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Levi-Civita and generalized Tanaka–Webster covariant derivatives for real hypersurfaces in complex two-plane Grassmannians

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Abstract It is known that submanifolds in Kaehler manifolds have many kinds of connections. Among them, we consider two connections, that is, Levi-Civita and Tanaka–Webster connections for real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. When they are equal to each other, we give some characterizations in $G_2(\mathbb{C}^{m+2})$.

Keywords Real hypersurfaces, Complex two-plane Grassmannians, Hopf hypersurface, $\mathfrak{D}^{\perp}$-invariant hypersurface, Levi-Civita connection, Generalized Tanaka–Webster connection

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1 Introduction

The study of real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ was initiated by Berndt and Suh [1]. Let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This set can be identified with the homogeneous space $SU(m+2)/S(U(2)\times U(m))$. From this, we know that $G_2(\mathbb{C}^{m+2})$ becomes the unique compact, irreducible, Riemannian manifold being equipped with both a Kaehler structure J and a quaternionic Kaehler structure $\mathfrak{J}$ not containing J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique

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compact, irreducible, Kaehler, quaternionic Kaehler manifold which is not a hyper-Kaehler manifold [1,2].

For real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$, we have the following two natural geometric conditions: the 1-dimensional distribution $[\xi] = \text{Span}\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M. Here the almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The *almost contact 3-structure* vector fields ξ_1, ξ_2, ξ_3 spanning the 3-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, where J_ν denotes a canonical local basis of the quaternionic Kaehler structure $\mathfrak{J}$ such that $T_x M = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

By using these two invariant conditions and the result in Alekseevskii [3], Berndt and Suh [1] proved the following:

Theorem A *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

The Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The 1-dimensional foliation of M by the integral curves of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the almost contact metric structure (ϕ, ξ, η, g) and the formula $\nabla_X \xi = \phi AX$ for any $X \in TM$ in Sect. 2, it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf. We will give a brief review of (ϕ, ξ, η, g) on M in Sect. 2.

On the other hand, when the distribution $\mathfrak{D}^\perp$ of a hypersurface M in $G_2(\mathbb{C}^{m+2})$ is invariant under the shape operator, we say that M is a $\mathfrak{D}^\perp$-*invariant hypersurface*. Moreover, we say that the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric*, when the Reeb vector field ξ on M is Killing. This means that the metric tensor g is invariant under the Reeb flow of ξ on M.

In [4], Berndt and Suh gave some equivalent conditions of isometric Reeb flow. They gave a characterization of real hypersurfaces of Type (A) in Theorem A in terms of the Reeb flow on M as follows:

Theorem B *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

In the proof of our Main Theorems, we will use that the Reeb flow on M is isometric if and only if the shape operator A commutes with the structure tensor field ϕ, that is, $A\phi = \phi A$. Related to this commuting property, recently, the authors gave many characterizations of model spaces of Type (A) in $G_2(\mathbb{C}^{m+2})$ mentioned in Theorems A and B (see [5,6]).

On the other hand, Suh [7] gave a characterization of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ in terms of the contact hypersurface. Moreover, as another characterization of one of Type (B) in $G_2(\mathbb{C}^{m+2})$ related to the Reeb vector field ξ Lee and Suh [8] obtained the following:

Theorem C *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

Usually, any submanifold in Kaehler manifolds has many kinds of connections. Among them, we consider two connections, namely, Levi-Civita and Tanaka–Webster connections for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$. In fact, $G_2(\mathbb{C}^{m+2})$ is a Riemannian symmetric space with Riemannian metric and Levi-Civita connection. Using the induced connection from the Levi-Civita connection, many geometers gave some characterizations for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ related to the covariant derivative ∇ of the shape operator on M ([9,10], etc). For real hypersurfaces in a Kaehler manifold, we consider a new affine connection $\widehat{\nabla}^{(k)}$ different from the Levi-Civita connection ∇, namely, the *generalized Tanaka–Webster connection* (in short, the *g-Tanaka–Webster connection*). It becomes a generalization of the well-known connection defined by Tanno [11]. Besides, it coincides with Tanaka–Webster connection if a real hypersurface in Kaehler manifolds satisfies $\phi A + A\phi = 2k\phi$ for a nonzero real number k. The Tanaka–Webster connection is defined as the canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold [12–14]. Using the generalized Tanaka–Webster connection, $\widehat{\nabla}^{(k)}$ defined in such a way that

$$\widehat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{*}$$

for any X, Y tangent to M, where ∇ denotes the Levi-Civita connection on M, A is the shape operator on M and k is a nonzero real number, the authors studied some characterizations of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ ([15,16], etc). The latter part of the generalized Tanaka–Webster connection $g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$ is denoted by $F_X Y$. Here the operator F_X is a kind of (1,1)-type tensor and said to be the *Tanaka–Webster operator*.

