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Classification of bifurcation diagrams for elliptic equations with exponential growth in a ball

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Abstract Let $B \subset \mathbb{R}^N$, $N \geq 3$, be the unit ball. We study the global bifurcation diagram of the solutions of

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \\ u > 0 & \text{in } B, \end{cases}$$

where $f(u) = e^u + g(u)$ and $g(u)$ is a lower order term. The solution set is a curve $\mathcal{C}$ parametrized by the L^∞-norm of the solution. We show that this problem has the singular solution (λ^*, u^*) and that the curve $\mathcal{C}$ has infinitely many turning points around λ^* if $3 \leq N \leq 9$. We show that under a certain condition on g, the curve $\mathcal{C}$ has no turning point if $N \geq 10$. We also study the Morse index of u^*.

Keywords Bifurcation diagram · Exponential growth · Intersection number · Elliptic Dirichlet problem · Singular solution

Mathematics Subject Classification (2000) 35B32 · 35J61 · 35B33 · 35J25

1 Introduction and main results

Let B be a unit ball in $\mathbb{R}^N$, $N \geq 3$. In this paper, we are interested in the global bifurcation diagram of the semilinear elliptic equation with exponential growth

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$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \\ u > 0 & \text{in } B, \end{cases} \tag{1.1}$$

where

$$f(u) = e^u + g(u), \tag{1.2}$$

$g(u)$ is a lower order term, and λ is a nonnegative constant. The precise assumptions on g are given in (f1) and (f2) below. The positive solution of (1.1) is radial, because of the symmetry result of Gidas et al. [9]. The problem (1.1) can be reduced to the following ODE:

$$\begin{cases} u_{rr} + \frac{N-1}{r} u_r + \lambda f(u) = 0, & 0 < r < 1, \\ u(r) > 0, & 0 \le r < 1, \\ u(1) = 0. \end{cases} \tag{1.3}$$

We see in Sect. 3 that the set of the solutions can be parametrized by the L^∞-norm of the solution. Let u be a solution of (1.3) and let $\gamma := u(0) = \|u\|_\infty$. Then λ becomes a graph of γ, i.e., $\lambda(\gamma)$. We assume $f(0) > 0$. Then the set of the positive solutions of (1.3) is a curve $\{(\lambda(\gamma), u(r, \gamma))\}$, which we call the branch, such that it emanates from $(\lambda, \gamma) = (0, 0)$.

We recall known results of the case $f(u) = e^u$. If $3 \le N \le 9$, then the branch has infinitely many turning points around $\lambda_N^* := 2(N-2)$, (1.3) has infinitely many solutions for $\lambda = \lambda_N^*$, and it blows up at λ_N^*. We call this property of the branch *Type I*. This property for $N = 3$ was found by Gel'fand [8]. If $N \ge 10$, then the branch consists only of the minimal solutions in $0 < \lambda < \lambda_N^*$ and it blows up at λ_N^*. We call this property *Type II*. When $N \ge 3$, the singular solution $u_N^* := -2 \log r$ exists for $\lambda = \lambda_N^*$. Schematic pictures of these bifurcation diagrams can be found in Fig. 1 of [13]. The case where $N \ge 4$ was studied by Joseph and Lundgren [14]. See the introduction of [13] for a survey of the case $f(u) = e^u$. When $f(u) = e^u$, there is a special change of variables such that (1.3) can be transformed into the autonomous system of differential equations of the first order. These results were proved by phase plane analysis. However, we cannot expect to find such a change of variables for a general nonlinearity. Our purpose is to show that for a rather general nonlinearity with exponential growth, the branch of the positive solutions of (1.3) is of Type I (resp. Type II) if $3 \le N \le 9$ (resp. $N \ge 10$).

We state assumptions of f:

$$f \in C^1([0,\infty)) \text{ and } f(u) > 0 \text{ in } [0, \infty), \tag{f1}$$

$$\begin{aligned} &f(u) = e^u + g(u), \text{ where there are constants } u_0 > 0, \delta > 0, C_0 > 0 \\ &\quad \text{such that } |g(u)| \le C_0 e^{(1-\delta)u} \ (u > u_0) \text{ and } |g'(u)| \le C_0 e^{(1-\delta)u} \ (u > u_0). \end{aligned} \tag{f2}$$

The first main result of the paper is the following:

Theorem A *Assume that* (f1) *and* (f2) *hold. Then* (1.3) *has a one-parameter family of regular solutions,* $\mathcal{C} := \{(\lambda(\gamma), u(r, \gamma))\}_{\gamma > 0}$*, such that the following hold:*

(i) *$\mathcal{C}$ contains all regular solutions of* (1.3)*,*
(ii) $\lambda(\gamma) \in C^1(0, \infty)$, $\lim_{\gamma \downarrow 0} \lambda(\gamma) = 0$*, and* $\lambda(\gamma) > 0$ $(0 < \gamma < \infty)$*,*
(iii) *there is* $\lambda^* > 0$*, which is given in Proposition 1 below, such that* $\lambda(\gamma) \to \lambda^*$ $(\gamma \to +\infty)$*,*
(iv) *If* $3 \le N \le 9$*, then* $\lambda(\gamma)$ *oscillates around* λ^* *as* $\gamma \to +\infty$*. Therefore, the branch* $\mathcal{C}$ *is of Type I. In particular,* (1.3) *has infinitely many solutions for* $\lambda = \lambda^*$*.*

In the proof of Theorem A, a singular solution plays an important role. We mention the existence of the singular solution of (1.3).

Proposition 1 *Assume that* (f1) *and* (f2) *hold. Then* (1.3) *has a singular solution* (λ^*, u^*) *such that*

$$u^*(r) = -2\log r - \log\lambda^* + \kappa + O(r^{2\delta}) \quad (r \to 0), \tag{1.4}$$

where $\kappa := \log\lambda_N^*$, δ *is the constant in* (f2)*, and* λ^* *is the same value as in Theorem* A *(iii). Moreover,* $u^* \in H^1(B)$.

Note that if $f(u) = e^u$, then $(\lambda^*, u^*) = (\lambda_N^*, u_N^*)$.

Brezis and Vázques [3] studied (1.3) when

$$\begin{aligned} &f \text{ is a continuous, positive, increasing, and convex function on } [0,\infty)\\ &\quad\text{such that } f(t)/t \to \infty \text{ as } t \to \infty. \end{aligned} \tag{1.5}$$

Under these conditions, there is an external value of $\lambda > 0$ such that (1.3) has a minimal solution. They studied the corresponding extremal solution when it is unbounded, i.e., the singular solution. Among other things, they have shown that

Proposition 2 (Brezis and Vazquez [3, Theorem 3.1]) *Suppose that* (1.5) *holds. If* (λ^*, u^*) *is a singular solution of* (1.3)*, if* $u^* \in H^1(B)$*, and if* u^* *is stable in the sense where*

$$\int_B \left(|\nabla\phi|^2 - \lambda^* f'(u^*)\phi^2\right) dx \ge 0 \ \text{ forall } \ \phi \in C_0^1(B), \tag{1.6}$$

then (λ^*, u^*) *is the extremal solution which indicates that the bifurcation diagram of* (1.3) *is of Type II. In particular, the branch does not have a turning point.*

Let $m(u)$ denote the Morse index of u in the space of radial functions, i.e., the number of the negative eigenvalues of the associated eigenvalue problem

$$\begin{cases} \Delta\phi + \lambda f'(u)\phi = -\mu\phi & \text{in } B,\\ \phi = 0 & \text{on } \partial B,\\ \phi \text{ is radial.} \end{cases}$$

Roughly speaking, Proposition 2 says that if $u^* \in H^1(B)$ and if $m(u^*) = 0$, then the bifurcation diagram is of Type II.

Next, we assume the following:

$$\begin{aligned} &f(u) = e^u + g(u),\ g \in C^1([0,\infty)),\ g(u) > 0 \text{ in } (0,\infty),\\ &\quad -e^u < g'(u) \le \tfrac{N-10}{8}e^u \text{ in } (0,\infty),\ \text{and } g''(u) > -e^u \text{ in } (0,\infty), \end{aligned} \tag{f1$'$}$$

instead of (f1). Note that (f1) holds if (f1′) holds. The second main result is the following:

Theorem B *Assume that* $N \ge 10$ *and that (f1′) and* (f2) *hold. Then the singular solution* (λ^*, u^*) *satisfies* (1.6)*. Therefore,* $m(u^*) = 0$ *and the bifurcation diagram of* (1.3) *is of Type II.*

Because of Theorems A and B, the bifurcation diagram has qualitatively the same property as the case $f(u) = e^u$ if all assumptions of Theorems A and B are satisfied. Several examples are given in Sect. 9.

