heisenberg–Reiter groups · Spectral multipliers · Sublaplacians · Mihlin–Hörmander multipliers · Singular integral operators

Mathematics Subject Classification (2000) 43A22 · 42B15

1 Introduction

Let L be a homogeneous sublaplacian on a stratified Lie group G of homogeneous dimension Q. Since L is a positive self-adjoint operator on $L^2(G)$, a functional calculus for L is defined via the spectral theorem and, for all Borel functions $F : \mathbb{R} \to \mathbb{C}$, the operator $F(L)$ is bounded on $L^2(G)$ whenever the "spectral multiplier" F is bounded. As for the L^p-boundedness for $p \neq 2$ of $F(L)$, a sufficient condition in terms of smoothness properties of the multiplier F is given by a theorem of Mihlin–Hörmander type due to Christ [4] and Mauceri and Meda [20]: the operator $F(L)$ is of weak type (1, 1) and bounded on $L^p(G)$ for all $p \in]1, \infty[$ whenever

$$\|F\|_{MW_2^s} := \sup_{t>0} \|F(t\cdot)\,\eta\|_{W_2^s} < \infty$$

A. Martini (✉)
Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel,
Ludewig-Meyn-Str. 4, 24118 Kiel, Germany
e-mail: martini@math.uni-kiel.de

for some $s > Q/2$, where $W_2^s(\mathbb{R})$ is the L^2 Sobolev space of fractional order s, and $\eta \in C_c^\infty(]0,\infty[)$ is a nontrivial auxiliary function.

A natural question that arises is whether the smoothness condition $s > Q/2$ is sharp. This is clearly true when G is abelian, so Q coincides with the topological dimension d of G, and L is essentially the Laplace operator on $\mathbb{R}^d$. Take, however, the smallest nonabelian example of a stratified group, that is, the Heisenberg group H_1, which is defined by endowing $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with the group law

$$(x, y, u) \cdot (x', y', u') = \left(x + x', y + y', u + u' + \left(xy' - x'y\right)/2\right) \tag{1}$$

and with the automorphic dilations

$$\delta_t(x, y, u) = \left(tx, ty, t^2 u\right). \tag{2}$$

H_1 is a 2-step stratified group, and the homogeneous dimension of H_1 is 4. Nevertheless, a result by Müller and Stein [23] and Hebisch [12] shows that, for a homogeneous sublaplacian on H_1, the smoothness condition on the multiplier can be pushed down to $s > d/2$, where $d = 3$ is the topological dimension of H_1 (in [23], it is also proved that the condition $s > d/2$ is sharp). Such an improvement of the Christ–Mauceri–Meda theorem holds not only for H_1, but for the larger class of Métivier groups (and for direct products of Métivier and abelian groups), and also for differential operators other than sublaplacians (see, e.g., [13,17]); moreover, as shown subsequently by Cowling and Sikora [5] (see also [6]), the sharp result on H_1 can be obtained by transplantation from an analogous result for a distinguished sublaplacian on the (nonstratified) group SU_2 (which in turn improves, in the case of SU_2, an extension of the Christ–Mauceri–Meda theorem to spaces of homogeneous type [1,7,11]). However, it is still an open question whether, for a general stratified Lie group (or even for a general 2-step stratified group), the homogeneous dimension in the smoothness condition can be replaced by the topological dimension.

The aim of this paper is to extend the class of the 2-step stratified groups and sublaplacians for which the smoothness condition in the multiplier theorem can be pushed down to half the topological dimension.

Take for instance the Heisenberg–Reiter group H_{d_1,d_2} (cf. [27]), defined by endowing $\mathbb{R}^{d_2 \times d_1} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with the group law (1) and the automorphic dilations (2); here, however, $\mathbb{R}^{d_2 \times d_1}$ is the set of the real $d_2 \times d_1$ matrices, and the products xy', $x'y$ in (1) are interpreted in the sense of matrix multiplication. H_{d_1,d_2} is a 2-step stratified group of homogeneous dimension $Q = d_1 d_2 + d_1 + 2d_2$ and topological dimension $d = d_1 d_2 + d_1 + d_2$. Despite the formal similarity with H_1, the group H_{d_1,d_2} does not fall into the class of Métivier groups, unless $d_2 = 1$ (in fact, $\mathrm{H}_{d_1,1}$ is the $(2d_1+1)$-dimensional Heisenberg group H_{d_1}). Nevertheless, the technique presented here allows one to handle the case $d_2 > 1$ too.

Namely, let $X_{1,1}, \dots, X_{d_2,d_1}, Y_1, \dots, Y_{d_1}, U_1, \dots, U_{d_2}$ be the left-invariant vector fields on H_{d_1,d_2} extending the standard basis of $\mathbb{R}^{d_2 \times d_1} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ at the identity, and define the homogeneous sublaplacian L by

$$L = -\sum_{j=1}^{d_1} \sum_{k=1}^{d_2} X_{k,j}^2 - \sum_{j=1}^{d_1} Y_j^2.$$

Then, a particular instance of our main result reads as follows.

Theorem 1 *Suppose that a function* $F : \mathbb{R} \to \mathbb{C}$ *satisfies*

$$\|F\|_{MW_2^s} < \infty$$

for some $s > d/2$. *Then, the operator* $F(L)$ *is of weak type* $(1, 1)$ *and bounded on* $L^p(\mathrm{H}_{d_1,d_2})$ *for all* $p \in]1, \infty[$.

To the best of our knowledge, this result is new, at least in the case $d_2 > d_1$. In fact, in the case $d_2 \le d_1$, the extension described in [17] of the technique of [12,13] would give the same result. However, the technique presented here is different, and yields the result irrespective of the parameters d_1, d_2.

The left quotient of H_{d_1,d_2} by the subgroup $\mathbb{R}^{d_2 \times d_1} \times \{0\} \times \{0\}$ gives a homogeneous space diffeomorphic to $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, and the sublaplacian L corresponds in the quotient to a Grushin operator. In recent joint works with Sikora [18] and Müller [14], we proved for these Grushin operators on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ a sharp spectral multiplier theorem of Mihlin–Hörmander type, where the smoothness requirement is again half the topological dimension of the ambient space.

The proofs in [14,18] rely heavily on properties of the eigenfunction expansions for the Hermite operators. Since a homogeneous sublaplacian on a 2-step stratified group reduces to a Hermite operator in almost all irreducible unitary representations of the group, it is conceivable that an adaptation of the methods of [14,18] may give an improvement to the multiplier theorem for 2-step stratified groups, even outside of the Métivier setting. A first result in this direction is shown in [19], where the free 2-step nilpotent Lie group $N_{3,2}$ on three generators is considered, and properties of Laguerre polynomials are exploited (somehow in the spirit of [21,23,24]). The argument presented here refines and extends the one in [19].

Theorem 1 above is just a particular case of the result presented here, and we refer the reader to the next section for a precise statement. We remark that the analog of Theorem 1 holds on H_{d_1,d_2} when the sublaplacian L has the more general form

$$L = -\sum_{j=1}^{d_1} \sum_{k,k'=0}^{d_2} a^j_{k,k'} X_{k,j} X_{k',j} \tag{3}$$

where $X_{0,j} = Y_j$ and $(a^j_{k,k'})_{k,k'=0,\dots,d_2}$ is a positive-definite symmetric matrix for all $j \in \{1, \dots, d_1\}$. Other groups can be considered too, e.g., the complexification of a Heisenberg–Reiter group, or the quotient of the direct product of $\mathrm{H}_{1,3}$ and $N_{3,2}$ given by identifying the respective centers.

2 The general setting

Let G be a connected, simply connected nilpotent Lie group of step 2. Recall that, via exponential coordinates, G may be identified with its Lie algebra $\mathfrak{g}$, that is, the tangent space of G at the identity. In turn, $\mathfrak{g}$ may be identified with the Lie algebra of left-invariant vector fields on G. We refer to [9] for the basic definitions and further details.

Let $\mathfrak{g}$ be decomposed as $\mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$, and let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{v}$. The sublaplacian L associated with the inner product is defined by $L = -\sum_j X_j^2$, where $\{X_j\}_j$ is any orthonormal basis of $\mathfrak{v}$. Note that, vice versa, by the Poincaré–Birkhoff–Witt theorem, any second-order operator L of the form $-\sum_j X_j^2$ for some basis $\{X_j\}_j$ of $\mathfrak{g}$ modulo $\mathfrak{z}$ determines uniquely a linear complement $\mathfrak{v} = \mathrm{span}\{X_j\}_j$ of $\mathfrak{z}$ and an inner product on $\mathfrak{v}$ such that $\{X_j\}_j$ is orthonormal.

Let $\mathfrak{z}^*$ be the dual of $\mathfrak{z}$ and, for all $\eta \in \mathfrak{z}^*$, define J_η as the linear endomorphism of $\mathfrak{v}$ such that $\eta([z, z']) = \langle J_\eta z, z' \rangle$ for all $z, z' \in \mathfrak{v}$. Clearly, J_η is skewadjoint with respect to the inner product; hence, J_η^2 is self-adjoint and negative semidefinite, with even rank, for all $\eta \in \mathfrak{z}^*$. Set moreover $\dot{\mathfrak{z}} = \mathfrak{z}^* \setminus \{0\}$.

Assumption (A) *There exist integers $r_1, \dots, r_{d_1} > 0$ and an orthogonal decomposition $\mathfrak{v} = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_{d_1}$ such that, if $P_1, \dots, P_{d_1}$ are the corresponding orthogonal projections, then $J_\eta P_j = P_j J_\eta$ and $J_\eta^2 P_j$ has rank $2r_j$ and a unique nonzero eigenvalue for all $\eta \in \dot{\mathfrak{z}}$ and all $j \in \{1, \dots, d_1\}$.*

Note that from Assumption (A) it follows that $J_\eta \neq 0$ for all $\eta \in \dot{\mathfrak{z}}$. Therefore $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$, that is, the decomposition $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ is a stratification of $\mathfrak{g}$, and the sublaplacian L is hypoelliptic.

In fact, J_η has constant rank $2(r_1 + \cdots + r_k)$ for all $\eta \in \dot{\mathfrak{z}}$. If J_η is invertible for all $\eta \in \dot{\mathfrak{z}}$, then G is a Métivier group, and if in particular $J_\eta^2 = -|\eta|^2 \mathrm{id}_\mathfrak{v}$ for some inner product norm $|\cdot|$ on $\mathfrak{z}^*$, then G is an H-type group. The main novelty of our Assumption (A) is that it allows J_η to have a nonzero kernel when $\eta \in \dot{\mathfrak{z}}$, although the dimension of the kernel must be constant.

The fact that J_η has constant rank for $\eta \in \dot{\mathfrak{z}}$ depends only on the algebraic structure of G. What depends on the inner product, that is, on the sublaplacian L, are the values and multiplicities of the eigenvalues of the J_η. The above Assumption (A) asks for a sort of simultaneous diagonalizability of the J_η.

Under our Assumption (A) on the group G and the sublaplacian L, we are able to prove the following multiplier theorem.

Theorem 2 *Suppose that a function $F : \mathbb{R} \to \mathbb{C}$ satisfies*

$$\|F\|_{MW_2^s} < \infty$$

for some $s > (\dim G)/2$. Then, the operator $F(L)$ is of weak type $(1, 1)$ and bounded on $L^p(G)$ for all $p \in]1, \infty[$.

The previously mentioned Heisenberg–Reiter groups H_{d_1,d_2} satisfy Assumption (A), where the inner product is determined by the sublaplacian (3), and the orthogonal decomposition of the first layer is given by the natural isomorphism $\mathbb{R}^{d_2 \times d_1} \times \mathbb{R}^{d_1} \cong (\mathbb{R}^{d_2} \times \mathbb{R})^{d_1}$. Other examples are the free 2-step nilpotent Lie group $N_{3,2}$ on 3 generators, considered in [19], and its complexification $N_{3,2}^{\mathbb{C}}$. Moreover, if G_1 and G_2 satisfy Assumption (A), and their centers have the same dimension, then the quotient of $G_1 \times G_2$ given by any linear identification of the centers satisfy Assumption (A). Note that the direct product $G_1 \times G_2$ itself does not satisfy Assumption (A), but an adaptation of the argument presented here allows one to consider that case too. We postpone to the end of this paper a more detailed discussion of these remarks.