On the other hand, there are many results for the classification problem of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ related to the structure Jacobi operator and Ricci tensor, for example, [17–24] and so on. Recently, Pérez and Suh [25] investigated the Levi-Civita and g-Tanaka–Webster covariant derivatives for the shape operator or the structure Jacobi operator of real hypersurfaces in complex projective space $\mathbb{C}P^m$. In particular, the authors [25] gave the result about the shape operator as follows:

Theorem D *There exist no real hypersurfaces M in $\mathbb{C}P^m$, $m \geq 2$ such that $\nabla A = \widehat{\nabla}^{(k)} A$.*

Motivated by Theorem D, in this paper, we study a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ whose Levi-Civita covariant derivative coincides with generalized Tanaka–Webster derivative for the shape operator of M, that is,

$$(\nabla_X A)\, Y = \left(\widehat{\nabla}^{(k)}_X A\right) Y \tag{C-1}$$

for arbitrary tangent vector fields X and Y on M.

The condition (C-1) has a geometric meaning such that the shape operator A commutes with the Tanaka–Webster operator F_X, that is, $A \cdot F_X = F_X \cdot A$. This meaning gives any eigenspaces of the shape operator A are invariant by the Tanaka–Webster operator F_X.

From such a point of view, in Sect. 3, we prove that a real hypersurface in Kaehler manifolds satisfying (C-1) must be Hopf. Then from this result, we assert the following:

Theorem 1 *There does not exist any real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, satisfying* (C-1).

First, if we restrict $X = \xi$ in (C-1), then the following condition (C-2) along the Reeb vector field ξ becomes a generalized condition weaker than the condition (C-1). This also has a geometric meaning that any eigenspaces of the shape operator A are invariant by the restricted Tanaka–Webster operator F_ξ in the direction of the Reeb vector field ξ. Thus, we assert the following:

Theorem 2 *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If M satisfies*

$$\left(\nabla_\xi A\right) Y = \left(\widehat{\nabla}^{(k)}_\xi A\right) Y \tag{C-2}$$

for any tangent vector field Y on M, then M is locally congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

As a second, let us consider a distribution $\mathfrak{D}^\perp$ spanned by $\{\xi_1, \xi_2, \xi_3\}$. Accordingly, if we consider the condition (C-1) to the distribution $\mathfrak{D}^\perp$, the derivatives of the shape operator A of M along the distribution $\mathfrak{D}^\perp$ becomes a condition more weaker than (C-1). Obviously, this has a geometric meaning that any eigenspaces of the shape operator A are invariant by the restricted Tanaka–Webster operator F_{ξ_ν}, $\nu = 1, 2, 3$, along the distribution $\mathfrak{D}^\perp$. Then we have the following:

Theorem 3 *There does not exist a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, satisfying*

$$\left(\nabla_{\xi_\nu} A\right) Y = \left(\widehat{\nabla}^{(k)}_{\xi_\nu} A\right) Y, \quad \nu = 1, 2, 3 \tag{C-3}$$

for any tangent vector field Y on M.

Finally, we consider a distribution $\mathfrak{D}$ which is an orthogonal complement of $\mathfrak{D}^\perp$ in TM. Then by restricting the condition (C-1) to the distribution $\mathfrak{D}$, we get the following condition (C-4), which becomes another condition more weaker than (C-1). Using this geometric notion, we get:

Theorem 4 *There does not exist a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with*

$$(\nabla_X A)\, Y = \left(\widehat{\nabla}^{(k)}_X A\right) Y \tag{C-4}$$

for all vector fields $X \in \mathfrak{D}$ and Y on M.

In this paper, we refer to [1,2,4,26] for Riemannian geometric structures of $G_2(\mathbb{C}^{m+2})$, and [11,13–16,27] for generalized Tanaka–Webster connection of real hypersurfaces in Kaehler manifolds.