The third result is about the Morse index of the singular solution u^* given in Proposition 1.

Theorem C *Assume that* (f1) *and* (f2) *hold. Let* (λ^*, u^*) *be a singular solution given in Proposition* 1. *Then*

$$m(u^*)\begin{cases} = \infty & (3 \le N \le 9),\\ < \infty & (N \ge 11).\end{cases}$$

When $N = 10$, we need the second term of the asymptotic expansion of the singular solution in order to calculate the Morse index. Then, we have to impose an additional assumption on g. We do not pursue the case $N = 10$ in this paper.

We give an example such that the exact singular solution can be obtained.

Corollary D *Let*

$$f(u) := \frac{(e^{\frac{u}{2}}+1)^3}{e^{\frac{u}{2}}}\left(1+\frac{1}{(N-2)e^{\frac{u}{2}}}\right). \tag{1.7}$$

Then (1.3) *has the singular solution*

$$(\lambda^*, u^*) = \left(\frac{N-2}{2}, -2\log\frac{r}{2-r}\right) \tag{1.8}$$

and

$$\text{the branch is of}\begin{cases} \text{Type I and } m(u^*) = \infty & \text{if } 3 \le N \le 9,\\ \text{Type II and } m(u^*) = 0 & \text{if } N \ge 10.\end{cases} \tag{1.9}$$

We will see in Sect. 9 that (f1$'$) does not hold for $N = 10$, but all assumptions of Proposition 2 hold. Hence, the bifurcation diagram is of Type II. This example indicates that (f1$'$) is not a necessary condition for the bifurcation diagram to be of Type II.

Let us mention technical details. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We recall known results of the Dirichlet problem

$$\begin{cases} \Delta u + \lambda u + u^p = 0 & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases} \tag{1.10}$$

The study of this equation was initiated by Brezis and Nirenberg [2]. They studied the critical case, i.e., $p = p_S := (N+2)/(N-2)$, $N \ge 3$. The study of the problem (1.10) in the supercritical case $p > p_S$ started after this work. Let $\Omega = B$. Since the solution of (1.10) is radial, (1.10) can be reduced to the ODE

$$\begin{cases} u_{rr} + \frac{N-1}{r}u_r + \lambda u + u^p = 0, & 0 < r < 1,\\ u(r) > 0, & 0 \le r < 1,\\ u(1) = 0.\end{cases} \tag{1.11}$$

Let $u(r, \gamma)$ be the solution of (1.11) such that $(u(0, \gamma), u_r(0, \gamma)) = (\gamma, 0)$. Then the branch of the positive solutions can be described as $\{(\lambda(\gamma), u(r, \gamma))\}$. In [16] Merle and Peletier showed that (1.11) has a singular solution (λ^*, u^*) and that $\lambda(\gamma) \to \lambda^*$ $(\gamma \to \infty)$ and $u(r, \gamma) \to u^*(r)$ $(\gamma \to \infty)$ in $C^1_{\mathrm{loc}}(B\backslash\{O\}) \cap H^1(B) \cap L^{p+1}(B)$. We also construct a singular solution of (1.3). Since the proof is similar to that of [16], the proof is shown in Sect. 10.

We show in Sect. 3 that the branch of the solutions of (1.3) is parametrized by the L^∞-norm, using the implicit function theorem. Moreover, it is shown that there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty}$ of solutions of (1.3) such that $\|u_n\|_\infty \to \infty$ $(n \to \infty)$.

In Sect. 4, we prove the convergence of the regular solution to the singular solution as $\|u\|_\infty \to \infty$. The proof is based on that of [16, Theorem B]. We use a scaling argument which is different from one used in [16], since the equation has a different scale invariance.

In Sect. 5, we show that the branch has infinitely many turning points when $3 \le N \le 9$. The problem (1.11) has the same phenomenon. See [4,7,12] for (1.11) and [6,17] for equations with rather general nonlinearities. We use the intersection number of the regular and singular solutions. This method was also used by Guo and Wei [12] and the author [17] in the study of the elliptic equation with supercritical exponent. Let $(\lambda(\gamma), u(r,\gamma))$ be the regular solution of (1.3) such that $(u(0,\gamma), u_r(0,\gamma)) = (\gamma, 0)$, and let $(\lambda^*, u^*(r))$ be the singular solution of (1.3). Let $\hat{u}(s,\gamma) := u(r,\gamma)$ $(s := \sqrt{\lambda(\gamma)}r)$, and let $\hat{u}^*(s) := u^*(r)$ $(s := \sqrt{\lambda^*}r)$. Then $\hat{u}$ and $\hat{u}^*$ satisfy

$$\begin{cases} \hat{u}_{ss} + \frac{N-1}{s}\hat{u}_s + f(\hat{u}) = 0, & 0 < s < \sqrt{\lambda}, \\ \hat{u}(s) > 0, & 0 < s < \sqrt{\lambda}, \\ \hat{u}(\sqrt{\lambda}) = 0, \end{cases} \tag{1.12}$$

where λ reads as λ^* if $\hat{u}^*$ is considered. Note that two intervals $[0, \sqrt{\lambda(\gamma)}]$ and $[0, \sqrt{\lambda^*}]$ may not be equal. We define $\tilde{u}(\rho,\gamma) := \hat{u}(s,\gamma) - \gamma$ $(\rho := e^{\gamma/2}s)$ and $\tilde{u}^*(\rho,\gamma) := \hat{u}^*(s) - \gamma$. Note that $\tilde{u}(0,\gamma) = 0$. Taking the limit as $\gamma \to \infty$, we show that $\tilde{u}(\rho,\gamma)$ and $\tilde{u}^*(\rho,\gamma)$ converge to the regular solution $\bar{u}(\rho, 0)$ and the singular solution $\bar{u}^*(\rho)$ in $C^1_{\text{loc}}(0,\infty)$, respectively. Here, $\bar{u}(\rho,\alpha)$ is the solution of the problem

$$\begin{cases} \bar{u}_{\rho\rho} + \frac{N-1}{\rho}\bar{u}_\rho + e^{\bar{u}} = 0, & 0 < \rho < \infty, \\ \bar{u}(0,\alpha) = \alpha, \\ \bar{u}_\rho(0,\alpha) = 0, \end{cases} \tag{1.13}$$

and

$$\bar{u}^*(\rho) := -2\log\rho + \kappa \tag{1.14}$$

which satisfies the equation in (1.13). We define the zero number of the function $v(r)$ on the interval I by

$$\mathcal{Z}_I[v(\cdot)] := \sharp\{r \in I;\ v(r) = 0\}.$$

Then the intersection number of $\bar{u}(\cdot,0) - \bar{u}^*(\cdot)$ can be written as $\mathcal{Z}_I[\bar{u}(\cdot,0) - \bar{u}^*(\cdot)]$. It is well known that

$$\mathcal{Z}_{[0,\infty)}[\bar{u}(\cdot,0) - \bar{u}^*(\cdot)] = \infty \tag{1.15}$$

provided that $3 \le N \le 9$. In Sect. 2, we briefly prove (1.15). In Sect. 4, we prove

$$\sqrt{\lambda(\gamma)}e^{\gamma/2} \to \infty \quad (\gamma \to \infty). \tag{1.16}$$

Let

$$\hat{\lambda}(\gamma) := \min\{\sqrt{\lambda^*}, \sqrt{\lambda(\gamma)}\} \quad \text{and} \quad I_\gamma := [0, \hat{\lambda}(\gamma)]. \tag{1.17}$$

Using (1.15) and (1.16), we will prove

$$\mathcal{Z}_{I_\gamma}[\hat{u}(\cdot,\gamma) - \hat{u}^*(\cdot)] \to \infty \quad (\gamma \to \infty). \tag{1.18}$$

Because of the uniqueness of the solution of the ODE of the second order, each zero of $\hat{u}(s,\gamma) - \hat{u}^*(s)$ is simple. Thus, the intersection number on I_γ is preserved unless a zero enters I_γ from the boundary ∂I_γ. Since $\hat{u}(0,\gamma) - \hat{u}^*(0) = -\infty$, (1.18) indicates that a simple zero enters I_γ from $\hat{\lambda}(\gamma)$ infinitely many times. Therefore, $\hat{\lambda}(\gamma)$ oscillates around

λ^* as $\gamma \to \infty$, otherwise the sign of $\hat{u}(\hat{\lambda}(\gamma), \gamma) - \hat{u}^*(\hat{\lambda}(\gamma))$ does not change hence a zero cannot enter I_γ.