From now on, unless otherwise specified, we assume that G and L are a 2-step stratified group and a homogeneous sublaplacian on G satisfying Assumption (A). Since L is a left-invariant operator, so is any operator of the form $F(L)$. Let $\mathcal{K}_{F(L)}$ denote the convolution kernel of $F(L)$. As shown, e.g., by [17, Theorem 4.6], the previous theorem is a consequence of the following estimate.

Proposition 3 *For all $s > (\dim G)/2$, there exists a weight $w_s : G \to [1, \infty[$ such that $w_s^{-1} \in L^2(G)$ and, for all compact sets $K \subseteq \mathbb{R}$ and for all functions $F : \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} F \subseteq K$,*

$$\|w_s \, \mathcal{K}_{F(L)}\|_2 \leq C_{K,s} \|F\|_{W_2^s}; \tag{4}$$

in particular,

$$\|\mathcal{K}_{F(L)}\|_1 \leq C_{K,s} \|F\|_{W_2^s}. \tag{5}$$

The rest of the paper, except for the last section, is devoted to the proof of this estimate.

3 The joint functional calculus

Let $d_2 = \dim \mathfrak{z}$, and let $U_1, \ldots, U_{d_2}$ be any basis of the center $\mathfrak{z}$. Let moreover the "partial sublaplacian" L_j be defined as $L_j = -\sum_l X_{j,l}^2$, where $\{X_{j,l}\}_l$ is any orthonormal basis of $\mathfrak{v}_j$, for all $j \in \{1, \ldots, d_1\}$; in particular $L = L_1 + \cdots + L_{d_1}$. Then, the left-invariant differential operators

$$L_1, \ldots, L_{d_1}, -iU_1, \ldots, -iU_{d_2} \tag{6}$$

are essentially self-adjoint and commute strongly; hence, they admit a joint functional calculus (see, e.g., [16]). Therefore, if $\mathbf{L}$ and $\mathbf{U}$ denote the "vectors of operators" $(L_1, \ldots, L_{d_1})$ and $(-iU_1, \ldots, -iU_{d_2})$, and if we identify $\mathfrak{z}^*$ with $\mathbb{R}^{d_2}$ via the dual basis of $U_1, \ldots, U_n$, then, for all bounded Borel functions $H : \mathbb{R}^{d_1} \times \mathfrak{z}^* \to \mathbb{C}$, the operator $H(\mathbf{L}, \mathbf{U})$ is defined and bounded on $L^2(G)$. Moreover, $H(\mathbf{L}, \mathbf{U})$ is left-invariant, and we can express its convolution kernel $\mathcal{K}_{H(\mathbf{L},\mathbf{U})}$ in terms of Laguerre functions.

Namely, for all $n, k \in \mathbb{N}$, let

$$L_n^{(k)}(t) = \frac{t^{-k}e^t}{n!} \left(\frac{d}{dt}\right)^n \left(t^{k+n}e^{-t}\right)$$

be the n-th Laguerre polynomial of type k, and define

$$\mathcal{L}_n^{(k)}(t) = (-1)^n e^{-t} L_n^{(k)}(2t).$$

Note that, by Assumption (A), for all $\eta \in \dot{\mathfrak{z}}$ and $j \in \{1, \ldots, d_1\}$,

$$J_\eta^2 P_j = -\left(b_j^\eta\right)^2 P_j^\eta$$

for some orthogonal projection P_j^η of rank $2r_j$ and some $b_j^\eta > 0$. Set moreover

$$\bar{P}_j^\eta = P_j - P_j^\eta.$$

Modulo reordering the $\mathfrak{v}_j$ in the decomposition of $\mathfrak{v}$, we may suppose that there exists $\tilde{d}_1 \in \{0, \ldots, d_1\}$ such that $\dim \mathfrak{v}_j > 2r_j$ if $j \le \tilde{d}_1$, and $\dim \mathfrak{v}_j = 2r_j$ if $j > \tilde{d}_1$. In particular, $\bar{P}_j^\eta = 0$ and $P_j^\eta = P_j$ for all $j > \tilde{d}_1$ and $\eta \in \dot{\mathfrak{z}}$. We will also use the abbreviations $r = (r_1, \ldots, r_{d_1})$, $\mathbb{R}^r = \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_{d_1}}$, $\mathbb{N}^r = \mathbb{N}^{r_1} \times \cdots \times \mathbb{N}^{r_{d_1}}$, $|r| = r_1 + \cdots + r_{d_1}$. Moreover $\langle \cdot, \cdot \rangle$ will also denote the duality pairing $\mathfrak{z}^* \times \mathfrak{z} \to \mathbb{R}$.

Proposition 4 *Let $H : \mathbb{R}^{d_1} \times \mathfrak{z}^* \to \mathbb{C}$ be in the Schwartz class, and set*

$$\begin{aligned} m(n, \mu, \eta) = H\Big(&(2n_1 + r_1)b_1^\eta + \mu_1, \ldots, \left(2n_{\tilde{d}_1} + r_{\tilde{d}_1}\right) b_{\tilde{d}_1}^\eta + \mu_{\tilde{d}_1}, \\ &\left(2n_{\tilde{d}_1+1} + r_{\tilde{d}_1+1}\right) b_{\tilde{d}_1+1}^\eta, \ldots, \left(2n_{d_1} + r_{d_1}\right) b_{d_1}^\eta, \eta\Big) \end{aligned} \tag{7}$$

for all $n \in \mathbb{N}^{d_1}$, $\mu \in \mathbb{R}^{\tilde{d}_1}$, $\eta \in \dot{\mathfrak{z}}$. Then, for all $(z, u) \in G$,

$$\begin{aligned} \mathcal{K}_{H(\mathbf{L},\mathbf{U})}(z, u) = \frac{2^{|r|}}{(2\pi)^{\dim G}} \int_{\dot{\mathfrak{z}}} \int_{\mathfrak{v}} \sum_{n \in \mathbb{N}^{d_1}} m\left(n, \left(|\bar{P}_1^\eta \xi|^2, \ldots, |\bar{P}_{\tilde{d}_1}^\eta \xi|^2\right), \eta\right) \\ \times \left[\prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(r_j - 1)}\left(|P_j^\eta \xi|^2 / b_j^\eta\right)\right] e^{i\langle \xi, z\rangle} e^{i\langle \eta, u\rangle}\, d\xi\, d\eta. \end{aligned} \tag{8}$$

Proof For all $\eta \in \dot{\mathfrak{z}}$ and $j \in \{1, \ldots, d_1\}$, let $E^\eta_{j,1}, \bar{E}^\eta_{j,1}, \ldots, E^\eta_{j,r_j}, \bar{E}^\eta_{j,r_j}$ be an orthonormal basis of the range of P^η_j such that

$$J_\eta E^\eta_{j,l} = b^\eta_j \bar{E}^\eta_{j,l}, \qquad J_\eta \bar{E}^\eta_{j,l} = -b^\eta_j E^\eta_{j,l}, \qquad \text{for } l = 1, \ldots, r_j.$$

Hence, for all $z \in \mathfrak{v}$, $\eta \in \dot{\mathfrak{z}}$, and $j \in \{1, \ldots, d_1\}$, we can write

$$P^\eta_j z = \sum_{l=1}^{r_j} \left(z^\eta_{j,l} E^\eta_j + \bar{z}^\eta_{j,l} \bar{E}^\eta_{j,l} \right)$$

for some uniquely determined $z^\eta_{j,l}, \bar{z}^\eta_{j,l} \in \mathbb{R}$; set then $z^\eta_j = (z^\eta_{j,1}, \ldots, z^\eta_{j,r_j})$, $\bar{z}^\eta_j = (\bar{z}^\eta_{j,1}, \ldots, \bar{z}^\eta_{j,r_j})$, and moreover $z^\eta = (z^\eta_1, \ldots, z^\eta_{d_1})$ and $\bar{z}^\eta = (\bar{z}^\eta_1, \ldots, \bar{z}^\eta_{d_1})$.

For all $\eta \in \dot{\mathfrak{z}}$ and all $\rho \in \ker J_\eta$, an irreducible unitary representation $\pi_{\eta,\rho}$ of G on $L^2(\mathbb{R}^r)$ is defined by

$$\pi_{\eta,\rho}(z, u)\phi(v) = e^{i\langle \eta, u\rangle} e^{i\langle \rho, \bar{P}^\eta z\rangle} e^{i\sum_{j=1}^{d_1} b^\eta_j \langle v_j + z^\eta_j/2, \bar{z}^\eta_j\rangle} \phi(z^\eta + v)$$

for all $(z, u) \in G$, $v \in \mathbb{R}^r$, $\phi \in L^2(\mathbb{R}^r)$, where $\bar{P}^\eta = \bar{P}^\eta_1 + \cdots + \bar{P}^\eta_{\tilde{d}_1}$ is the orthogonal projection onto $\ker J_\eta$. Following, e.g., [2, §2], one can see that these representations are sufficient to write the Plancherel formula for the group Fourier transform of G, and the corresponding Fourier inversion formula:

$$f(z, u) = (2\pi)^{|r|-\dim G} \int_{\dot{\mathfrak{z}}} \int_{\ker J_\eta} \operatorname{tr}(\pi_{\eta,\rho}(z, u)\, \pi_{\eta,\rho}(f)) \prod_{j=1}^{d_1} \left(b^\eta_j\right)^{r_j} \, d\rho \, d\eta \tag{9}$$

for all $f : G \to \mathbb{C}$ in the Schwartz class and all $(z, u) \in G$, where $\pi_{\eta,\rho}(f) = \int_G f(g)\, \pi_{\eta,\rho}(g^{-1})\, dg$.

Fix $\eta \in \dot{\mathfrak{z}}$ and $\rho \in \ker J_\eta$. The operators (6) are represented in $\pi_{\eta,\rho}$ as

$$d\pi_{\eta,\rho}(L_j) = -\Delta^2_{v_j} + \left(b^\eta_j\right)^2 |v_j|^2 + |P_j \rho|^2, \qquad d\pi_{\eta,\rho}(-iU_k) = \eta_k, \tag{10}$$

for all $j \in \{1, \ldots, d_1\}$ and $k \in \{1, \ldots, d_2\}$, where $v_j \in \mathbb{R}^{r_j}$ denotes the j-th component of $v \in \mathbb{R}^r$, and Δ_{v_j} denotes the corresponding partial Laplacian. Let h_ℓ denote the ℓ-th Hermite function, that is,

$$h_\ell(t) = (-1)^\ell (\ell!\, 2^\ell \sqrt{\pi})^{-1/2} e^{t^2/2} \left(\frac{d}{dt}\right)^\ell e^{-t^2},$$

and, for all $\omega \in \mathbb{N}^r$, define $\tilde{h}_{\eta,\omega} : \mathbb{R}^r \to \mathbb{R}$ by

$$\tilde{h}_{\eta,\omega} = \tilde{h}_{\eta,\omega,1} \otimes \cdots \otimes \tilde{h}_{\eta,\omega,d_1}, \qquad \tilde{h}_{\eta,\omega,j}(v_j) = \left(b^\eta_j\right)^{r_j/4} \prod_{l=1}^{r_j} h_{\omega_{j,l}} \left(\left(b^\eta_j\right)^{1/2} v_{j,l} \right),$$

for all $j \in \{1, \ldots, d_1\}$, where $\omega_{j,l}$ and $v_{j,l}$ denote the l-th components of $\omega_j \in \mathbb{N}^{r_j}$ and $v_j \in \mathbb{R}^{r_j}$. Then, $\{\tilde{h}_{\eta,\omega}\}_{\omega \in \mathbb{N}^r}$ is a complete orthonormal system for $L^2(\mathbb{R}^r)$, made of joint eigenfunctions of the operators (10). In fact,

$$\begin{aligned} d\pi_{\eta,\rho}(L_j)\tilde{h}_{\eta,\omega} &= \left(\left(2|\omega_j| + r_j\right) b^\eta_j + |P_j \rho|^2 \right) \tilde{h}_{\eta,\omega}, \\ d\pi_{\eta,\rho}(-iU_k)\tilde{h}_{\eta,\omega} &= \eta_k \, \tilde{h}_{\eta,\omega}, \end{aligned} \tag{11}$$

where $|\omega_j| = \omega_{j,1} + \cdots + \omega_{j,r_j}$; it should be observed that $P_j\rho = 0$ if $j > \tilde{d}_1$.