2 Key Lemmas

Let M be a real hypersurface in Kaehler manifolds $(\tilde{M}, \tilde{g})$. The induced Riemannian metric on M is denoted by g. In addition, $\tilde{\nabla}$ and ∇ denote the Levi-Civita connections of $\tilde{M}$ and M, respectively. Let N be a local unit normal vector field of M and A the shape operator of M with respect to N.

From the Kaehler structure J of $\tilde{M}$, we have a tensor field ϕ of type (1,1) on M, given by

$$g(\phi X, Y) = \tilde{g}(JX, Y)$$

for all tangent vector fields X of M. Moreover, we obtain the unit tangent vector field ξ and the 1-form η of M defined by

$$\xi = -JN \text{ and } \eta(X) = g(X, \xi) = \tilde{g}(JX, N),$$

respectively. It implies that $\phi^2 X = -X + \eta(X)\xi$, $\eta(\xi) = 1$, $\phi\xi = 0$ and

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX,$$

together with Gauss and Weingarten formulas. Thus, the Kaehler structure J of $\tilde{M}$ induces an almost contact metric structure (ϕ, ξ, η, g) on M.

Now let us assume that a real hypersurface M in $\tilde{M}$ satisfies

$$(\nabla_X A)\, Y = \left(\widehat{\nabla}^{(k)}_X A\right) Y \tag{C-1}$$

for all tangent vector fields X and Y on M.

From the definition of the g-Tanaka–Webster connection (*), we have

$$\begin{aligned}(\widehat{\nabla}^{(k)}_X A)Y &= \widehat{\nabla}^{(k)}_X (AY) - A(\widehat{\nabla}^{(k)}_X Y)\\ &= (\nabla_X A)Y + g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY\\ &\quad -g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y.\end{aligned}$$

Therefore, the condition (C-1) can be written as

$$\begin{aligned}&g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY\\ &\quad -g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y = 0\end{aligned} \tag{2.1}$$

for all tangent vector fields X and Y on M.

In a situation like this, we prove

Lemma 2.1 *Let M be a real hypersurface in a Kaehler manifold $\tilde{M}$ with the condition* (C-1). *Then M becomes a Hopf hypersurface.*

Proof The purpose of this lemma is to show that the structure vector field ξ is principal. In order to prove this, let us suppose that there is a point where the Reeb vector field ξ is not principal. Then there exists a neighborhood $\mathfrak{U}$ of this point, on which we can define a unit vector field U orthogonal to ξ in such a way that

$$\beta U = A\xi - g(A\xi, \xi)\xi = A\xi - \alpha\xi$$

where β denotes the length of vector filed $A\xi - \alpha\xi$ and $\beta(x) \neq 0$ for any point x in $\mathfrak{U}$. Hereafter, unless otherwise stated, let us continue our discussion on this neighborhood $\mathfrak{U}$.

Taking $X = Y = \xi$ in (2.1), we get $\beta(\alpha + k)\phi U = \beta A\phi U$. Since $\beta \neq 0$, it follows that

$$A\phi U = (\alpha + k)\phi U. \tag{2.2}$$

Moreover, putting $X = Y = U$ in (2.1), we have $-\beta\phi AU = 0$. It implies that

$$AU = \beta\xi, \tag{2.3}$$

together with $\beta \neq 0$ and $\phi^2 AU = -AU + \eta(AU)\xi = -AU + \beta\xi$.

Replacing Y by U in (2.1), we have

$$-\beta\phi AX - g(\phi AX, U)A\xi + k\eta(X)A\phi U = 0 \tag{2.4}$$

for any tangent vector field X on M. Substituting $X = \xi$ in the above equation, we get

$$\big(-\beta^2 + k(\alpha + k)\big)\phi U = 0$$

together with $\phi A\xi = \beta\phi U$ and (2.2). Taking the inner product with ϕU, it turns to

$$\alpha + k = \frac{\beta^2}{k} \tag{2.5}$$

because k is nonzero real number from the definition of g-Tanaka–Webster connection on real hypersurfaces in Kaehler manifolds.