We prove Theorem B in Sect. 7. In general, the singular solution cannot be written explicitly. Hence, we compare $\hat{u}^*(s)$ with $\bar{u}^*(s)$ and show that $\hat{u}^*(s) \le \bar{u}^*(s)$ under the condition (f1′). The technique used in the proof of lemma 6, which is a key of the proof of Theorem B, was devised by Gui [10,11] and was extended by Bae and Ni [1]. Using this inequality, we check the assumptions of Proposition 2.

We prove Theorem C in Sect. 8. In the proof of the case $N \ge 11$, we use Hardy's inequality and (1.4). When $3 \le N \le 9$, we can find an arbitrary large number of unstable directions and show that $m(u^*) = \infty$.

This paper consists of ten sections. In Sect. 2, we recall known results of the case $f(u) = e^u$. In Sects. 3, 4, and 5, we prove (ii), (iii), and (iv) of Theorem A, respectively. The other assertions of Theorem A are proved in Sect. 6. In Sects. 7 and 8, we prove Theorems B and C, respectively. Several examples including Corollary D are given in Sect. 9. In Sect. 10, we briefly prove the existence of the singular solution (Proposition 1).

2 Preliminaries

We recall known results about the branch of the positive solutions of the Gel'fand problem. See [14,18,19] for details of the facts in this section. We study the equation

$$\bar{u}'' + \frac{N-1}{\rho}\bar{u}' + e^{\bar{u}} = 0, \quad 0 < \rho < \infty, \tag{2.1}$$

where the prime stands for the derivative. It is well known that (1.14) is a singular solution of (2.1). Next, we consider regular solutions of (2.1). Let $\bar{u}(\rho, \alpha)$ be the solution of (1.13). We change variables to $t := \log \rho$ and $y(t) := \bar{u}(\rho, \alpha) - \bar{u}^*(\rho)$. Then $y(t)$ satisfies

$$\begin{cases} y'' + (N-2)y' + 2(N-2)(e^y - 1) = 0, & -\infty < t < \infty, \\ \lim_{t \to -\infty} (y(t) - 2t + \kappa) = \alpha, \\ \lim_{t \to -\infty} e^{-t}(y'(t) - 2) = 0. \end{cases} \tag{2.2}$$

The singular solution $\bar{u}^*(\rho)$ is transformed into $y^*(t)(:= \bar{u}^*(\rho) - \bar{u}^*(\rho)) = 0$. The problem (2.2) becomes the following:

$$\begin{cases} y' = z, \\ z' = -(N-2)z - 2(N-2)(e^y - 1), \\ \lim_{t \to -\infty} (y(t) - 2t + \kappa) = \alpha, \\ \lim_{t \to -\infty} e^{-t}(z(t) - 2) = 0. \end{cases} \tag{2.3}$$

The problem (2.3) has a unique solution $(y(t), z(t))$. This system has the Lyapunov function

$$E(y, z) := \frac{z^2}{2} + 2(N-2)(e^y - y).$$

Then $\frac{\mathrm{d}}{\mathrm{d}t} E(y(t), z(t)) = -(N-2)(z(t))^2 \le 0$. The orbit $\{(y(t), z(t));\ -\infty < t < \infty\}$ in the (y, z)-plane starts along the line $z = 2$ at $t = -\infty$ and converges to the origin. When $3 \le N \le 9$, the origin is a stable spiral and the orbit rotates clockwise around the origin.

Therefore, there is $\{t_j\}_{j=1}^{\infty}$ ($t_1 < t_2 < \cdots$) such that $y(t_j) = 0$ ($j \in \{1, 2, \ldots\}$) and

$$z(t_2) < z(t_4) < \cdots < z(t_{2j}) < \cdots < 0 < \cdots < z(t_{2j-1}) < \cdots < z(t_3) < z(t_1).$$

This means that $y(t)$ oscillates around 0 infinitely many times. Since $y(t) = \bar{u}(\rho, \alpha) - \bar{u}^*(\rho)$, the intersection number of $\bar{u}$ and $\bar{u}^*$ is ∞. When $N \geq 11$ (resp. $N = 10$), the origin is a stable node (resp. a stable star). The orbit does not cross the z-axis, and it converges to the origin.

$\bar{u}(e^{\alpha/2}\rho, 0) + \alpha$ also satisfies (1.13), hence it follows from the uniqueness of the solution that $\bar{u}(\rho, \alpha) = \bar{u}(e^{\alpha/2}\rho, 0) + \alpha$. This transformation does not change the singular solution, i.e., $\bar{u}^*(\rho) = \bar{u}^*(e^{\alpha/2}\rho) + \alpha$.

Proposition 3 *Let $\bar{u}(\rho, \alpha)$ be the regular solution of* (1.13)*, and let $\bar{u}^*(\rho)$ be the singular solution of* (2.1) *given by* (1.14)*. Then $\bar{u}(\rho, \alpha) = \bar{u}(e^{\alpha/2}\rho, 0) + \alpha$ and $\bar{u}^*(\rho) = \bar{u}^*(e^{\alpha/2}\rho) + \alpha$. Moreover,*

$$\mathcal{Z}_{[0,\infty)}[\bar{u}(\,\cdot\,, \alpha) - \bar{u}^*(\,\cdot\,)] = \begin{cases} +\infty & (3 \leq N \leq 9), \\ 0 & (N \geq 10). \end{cases}$$

We consider the case $N \geq 10$. As mentioned in Sect. 1, the positive branch of (1.3) consists only of the minimal solutions when $f(u) = e^u$. In particular, for each fixed $\lambda > 0$, (1.3) has at most one solution. Let $\beta > \alpha$. We suppose that there is $\rho_0 > 0$ such that $\bar{u}(\rho, \beta) > \bar{u}(\rho, \alpha)$ for $0 \leq \rho < \rho_0$ and $\bar{u}(\rho_0, \beta) = \bar{u}(\rho_0, \alpha)$. Let $\gamma := \bar{u}(\rho_0, \beta) (= \bar{u}(\rho_0, \alpha))$. Let $u_0(\rho) := \bar{u}(\rho_0\rho, \beta) - \gamma$ and $u_1(\rho) := \bar{u}(\rho_0\rho, \alpha) - \gamma$. Then both u_0 and u_1 satisfy $\Delta u + \rho_0^2 e^{\gamma} e^u = 0$. Moreover, $u_0(1) = u_1(1) = 0$. Since $u_0(0) = \beta - \gamma \neq \alpha - \gamma = u_1(0)$, (1.3) has two solutions for $\lambda = \rho_0^2 e^{\gamma}$, which is a contradiction. Therefore, we obtain the following:

Proposition 4 *Assume that $N \geq 10$. If $\beta > \alpha$, then $\bar{u}(\rho, \beta) > \bar{u}(\rho, \alpha)$ for $\rho \geq 0$.*

3 Parametrization results of the branch

3.1 Assumption (f2)

We transform (f2) into (f2′) below.

Proposition 5 *If* (f2) *holds, then the following (f2′) holds:*

$$f(u) = e^u + g(u), \text{ where there are constants } \delta > 0, C_0 > 0 \text{ such that } \max_{0 \leq \rho \leq u} |g(\rho)| \leq C_0 e^{(1-\delta)u} \ (u \geq 0) \text{ and } \max_{0 \leq \rho \leq u} |g'(\rho)| \leq C_0 e^{(1-\delta)u} \ (u \geq 0). \tag{f2′}$$

Proof of Proposition 5 We take $C_0 > 0$ large enough such that

$$\max_{0 \leq \rho \leq u_0} \{|g(\rho)|, |g'(\rho)|\} \leq C_0,$$

where u_0 appears in (f2). Then (f2′) holds. □

3.2 Parametrization of the positive branch

Let $\mathcal{C}$ denote the branch consisting of the positive regular solutions of (1.3). We recall known properties of $\mathcal{C}$.