Since $H : \mathbb{R}^{d_1} \times \mathfrak{z}^* \to \mathbb{C}$ is in the Schwartz class, $\mathcal{K}_{H(\mathbf{L},\mathbf{U})} : G \to \mathbb{C}$ is in the Schwartz class too (see [3, Theorem 5.2] or [15, §4.2]). Moreover,

$$\pi_{\eta,\rho}\left(\mathcal{K}_{H(\mathbf{L},\mathbf{U})}\right)\tilde{h}_{\eta,\omega} = m\left(\left(|\omega_1|,\ldots,|\omega_{d_1}|\right), \left(|P_1\rho|^2,\ldots,|P_{\tilde{d}_1}\rho|^2\right), \eta\right)\tilde{h}_{\eta,\omega}$$

by (11) and [22, Proposition 1.1]; hence, if $\varphi_{\eta,\rho,\omega}(z,u) = \langle \pi_{\eta,\rho}(z,u)\tilde{h}_{\eta,\omega}, \tilde{h}_{\eta,\omega}\rangle$ is the corresponding diagonal matrix coefficient of $\pi_{\eta,\rho}$, then

$$\langle \pi_{\eta,\rho}(z,u)\,\pi_{\eta,\rho}\left(\mathcal{K}_{H(\mathbf{L},\mathbf{U})}\right)\tilde{h}_{\eta,\omega}, \tilde{h}_{\eta,\omega}\rangle = m\left((|\omega_j|)_{j\le d_1}, \left(|P_j\rho|^2\right)_{j\le \tilde{d}_1}, \eta\right)\varphi_{\eta,\rho,\omega}(z,u).$$

Therefore, (9) gives that

$$\mathcal{K}_{H(\mathbf{L},\mathbf{U})}(z,u) = (2\pi)^{|r|-\dim G}\int_{\mathfrak{z}}\int_{\ker J_\eta}\sum_{n\in\mathbb{N}^{d_1}} m\left(n, \left(|P_j\rho|^2\right)_{j\le\tilde{d}_1}, \eta\right)\psi_{\eta,\rho,n}(z,u)\prod_{j=1}^{d_1}\left(b_j^\eta\right)^{r_j} d\rho\, d\eta, \tag{12}$$

where

$$\psi_{\eta,\rho,n}(z,u) = \sum_{\substack{\omega\in\mathbb{N}^r \\ |\omega_1|=n_1,\ldots,|\omega_{d_1}|=n_{d_1}}} \varphi_{\eta,\rho,\omega}(z,u).$$

On the other hand,

$$\varphi_{\eta,\rho,\omega}(z,u) = e^{i\langle\eta,u\rangle}e^{i\langle\rho,\bar{P}^\eta z\rangle}\prod_{j=1}^{d_1}\prod_{l=1}^{r_j}\Bigg[\left(b_j^\eta\right)^{1/2} \times\int_{\mathbb{R}} e^{ib_j^\eta s\bar{z}_{j,l}^\eta} h_{\omega_{j,l}}\left(\left(b_j^\eta\right)^{1/2}(s+z_{j,l}^\eta/2)\right) h_{\omega_{j,l}}\left(\left(b_j^\eta\right)^{1/2}(s-z_{j,l}^\eta/2)\right) ds\Bigg].$$

The last integral is essentially the Fourier–Wigner transform of a pair of Hermite functions, whose bidimensional Fourier transform is a Fourier–Wigner transform too [10, formula (1.90)]. The parity properties of the Hermite functions then yield

$$\varphi_{\eta,\rho,\omega}(z,u) = e^{i\langle\eta,u\rangle}e^{i\langle\rho,\bar{P}^\eta z\rangle}\prod_{j=1}^{d_1}\prod_{l=1}^{j}\Bigg[\frac{(-1)^{\omega_{j,l}}}{\pi\, b_j^\eta}\int_{\mathbb{R}\times\mathbb{R}} e^{i\theta_1 z_{j,l}^\eta}e^{i\theta_2\bar{z}_{j,l}^\eta} \times\int_{\mathbb{R}} e^{it\left(2\theta_1/\left(b_j^\eta\right)^{1/2}\right)} h_{\omega_{j,l}}\left(t+\theta_2/\left(b_j^\eta\right)^{1/2}\right) h_{\omega_{j,l}}\left(t-\theta_2/\left(b_j^\eta\right)^{1/2}\right) dt\, d\theta_1\, d\theta_2\Bigg].$$

Since the Fourier–Wigner transform of a pair of Hermite functions can be expressed in terms of Laguerre polynomials (see [10, Theorem 1.104] or [26, Theorem 1.3.4]), we obtain that

$$\varphi_{\eta,\rho,\omega}(z,u) = \frac{e^{i\langle\eta,u\rangle}e^{i\langle\rho,\bar{P}^\eta z\rangle}}{\pi^{|r|}}\int_{\mathbb{R}^r\times\mathbb{R}^r} e^{i\langle\zeta_1,z^\eta\rangle}e^{i\langle\zeta_2,\bar{z}^\eta\rangle} \times\prod_{j=1}^{d_1}\left[\left(b_j^\eta\right)^{-r_j}\prod_{l=1}^{r_j}\mathcal{L}_{\omega_{j,l}}^{(0)}\left(\left(\zeta_{1,j,l}^2+\zeta_{2,j,l}^2\right)/b_j^\eta\right)\right] d\zeta_1\, d\zeta_2$$

Consequently, for all $n \in \mathbb{N}^{d_1}$,

$$\psi_{\eta,\rho,n}(z,u) = \frac{e^{i\langle \eta,u\rangle} e^{i\langle \rho, \bar{P}^\eta z\rangle}}{\pi^{|r|}} \int_{\mathbb{R}^r \times \mathbb{R}^r} e^{i\langle \zeta_1, z^\eta\rangle} e^{i\langle \zeta_2, \bar{z}^\eta\rangle}$$
$$\times \prod_{j=1}^{d_1} \left[\left(b_j^\eta\right)^{-r_j} \mathcal{L}_{n_j}^{(r_j-1)} \left(\left(|\zeta_{1,j}|^2 + |\zeta_{2,j}|^2\right) / b_j^\eta \right) \right] d\zeta_1 \, d\zeta_2 \tag{13}$$

[9, §10.12, formula (41)]. The conclusion then follows by plugging (13) into (12) and performing a change of variable by rotation in the inner integrals. □

4 A weighted Plancherel estimate

Proposition 4 expresses the convolution kernel $\mathcal{K}_{H(\mathbf{L},\mathbf{U})}$ as the inverse Fourier transform of a function of the multiplier H. Due to the properties of the Fourier transform, it is not unreasonable to think that multiplying the kernel by a polynomial weight might correspond to taking derivatives of the multiplier. As a matter of fact, the presence of the Laguerre expansion leads us to consider both "discrete" and "continuous" derivatives of the reparametrization $m : \mathbb{N}^{d_1} \times \mathbb{R}^{\tilde{d}_1} \times \mathfrak{z} \to \mathbb{C}$ of the multiplier H given by (7).

For convenience, set $\mathcal{L}_n^{(k)} = 0$ for all $n < 0$. From the properties of Laguerre polynomials (see, e.g., [9, §10.12]), one can easily derive the following identities.

Lemma 5 *For all $k, n, m \in \mathbb{N}$ and $t \in \mathbb{R}$,*

$$\mathcal{L}_n^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) + \mathcal{L}_n^{(k+1)}(t), \tag{14}$$

$$\frac{d}{dt}\mathcal{L}_n^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) - \mathcal{L}_n^{(k+1)}(t), \tag{15}$$

$$\int_0^\infty \mathcal{L}_n^{(k)}(t)\, \mathcal{L}_m^{(k)}(t)\, t^k \, dt = \begin{cases} \frac{(n+k)!}{2^{k+1} n!} & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Let $e_1, \dots, e_{d_1}$ denote the standard basis of $\mathbb{R}^{d_1}$. We introduce some operators on functions $f : \mathbb{N}^{d_1} \times \mathbb{R}^{\tilde{d}_1} \times \mathfrak{z} \to \mathbb{C}$:

$$\tau_j f(n,\mu,\eta) = f(n + e_j, \mu, \eta),$$
$$\delta_j f(n,\mu,\eta) = f(n + e_j, \mu, \eta) - f(n,\mu,\eta),$$
$$\partial_{\mu_l} f(n,\mu,\eta) = \frac{\partial}{\partial \mu_l} f(n,\mu,\eta),$$
$$\partial_{\eta_k} f(n,\mu,\eta) = \frac{\partial}{\partial \eta_k} f(n,\mu,\eta)$$

for all $j \in \{1, \dots, d_1\}$, $l \in \{1, \dots, \tilde{d}_1\}$, $k \in \{1, \dots, d_2\}$.

For all $h \in \mathbb{N}$ and all multiindices $\alpha \in \mathbb{N}^h$, we denote by $|\alpha|$ the length $\alpha_1 + \dots + \alpha_h$ of α. Inequalities between multiindices, such as $\alpha \le \alpha'$, shall be interpreted componentwise. Set moreover $(\alpha)_+ = ((\alpha_1)_+, \dots, (\alpha_h)_+)$, where $(\ell)_+ = \max\{\ell, 0\}$.

A function $\Psi : \dot{\mathfrak{z}} \times \mathfrak{v} \to \mathbb{C}$ will be called *multihomogeneous* if there exist $h_0, h_1, \dots, h_{d_1} \in \mathbb{R}$ such that

$$\Psi\left(\lambda_0 \eta, \sum_{j=1}^{d_1} \lambda_j P_j \xi\right) = \lambda_0^{h_0} \lambda_1^{h_1} \dots \lambda_{d_1}^{h_{d_1}} \Psi(\eta, \xi)$$

for all $\eta \in \dot{\mathfrak{z}}$, $\xi \in \mathfrak{v}$, $\lambda_0, \lambda_1, \dots, \lambda_{d_1} \in]0, \infty[$; the homogeneity degrees $h_0, h_1, \dots, h_{d_1}$ of Ψ will also be denoted as $\deg_{\mathfrak{z}} \Psi, \deg_{\mathfrak{v}_1} \Psi, \dots, \deg_{\mathfrak{v}_{d_1}} \Psi$. Note that, if Ψ is multihomogeneous and continuous, then $\deg_{\mathfrak{v}_j} \Psi \geq 0$ for all $j \in \{1, \dots, d_1\}$.