On the other hand, putting $X = \phi U$ in (2.4), we get

$$2\beta(\alpha + k)U + \alpha(\alpha + k)\xi = 0 \tag{2.6}$$

from (2.2) and $\phi^2 U = -U$. Taking the inner product with ξ, we obtain $\alpha(\alpha + k) = 0$. By (2.5), this equation is written as $\frac{\alpha\beta^2}{k} = 0$. Since $k \neq 0$ and $\beta \neq 0$, we have $\alpha = 0$. Moreover, taking the inner product of (2.6) with U, we have $\beta(\alpha + k) = 0$. It follows that $\beta = 0$, together with $\alpha = 0$ and $k \neq 0$, which gives a contradiction. This is, the set $\mathfrak{U}$ should be empty. Thus, there does not exist such an open neighborhood $\mathfrak{U}$ in M, which means that the structure vector field ξ is principle. Hence, M must be Hopf under our assumption.

By means of Lemma 2.1, the condition (C-1) implies

$$\begin{aligned} &g(\phi AX, AY)\xi - \alpha\eta(Y)\phi AX - k\eta(X)\phi AY \\ &-\alpha g(\phi AX, Y)\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y = 0 \end{aligned} \tag{2.7}$$

for all tangent vector fields X and Y on M. Moreover, putting $Y = \xi$ in the above equation, we obtain $A\phi AX = \alpha\phi AX$ for any tangent vector field X on M. From this, the Eq. (2.7) is reduced to

$$k\eta(X)(A\phi - \phi A)Y = 0$$

for all tangent vector fields X and Y on M. By the definition of generalized Tanaka–Webster connection for real hypersurfaces in a Kaehler manifold, it follows that

$$\eta(X)(A\phi - \phi A)Y = 0$$

for all tangent vector fields X and Y on M.

Summing up above discussions, we assert the following

Lemma 2.2 *Let M be a real hypersurface in a Kaehler manifold $\tilde{M}$ with the condition* (C-1). *Then we have*

$$A\phi AX = \alpha\phi AX, \tag{2.8}$$

$$\eta(X)(A\phi - \phi A)Y = 0 \tag{2.9}$$

for all tangent vector fields X, Y on M.

3 Proof of Theorem 1

From now on, we will prove Theorem 1 in the introduction by using the above two Lemmas which are induced from our condition (C-1).

In fact, since M is a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with the property (C-1), M becomes a Hopf hypersurface (Lemma 2.1). From this, we have

$$\eta(X)(A\phi - \phi A)Y = 0, \tag{3.1}$$

because k is a nonzero constant (Lemma 2.2).

Putting $X = \xi$ in (3.1), it follows that $A\phi - \phi A = 0$. On the other hand, Berndt and Suh [4] gave a characterization of real hypersurfaces of Type (A) in $G_2(\mathbb{C}^{m+2})$ when the shape operator A of M commutes with the structure tensor ϕ of M. By virtue of this result, we assert that if M is a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying (C-1), then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Let us check that whether the model space M_A of Type (A) satisfies the condition (C-1). In order to do this, let us assume that the shape operator A of M_A satisfies the condition (C-1). According to Proposition 3 given in [1], the Eq. (2.8) implies

$$\beta(\beta-\alpha)=0 \tag{3.2}$$

if $X=\xi_2$. But it does not hold, because $\beta(\beta-\alpha)=2$ where $\alpha=\sqrt{8}\cot(2\sqrt{2}r)$ and $\beta=\sqrt{2}\cot(\sqrt{2}r)$, $r\in(0,\pi/2\sqrt{2})$. It completes the proof of Theorem 1. □

4 Proofs of Theorems 2 and 3

In this section, we investigate Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying the property (C-2) and (C-3) which are weaker than (C-1), respectively. On the other hand, $G_2(\mathbb{C}^{m+2})$ is equipped with both a Kaehler and a quaternionic Kaehler structure. By applying these two structures to the normal vector field N of M in $G_2(\mathbb{C}^{m+2})$, we get 1- and 3-dimensional distributions on M. For the sake of convenience, we denote $[\xi]=\text{Span}\{\xi\}$ or $\mathfrak{D}^{\perp}=\text{Span}\{\xi_1,\xi_2,\xi_3\}$, respectively. For these two distributions, we define a new distribution $\mathfrak{F}$ given by $\mathfrak{F}=[\xi]\cup\mathfrak{D}^{\perp}$. If we restrict $X\in\mathfrak{F}$ in (C-1), then it becomes a new weaker condition for (C-1). Accordingly, we also consider this case.