Proposition 6 *Suppose that* (f1) *holds. Let* $(\lambda_0, u_0(r)) \in \mathcal{C}$. *Then* $\mathcal{C}$ *can be locally parametrized by* $\|u\|_\infty = u(0)$. *Specifically, there are a* C^1*-map* $(\lambda(\gamma), u(r,\gamma))$ *and a neighborhood* $\mathcal{U}$ *of* (λ_0, u_0) *such that* $(\lambda(\gamma_0), u(r,\gamma_0)) = (\lambda_0, u_0(r))$ *and* $\mathcal{C} \cap \mathcal{U} = \{(\lambda(\gamma), u(r,\gamma)); u(0,\gamma) = \gamma,\ |\gamma - \gamma_0| < \varepsilon\}$.

This proposition was proven by Korman [15, Theorem 2.1]. However, we give a proof for readers' convenience.

Proof of Proposition 6 Let $u(r,\gamma)$ be the unique solution of

$$\begin{cases} u'' + \dfrac{N-1}{r}u' + \lambda f(u) = 0, & 0 < r < 1, \\ u(0,\gamma) = \gamma,\ u'(0,\gamma) = 0,\ u(1,\gamma) = 0, & \\ u(r,\gamma) > 0, & 0 \le r < 1. \end{cases} \tag{3.1}$$

We change variables to $\hat{u}(s,\gamma) := u(r,\gamma)$ and $s = \sqrt{\lambda}r$. Then $\hat{u}$ satisfies

$$\begin{cases} \hat{u}'' + \dfrac{N-1}{s}\hat{u}' + f(\hat{u}) = 0, & 0 < s < \sqrt{\lambda}, \\ \hat{u}(0,\gamma) = \gamma,\ \hat{u}'(0,\gamma) = 0,\ \hat{u}(\sqrt{\lambda},\gamma) = 0, & \\ \hat{u}(s,\gamma) > 0, & 0 \le s < \sqrt{\lambda}. \end{cases} \tag{3.2}$$

We define $h(\lambda,\gamma) := \hat{u}(\sqrt{\lambda},\gamma)$. Then $h(\lambda_0,\gamma_0) = 0$ and $h \in C^1$ in a neighborhood of (λ_0,γ_0). Let B_R denote the ball of radius R. Differentiating h with respect to λ, we have $h_\lambda(\lambda, h) = \hat{u}_s(\sqrt{\lambda},\gamma)/(2\sqrt{\lambda})$. Since $-\Delta\hat{u} = f(\hat{u}) > 0$ in $B_{\sqrt{\lambda_0}}$ and 0 is the minimum of $\hat{u}(\,\cdot\,,\gamma_0)$ in $\overline{B_{\sqrt{\lambda_0}}}$, Hopf's boundary point lemma tells us that $\hat{u}_s(\sqrt{\lambda_0},\gamma_0) < 0$. Thus, $h_\lambda(\lambda_0,\gamma_0) < 0$. The implicit function theorem says that there are a C^1-function $\lambda = \lambda(\gamma)$ and a small $\varepsilon > 0$ such that $h(\lambda(\gamma),\gamma) = 0$ for $\gamma \in (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$ and $\lambda(\gamma_0) = \lambda_0$. This means that all the solutions of (3.1) in a neighborhood of (λ_0,γ_0) are $\{(\lambda(\gamma), u(r,\gamma))\}_{|\gamma-\gamma_0|<\varepsilon}$. □

Because of (f1), $f(0) > 0$, hence $\mathcal{C}$ emanates from $(\lambda, u) = (0,0)$. We extend $\mathcal{C}$. Specifically, we show the global parametrization result of $\mathcal{C}$ under the condition $f(t) \ge C_1 t$ $(t \ge 0)$.

Proposition 7 *Suppose that* (f1) *and* (f2) *hold. Then the branch* $\mathcal{C}$ *can be globally parametrized by* $\|u\|_\infty$, *it is unbounded in the positive direction of* $\|u\|_\infty$, *and there is* $C_0 > 0$ *such that* $\mathcal{C} \subset \{(\lambda, u);\ 0 < \lambda < C_0\}$. *Specifically,* $\mathcal{C} := \{(\lambda(\gamma), u(r,\gamma));\ u(0,\gamma) = \gamma,\ 0 < \gamma < \infty\}$ *and* $0 < \lambda(\gamma) < C_0$.

This proposition was essentially proved by Crandall and Rabinowitz [5, Theorem 1.1].

Because of Proposition 7, we can choose a sequence $\{\gamma_n\}_{n=1}^\infty$ diverging to $+\infty$ such that $(\lambda(\gamma_n), u(r,\gamma_n))$ is a solution of (3.1). The sequence $\{(\lambda(\gamma_n), u(r,\gamma_n))\}_{n=1}^\infty$ is used for the scaling argument in the proof of Theorem A.

4 Convergence to the singular solution

We consider the initial value problem

$$\begin{cases} \hat{u}_{ss} + \frac{N-1}{s}\hat{u} + e^{\hat{u}} + g(\hat{u}) = 0, & 0 < s < \infty, \\ \hat{u}(0) = \gamma, & \\ \hat{u}_s(0) = 0. & \end{cases} \tag{4.1}$$

Let $\hat{u}(s,\gamma)$ be the solution of (4.1). Let $\rho := e^{\gamma/2}s$, and let $\tilde{u}(\rho,\gamma) := \hat{u}(s,\gamma) - \gamma$. Then, $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_{\rho\rho} + \frac{N-1}{\rho}\tilde{u}_\rho + e^{\tilde{u}} + e^{-\gamma}g(\tilde{u}+\gamma) = 0, & 0 < \rho < \infty, \\ \tilde{u}(0) = 0, \\ \tilde{u}_\rho(0) = 0. \end{cases} \tag{4.2}$$

Lemma 1 *Let $\bar{u}(\rho,\alpha)$ be the solution of* (1.13). *Then*

$$\tilde{u}(\rho,\gamma) \to \bar{u}(\rho,0) \ \ in \ \ C^1_{\mathrm{loc}}[0,\infty) \ \ as \ \ \gamma \to \infty.$$

Proof Because of (f2′),

$$e^{-\gamma}g(\tilde{u}+\gamma) \le e^{-\gamma}C_0 e^{(1-\delta)(\tilde{u}+\gamma)}. \tag{4.3}$$

We see that if $\gamma > 0$ is large, then

$$C_0 e^{-\delta\gamma} \le 3. \tag{4.4}$$

We easily see that

$$\tilde{u}(\rho,\gamma) \le 0 \ \text{ for } \ 0 \le \rho < \infty. \tag{4.5}$$

Using (4.3) and (4.4), we have

$$\begin{aligned} \tilde{u}_{\rho\rho} + \frac{N-1}{\rho}\tilde{u}_\rho &= -e^{\tilde{u}} - e^{-\gamma}g(\tilde{u}+\gamma) \\ &\ge -e^{\tilde{u}} - C_0 e^{-\delta\gamma} e^{(1-\delta)\tilde{u}} \\ &\ge -e^{\tilde{u}} - 3e^{(1-\delta)\tilde{u}} \\ &\ge -4e^{\tilde{u}}. \end{aligned}$$

Therefore, $(\rho^{N-1}\tilde{u}_\rho)_\rho \ge -4e^{\tilde{u}}\rho^{N-1}$. Integrating this inequality, we have

$$\begin{aligned} \tilde{u}_\rho(\rho) &\ge -\frac{4}{\rho^{N-1}}\int_0^\rho e^{\tilde{u}}\eta^{N-1}\mathrm{d}\eta \\ &\ge -\frac{4}{\rho^{N-1}}\int_0^\rho \eta^{N-1}\mathrm{d}\eta \\ &= -\frac{4\rho}{N}, \end{aligned}$$

where we use (4.5). Integrating this inequality, we have

$$\tilde{u}(\rho) \ge -\frac{2\rho^2}{N}. \tag{4.6}$$

Because of (4.6) and (4.5),

$$-\frac{2\rho^2}{N} \le \tilde{u}(\rho,\gamma) \le 0. \tag{4.7}$$

Thus, for each $\rho_0 > 0$, $\tilde{u}(\rho,\gamma)$ is bounded in $0 \le \rho \le \rho_0$. Since $|e^{-\gamma}g(\tilde{u}(\rho,\gamma)+\gamma)| \le C_0 e^{-\delta\gamma+(1-\delta)\tilde{u}(\rho,\gamma)}$, $e^{-\gamma}g(\tilde{u}(\rho,\gamma)+\gamma)$ uniformly converges to 0 in $0 \le \rho \le \rho_0$ as $\gamma \to +\infty$. This indicates that

$$\tilde{u}(\rho,\gamma) \to \bar{u}(\rho,0) \ \text{ in } \ C^1[0,\rho_0] \ \text{ as } \ \gamma \to \infty. \tag{4.8}$$