Proposition 6 *Let $H : \mathbb{R}^{d_1} \times \mathfrak{z}^* \to \mathbb{C}$ be smooth and compactly supported in $\mathbb{R}^{d_1} \times \dot{\mathfrak{z}}$, and let $m(n, \mu, \eta)$ be defined by (7). For all $\alpha \in \mathbb{N}^{d_2}$,*

$$u^\alpha \; \mathcal{K}_{H(\mathbf{L},\mathbf{U})}(z, u) = \sum_{\iota \in I_\alpha} \int_{\dot{\mathfrak{z}}} \int_{\mathfrak{v}} \sum_{n \in \mathbb{N}^{d_1}} \partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \delta^{\beta^\iota} m\left(n, \left(|\bar{P}_j^\eta \xi|^2\right)_{j \leq \tilde{d}_1}, \eta\right)$$
$$\times \Psi_\iota(\eta, \xi) \left[\prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(r_j - 1 + \beta_j^\iota)} \left(|P_j^\eta \xi|^2 / b_j^\eta\right)\right] e^{i\langle \xi, z\rangle} e^{i\langle \eta, u\rangle} \, d\xi \, d\eta,$$

for almost all $(z, u) \in G$, where I_α is a finite set and, for all $\iota \in I_\alpha$,

- $\gamma^\iota \in \mathbb{N}^{d_2}$, $\theta^\iota \in \mathbb{N}^{\tilde{d}_1}$, $\beta^\iota \in \mathbb{N}^{d_1}$, $\gamma^\iota \leq \alpha$,
- $\Psi_\iota = \Psi_{\iota,0} \Psi_{\iota,1} \dots \Psi_{\iota,d_1}$, *where $\Psi_{\iota,j} : \dot{\mathfrak{z}} \times \mathfrak{v} \to \mathbb{C}$ is smooth and multihomogeneous for all $j \in \{0, \dots, d_1\}$,*
- $\deg_{\mathfrak{z}} \Psi_\iota = |\gamma^\iota| - |\alpha| - |\beta^\iota|$ *and* $\deg_{\mathfrak{v}_j} \Psi_\iota = 2\beta_j^\iota + 2\theta_j^\iota$ *for all $j \in \{1, \dots, d_1\}$,*
- *for all $j \in \{1, \dots, d_1\}$, $\Psi_{\iota,j}(\eta, \xi)$ is a product of factors of the form $|P_j^\eta \xi|^2$ or $\partial_{\eta_k} |P_j^\eta \xi|^2$ for $k \in \{1, \dots, d_2\}$,*
- $|\gamma^\iota| + |\theta^\iota| + |\beta^\iota| + \sum_{j=1}^{d_1} (\beta_j^\iota - (\deg_{\mathfrak{v}_j} \Psi_{\iota,j})/2)_+ \leq |\alpha|$.

Proof By Proposition 4 and the properties of the Fourier transform, we are reduced to proving that, for all $\alpha \in \mathbb{N}^{d_2}$, $\eta \in \dot{\mathfrak{z}}$, $\xi \in \mathfrak{v}$,

$$\left(\frac{\partial}{\partial \eta}\right)^\alpha \sum_{n \in \mathbb{N}^{d_1}} m\left(n, \left(|\bar{P}_j^\eta \xi|^2\right)_{j \leq \tilde{d}_1}, \eta\right) \prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(r_j - 1)} \left(|P_j^\eta \xi|^2 / b_j^\eta\right)$$
$$= \sum_{\iota \in I_\alpha} \sum_{n \in \mathbb{N}^{d_1}} \partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \delta^{\beta^\iota} m\left(n, \left(|\bar{P}_j^\eta \xi|^2\right)_{j \leq \tilde{d}_1}, \eta\right) \Psi_\iota(\eta, \xi) \prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(r_j - 1 + \beta_j^\iota)} \left(|P_j^\eta \xi|^2 / b_j^\eta\right),$$

where I_α, γ^ι, θ^ι, β^ι, Ψ_ι are as in the above statement.

This is easily proved by induction on $|\alpha|$. For $|\alpha| = 0$, it is trivially verified. For the inductive step, one applies Leibniz' rule and exploits the following observations:

- when a derivative ∂_{η_k} hits a Laguerre function, by the identity (15) and summation by parts, the type of the Laguerre function is increased by 1, as well as the corresponding component of β^ι;
- for all $j \in \{1, \dots, d_1\}$, $b_j^\eta = \sqrt{\operatorname{tr}(-J_\eta^2 P_j)/(2r_j)}$ is a smooth function of $\eta \in \dot{\mathfrak{z}}$, homogeneous of degree 1;
- for all $j \in \{1, \dots, d_1\}$, $P_j^\eta = -J_\eta^2 P_j / (b_j^\eta)^2$ is a smooth function of $\eta \in \dot{\mathfrak{z}}$, homogeneous of degree 0, and in fact it is constant if $j > \tilde{d}_1$;

– for all $j \in \{1, \dots, \tilde{d}_1\}$, $|P_j^\eta \xi|^2 = \langle P_j^\eta P_j \xi, P_j \xi \rangle$ is a smooth bihomogeneous function of $(\eta, P_j \xi) \in \mathfrak{z} \times \mathfrak{v}_j$ of bidegree $(0, 2)$, and moreover

$$|\bar{P}_j^\eta \xi|^2 = |P_j \xi|^2 - |P_j^\eta \xi|^2, \qquad \partial_{\eta_k} |\bar{P}_j^\eta \xi|^2 = -\partial_{\eta_k} |P_j^\eta \xi|^2,$$
$$\partial_{\eta_k} \left(|P_j^\eta \xi|^2 / b_j^\eta \right) = |P_j^\eta \xi|^2 \partial_{\eta_k} \left(1/b_j^\eta \right) + \left(\partial_{\eta_k} |P_j^\eta \xi|^2 \right) / b_j^\eta$$

for all $k \in \{1, \dots, d_2\}$.

The conclusion follows. □

Note that, for all $j \in \{1, \dots, d_1\}$, $\mu \in \mathbb{R}^{\tilde{d}_1}$, $\eta \in \mathfrak{z}$, the quantities $\tau_j f(\cdot, \mu, \eta)$, $\delta_j f(\cdot, \mu, \eta)$ depend only on $f(\cdot, \mu, \eta)$; in other words, τ_j and δ_j can be considered as operators on functions $\mathbb{N}^{d_1} \to \mathbb{C}$.

The following lemma exploits the orthogonality properties (16) of the Laguerre functions, together with (14), and shows that a mismatch between the type of the Laguerre function and the exponent of the weight attached to the measure may be turned in some cases into discrete differentiation.

Lemma 7 *For all $h, k \in \mathbb{N}^{d_1}$ and all compactly supported $f : \mathbb{N}^{d_1} \to \mathbb{C}$,*

$$\int_{]0,\infty[^{d_1}} \Big| \sum_{n \in \mathbb{N}^{d_1}} f(n) \prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(k_j)}(t_j) \Big|^2 t^h \, dt$$
$$\leq C_{h,k} \sum_{n \in \mathbb{N}^{d_1}} |\delta^{(k-h)_+} f(n)|^2 \prod_{j=1}^{d_1} (1 + n_j)^{h_j + 2(k_j - h_j)_+}.$$

Proof Via an inductive argument, we may reduce to the case $d_1 = 1$.

Note that, if f is compactly supported, then $\tau^l f$ is null for all sufficiently large $l \in \mathbb{N}$. Hence, the operator $1 + \tau$, when restricted to the set of compactly supported functions, is invertible, with inverse given by

$$(1 + \tau)^{-1} f = \sum_{l \in \mathbb{N}} (-1)^l \tau^l f.$$

Then by (14), we deduce that, for all $k \in \mathbb{N}$,

$$\sum_{n \in \mathbb{N}} f(n) \, \mathcal{L}_n^{(k)}(t) = \sum_{n \in \mathbb{N}} (1 + \tau) f(n) \, \mathcal{L}_n^{(k+1)}(t),$$
$$\sum_{n \in \mathbb{N}} f(n) \, \mathcal{L}_n^{(k+1)}(t) = \sum_{n \in \mathbb{N}} (1 + \tau)^{-1} f(n) \, \mathcal{L}_n^{(k)}(t),$$

and consequently, for all $h, k \in \mathbb{N}$,

$$\sum_{n \in \mathbb{N}} f(n) \, \mathcal{L}_n^{(k)}(t) = \sum_{n \in \mathbb{N}} (1 + \tau)^{h-k} f(n) \, \mathcal{L}_n^{(h)}(t)$$

Thus, the orthogonality properties (16) of the Laguerre functions give us that

$$\int_0^\infty \Big| \sum_{n \in \mathbb{N}} f(n) \, \mathcal{L}_n^{(k)}(t) \Big|^2 t^h \, dt \leq C_{h,k} \sum_{n \in \mathbb{N}} |(1 + \tau)^{h-k} f(n)|^2 \, \langle n \rangle^h,$$

where $\langle n \rangle = 1 + n$.

In the case $h \geq k$, $(1+\tau)^{h-k}$ is given by the finite sum

$$(1+\tau)^{h-k} = \sum_{\ell=0}^{h-k} \binom{h-k}{\ell} \tau^{\ell},$$

and the conclusion follows immediately by the triangular inequality.

In the case $h < k$, instead, since $\delta = \tau - 1$, from the identity $1 - \tau^2 = (1-\tau)(1+\tau)$, we deduce that

$$(1+\tau)^{h-k} = (-\delta)^{k-h}(1-\tau^2)^{h-k} = (-1)^{k-h} \sum_{\ell \geq 0} \binom{\ell+k-h-1}{\ell} \delta^{k-h} \tau^{2\ell},$$

hence

$$\begin{aligned}
\sum_{n\in\mathbb{N}} |(1+\tau)^{h-k} f(n)|^2 \langle n\rangle^h &= \sum_{n\in\mathbb{N}} \left| \sum_{\ell\geq 0} \binom{\ell+k-h-1}{\ell} \delta^{k-h} f(n+2\ell) \right|^2 \langle n\rangle^h \\
&\leq C_{h,k} \sum_{n\in\mathbb{N}} \left| \sum_{\ell\geq n} \langle \ell\rangle^{k-h-1} \delta^{k-h} f(\ell) \right|^2 \langle n\rangle^h \\
&\leq C_{h,k} \sum_{n\in\mathbb{N}} \langle n\rangle^{-1/2} \sum_{\ell\geq n} |\langle \ell\rangle^{k-h-1/4} \delta^{k-h} f(\ell)|^2 \langle n\rangle^h \\
&\leq C_{h,k} \sum_{\ell\in\mathbb{N}} \langle \ell\rangle^{2k-2h-1/2} |\delta^{k-h} f(\ell)|^2 \sum_{n=0}^{\ell} \langle n\rangle^{h-1/2} \\
&\leq C_{h,k} \sum_{\ell\in\mathbb{N}} \langle \ell\rangle^{2k-h} |\delta^{k-h} f(\ell)|^2,
\end{aligned}$$

by the Cauchy–Schwarz inequality, and we are done. □

Let $|\cdot|$ denote any Euclidean norm on $\mathfrak{z}^*$. The previous lemma, together with Plancherel's formula for the Fourier transform, yields the following L^2-estimate.

Proposition 8 *Under the hypotheses of Proposition 6, for all* $\alpha \in \mathbb{N}^{d_2}$,

$$\begin{aligned}
\int_G |u^{\alpha}\, \mathcal{K}_{H(\mathbf{L},\mathbf{U})}(z,u)|^2 \,dz\,du \leq C_{\alpha} \sum_{\iota\in\tilde{I}_{\alpha}} \int_{\dot{\mathfrak{z}}} \int_{[0,\infty[^{\tilde{d}_1}} \sum_{n\in\mathbb{N}^{d_1}} |\partial_{\eta}^{\gamma^{\iota}} \partial_{\mu}^{\theta^{\iota}} \delta^{\beta^{\iota}} m(n,\mu,\eta)|^2 \\
\times |\eta|^{2|\gamma^{\iota}|-2|\alpha|-2|\beta^{\iota}|+|a^{\iota}|+d_1} (1+n_1)^{a_1^{\iota}} \dots (1+n_{d_1})^{a_{d_1}^{\iota}} \,d\sigma_{\iota}(\mu)\,d\eta, \qquad (17)
\end{aligned}$$

where $\tilde{I}_{\alpha}$ *is a finite set and, for all* $\iota \in \tilde{I}_{\alpha}$,

- $\gamma^{\iota} \in \mathbb{N}^{d_2}$, $\theta^{\iota} \in \mathbb{N}^{\tilde{d}_1}$, $a^{\iota}, \beta^{\iota} \in \mathbb{N}^{d_1}$,
- $\gamma^{\iota} \leq \alpha$, $|\gamma^{\iota}| + |\theta^{\iota}| + |\beta^{\iota}| \leq |\alpha|$,
- σ_{ι} *is a regular Borel measure on* $[0,\infty[^{\tilde{d}_1}$.