First, we assume that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying

$$(\nabla_\xi A)Y=(\widehat{\nabla}^{(k)}_\xi A)Y \tag{C-2}$$

for any vector field $Y\in TM$.

Under our assumptions, this condition means that the structure tensor field ϕ commutes with the shape operator A of M. In fact, putting $X=\xi$ in (2.1), it follows that for any tangent vector field Y on M

$$\phi AY-A\phi Y=0,$$

because M is Hopf and k is a nonzero real number. By Theorem B, we assert our Theorem 2 in the introduction. □

Next, we observe the following condition of covariant derivatives with respect to the Levi-Civita and g-Tanaka–Webster connections for shape operator A on Hopf hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ given by

$$(\nabla_{\xi_\nu}A)Y=(\widehat{\nabla}^{(k)}_{\xi_\nu}A)Y,\quad \nu=1,2,3 \tag{C-3}$$

for any tangent vector field Y on M.

According to (2.1), the condition (C-3) is equal to

$$\begin{aligned}&g(\phi A\xi_\nu,AY)\xi-\alpha\eta(Y)\phi A\xi_\nu-k\eta(\xi_\nu)\phi AY\\&-\alpha g(\phi A\xi_\nu,Y)\xi+\eta(Y)A\phi A\xi_\nu+k\eta(\xi_\nu)A\phi Y=0\end{aligned} \tag{4.1}$$

where Y is any tangent vector field on M and $\nu=1,2,3$.

Putting $Y=\xi$ in (4.1), we have that

$$A\phi A\xi_\nu=\alpha\phi A\xi_\nu,\quad \nu=1,2,3. \tag{4.2}$$

From this, (4.1) can be written as

$$\eta(\xi_\nu)(A\phi-\phi A)Y=0$$

for any vector field $Y\in TM$ and $\nu=1,2,3$.

By virtue of this equation, we have the following two cases:

- **Case 1** $\eta(\xi_\nu) = 0, \quad \nu = 1, 2, 3$ and
- **Case 2** $A\phi = \phi A$.

First, we consider the case $\eta(\xi_\nu) = 0$ for any $\nu = 1, 2, 3$. It means that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$. By Theorem C, it implies that M is of Type (B) in Theorem A given in the introduction.

On the other hand, due to Berndt and Suh's classification [1], all the principal curvatures on a model space of Type (B) are given as follows: $\alpha = -2\tan(2r)$, $\beta = 2\cot(2r)$, $\gamma = 0$, $\lambda = \cot(r)$ and $\mu = -\tan(r)$ for some $r \in (0, \pi/4)$. Since $\gamma = 0$, we get

$$\begin{aligned}\left(\widehat{\nabla}^{(k)}_{\xi_\nu} A\right)\xi - (\nabla_{\xi_\nu} A)\xi &= A\phi A\xi_\nu - \alpha\phi A\xi_\nu \\ &= -\alpha\beta\phi\xi_\nu\end{aligned}$$

for $\nu = 1, 2, 3$. In fact, since $\alpha = -2\tan(2r)$, $\beta = 2\cot(2r)$ for some $r \in (0, \pi/4)$, the constant $\alpha\beta$ must be nonzero. It means that the model space of Type (B) does not satisfy our condition (C-3).

Next we consider the remain case that the structure tensor ϕ commutes with the shape operator A of M. By virtue of Theorem B, we see that M must be a real hypersurface of Type (A) in $G_2(\mathbb{C}^{m+2})$.

From now on, let us check the converse problem, that is, whether a model space M_A of Type (A) satisfies the condition (C-3) or not. In fact, we suppose that M_A has the condition (C-3), that is, M_A satisfies (4.2). For $\nu = 2$, it becomes $\beta(\beta - \alpha) = 0$. In the proof of Theorem 1, we get $\beta(\beta - \alpha) = 2$, because $\alpha = \sqrt{8}\cot(2\sqrt{2}r)$ and $\beta = \sqrt{2}\cot\sqrt{2}r$ where $r \in (0, \pi/2\sqrt{2})$. Hence, we assert that M_A does not satisfy the condition (C-3).