We can choose ρ_0 arbitrarily large. We obtain the conclusion. □

We change variables $t := \log s$ and $y(t,\gamma) := \hat{u}(s,\gamma) + 2t - \kappa$. Then $y(t,\gamma)$ satisfies

$$\begin{cases} y_{tt} + (N-2)y_t + 2(N-2)(e^y - 1) \\ zw + e^{2t} g(y - 2t + \kappa) = 0, \quad -\infty < t < \log\sqrt{\lambda(\gamma)}, \\ \lim\limits_{t\to-\infty} (y(t) - 2t + \kappa) = \gamma, \\ \lim\limits_{t\to-\infty} e^{-t}(y'(t) - 2) = 0. \end{cases} \tag{4.9}$$

Let

$$\tau := t + \frac{\gamma}{2} \quad \text{and} \quad \hat{y}(\tau,\gamma) := y(t,\gamma).$$

Then $\hat{y}$ satisfies

$$\begin{cases} \hat{y}_{\tau\tau} + (N-2)\hat{y}_\tau + 2(N-2)(e^{\hat{y}} - 1) \\ zw + e^{2\tau-\gamma} g(\hat{y} - 2\tau + \gamma + \kappa) = 0, \quad -\infty < \tau < \frac{\gamma}{2} + \log\sqrt{\lambda(\gamma)}, \\ \lim\limits_{\tau\to-\infty} (\hat{y}(\tau) - 2\tau + \kappa) = 0, \\ \lim\limits_{\tau\to-\infty} e^{-\tau}(\hat{y}(\tau) - 2) = 0. \end{cases}$$

Let $\check{y}(t,\gamma)$ be the solution of (4.9) with $g \equiv 0$, and let $\bar{y}(\tau) := \check{y}(t,\gamma)$. Then, $\bar{y}$ satisfies

$$\begin{cases} \bar{y}_{\tau\tau} + (N-2)\bar{y}_\tau + 2(N-2)(e^{\bar{y}} - 1) = 0, \qquad -\infty < \tau < \frac{\gamma}{2} + \log\sqrt{\lambda(\gamma)}, \\ \lim\limits_{\tau\to-\infty} (\bar{y}(\tau) - 2\tau + \kappa) = 0, \\ \lim\limits_{\tau\to-\infty} e^{-\tau}(\bar{y}(\tau) - 2) = 0. \end{cases} \tag{4.10}$$

Because of Lemma 1, for each $\rho_0 > 0$, (4.8) holds. Since $\tilde{u}(\rho,\gamma) = \hat{u}(s,\gamma) - \gamma = y(t,\gamma) - 2t + \kappa - \gamma = \hat{y}(\tau,\gamma) - 2\tau + \kappa$,

$$\hat{y}(\tau,\gamma) - 2\tau + \kappa = \tilde{u}(\rho,\gamma) \to \bar{u}(\rho,\gamma) = \bar{y}(\tau) - 2\tau + \kappa$$

uniformly on $-\infty < \tau < \log\rho_0$ as $\gamma \to +\infty$ and

$$e^{-\tau}(\hat{y}_\tau(\tau,\gamma) - 2) = \tilde{u}_\rho(\rho,\gamma) \to \bar{u}_\rho(\rho,\gamma) = e^{-\tau}(\bar{y}_\tau(\tau) - 2)$$

uniformly on $-\infty < \tau < \log\rho_0$ as $\gamma \to +\infty$. Since $\rho_0 > 0$ can be chosen arbitrarily large, we have

Corollary 1 *For each $\tau_0 > 0$,*

$$\hat{y}(\tau,\gamma) \to \bar{y}(\tau), \quad \hat{y}_\tau(\tau,\gamma) \to \bar{y}_\tau(\tau) \quad \textit{as} \quad \gamma \to +\infty$$

uniformly on the interval $-\infty < \tau < \tau_0$.

When the solution $\bar{y}(\tau)$ of (4.10) is defined on the whole interval $\mathbb{R}$, (4.10) says that $(\bar{y}(\tau), \bar{y}_\tau(\tau)) \to (0,0)$ as $\tau \to +\infty$. This fact and Corollary 1 indicate that $(y(t,\gamma), y_t(t,\gamma))$ approaches $(0,0)$ as $\gamma \to \infty$ along $t = \tau_0 - \frac{\gamma}{2}$ provided that τ_0 is chosen large enough. Let $z(t,\gamma) := y_t(t,\gamma)$. Then (y,z) satisfies

$$\begin{cases} y_t = z \\ z_t = -(N-2)z - 2(N-2)(e^y - 1) - e^{2t} g(y - 2t + \kappa). \end{cases} \tag{4.11}$$

In the next lemma, we prove the following: If there is large $t_0 > 0$ such that $(y(-t_0,\gamma), z(-t_0,\gamma))$ is in a neighborhood of $(0,0)$, then there exists $T (> -t_0)$ independent of γ such that $(y(t,\gamma), z(t,\gamma))$ stays in the neighborhood for $-t_0 < t < T$.

Lemma 2 *Let* $\Gamma_\varepsilon := \{(y,z)\in\mathbb{R}^2;\ 2(N-2)(e^y-1-y)+\frac{z^2}{2}\le\varepsilon\}$. *For each small* $\varepsilon>0$, *there is a large* $t_0>0$ *such that* $(y(-t_0,\gamma), y_t(-t_0,\gamma))\in\Gamma_\varepsilon$ *provided that* $\gamma>0$ *is large. Moreover, there is* T_ε *independent of* t_0 *such that* $(y(t,\gamma), y_t(t,\gamma))\in\Gamma_{2\varepsilon}$ $(-t_0\le t\le T_\varepsilon)$.

Proof Let $H(y,z) := 2(N-2)(e^y-1-y)+\frac{z^2}{2}$ and $G(u)=\int_0^u g(s)\mathrm{d}s$. We define $E(y,z,t)$ by $E(y,z,t) := H(y,z)+e^{2t}G(y-2t+\kappa)$. By direct calculation, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(y(t),z(t),t) &= -(N-2)(y_t(t))^2+2e^{2t}(G(y(t)-2t+\kappa)-g(y(t)-2t+\kappa))\\ &\le 2e^{2t}(G(y(t)-2t+\kappa)-g(y(t)-2t+\kappa)).\end{aligned}\tag{4.12}$$

Let $\varepsilon>0$ be small such that $\Gamma_{2\varepsilon}\subset\{|y|<1\}$. Note that Γ_ε is a neighborhood of $(0,0)$. We choose $T\in\mathbb{R}$ such that

$$\max\left\{\frac{C_0(2-\delta)}{\delta(1-\delta)}e^{(1-\delta)(1+\kappa)}e^{2\delta T}, \frac{C_0}{1-\delta}e^{(1-\delta)(1+\kappa)}e^{2\delta T}\right\}\le\frac{\varepsilon}{4}.\tag{4.13}$$

Because of Corollary 1, there is large $t_0>0$ ($t_0>-T$) such that if $\gamma>0$ is large, then

$$(y(-t_0,\gamma), z(-t_0,\gamma))\in\Gamma_\varepsilon.\tag{4.14}$$

We show that $(y(t),z(t))\in\Gamma_{2\varepsilon}$ $(-t_0\le t\le T)$. Suppose the contrary, we assume that

$$(y(t),z(t))\in\Gamma_{2\varepsilon}\ \ (-t_0\le t<T)\quad\text{and}\quad (y(T),z(T))\notin\Gamma_{2\varepsilon}.\tag{4.15}$$