Proof Note that, for all $j \in \{1,\dots,d_1\}$,

$$\partial_{\eta_k}\left(|P_j^{\eta}\xi|^2\right) = 2\left\langle \left(\partial_{\eta_k} P_j^{\eta}\right) P_j\xi, P_j^{\eta}\xi \right\rangle \leq C|\eta|^{-1} |P_j^{\eta}\xi| |P_j\xi|;$$

consequently, if $\Psi_{\iota}, \Psi_{\iota,j}, \gamma^{\iota}, \theta^{\iota}, \beta^{\iota}$ are as in the statement of Proposition 6, then

$$|\Psi_{\iota,j}(\eta,\xi)|^2 \leq C_{\iota} |\eta|^{2\deg_{\mathfrak{z}} \Psi_{\iota,j}} |P_j^{\eta}\xi|^{\deg_{\mathfrak{v}_j} \Psi_{\iota,j}} |P_j\xi|^{\deg_{\mathfrak{v}_j} \Psi_{\iota,j}}$$

for all $j \in \{1, \dots, d_1\}$, hence

$$|\Psi_\iota(\eta,\xi)|^2 \le C_\iota |\eta|^{2\deg_{\mathfrak{z}} \Psi_\iota} \prod_{j=1}^{d_1} |P_j^\eta \xi|^{\deg_{\mathfrak{v}_j} \Psi_{\iota,j}} |P_j \xi|^{2\deg_{\mathfrak{v}_j} \Psi_{\iota,0} + \deg_{\mathfrak{v}_j} \Psi_{\iota,j}}$$
$$\le C_\iota |\eta|^{2|\gamma^\iota| - 2|\alpha| - 2|\beta^\iota|} \prod_{j=1}^{d_1} \sum_{h_j = (\deg_{\mathfrak{v}_j} \Psi_{\iota,j})/2}^{2\theta_j^\iota + 2\beta_j^\iota} |P_j^\eta \xi|^{2h_j} |\bar{P}_j^\eta \xi|^{4\theta_j^\iota + 4\beta_j^\iota - 2h_j},$$

and moreover, for all $h \in \mathbb{N}^{d_1}$, if $h_j \ge (\deg_{\mathfrak{v}_j} \Psi_{\iota,j})/2$ for all $j \in \{1, \dots, d_1\}$, then

$$|\gamma^\iota| + |\theta^\iota| + |\beta^\iota| + \sum_{j=1}^{\tilde{d}_1} \left(\beta_j^\iota - h_j\right)_+ \le |\alpha|.$$

By Proposition 6, Plancherel's formula and the triangular inequality, we then obtain that the left-hand side of (17) is majorized by a finite sum of terms of the form

$$\int_{\mathfrak{z}} \int_{\mathfrak{v}} \left| \sum_{n \in \mathbb{N}^{d_1}} \partial_\eta^\gamma \partial_\mu^\theta \delta^\beta m \left(n, \left(|\bar{P}_j^\eta \xi|^2\right)_{j \le \tilde{d}_1}, \eta\right) \prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(r_j - 1 + \beta_j)} \left(|P_j^\eta \xi|^2 / b_j^\eta\right) \right|^2$$
$$\times |\eta|^{2|\gamma| - 2|\alpha| - 2|\beta|} \prod_{j=1}^{d_1} |P_j^\eta \xi|^{2h_j} \prod_{j=1}^{\tilde{d}_1} |\bar{P}_j^\eta \xi|^{2k_j} \, d\xi \, d\eta, \tag{18}$$

where $\gamma \in \mathbb{N}^{d_2}$, $\theta, k \in \mathbb{N}^{\tilde{d}_1}$, $\beta, h \in \mathbb{N}^{d_1}$ and $|\gamma| + |\theta| + |\beta + (\beta - h)_+| \le |\alpha|$. Simple changes of variables (rotation, polar coordinates and rescaling) allow one to rewrite (18) as a constant times

$$\int_{\mathfrak{z}} \int_{]0,\infty[^{\tilde{d}_1}} \int_{]0,\infty[^{d_1}} \left| \sum_{n \in \mathbb{N}^{d_1}} \partial_\eta^\gamma \partial_\mu^\theta \delta^\beta m(n, \mu, \eta) \prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(r_j - 1 + \beta_j)}(t_j) \right|^2 \prod_{j=1}^{d_1} t_j^{r_j - 1 + h_j} \, dt$$
$$\times |\eta|^{2|\gamma| - 2|\alpha| - 2|\beta|} \prod_{j=1}^{d_1} \left(b_j^\eta\right)^{h_j + r_j} \prod_{j=1}^{\tilde{d}_1} \mu_j^{k_j + (\dim \mathfrak{v}_j - 2r_j)/2} \frac{d\mu}{\mu_1 \cdots \mu_{\tilde{d}_1}} \, d\eta.$$

By exploiting the fact that the b_j^η are smooth functions of $\eta \in \mathfrak{z}$, homogeneous of degree 1 (see the proof of Proposition 6), and applying Lemma 7 to the inner integral, the last quantity is majorized by

$$C \int_{\mathfrak{z}} \int_{]0,\infty[^{\tilde{d}_1}} \sum_{n \in \mathbb{N}^{d_1}} |\partial_\eta^\gamma \partial_\mu^\theta \delta^{\beta + (\beta - h)_+} m(n, \mu, \eta)|^2 \prod_{j=1}^{d_1} (1 + n_j)^{r_j - 1 + h_j + 2(\beta_j - h_j)_+}$$
$$\times |\eta|^{2|\gamma| - 2|\alpha| - 2|\beta| + |h| + |r|} \prod_{j=1}^{\tilde{d}_1} \mu_j^{k_j + (\dim \mathfrak{v}_j - 2r_j)/2} \frac{d\mu}{\mu_1 \dots \mu_{\tilde{d}_1}} \, d\eta,$$

and since the exponents $k_j + (\dim \mathfrak{v}_j - 2r_j)/2$ are strictly positive, while

$$-2|\beta| + |h| + |r| = -2|\beta + (\beta - h)_+| + \sum_{j=1}^{d_1} \left(r_j - 1 + h_j + 2(\beta_j - h_j)_+\right) + d_1$$

and $|\gamma| + |\theta| + |\beta + (\beta - h)_+| \leq |\alpha|$, the conclusion follows by suitably renaming the multiindices. □

5 From discrete to continuous

Via the fundamental theorem of integral calculus, finite differences can be estimated by continuous derivatives. The next lemma is a multivariate analog of [19, Lemma 6], and we omit the proof (see also [18, Lemma 7]).

Lemma 9 *Let $f : \mathbb{N}^{d_1} \to \mathbb{C}$ have a smooth extension $\tilde{f} : [0,\infty[^{d_1} \to \mathbb{C}$, and let $\beta \in \mathbb{N}^{d_1}$. Then,*

$$\delta^\beta f(n) = \int_{J_\beta} \partial^\beta \tilde{f}(n+s)\, d\nu_\beta(s)$$

for all $n \in \mathbb{N}$, where $J_\beta = \prod_{j=1}^{d_1} \left[0, \beta_j\right]$, and ν_β is a Borel probability measure on J_β. In particular,

$$|\delta^\beta f(n)|^2 \leq \int_{J_\beta} |\partial^\beta \tilde{f}(n+s)|^2\, d\nu_\beta(s)$$

for all $n \in \mathbb{N}^{d_1}$.

We give now a simplified version of the right-hand side of (17), in the case we restrict to the functional calculus of L alone. In order to avoid issues of divergent series, it is, however, convenient at first to truncate the multiplier along the spectrum of $\mathbf{U}$.

Lemma 10 *Let $\chi \in C_c^\infty(\mathbb{R})$ be supported in $[1/2, 2]$, $K \subseteq \mathbb{R}$ be compact and $M \in]0,\infty[$. If $F : \mathbb{R} \to \mathbb{C}$ is smooth and supported in K, and $F_M : \mathbb{R} \times \mathfrak{z}^* \to \mathbb{C}$ is given by*

$$F_M(\lambda, \eta) = F(\lambda)\, \chi(|\eta|/M),$$

then, for all $r \in [0,\infty[$,

$$\int_G ||u|^r\, \mathcal{K}_{F_M(L,\mathbf{U})}(z,u)|^2\, dz\, du \leq C_{K,\chi,r}\, M^{d_2 - 2r} \|F\|^2_{W_2^r}.$$

Proof We may restrict to the case $r \in \mathbb{N}$, the other cases being recovered a posteriori by interpolation. Hence, we need to prove that

$$\int_G |u^\alpha\, \mathcal{K}_{F_M(L,\mathbf{U})}(z,u)|^2\, dz\, du \leq C_{K,\chi,\alpha}\, M^{d_2 - 2|\alpha|} \|F\|^2_{W_2^{|\alpha|}} \tag{19}$$

for all $\alpha \in \mathbb{N}^{d_1}$. On the other hand, if m is defined by

$$m(n,\mu,\eta) = F\left(\sum_{j=1}^{d_1} b_j^\eta \langle n_j \rangle_j + |\mu|_\Sigma\right) \chi(|\eta|/M), \tag{20}$$

where $\langle \ell \rangle_j = 2\ell + r_j$ and $|\mu|_\Sigma = \sum_{j=1}^{\tilde{d}_1} \mu_j$, then the left-hand side of (19) is majorized by the right-hand side of (17), and we are reduced to proving that

$$\sum_{n \in \mathbb{N}^{d_1}} \int_{\dot{\mathfrak{z}}} \int_{[0,\infty[^{\tilde{d}_1}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \delta^{\beta^\iota} m(n,\mu,\eta)|^2 \, |\eta|^{2|\gamma^\iota|-2|\alpha|-2|\beta^\iota|+|a^\iota|+d_1} \times (1+n_1)^{a_1^\iota} \dots (1+n_{d_1})^{a_{d_1}^\iota} \, d\sigma_\iota(\mu)\, d\eta \le C_{K,\chi,\alpha} \, M^{d_2-2|\alpha|} \|F\|^2_{W_2^{|\alpha|}} \tag{21}$$

for all $\iota \in \tilde{I}_\alpha$, where $\tilde{I}_\alpha$, γ^ι, θ^ι, β^ι, a^ι, σ_ι are as in Proposition 8.

Note that the right-hand side of (20) makes sense for all $n \in \mathbb{R}^{d_1}$ and defines a smooth extension of m, which we still denote by m by a slight abuse of notation. Hence, by Lemma 9,

$$|\partial_\eta^{\gamma_\iota} \partial_\mu^{\theta^\iota} \delta^{\beta^\iota} m(n,\mu,\eta)|^2 \le \int_{J_\iota} |\partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \partial_n^{\beta^\iota} m(n+s,\mu,\eta)|^2 \, d\nu_\iota(s), \tag{22}$$

where $J_\iota = \prod_{j=1}^{d_1} \left[0, \beta_j^\iota\right]$ and ν_ι is a suitable probability measure on J_ι. Moreover, the measure σ_ι in (21) is finite on compacta, and the right-hand side of (22) vanishes when $|\mu|_\Sigma > \max K$, because $\operatorname{supp} F \subseteq K$. Consequently, (21) will be proved if we show that

$$\sum_{n \in \mathbb{N}^{d_1}} \int_{\dot{\mathfrak{z}}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \partial_n^{\beta^\iota} m(n+s,\mu,\eta)|^2 \, |\eta|^{2|\gamma^\iota|-2|\alpha|-2|\beta^\iota|+|a^\iota|+d_1} \times (1+n_1)^{a_1^\iota} \dots (1+n_{d_1})^{a_{d_1}^\iota} \, d\eta \le C_{K,\chi,\alpha} \, M^{d_2-2|\alpha|} \|F\|^2_{W_2^{|\alpha|}} \tag{23}$$

for all $s \in J_\iota$ and $\mu \in [0, \max K]^{\tilde{d}_1}$, uniformly in s and μ.