Summing up these subcases, we give a complete proof of Theorem 3. □

As mentioned above, the distribution $\mathfrak{F}$ is defined by $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$. From the structure of $\mathfrak{F}$ and the proofs of Theorems 2 and 3, we naturally obtain

Corollary 4.1 *There does not exist a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with*

$$(\nabla_X A)\, Y = \left(\widehat{\nabla}^{(k)}_X A\right) Y$$

for any $X \in \mathfrak{F}$ and $Y \in TM$.

5 Proof of Theorem 4

In this section, we observe the condition

$$(\nabla_X A)\, Y = \left(\widehat{\nabla}^{(k)}_X A\right) Y \tag{C-4}$$

for all tangent vector fields $X \in \mathfrak{D}$ and $Y \in TM$. Putting $Y = \xi$ in (2.1) and using the assumption that M is Hopf, we obtain

$$A\phi AX = \alpha\phi AX \tag{5.1}$$

for any tangent vector field $X \in \mathfrak{D}$. Thus, the condition (C-4) is equal to

$$\eta(X)(A\phi - \phi A)Y = 0 \tag{5.2}$$

for any $X \in \mathfrak{D}$ and $Y \in TM$. From this, we have the following two cases:

- **Case 1** $\quad A\phi = \phi A$ and

- **Case 2** $\eta(X) = 0$ for any $X \in \mathfrak{D}$.

For the first case $A\phi = \phi A$, we know that M becomes a model space of Type (A) by Theorem B in the introduction.

Now let us consider the remaining case $\eta(X) = 0$ for any $X \in \mathfrak{D}$. It means that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^{\perp}$. Thus, without loss of generality we may put $\xi = \xi_1$. Under these assumptions, we now prove that M becomes to be a $\mathfrak{D}^{\perp}$-invariant hypersurface, that is, $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$.

Since M is Hopf, we have the following formula given by Berndt and Suh [4]:

$$\begin{aligned}2A\phi AX = \alpha A\phi X + \alpha\phi AX + 2\phi X + 2\sum_{\nu=1}^{3}\Big\{&\eta_\nu(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi_\nu \\ &+\eta_\nu(\xi)\phi_\nu X - 2\eta(X)\eta_\nu(\xi)\phi\xi_\nu - 2\eta_\nu(\phi X)\eta_\nu(\xi)\xi\Big\}\end{aligned}$$

for any tangent vector field X on M. It can be written as

$$2A\phi AX = \alpha A\phi X + \alpha\phi AX + 2\phi X + 2\phi_1 X \tag{5.3}$$

for any $X \in \mathfrak{D}$ and $\xi = \xi_1$. Substituting (5.1) into (5.3), we get

$$\alpha(A\phi - \phi A)X = -2(\phi X + \phi_1 X) \tag{5.4}$$

for any $X \in \mathfrak{D}$.

Let $\{e_1, e_2, \cdots, e_{4m-4}, e_{4m-3} = \xi, e_{4m-2} = \xi_2, e_{4m-1} = \xi_3\}$ be an orthonormal basis for T_xM, $x \in M$. Then for any tangent vector field Y on M it follows that

$$\begin{aligned}\alpha(A\phi - \phi A)Y &= \sum_{i=1}^{4m-1} g(\alpha(A\phi - \phi A)Y, e_i)e_i \\ &= \sum_{i=1}^{4m-4} g(\alpha(A\phi - \phi A)Y, e_i)e_i + \sum_{\nu=1}^{3} g(\alpha(A\phi - \phi A)Y, \xi_\nu)\xi_\nu \\ &= \sum_{i=1}^{4m-4} g(\alpha(A\phi - \phi A)e_i, Y)e_i + \sum_{\nu=1}^{3} g(\alpha(A\phi - \phi A)Y, \xi_\nu)\xi_\nu.\end{aligned}$$

Putting $Y = e_k \in \mathfrak{D}$ $(k = 1, 2, \cdots, 4m-4)$, this equation can be changed

$$\alpha(A\phi - \phi A)e_k = \sum_{i=1}^{4m-4} g(\alpha(A\phi - \phi A)e_i, e_k)e_i + \sum_{\nu=1}^{3} g(\alpha(A\phi - \phi A)e_k, \xi_\nu)\xi_\nu.$$