Integrating (4.12) over $[-t_0,T]$, we have

$$\begin{aligned}&E(y(T),y(T),T)-E(y(-t_0),z(-t_0),-t_0)\\ &\le 2\int_{-t_0}^{T} e^{2t}(G(y(t)-2t+\kappa)-g(y(t)-2t+\kappa))\mathrm{d}t\\ &\le 2\int_{-t_0}^{T} e^{2t}(|G(y(t)-2t+\kappa)|+|g(y(t)-2t+\kappa)|)\mathrm{d}t\\ &\le 2\int_{-t_0}^{T} e^{2t}\left(\frac{C_0}{1-\delta}e^{(1-\delta)(y(t)-2t+\kappa)}+C_0e^{(1-\delta)(y(t)-2t+\kappa)}\right)\mathrm{d}t\\ &=\frac{2C_0(2-\delta)}{1-\delta}\int_{-t_0}^{T}e^{(1-\delta)(y(t)+\kappa)+2\delta t}\mathrm{d}t\\ &=\frac{C_0(2-\delta)}{\delta(1-\delta)}e^{(1-\delta)(1+\kappa)}\left(e^{2\delta T}-e^{-2\delta t_0}\right),\end{aligned}$$

where we use

$$|g(u)|\le C_0e^{(1-\delta)u},\quad |G(u)|\le\frac{C_0}{1-\delta}e^{(1-\delta)u},\quad\text{and}\quad |y(t)|\le 1.$$

Using

$$\begin{aligned}|e^{-2t_0}G(y(-t_0)+2t_0+\kappa)| &\le \frac{C_0}{1-\delta}e^{(1-\delta)(1+\kappa)-2\delta t_0}\le\frac{\varepsilon}{4}\quad\text{and}\\ |e^{2T}G(y(T)-2T+\kappa)| &\le\frac{C_0}{1-\delta}e^{(1-\delta)(1+\kappa)+2\delta T}\le\frac{\varepsilon}{4},\end{aligned}$$

we have

$$\begin{aligned}H(y(T), z(T)) &\le H(y(-t_0), z(-t_0)) + e^{-2t_0} G(y(-t_0) + 2t_0 + \kappa)\\ &\quad zw - e^{2T} G(y(T) - 2T + \kappa)\\ &\quad zw + \frac{C_0(2-\delta)}{\delta(1-\delta)} e^{(1-\delta)(1+\kappa)} \left(e^{2\delta T} - e^{-2\delta t_0}\right)\\ &\le \varepsilon + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}\\ &= \frac{7\varepsilon}{4},\end{aligned}$$

where we use (4.13) and (4.14). Hence, $(y(T), z(T)) \in \Gamma_{7\varepsilon/4} \subset \Gamma_{2\varepsilon}$, which contradicts to (4.15). The conclusion of the lemma holds. □

Lemma 3 *Let (λ^*, u^*) be the singular solution given in Proposition 1. Then, as $\gamma \to \infty$,*

$$\lambda(\gamma) \to \lambda^* \quad \text{and} \quad u(r, \gamma) \to u^*(r) \quad \text{in} \quad C^1_{\mathrm{loc}}(0, 1].$$

Proof Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence diverging to $+\infty$, and let $y(t, \gamma_n)$ be a solution of (4.9). We define $z(t, \gamma_n)$ by $z(t, \gamma_n) := y_t(t, \gamma_n)$. By Lemma 2, we see that $(y(t, \gamma_n), z(t, \gamma_n))$ is uniformly bounded in $(C^0[-t_0, T])^2$. Because y and y_t satisfy (4.9), $y_{tt}(t, \gamma_n)$ is also uniformly bounded in $C^0[-t_0, T]$. Differentiating the equation in (4.9), we see that $y_{ttt}(t, \gamma_n)$ is uniformly bounded. By Arzelá–Ascoli theorem, we see that there is a subsequence, which is still denoted by $\{(y(t, \gamma_n), z(t, \gamma_n))\}$, such that $(y(t, \gamma_n), z(t, \gamma_n))$ converges to some pair of functions $(y_*(t), z_*(t))$ in $(C^1[-t_0, T])^2$. Since $t_0 > 0$ can be arbitrary large, we see that $(y(t, \gamma_n), z(t, \gamma_n)) \to (y_*(t), z_*(t))$ in $(C^1_{\mathrm{loc}}(-\infty, T))^2$. Since $0 < \lambda(\gamma_n) < C_0$ (Proposition 7), there is $\lambda_* \in [0, C_0]$ such that $\lambda(\gamma_n) \to \lambda_*$. Since $(y(t, \gamma_n), z(t, \gamma_n))$ satisfies (4.11), (y_*, z_*) also satisfies (4.11). In order to show that (y_*, z_*, λ_*) is a solution of the problem

$$\begin{cases} y_t = z,\\ z_t = -(N-2)z - 2(N-2)(e^y - 1) - e^{2t} g(y - 2t + \kappa),\\ y(t) - 2t + \kappa > 0, \quad -\infty < t < \log\sqrt{\lambda_*},\\ \lim\limits_{t \to -\infty} y(t) = 0,\\ y(\log\sqrt{\lambda_*}) - 2\log\sqrt{\lambda_*} + \kappa = 0,\end{cases}$$

we show later that

$$y_*(t) \to 0 \quad \text{as} \quad t \to -\infty. \tag{4.16}$$

In the proof of Proposition 1, we show that this problem has a unique solution (y^*, z^*, λ^*). Thus, $y_* = y^*$ and $\lambda_* = \lambda^*$. Since $y(t, \gamma) \to y^*(t)$ in $C^1_{\mathrm{loc}}(-\infty, T]$, it follows from the uniqueness of the solution to the ODE that $y(t, \gamma_n) \to y^*(t)$ in $C^1_{\mathrm{loc}}(-\infty, \lambda^*)$. Therefore, $\hat{u}(s, \gamma_n) \to \hat{u}^*(s)$ in $C^1_{\mathrm{loc}}(0, \lambda^* + 1)$, which also implies that $\lambda(\gamma_n) \to \lambda^*$ $(n \to \infty)$. Since $u(r, \gamma_n) = \hat{u}(s, \gamma_n)$ $(\sqrt{\lambda(\gamma_n)}r = s)$, $u(r, \gamma_n) \to u^*(r)$ in $C^1_{\mathrm{loc}}(0, \lambda^*]$. The conclusion of the lemma holds.

We prove (4.16) by contradiction. Suppose the contrary, there is a sequence t_k such that $t_k \to -\infty$ and $(y_*(t_k), z_*(t_k)) \notin \Gamma_\delta$ for all $k \ge 1$. We choose $\varepsilon = \delta/4$. By Corollary 1 there exist large $\tau_0 > 0$ and large $\gamma > 0$ such that $\left(y\left(\tau_0 - \frac{\gamma}{2}\right), z\left(\tau_0 - \frac{\gamma}{2}\right)\right) \in \Gamma_\varepsilon$. By Lemma 3, we see that $(y(t), z(t)) \in \Gamma_{2\varepsilon} \subset \Gamma_\delta$ in $(\tau_0 - \frac{\gamma}{2}, T)$, where T is independent of γ. Since $\gamma > 0$ is large enough, the interval $(\tau_0 - \frac{\gamma}{2}, T]$ can be made to include an element of $\{t_k\}$. We obtain a contradiction. □

5 Oscillation of the branch

Lemma 4 *Suppose that* $3 \leq N \leq 9$ *and that* (f1) *and* (f2) *are satisfied. Then* (1.18) *holds.*

Proof We show that

$$\tilde{u}(\rho, \gamma) \to \bar{u}(\rho, 0) \text{ in } C^1_{\mathrm{loc}}(0, \infty) \text{ as } \gamma \to +\infty. \tag{5.1}$$

For each bounded interval I, there is a constant $C > 0$ independent of γ such that $\|\tilde{u}(\cdot, \gamma)\|_{C^0(I)} < C$, because of (4.7). Then,

$$|e^{-\gamma} g(\tilde{u}(\rho, \gamma) + \gamma)| \leq C_0 e^{(1-\delta)\tilde{u}(\rho, \gamma) - \delta\gamma} \to 0 \text{ in } C^0(I) \ (\gamma \to \infty).$$

Since $\tilde{u}$ satisfies

$$\tilde{u}_{\rho\rho} + \frac{N-1}{\rho}\tilde{u}_\rho + e^{\tilde{u}} + e^{-\gamma} g(\tilde{u} + \gamma) = 0,$$

$\tilde{u}(\rho, \gamma)$ converges to $\bar{u}(\rho, 0)$ in $C^1(I)$ as $\gamma \to \infty$. Since I can be arbitrarily chosen, (5.1) holds.