As observed in the proof of Proposition 6, the b_j^η are positive, smooth functions of $\eta \in \dot{\mathfrak{z}}$, homogeneous of degree 1; therefore, for all $n \in \mathbb{N}^{d_1}$, $j \in \{1,\dots,d_1\}$, $\eta \in \dot{\mathfrak{z}}$, $s \in [0,\infty[^{d_1}$, $\mu \in [0,\infty[^{\tilde{d}_1}$,

$$|\eta|(1+n_j) \sim b_j^\eta \langle n_j \rangle_j \le \sum_{l=1}^{d_1} b_l^\eta \langle n_l + s_l \rangle_l + |\mu|_\Sigma, \tag{24}$$

and the last quantity is bounded by the constant $\max K$ whenever $(n+s,\mu,\eta) \in \operatorname{supp} m$, because $\operatorname{supp} F \subseteq K$. Hence, the factors $|\eta|(1+n_j)$ in the left-hand side of (23) can be discarded, that is, we are reduced to proving (23) in the case $a^\iota = 0$.

From (20), it follows immediately that

$$\partial_\mu^{\theta^\iota} \partial_n^{\beta^\iota} m(n,\mu,\eta) = F^{(|\theta^\iota|+|\beta^\iota|)}\left(\sum_{j=1}^{d_1} b_j^\eta \langle n_j \rangle_j + |\mu|_\Sigma\right) \chi(|\eta|/M) \prod_{j=1}^{d_1} (2b_j^\eta)^{\beta_j^\iota}$$

and then it is easily proved inductively that

$$\partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \partial_n^{\beta^\iota} m(n,\mu,\eta) = \sum_{\substack{\upsilon \in \mathbb{N}^{d_1} \\ |\upsilon| \le |\gamma^\iota|}} \sum_{q=0}^{|\gamma^\iota|-|\upsilon|} F^{(|\theta^\iota|+|\beta^\iota|+|\upsilon|)}\left(\sum_{j=1}^{d_1} b_j^\eta \langle n_j \rangle_j + |\mu|_\Sigma\right) \times \Psi_{\iota,\upsilon,q}(\eta) \, M^{-q} \, \chi^{(q)}(|\eta|/M) \prod_{j=1}^{d_1} \langle n_j \rangle_j^{\upsilon_j}$$

where $\Psi_{\iota,\upsilon,q} : \dot{\mathfrak{z}} \to \mathbb{R}$ is smooth and homogeneous of degree $|\beta^\iota| + |\upsilon| + q - |\gamma^\iota|$. By exploiting again (24) and the fact that supp $F \subseteq K$, we can majorize the factors $\langle n_j \rangle_j$ in the right-hand side by $|\eta|^{-1} \sim M^{-1}$ and obtain that

$$|\partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \partial_n^{\beta^\iota} m(n,\mu,\eta)|^2 \le C_{K,\chi,\alpha} M^{2|\beta^\iota|-2|\gamma^\iota|} \tilde{\chi}(|\eta|/M) \times \sum_{\upsilon=0}^{|\gamma^\iota|} \left| F^{(|\beta^\iota|+|\theta^\iota|+\upsilon)} \left(\sum_{j=1}^{d_1} b_j^\eta \langle n_j \rangle_j + |\mu|_\Sigma \right) \right|^2,$$

where $\tilde{\chi}$ is the characteristic function of $[1/2, 2]$. Hence, the left-hand side of (23), when $a^\iota = 0$, is majorized by

$$C_{K,\chi,\alpha} M^{d_1 - 2|\alpha|} \times \sum_{\upsilon=0}^{|\gamma^\iota|} \int_{\dot{\mathfrak{z}}} \sum_{n \in \mathbb{N}^{d_1}} \left| F^{(|\beta^\iota|+|\theta^\iota|+\upsilon)} \left(\sum_{j=1}^{d_1} b_j^\eta \langle n_j + s_j \rangle_j + |\mu|_\Sigma \right) \right|^2 \tilde{\chi}(|\eta|/M)\, d\eta.$$

Let S denote the unit sphere in $\mathfrak{z}^*$. By passing to polar coordinates and exploiting the homogeneity of the b_j^η, the integral in the above formula is majorized by

$$C \int_S \int_0^\infty \sum_{n \in \mathbb{N}^{d_1}} \left| F^{(|\beta^\iota|+|\theta^\iota|+\upsilon)} \left(\rho \sum_{j=1}^{d_1} b_j^\omega \langle n_j + s_j \rangle_j + |\mu|_\Sigma \right) \right|^2 \tilde{\chi}(\rho/M) \rho^{d_2} \frac{d\rho}{\rho}\, d\omega$$

$$\le C M^{d_2} \int_0^\infty |F^{(|\beta^\iota|+|\theta^\iota|+\upsilon)}(\rho + |\mu|_\Sigma)|^2 \int_S \sum_{n \in \mathbb{N}^{d_1}} \tilde{\chi}(\rho/(M \langle n \rangle_{\omega,s}))\, d\omega \frac{d\rho}{\rho} \tag{25}$$

where $\langle n \rangle_{\omega,s} = \sum_{j=1}^{d_1} b_j^\omega \langle n_j + s_j \rangle_j \sim 1 + |n|$ uniformly in $\omega \in S$ and $s \in J_\iota$. Since $\tilde{\chi}(\rho/(M\langle n \rangle_{\omega,s}))$ vanishes unless $\langle n \rangle_{\omega,s} \sim \rho/M$, the sum in the right-hand side of (25) has at most $C_\iota(\rho/M)^{d_1}$ nonvanishing summands, and the integral on S is majorized by $C_\iota(\rho/M)^{d_1}$. In conclusion, the left-hand side of (23) is majorized by

$$C_{K,\chi,\alpha} M^{d_2 - 2|\alpha|} \sum_{\upsilon=0}^{|\gamma^\iota|} \int_0^\infty |F^{(|\beta^\iota|+|\theta^\iota|+\upsilon)}(\rho + |\mu|_\Sigma)|^2 \rho^{d_1 - 1}\, d\rho \le C_{K,\chi,\alpha} M^{d_2 - 2|\alpha|} \|F\|^2_{W_2^{|\alpha|}},$$

because $d_1 \ge 1$, supp $F \subseteq K$ and $|\beta^\iota| + |\theta^\iota| + |\gamma^\iota| \le |\alpha|$, and we are done. □

Proposition 11 *Let $F : \mathbb{R} \to \mathbb{C}$ be smooth and such that* supp $F \subseteq K$ *for some compact set $K \subseteq \mathbb{R}$. For all $r \in [0, d_2/2[$,*

$$\int_G \left| (1+|u|)^r\ \mathcal{K}_{F(L)}(z,u) \right|^2 dz\, du \le C_{K,r} \|F\|^2_{W_2^r}.$$

Proof Take $\chi \in C_c^\infty(]0,\infty[)$ such that supp $\chi \subseteq [1/2, 2]$ and $\sum_{k \in \mathbb{Z}} \chi(2^{-k} t) = 1$ for all $t \in]0,\infty[$. If F_M is defined for all $M \in]0,\infty[$ as in Lemma 10, then $\mathcal{K}_{F_M(L,\mathbf{U})}$ is given by the right-hand side of (8), where m is defined by (20), and moreover,

$$\sum_{j=1}^{d_1} b_j^\eta \langle n_j \rangle_j + |\mu|_\Sigma \ge C^{-1} |\eta|$$

for all $\eta \in \dot{\mathfrak{z}}$, $\mu \in [0, \infty[^{\tilde{d}_1}$ and $n \in \mathbb{N}^{d_1}$, therefore $F_M(L, \mathbf{U}) = 0$ whenever $M > 2C \max K$. Hence, if $k_K \in \mathbb{Z}$ is sufficiently large so that $2^{k_K} > 2C \max K$, then

$$F(L) = \sum_{k \in \mathbb{Z},\, k \leq k_K} F_{2^k}(L, \mathbf{U})$$

(with convergence in the strong sense). Consequently, an estimate for $\mathcal{K}_{F(L)}$ can be obtained, via Minkowski's inequality, by summing the corresponding estimates for $\mathcal{K}_{F_{2^k}}(L, \mathbf{U})$ given by Lemma 10. If $r < d/2$, then the series $\sum_{k \leq k_K} (2^k)^{d_2/2-r}$ converges, thus

$$\int_G \left| |u|^r \ \mathcal{K}_{F(L)}(z, u) \right|^2 \, dz\, du \leq C_{K,r} \|F\|^2_{W_2^r}.$$

The conclusion follows by combining the last inequality with the corresponding one for $r = 0$. □

Let $|\cdot|_\delta$ be a δ_t-homogeneous norm on G; take, e.g., $|(z, u)|_\delta = |z| + |u|^{1/2}$. Interpolation then allows us to improve the standard weighted estimate for a homogeneous sublaplacian on a stratified group.

Proposition 12 *Let $F : \mathbb{R} \to \mathbb{C}$ be smooth and such that* supp $F \subseteq K$ *for some compact set $K \subseteq \mathbb{R}$. For all $r \in [0, d_2/2[$, $\alpha \geq 0$ and $\beta > \alpha + r$,*

$$\int_G \left| (1 + |(z, u)|_\delta)^\alpha \ (1 + |u|)^r \ \mathcal{K}_{F(L)}(z, u) \right|^2 \, dz\, du \leq C_{K,\alpha,\beta,r} \|F\|^2_{W_2^\beta}. \tag{26}$$

Proof Note that $1 + |u| \leq C(1 + |(z, u)|_\delta)^2$. Hence, in the case $\alpha \geq 0$, $\beta > \alpha + 2r$, the inequality (26) follows by the mentioned standard estimate (see [21, Lemma 1.2] or [17, Theorem 2.7]). On the other hand, if $\alpha = 0$ and $\beta \geq r$, then (26) is given by Proposition 11. The full range of α and β is then obtained by interpolation. □

We can finally prove the crucial estimate.

Proof of Proposition 3 Take $r \in](\dim G)/2 + d_2/2 - s, d_2/2[$. Then,

$$s - r > (\dim G)/2 + d_2/2 - 2r = (\dim \mathfrak{v})/2 + d_2 - 2r,$$

hence we can find $\alpha_1 > (\dim \mathfrak{v})/2$ and $\alpha_2 > d_2 - 2r$ such that $s - r > \alpha_1 + \alpha_2$. Set $w_s(z, u) = (1 + |(z, u)|_\delta)^\alpha \ (1 + |u|)^r$. The L^2-estimate (4) then follows from Proposition 12. On the other hand, for all $(z, u) \in G$,

$$w_s^{-2}(z, u) \leq C_s (1 + |z|)^{-2\alpha_1} \ (1 + |u|)^{-\alpha_2 - 2r},$$

and the right-hand side is integrable over $G \cong \mathfrak{v} \times \mathfrak{z}$ since $2\alpha_1 > \dim \mathfrak{v}$ and $\alpha_2 + 2r > d_2 = \dim \mathfrak{z}$. Therefore, $w_s^{-1} \in L^2(G)$, and the L^1-estimate (5) follows from (4) and Hölder's inequality. □

6 Remarks on the validity of the assumption and direct products

In this section, we do no longer suppose that G and L are a 2-step stratified Lie group and a sublaplacian satisfying Assumption (A).

As observed in Sect. 2, a necessary condition for the validity of Assumption (A) is that the skewadjoint endomorphism J_η of the first layer $\mathfrak{v}$ has constant rank for η ranging in $\dot{\mathfrak{z}} = \mathfrak{z}^* \setminus \{0\}$. Here, we show that this condition is also sufficient when the rank is minimal.

Proposition 13 *Let G be a 2-step nilpotent Lie group, with Lie algebra $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$, and let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{v}$. Suppose that the skewadjoint endomorphism J_η of $\mathfrak{v}$ has rank 2 for all $\eta \in \dot{\mathfrak{z}}$. Then, G satisfies Assumption* (A) *with the sublaplacian L associated to the given inner product, and also with any other sublaplacian associated to an inner product on a complement of $\mathfrak{z}$.*

Let moreover $G_{\mathbb{C}}$ be the complexification of G, considered as a real 2-step group, with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{v}_{\mathbb{C}} \oplus \mathfrak{z}_{\mathbb{C}}$, and let $\mathfrak{v}_{\mathbb{C}}$ be endowed with the real inner product induced by the inner product on $\mathfrak{v}$. Then, $G_{\mathbb{C}}$, with the sublaplacian associated to the given inner product, satisfies Assumption (A).