From (5.4), it follows that

$$\begin{aligned}
-2(\phi e_k + \phi_1 e_k) &= \alpha(A\phi - \phi A)e_k \\
&= \sum_{i=1}^{4m-4} g(\alpha(A\phi - \phi A)e_i, e_k)e_i + \sum_{\nu=1}^{3} g(\alpha(A\phi - \phi A)e_k, \xi_\nu)\xi_\nu \\
&= \sum_{i=1}^{4m-4} g(-2(\phi e_i + \phi_1 e_i), e_k)e_i + \sum_{\nu=1}^{3} g(-2(\phi e_k + \phi_1 e_k), \xi_\nu)\xi_\nu \\
&= -2\sum_{i=1}^{4m-4} g(\phi e_i, e_k)e_i - 2\sum_{i=1}^{4m-4} g(\phi_1 e_i, e_k)e_i \\
&= 2\sum_{i=1}^{4m-4} g(\phi e_k, e_i)e_i + 2\sum_{i=1}^{4m-4} g(\phi_1 e_k, e_i)e_i \\
&= 2\sum_{i=1}^{4m-4} g(\phi e_k, e_i)e_i + 2\sum_{\nu=1}^{3} g(\phi e_k, \xi_\nu)\xi_\nu \\
&\quad +2\sum_{i=1}^{4m-4} g(\phi_1 e_k, e_i)e_i + 2\sum_{\nu=1}^{3} g(\phi_1 e_k, \xi_\nu)\xi_\nu \\
&= 2\sum_{i=1}^{4m-1} g(\phi e_k, e_i)e_i + 2\sum_{i=1}^{4m-1} g(\phi_1 e_k, e_i)e_i \\
&= 2\phi e_k + 2\phi_1 e_k
\end{aligned}$$

where in the fourth and sixth equalities, we have used $g(\phi e_k, \xi_\nu) = g(\phi_1 e_k, \xi_\nu) = 0$ for any $\nu \pmod 3$ and nonzero real number k. Thus, we get

$$\phi X = -\phi_1 X \tag{5.5}$$

for any tangent vector field $X \in \mathfrak{D}$. Differentiating this equation covariantly in the direction of Y, we have

$$g(AX, Y) = 0$$

for all tangent vector fields $X \in \mathfrak{D}$ and $Y \in TM$, where we have used the formulas about the covariant derivative of structure tensors ϕ and ϕ_ν ($\nu = 1, 2, 3$). It implies that M must be a $\mathfrak{D}^\perp$-invariant hypersurface, if we restrict to $Y \in \mathfrak{D}^\perp$. Accordingly, for this case we can assert that M is locally congruent to model spaces of Type (A) by virtue of Theorem A in the introduction.

Summing up these cases, we consequently know that any Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfying the condition (C-4) is of Type (A).

Now it remains only to show that whether a real hypersurface M_A of Type (A) satisfies the condition (C-4) or not. To check this, let us assume that M_A has the condition $(\nabla_X A)Y = (\widehat{\nabla}^{(k)}_X A)Y$ for any $X \in \mathfrak{D}$ and $Y \in TM_A$. It is equivalent that

$$A\phi AX = \alpha\phi AX, \tag{5.6}$$

for $X \in \mathfrak{D}$ as observed in this section.

From the structure of the tangent vector space $T_x M_A$ for a model space of Type (A) at any point x on M_A, we see that the distribution $\mathfrak{D}$ is composed with two eigenspaces T_λ and

T_μ. In addition, since the eigenspace T_λ is given by $T_\lambda = \{X|\ X \perp \mathbb{H}\xi,\ JX = J_1X\}$ where $\mathbb{H}\xi$ denotes quaternionic span of ξ, we see that $\phi X \in T_\lambda$ for any $X \in T_\lambda$. Using these facts, the Eq. (5.6) is reformed as

$$(\lambda^2 - \alpha\lambda)\phi X = 0$$

for any $X \in T_\lambda \subset \mathfrak{D}$. From this, we get $\lambda^2 - \alpha\lambda = 0$.