We define $\tilde{u}^*(\rho, \gamma) := \hat{u}^*(s) - \gamma$ ($\rho := e^{\frac{\gamma}{2}} s$). Because of Proposition 1, $\hat{u}^*(s) = -2 \log s + \kappa + o(1)$ ($s \to 0$). Then, $\tilde{u}^*(\rho, \gamma) = 2 \log \rho + \kappa + o(1)$ ($e^{-\frac{\gamma}{2}} \rho \to 0$). For each bounded interval I, $e^{-\gamma} \rho \to 0$ in $C^0(I)$ ($\gamma \to \infty$). Thus,

$$\tilde{u}^*(\rho, \gamma) \to \bar{u}^*(\rho) \text{ in } C^1_{\mathrm{loc}}(0, \infty) \text{ as } \gamma \to \infty. \tag{5.2}$$

Here $\bar{u}^*(\rho)$ is defined by (1.14).

Lemma 3 says that $\lambda(\gamma) \to \lambda^*$ ($\gamma \to \infty$). Since $\lambda^* > 0$, $\sqrt{\lambda(\gamma)} e^{\frac{\gamma}{2}} \to \infty$ ($\gamma \to \infty$). Using (5.1), (5.2), and Proposition 3, we have

$$\mathcal{Z}_{[0, \min\{\sqrt{\lambda^*} e^{\frac{\gamma}{2}}, \sqrt{\lambda(\gamma)} e^{\frac{\gamma}{2}}\}]}[\tilde{u}(\cdot, \gamma) - \tilde{u}^*(\cdot, \gamma)] \to \infty \ (\gamma \to \infty).$$

Hence, we obtain (1.18). □

The main result of this section is the following:

Lemma 5 *Suppose that* $3 \leq N \leq 9$ *and that* (f1) *and* (f2) *are satisfied. The function* $\lambda(\gamma)$ *oscillates infinitely many times around* λ^* *as* $\gamma \to \infty$.

Proof We consider $\hat{u}(s, \gamma)$ and $\hat{u}^*(s)$. Let I_γ be given by (1.17). They satisfy the equation in (1.12) on I_γ. Because of the uniqueness of the solution of the ODE, if $\hat{u}(s, \gamma) - \hat{u}^*(s)$ has a zero, then it should be simple. We call $\mathcal{Z}_{I_\gamma}[\hat{u}(\cdot, \gamma) - \hat{u}^*(\cdot)]$ the intersection number. For each $\gamma > 0$, the intersection number is finite, otherwise zeros of $\hat{u}(s, \gamma) - \hat{u}^*(s)$ accumulate at some point and the accumulation point is a degenerate zero which contradicts to the uniqueness of the solution of the ODE. Since every zero of $\hat{u}(s, \gamma) - \hat{u}^*(s)$ is simple and the intersection number is finite, it follows from the implicit function theorem that each zero depends continuously on γ. The intersection number is preserved unless another zero enters I_γ from the boundary of I_γ. Since $\hat{u}(0, \gamma) - \hat{u}^*(0) = -\infty$, a zero cannot enter I_γ from $s = 0$. We prove the statement of the lemma by contradiction. Suppose the contrary, i,e.,

$$\text{there is } \gamma_0 > 0 \text{ such that } \lambda(\gamma) > \lambda^* \text{ forall } \gamma > \gamma_0. \tag{5.3}$$

Then, $I_\gamma = [0, \sqrt{\lambda^*}]$ and $\hat{u}(\sqrt{\lambda^*}, \gamma) - \hat{u}^*(\sqrt{\lambda^*}) > 0$. Therefore, a zero cannot enter I_γ from $s = \sqrt{\lambda^*}$, and the zero number does not increase. This contradicts to Lemma 4. (5.3) does not hold. We can similarly show that

$$\text{there does not exist } \gamma_0 > 0 \text{ such that } \lambda(\gamma) < \lambda^* \text{ forall } \gamma > \gamma_0. \tag{5.4}$$

By (5.3) and (5.4), we see that $\lambda(\gamma)$ oscillates infinitely many times around λ^* as $\gamma \to \infty$. □

6 Proof of Theorem A

Proof of Theorem A Let $\mathcal{C} := \{(\lambda(\gamma), u(r, \gamma))\}$ be the continuum of solutions of (1.3) constructed in Proposition 7. Since for each $\gamma > 0$, there is a unique $\lambda > 0$, which is $\lambda(\gamma)$, such that (1.10) holds. Hence, there is no solution except $\mathcal{C}$. Thus, (i) holds. Because of Proposition 6, $\lambda(\gamma) \in C^1(0, \infty)$. We easily see that $\lambda(0) = 0$ and $\lambda(\gamma) > 0$ $(0 < \gamma < \infty)$. (ii) holds. By Lemma 3, we see that (iii) holds. (iv) follows from Lemma 5. The proof is complete. □

7 Proof of Theorem B

Let $\bar{u}(s, \alpha)$ be the solution of (1.13), and let $\hat{u}(s, \gamma)$ be the solution of (4.1).

Lemma 6 *Suppose that* $N \geq 10$ *and that* (f1′) *and* (f2) *are satisfied. If* $0 < \gamma < \alpha$, *then* $\hat{u}(s, \gamma) < \bar{u}(s, \alpha)$ *for* $s > 0$.

Proof Suppose the contrary, i.e., there is $S > 0$ such that $\hat{u}(s, \gamma) < \bar{u}(s, \alpha)$ for $0 < s < S$ and $\hat{u}(S, \gamma) = \bar{u}(S, \alpha)$. Let $w_0(s) := \bar{u}(s, \alpha) - \hat{u}(s, \gamma)$, and let B_S denote the ball of radius S. Then

$$\begin{cases} \Delta w_0 + k_0 w_0 = g(\hat{u}) & \text{in } B_S, \\ w_0 > 0 & \text{in } B_S, \\ w_0(S) = 0, \end{cases}$$

where

$$k_0 := \frac{e^{\bar{u}(s,\alpha)} - e^{\hat{u}(s,\gamma)}}{\bar{u}(s, \alpha) - \hat{u}(s, \alpha)} < e^{\bar{u}(s,\gamma)} \text{ in } B_S$$

and $w_0'(S) \leq 0$. Let $w_1(s) := \bar{u}(s, \beta) - \bar{u}(s, \alpha)$. When $\beta > \alpha$, then we see by Proposition 4 that $w_1(s) > 0$ for $s > 0$. We have

$$\Delta w_1 + k_1 w_1 = 0 \text{ in } \mathbb{R}^N,$$

where

$$k_1 := \frac{e^{\bar{u}(s,\beta)} - e^{\bar{u}(s,\alpha)}}{\bar{u}(s, \beta) - \bar{u}(s, \alpha)} > e^{\bar{u}(s,\alpha)}.$$

By Green's identity, we have

$$\begin{aligned} \omega_N S^{N-1} w_1(S) w_0'(S) &= \int_{B_S} (w_1 \Delta w_0 - w_0 \Delta w_1) \\ &\geq \int_{B_S} (k_1 - k_0) w_0 w_1 > 0, \end{aligned}$$

where ω_N denotes the surface area of the unit sphere in $\mathbb{R}^N$ and we used the inequality $g > 0$ in $(0, \infty)$. This implies that $w_0'(S) > 0$, which contradicts that $w_0'(S) \leq 0$. □

Lemma 7 *Suppose that* $N \geq 10$ *and that* (f1$'$) *and* (f2) *are satisfied.* $\hat{u}^*(s) \leq \bar{u}^*(s)$ *for* $0 < s \leq \sqrt{\lambda_N^*}$.