Proof From the normal form for skewadjoint endomorphisms, it follows immediately that, if J_η has rank 2, then J_η^2 has exactly one nonzero eigenvalue, and Assumption (A) is trivially verified. Moreover, if $\mathfrak{v}$ is identified with $\mathfrak{g}/\mathfrak{z}$, then $\ker J_\eta$ corresponds to the subspace

$$N_\eta = \big\{x + \mathfrak{z} \,:\, x \in \mathfrak{g} \text{ and } \eta([x, x']) = 0 \text{ for all } x' \in \mathfrak{g}\big\}$$

of $\mathfrak{g}/\mathfrak{z}$; hence, the rank condition on J_η can be rephrased by saying that N_η has codimension 2 for all $\eta \in \dot{\mathfrak{z}}$, and this condition does not depend on the sublaplacian L chosen on G.

Let $R(J_\eta)$ denote the range of J_η. We show now that, for all $\eta, \eta' \in \dot{\mathfrak{z}}$, the intersection $R(J_\eta) \cap R(J_{\eta'})$ is nontrivial. If it were trivial, since $J_{\eta+\eta'} = J_\eta + J'_\eta$, we would have $\ker J_{\eta+\eta'} = \ker J_\eta \cap \ker J_{\eta'}$, hence

$$R(J_{\eta+\eta'}) = (\ker J_{\eta+\eta'})^\perp = R(J_\eta) \oplus R(J_{\eta'}),$$

thus $J_{\eta+\eta'}$ would have rank 4, contradiction.

Consider now the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$. Via the linear identifications $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \times \mathfrak{g}$, $\mathfrak{z}_{\mathbb{C}}^* = \mathfrak{z}^* \times \mathfrak{z}^*$, $\mathfrak{v}_{\mathbb{C}} = \mathfrak{v} \times \mathfrak{v}$, the skewsymmetric endomorphism $\tilde{J}_\eta$ of the first layer $\mathfrak{v}_{\mathbb{C}}$ corresponding to the element $\eta = (\eta_R, \eta_I) \in \mathfrak{z}_{\mathbb{C}}^*$ is given by

$$\tilde{J}_\eta(x_R, x_I) = \big(J_{\eta_R} x_R + J_{\eta_I} x_I,\, J_{\eta_I} x_R - J_{\eta_R} x_I\big)\,. \tag{27}$$

Take now $\eta = (\eta_R, \eta_I) \in \dot{\mathfrak{z}}_{\mathbb{C}}$; we want to show that $\tilde{J}_\eta^2$ has rank 4 and a unique nonzero eigenvalue. We distinguish several cases.

If $\eta_I = 0$, then $\tilde{J}_\eta = J_{\eta_R} \times (-J_{\eta_R})$, hence $\tilde{J}_\eta^2 = J_{\eta_R}^2 \times J_{\eta_R}^2$ satisfies the condition. The same argument gives the conclusion in the case $\eta_R = 0$.

If both $\eta_R, \eta_I \in \dot{\mathfrak{z}}$, then $R(J_{\eta_R}) \cap R(J_{\eta_I}) \neq 0$, hence $\dim(R(J_{\eta_R}) \cap R(J_{\eta_I}))$ is either 2 or 1. In the first case, $R(J_{\eta_R}) = R(J_{\eta_I})$, so J_{η_R} and J_{η_I} commute and (27) implies that

$$\tilde{J}_\eta^2 = \Big(J_{\eta_R}^2 + J_{\eta_I}^2\Big) \times \Big(J_{\eta_R}^2 + J_{\eta_I}^2\Big)\,;$$

since $J_{\eta_R}^2$ and $J_{\eta_I}^2$ are negative multiples of the same orthogonal projection, the conclusion follows.

Suppose now that $R(J_{\eta_R}) \cap R(J_{\eta_I}) = \mathbb{R}x$ for some unit vector $x \in \mathfrak{v}$, and set $y_R = J_{\eta_R} x$, $y_I = J_{\eta_I} x$, $b_R = |y_R|$, $b_I = |y_I|$; in particular, $J_{\eta_R}^2 x = -b_R^2 x$ and $J_{\eta_I}^2 x = -b_I^2 x$. Since J_{η_R} and J_{η_I} are skewadjoint and of rank 2, necessarily $J_{\eta_R} x, J_{\eta_I} x \in x^\perp$ and $J_{\eta_R}(x^\perp) = J_{\eta_I}(x^\perp) = \mathbb{R}x$, therefore $J_{\eta_R} J_{\eta_I} x$ and $J_{\eta_I} J_{\eta_R} x$ are both multiples of x; on the other hand,

$$\langle J_{\eta_R} J_{\eta_I} x, x\rangle = -\langle J_{\eta_I} x, J_{\eta_R} x\rangle = \langle x, J_{\eta_I} J_{\eta_R} x\rangle,$$

hence $J_{\eta_R} J_{\eta_I} x = J_{\eta_I} J_{\eta_R} x$. This identity, together with (27), allows us easily to show that

$$\tilde{J}_\eta(x, 0) = (y_R, y_I), \qquad \tilde{J}_\eta(y_R, y_I) = -(b_R^2 + b_I^2)(x, 0),$$
$$\tilde{J}_\eta(0, x) = (y_I, -y_R), \qquad \tilde{J}_\eta(y_I, -y_R) = -(b_R^2 + b_I^2)(0, x).$$

Note that $b_R^2 + b_I^2$ is the squared norm of both (y_R, y_I) and $(y_I, -y_R)$. Hence, we would be done if we knew that $R(\tilde{J}_\mu)$ coincides with the linear span W of $(x, 0)$, $(0, x)$, (y_R, y_I), $(y_I, -y_R)$.

In fact, we just need to show that $R(\tilde{J}_\eta)$ is contained in W, or equivalently, that $W^\perp$ is contained in $\ker \tilde{J}_\eta$. On the other hand, if $v = (v_R, v_I) \in W^\perp$, then $v_R, v_I \in x^\perp$ and moreover

$$\langle v_R, y_R\rangle + \langle v_I, y_I\rangle = 0, \qquad \langle v_R, y_I\rangle - \langle v_I, y_R\rangle = 0,$$

hence $J_{\eta_R} v_R, J_{\eta_R} v_I, J_{\eta_I} v_R, J_{\eta_I} v_I \in \mathbb{R}x$, and

$$\langle J_{\eta_R} v_R, x\rangle = -\langle v_R, y_R\rangle = \langle v_I, y_I\rangle = -\langle J_{\eta_I} v_I, x\rangle,$$
$$\langle J_{\eta_I} v_R, x\rangle = -\langle v_R, y_I\rangle = -\langle v_I, y_R\rangle = \langle J_{\eta_R} v_I, x\rangle,$$

therefore $J_{\eta_R} v_R = -J_{\eta_I} v_I$ and $J_{\eta_I} v_R = J_{\eta_R} v_I$, from which it follows immediately that $\tilde{J}_\eta(v_R, v_I) = 0$. □

The next proposition shows how groups and sublaplacians satisfying Assumption (A) may be "glued together", so to give a higher-dimensional group and a sublaplacian that satisfy Assumption (A) too.

Proposition 14 *Suppose that, for $j = 1, 2$, the sublaplacian L_j on the 2-step stratified Lie group G_j satisfies Assumption* (A). *Suppose further that the centers of G_1 and G_2 have the same dimension. Let G be the quotient of $G_1 \times G_2$ given by any linear identification of the respective centers, and let $L = L_1^\sharp + L_2^\sharp$, where $L_j^\sharp$ is the pushforward of L_j to G. Then, the sublaplacian L on the group G satisfies Assumption* (A).

Proof Let $\mathfrak{g}_j$ be the Lie algebra of G_j, and let $\mathfrak{v}_j$ and $\langle\cdot,\cdot\rangle_j$ be the linear complement of the center $\mathfrak{z}_j$ and the inner product on $\mathfrak{v}_j$ determined by the sublaplacian L_j; denote moreover by $J_{j,\eta}$ the skewadjoint endomorphism of $\mathfrak{v}_j$ determined by $\eta \in \mathfrak{z}_j^*$.

The linear identification of the centers of G_1 and G_2 corresponds to a linear isomorphism $\phi : \mathfrak{z}_1 \to \mathfrak{z}_2$, and the Lie algebra $\mathfrak{g}$ of the quotient G can be identified with $\mathfrak{v}_1 \times \mathfrak{v}_2 \times \mathfrak{z}_2$, with Lie bracket

$$\left[(v_1, v_2, z), (v_1', v_2', z')\right] = \left(0, 0, \phi\left(\left[v_1, v_1'\right]\right) + \left[v_2, v_2'\right]\right).$$

Then, the sublaplacian L on G corresponds to the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{v}_1 \times \mathfrak{v}_2$ defined by

$$\langle (v_1, v_2), (v_1', v_2')\rangle = \langle v_1, v_1'\rangle_1 + \langle v_2, v_2'\rangle_2.$$

In particular, if $\phi^* : \mathfrak{z}_2^* \to \mathfrak{z}_1^*$ denotes the adjoint map of $\phi : \mathfrak{z}_1 \to \mathfrak{z}_2$, then it is easily checked that the skewadjoint endomorphism of the first layer $\mathfrak{v}_1 \times \mathfrak{v}_2$ of $\mathfrak{g}$ corresponding to an element η of the dual $\mathfrak{z}_2^*$ of the center of $\mathfrak{g}$ is given by $J_\eta = J_{1,\phi^*\eta} \times J_{2,\eta}$. Hence, the orthogonal decomposition of $\mathfrak{v}_1 \times \mathfrak{v}_2$ giving the "simultaneous diagonalization" of the J_η for all $\eta \in \dot{\mathfrak{z}}_2$ (in the sense of Sect. 2) is simply obtained by juxtaposing the corresponding orthogonal decompositions of $\mathfrak{v}_1$ and $\mathfrak{v}_2$. □

Note that the direct product $G_1 \times G_2$ itself need not satisfy Assumption (A), even if the factors G_1 and G_2 do. However, a functional-analytic argument, as in [24, §4], can be used to deal with that case.

The key step in our proof of Theorem 2 is the weighted L^2-estimate (4) of Proposition 3. Let us now turn the conclusion of Proposition 3 into an assumption on a homogeneous sublaplacian L on a stratified group G.

Assumption (B_t). For all $s > t$, there exist a weight $w_s : G \to [1, \infty[$ such that $w_s^{-1} \in L^2(G)$ and, for all compact sets $K \subseteq \mathbb{R}$ and all Borel functions $F : \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} F \subseteq K$,

$$\|w_s \, \mathcal{K}_{F(L)}\|_{L^2(G)} \le C_{K,s} \|F\|_{W_2^s(\mathbb{R})}. \tag{28}$$

Our Proposition 3 can then be rephrased by saying that Assumption (A) implies Assumption (B_t) for $t = (\dim G)/2$. Note, on the other hand, that Assumption (B_t) makes sense for homogeneous sublaplacians on stratified groups G of step other than 2. In fact, every homogeneous sublaplacian on a stratified group of homogeneous dimension Q satisfies Assumption (B_t) for $t = Q/2$, by [21, Lemma 1.2] (suitably extended so to admit multipliers that do not vanish in a neighborhood of the origin of $\mathbb{R}$; see, e.g., [24, Lemma 3.1] for the 1-dimensional case, and [17, Theorem 2.7] for the higher-dimensional case).

Differently from Assumption (A), the new Assumption (B_t) "behaves well" under direct products.

Proposition 15 *For $j = 1, \dots, n$, let L_j be a homogeneous sublaplacian on a stratified Lie group G_j satisfying Assumption* (B_{t_j}) *for some $t_j > 0$. Let $G = G_1 \times \cdots \times G_n$ and $L = L_1^\sharp + \cdots + L_n^\sharp$, where $L_j^\sharp$ is the pushforward to G of the operator L_j. Then, the sublaplacian L on G satisfies Assumption* (B_t)*, where $t = t_1 + \cdots + t_n$.*

Proof Take $s > t$. Then, we can choose $s_1, \dots, s_n$ such that $s_1 > t_1, \dots, s_n > t_n$ and $s = s_1 + \cdots + s_n$. Let then $w_{j,s_j} : G_j \to [1, \infty[$ be the weight corresponding to s_j given by Assumption (B_{t_j}) on G_j and L_j, for $j = 1, \dots, n$. In particular, $w_{j,s_j}^{-1} \in L^2(G_j)$ and, for all $\phi \in C_c^\infty(\mathbb{R})$, the map $F \mapsto \mathcal{K}_{(\phi F)(L_j)}$ is a bounded linear map of Hilbert spaces $W_2^{s_j}(\mathbb{R}) \to L^2(G_j, w_{j,s_j}^2(x_j)\, dx_j)$, where dx_j denotes the Haar measure on G_j.