On the other hand, from Proposition 3 in [1], we know that

$$\lambda^2 - \alpha\lambda = 2$$

where $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ and $\alpha = 2\sqrt{2}\cot(2\sqrt{2}r)$ for some $r \in (0, \pi/2\sqrt{2})$. This makes a contradiction, and therefore, we have Theorem 4 in the introduction. ◻

References

1. Berndt, J., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians. Monatsh. Math. **127**, 1–14 (1999)
2. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry I, II. Wiley, New York (1969)
3. Alekseevskii, D.V.: Compact quaternion spaces. Funct. Anal. Appl. **2**, 106–114 (1968)
4. Berndt, J., Suh, Y.J.: Real hypersurfaces with isometric Reeb flow in complex two-plane Grassmannians. Monatsh. Math. **137**, 87–98 (2002)
5. Jeong, I., Suh, Y.J.: Real hypersurfaces of Type A in complex two-plane Grassmannians related to commuting shape operator. Forum Math. **25**, 179–192 (2013)
6. Lee, H., Kim, S., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with certain commuting condition. Czechoslov. Math. J. **62**(137), 849–861 (2012)
7. Suh, Y.J.: Real hypersurfaces of Type B in complex two-plane Grassmannians. Monatsh. Math. **147**(4), 375–355 (2006)
8. Lee, H., Suh, Y.J.: Real hypersurfaces of type B in complex two-plane Grassmannians related to the Reeb vector. Bull. Korean Math. Soc. **47**(3), 551–561 (2010)
9. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel shape operator. Bull. Aust. Math. Soc. **68**, 493–502 (2003)
10. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel shape operator II. J. Korean Math. Soc. **41**(3), 535–565 (2004)
11. Tanno, S.: Variational problems on contact Riemannian manifolds. Trans. Am. Math. Soc. **314**(1), 349–379 (1989)
12. Blair, D.E.: Riemannian geometry of contact and symplectic manifolds. Birkhäuser, Boston (2002)
13. Tanaka, N.: On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, Japan J. Math. (N.S.) 2(1), 131–190 (1976)
14. Webster, S.M.: Pseudo-Hermitian structures on a real hypersurface. J. Diff. Geom. **13**, 25–41 (1978)
15. Jeong, I., Lee, H., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with generalized Tanaka-Webster parallel shape operator. Kodai Math. J. **34**(3), 352–366 (2011)
16. Jeong, I., Lee, H., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with generalized Tanaka-Webster $\mathfrak{D}^\perp$-parallel shape operator. Int. J. Geom. Methods Mod. Phys. **9**(4), 1250032 (20 p) (2012)
17. Jeong, I., Machado, C.J.G., Pérez, J.D., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with $\mathfrak{D}^\perp$-parallel structure Jacobi operator. Int. J. Math. **22**, 655–673 (2011)
18. Jeong, I., Pérez, J.D., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel structure Jacobi operator. Acta Math. Hung. **122**, 173–186 (2009)
19. Machado, C.J.G., Pérez, J.D.: Real hypersurfaces in complex two-plane Grassmannians some of whose Jacobi operators are ξ-invariant. International J. Math. **23**(3), 1250002 (12 pp) (2012)
20. Machado, C.J.G., Pérez, J.D., Jeong, I., Suh, Y.J.: $\mathfrak{D}$-parallelism of normal and structure Jacobi operators for hypersurfaces in complex two-plane Grassmannians. Ann. Mat. Pura Appl. (2014) (in press)
21. Pérez, J.D., Suh, Y.J.: The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians. J. Korean Math. Soc. **44**(1), 211–235 (2007)
22. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor. Proc. R. Soc. Edinb. Sect. A **142**, 1309–1324 (2012)

23. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with Reeb parallel Ricci tensor. J. Geom. Phys. **64**, 1–11 (2013)
24. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with harmonic curvature. J. Math. Pures Appl. **100**, 16–33 (2013)
25. Pérez, J.D., Suh, Y.J.: On g-Tanaka-Webster and covariant derivatives of a real hypersurfaces in a complex projective space (submitted)
26. Berndt, J.: Riemannian geometry of complex two-plane Grassmannian. Rend. Semin. Mat. Univ. Politec. Torino **55**, 19–83 (1997)
27. Kon, M.: Real hypersurfaces in complex space forms and the generalized Tanaka-Webster connection. In: Suh, Y.J., Berndt, J., Choi, Y.S. (eds.) Proceedings of the 13th International Workshop on Differential Geometry and Related Fields, pp. 145–159 (2010)