The operators $L_1^\sharp, \dots, L_n^\sharp$ are essentially self-adjoint and commute strongly, that is, they admit a joint spectral resolution and a joint functional calculus on $L^2(G)$, and moreover, for all bounded Borel functions $F_1, \dots, F_n : \mathbb{R} \to \mathbb{C}$,

$$\mathcal{K}_{(F_1 \otimes \cdots \otimes F_n)(L_1^\sharp, \dots, L_n^\sharp)} = \mathcal{K}_{F_1(L_1)} \otimes \cdots \otimes \mathcal{K}_{F_n(L_n)}$$

[16, Corollary 5.5]. Hence, for all $\phi_1, \dots, \phi_n \in C_c^\infty(\mathbb{R})$, if $\phi = \phi_1 \otimes \cdots \otimes \phi_n$, then the map $H \mapsto \mathcal{K}_{(\phi H)(L_1^\sharp, \dots, L_n^\sharp)}$ is the tensor product of the maps $F_j \mapsto \mathcal{K}_{(\phi_j F_j)(L_j)}$. Since these maps are bounded $W_2^{s_j}(\mathbb{R}) \to L^2(G_j, w_{j,s_j}^2(x_j)\, dx_j)$, the map $H \mapsto \mathcal{K}_{(\phi H)(L_1^\sharp, \dots, L_n^\sharp)}$ is bounded $S_2^{(s_1, \dots, s_n)} W(\mathbb{R}^n) \to L^2(G, w_s^2(x)\, dx)$, where $S_2^{(s_1, \dots, s_n)} W(\mathbb{R}^n) = W_2^{s_1}(\mathbb{R}) \otimes \cdots \otimes W_2^{s_n}(\mathbb{R})$ is the L^2 Sobolev space with dominating mixed smoothness [25] of order $(s_1, \dots, s_n)$, and $w_s = w_{1,s_1} \otimes \cdots \otimes w_{n,s_n}$ is the product weight on G. In particular, for all compact sets $K \subseteq \mathbb{R}$, if we choose the cutoffs $\phi_j \in C_c^\infty(\mathbb{R})$ so that $\phi_j|_K = 1$, then we deduce that, for all $H : \mathbb{R}^n \to \mathbb{C}$ with $\operatorname{supp} H \subseteq K^n$,

$$\|w_s \, \mathcal{K}_{H(L_1^\sharp, \dots, L_n^\sharp)}\|_{L^2(G)} \le C_{K,s} \|H\|_{S_2^{(s_1, \dots, s_n)} W(\mathbb{R}^n)}.$$

(cf. [17, Proposition 5.2]). Since

$$\begin{aligned} \|f\|^2_{S_2^{(s_1, \dots, s_n)} W(\mathbb{R}^n)} &\sim \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi_1|)^{2s_1} \dots (1 + |\xi_n|)^{2s_n} \, d\xi \\ &\le \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|)^{2s_1 + \cdots + 2s_n} \, d\xi \sim \|f\|^2_{W_2^s(\mathbb{R}^n)}, \end{aligned}$$

where $\hat{f}$ denotes the Euclidean Fourier transform of f, we see immediately that the estimate

$$\| w_s \, \mathcal{K}_{H(L_1^\sharp,\dots,L_n^\sharp)} \|_{L^2(G)} \leq C_{K,s_1,\dots,s_n} \| H \|_{W_2^s(\mathbb{R}^n)}, \tag{29}$$

holds true whenever $K \subseteq \mathbb{R}$ is compact and $H : \mathbb{R}^n \to \mathbb{C}$ is supported in K^n.

Take now a compact set $K \subseteq \mathbb{R}$ and choose a smooth cutoff $\eta_K \in C_c^\infty(\mathbb{R})$ such that $\eta_K|_{[0,\max K]} = 1$. Let $F : \mathbb{R} \to \mathbb{C}$ be such that $\operatorname{supp} F \subseteq K$, and define $H : \mathbb{R}^n \to \mathbb{C}$ by

$$H(\lambda_1, \dots, \lambda_n) = F(\lambda_1 + \dots + \lambda_n) \, \eta_K(\lambda_1) \dots \eta_K(\lambda_n)$$

for all $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. Then, $\operatorname{supp} H \subseteq (\operatorname{supp} \eta_K)^n$, and

$$F(\lambda_1 + \dots + \lambda_n) = H(\lambda_1, \dots, \lambda_n)$$

for all $(\lambda_1, \dots, \lambda_n) \in [0, \infty[^n$. Since the operators $L_1, \dots, L_n$ are nonnegative, the joint spectrum of $L_1^\sharp, \dots, L_n^\sharp$ is contained in $[0, \infty[^n$, hence

$$F(L) = F\left(L_1^\sharp + \dots + L_n^\sharp\right) = H\left(L_1^\sharp, \dots, L_n^\sharp\right).$$

Consequently, by (29) and the smoothness of the map $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 + \dots + \lambda_n$, we obtain that

$$\| w_s \, \mathcal{K}_{F(L)} \|_{L^2(G)} \leq C_{K,s} \| H \|_{W_2^s(\mathbb{R}^n)} \leq C_{K,s} \| F \|_{W_2^s(\mathbb{R})}.$$

Since clearly $w_s^{-1} = w_{1,s_1}^{-1} \otimes \dots \otimes w_{n,s_n}^{-1} \in L^2(G)$, we are done. □

The previous results, together with the known weighted estimates for abelian [24, Lemma 3.1] and Métivier [12,13,17] groups, then yield the following extension of Theorem 2.

Theorem 16 *For $j = 1, \dots, n$, suppose that L_j is a homogeneous sublaplacian on a stratified Lie group G_j. Suppose further that, for each $j \in \{1, \dots, n\}$, at least one of the following conditions holds:*

- *G_j and L_j satisfy Assumption* (A)*;*
- *G_j is a Métivier group;*
- *G_j is abelian.*

Let $G = G_1 \times \dots \times G_n$ and $L = L_1^\sharp + \dots + L_n^\sharp$, as in Proposition 15. If $F : \mathbb{R} \to \mathbb{C}$ satisfies

$$\| F \|_{M W_2^s} < \infty$$

for some $s > (\dim G)/2$, then $F(L)$ is of weak type $(1, 1)$ and bounded on $L^p(G)$ for all $p \in]1, \infty[$.

Acknowledgments I gratefully acknowledge the financial support of the Alexander von Humboldt Foundation. I am deeply grateful to Detlef Müller for stimulating discussions and his continuous encouragement. I am also grateful for the opportunity of visiting the Mathematical Institute of the University of Wrocław in the first bimester of 2011; numerous discussions, particularly with Jacek Dziubański and Waldemar Hebisch, have greatly contributed to my understanding of problems related to spectral multipliers.

References

1. Alexopoulos, G.: Spectral multipliers on Lie groups of polynomial growth. Proc. Am. Math. Soc. **120**(3), 973–979 (1994)
2. Astengo, F., Cowling, M., Di Blasio, B., Sundari, M.: Hardy's uncertainty principle on certain Lie groups. J. Lond. Math. Soc. (2) **62**(2), 461–472 (2000)
3. Astengo, F., Di Blasio, B., Ricci, F.: Gelfand pairs on the Heisenberg group and Schwartz functions. J. Funct. Anal. **256**(5), 1565–1587 (2009)
4. Christ, M.: L^p bounds for spectral multipliers on nilpotent groups. Trans. Am. Math. Soc. **328**(1), 73–81 (1991)
5. Cowling, M., Sikora, A.: A spectral multiplier theorem for a sublaplacian on SU(2). Math. Z. **238**(1), 1–36 (2001)
6. Cowling, M., Klima, O., Sikora, A.: Spectral multipliers for the Kohn sublaplacian on the sphere in $\mathbb{C}^n$. Trans. Am. Math. Soc. **363**(2), 611–631 (2011)
7. Duong, X.T., Ouhabaz, E.M., Sikora, A.: Plancherel-type estimates and sharp spectral multipliers. J. Funct. Anal. **196**(2), 443–485 (2002)
8. Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G.: Higher Transcendental Functions, vol. II. Robert E. Krieger Publishing Co., Inc., Melbourne (1981)
9. Folland, G.B., Stein, E.M.: Hardy Spaces on Homogeneous Groups, Mathematical Notes, vol. 28. Princeton University Press, Princeton (1982)
10. Folland, G.B.: Harmonic Analysis in Phase Space. Ann. Math. Stud. vol. 122. Princeton University Press, Princeton (1989)
11. Hebisch, W.: Functional calculus for slowly decaying kernels (1995). Preprint. Available on the web at http://www.math.uni.wroc.pl/~hebisch/
12. Hebisch, W.: Multiplier theorem on generalized Heisenberg groups. Colloq. Math. **65**(2), 231–239 (1993)
13. Hebisch, W., Zienkiewicz, J.: Multiplier theorem on generalized Heisenberg groups. II. Colloq. Math. **69**(1), 29–36 (1995)
14. Martini, A., Müller, D.: A sharp multiplier theorem for Grushin operators in arbitrary dimensions (2012). To appear in, Rev. Mat. Iberoam. arXiv:1210.3564
15. Martini, A.: Algebras of differential operators on Lie groups and spectral multipliers. Tesi di perfezionamento (PhD thesis), Scuola Normale Superiore, Pisa (2010). arXiv:1007.1119
16. Martini, A.: Spectral theory for commutative algebras of differential operators on Lie groups. J. Funct. Anal. **260**(9), 2767–2814 (2011)
17. Martini, A.: Analysis of joint spectral multipliers on Lie groups of polynomial growth. Ann. Inst. Fourier (Grenoble) **62**(4), 1215–1263 (2012)
18. Martini, A., Sikora, A.: Weighted Plancherel estimates and sharp spectral multipliers for the Grushin operators. Math. Res. Lett. **19**(5), 1075–1088 (2012)
19. Martini, A., Müller, D.: L^p spectral multipliers on the free group $N_{3,2}$. Studia Math. **217**(1), 41–55 (2013)
20. Mauceri, G., Meda, S.: Vector-valued multipliers on stratified groups. Rev. Mat. Iberoam. **6**(3–4), 141–154 (1990)
21. De Michele, L., Mauceri, G.: L^p multipliers on the Heisenberg group. Mich. Math. J. **26**(3), 361–371 (1979)
22. Müller, D.: A restriction theorem for the Heisenberg group. Ann. Math. (2) **131**(3), 567–587 (1990)
23. Müller, D., Stein, E.M.: On spectral multipliers for Heisenberg and related groups. J. Math. Pures Appl. (9) **73**(4), 413–440 (1994)
24. Müller, D., Ricci, F., Stein, E.M.: Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups. II. Math. Z. **221**(2), 267–291 (1996)
25. Schmeisser, H.J.: Recent developments in the theory of function spaces with dominating mixed smoothness. In: Nonlinear Analysis, Function Spaces and Applications. Proceedings of the Spring School held in Prague, May 30-June 6, 2006, vol. 8, pp. 145–204. Czech Academy of Sciences, Mathematical Institute, Praha (2007)
26. Thangavelu, S.: Lectures on Hermite and Laguerre Expansions, Mathematical Notes, vol. 42. Princeton University Press, Princeton (1993)
27. Torres Lopera, J.F.: The cohomology and geometry of Heisenberg–Reiter nilmanifolds. In: Differential geometry, Peñíscola 1985, Lecture Notes in Math., vol. 1209, pp. 292–301. Springer, Berlin (1986)