Proof Suppose the contrary, i.e., there is $s_0 \in (0, \sqrt{\lambda_N^*}]$ such that $\hat{u}^*(s_0) > \bar{u}^*(s_0)$. Because of Lemma 3, as $\alpha \to +\infty$,

$$\hat{u}(s, \alpha) \to \hat{u}^*(s) \text{ in } C^0_{\text{loc}}(0, \sqrt{\lambda_N^*}]$$

and

$$\bar{u}(s, \alpha + 1) \to \bar{u}^*(s) \text{ in } C^0_{\text{loc}}(0, \sqrt{\lambda_N^*}].$$

It follows from Lemma 6 that $\hat{u}(s, \alpha) < \bar{u}(s, \alpha + 1)$. Taking the limit, we have that $\hat{u}^*(s) \leq \bar{u}^*(s)$ for $s \in (0, \sqrt{\lambda_N^*}]$. We obtain a contradiction, because $\hat{u}^*(s_0) \leq \bar{u}^*(s_0)$. □

Proof of Theorem B We can easily show that (1.5) holds, using (f1$'$). It follows from Proposition 1 that $u^* \in H^1(B)$. All we have to do is to check (1.6). Because of Lemma 7, we have $\hat{u}^*(s) \leq \bar{u}^*(s)$. Therefore,

$$\begin{aligned} e^{\hat{u}^*} + g'(\hat{u}^*) &\leq e^{\hat{u}^*} + \frac{N-10}{8} e^{\hat{u}^*} \\ &= \frac{N-2}{8} e^{\hat{u}^*} \\ &\leq \frac{N-2}{8} e^{\bar{u}^*} \\ &= \frac{(N-2)}{8} \frac{2(N-2)}{s^2}, \end{aligned}$$

where we used $e^{\bar{u}^*} = \frac{2(N-2)}{s^2}$ and the inequality $g'(u) \leq \frac{N-10}{8} e^u$. Therefore,

$$\begin{aligned} &\int_0^1 \left(|\nabla \phi|^2 - \lambda^* \left(e^{u^*} + g'(u^*) \right) \phi^2 \right) r^{N-1} \mathrm{d}r \\ &= (\lambda^*)^{-\frac{N-2}{2}} \int_0^{\sqrt{\lambda^*}} \left(|\nabla \hat{\phi}|^2 - \left(e^{\hat{u}^*} + g'(\hat{u}^*) \right) \hat{\phi}^2 \right) s^{N-1} \mathrm{d}s \\ &\geq (\lambda^*)^{-\frac{N-2}{2}} \int_0^{\sqrt{\lambda^*}} \left(|\nabla \hat{\phi}|^2 - \frac{(N-2)^2}{4s^2} \hat{\phi}^2 \right) s^{N-1} \mathrm{d}s \geq 0, \end{aligned}$$

where $\hat{\phi}(s) := \phi(r)$, $s := \sqrt{\lambda^*} r$, and we use Hardy's inequality. We have checked (1.6). □

8 Proof of Theorem C

We study the Morse index of the singular solution.

Lemma 8 *Suppose that* $3 \leq N \leq 9$ *and that* (f1) *and* (f2) *are satisfied. Then* $m(u^*) = \infty$.

Proof Let $(\lambda^*, u^*(r))$ be the singular solution given by Proposition 1. Let $\hat{u}^*(s) := u^*(r)$, and let $s := \sqrt{\lambda^*} r$. Because of (1.4), for each small $\theta > 0$, there is a small $\rho_0 > 0$ such that

$$-2\log s + \kappa - \theta \le \hat{u}^*(s) \le -2\log s + \kappa + \theta \quad \text{for} \quad s \in (0, \rho_0).$$

Because of (f2′),

$$|g'(\hat{u}^*(s))| \le C_0 e^{(1-\delta)\hat{u}^*} \le C_0 \frac{\{2(N-2)\}^{1-\delta}}{s^{2(1-\delta)}} e^{(1-\delta)\theta} \quad \text{for} \quad s \in (0, \rho_0). \tag{8.1}$$

For each small $\varepsilon > 0$, there is $\rho_1 \in (0, \rho_0)$ such that

$$\begin{aligned} f'(\hat{u}^*) = e^{\hat{u}^*} + g'(\hat{u}^*) &\ge \frac{2(N-2)}{s^2}\left(e^{-\theta} - C_0\{2(N-2)\}^{-\delta} s^{2\delta} e^{(1-\delta)\theta}\right) \\ &\ge \frac{2(N-2)(1-\varepsilon)}{s^2} \quad \text{for} \quad s \in (0, \rho_1). \end{aligned}$$

Using this inequality, we have

$$\int_0^{\rho_1} \left(|\nabla\hat{\phi}|^2 - f'(\hat{u}^*)\hat{\phi}^2\right) s^{N-1} ds \le \int_0^{\rho_1} \left(|\nabla\hat{\phi}|^2 - \frac{2(N-2)(1-\varepsilon)}{s^2}\hat{\phi}^2\right) s^{N-1} ds. \tag{8.2}$$

When $3 \le N \le 9$, $2(N-2)(1-\varepsilon) > (N-2)^2/4$. Hence, there is a small $\varepsilon_0 > 0$ such that

$$-\frac{2(N-2)(1-\varepsilon)}{s^2} < -\left(\frac{(N-2)^2}{4} + \varepsilon_0^2\right)\frac{1}{s^2}. \tag{8.3}$$

We define $\hat{\phi}_j(s) := \hat{\phi}(s)\chi_j(s)$, where $\hat{\phi}(s) := s^{-\frac{N-2}{2}} \sin(\frac{\varepsilon_0}{2}\log s)$, $s_j := e^{-2\pi j/\varepsilon_0}$, and

$$\chi_j(s) := \begin{cases} 1 & (s \in [s_{j+1}, s_j]), \\ 0 & (s \notin [s_{j+1}, s_j]). \end{cases}$$

Then $\hat{\phi}_j$ satisfies

$$\hat{\phi}_j'' + \frac{N-1}{s}\hat{\phi}_j' + \frac{1}{s^2}\left(\frac{(N-2)^2}{4} + \frac{\varepsilon_0^2}{4}\right)\hat{\phi}_j = 0, \quad s_{j+1} < s < s_j.$$

This indicates that

$$\int_0^{\sqrt{\lambda^*}} \left(|\nabla\hat{\phi}_j|^2 - \frac{1}{s^2}\left(\frac{(N-2)^2}{4} + \varepsilon_0^2\right)\hat{\phi}_j^2\right) s^{N-1} ds = -\int_0^{\sqrt{\lambda^*}} \frac{3\varepsilon_0^2}{4}\hat{\phi}_j^2 s^{N-3} ds < 0. \tag{8.4}$$

Because of (8.2), (8.3), and (8.4), there is a large $j_0 > 0$ such that, for $j \geq j_0$,

$$\int_0^1 \left(|\nabla \phi_j|^2 - \lambda^* f'(u^*)\phi_j^2\right) r^{N-1} \mathrm{d}r \tag{8.5}$$

$$= (\lambda^*)^{-\frac{N-2}{2}} \int_0^{\sqrt{\lambda^*}} \left(|\nabla \hat{\phi}_j|^2 - f'(\hat{u}^*)\hat{\phi}_j^2\right) s^{N-1} \mathrm{d}s$$

$$< -(\lambda^*)^{-\frac{N-2}{2}} \int_0^{\sqrt{\lambda^*}} \frac{3\varepsilon_0^2}{4} \hat{\phi}_j^2 s^{N-3} \mathrm{d}s < 0, \tag{8.6}$$

where $\hat{\phi}_j(s) := \phi_j(r)$ and $s := \sqrt{\lambda^*} r$. Since $\int_0^1 \phi_j \phi_k r^{N-1} \mathrm{d}r = 0$ $(j \neq k)$, (8.6) indicates that $m(u^*) = \infty$. □

Lemma 9 *Suppose that $N \geq 11$ and that (f1) and (f2) holds. Then $m(u^*) < \infty$.*

Proof For each small $\varepsilon > 0$, there is a small $\rho_0 > 0$ such that $\hat{u}^*(s) \leq -2\log s + \kappa + \varepsilon$ for $s \in (0, \rho_0)$. For $s \in [0, \rho_0)$,

$$\begin{aligned} f'(\hat{u}^*) = e^{\hat{u}^*} + g'(\hat{u}^*) &\leq \frac{2(N-2)}{s^2} e^{\varepsilon} + C_0 e^{(1-\delta)\hat{u}^*} \\ &\leq \frac{2(N-2)}{s^2} e^{\varepsilon} + \frac{C_0\{2(N-2)\}^{1-\delta}}{s^{2(1-\delta)}} e^{(1-\delta)\varepsilon} \\ &\leq \frac{2(N-2)}{s^2} \left(e^{\varepsilon} + C_0(2(N-2))^{1-\delta} e^{(1-\delta)\varepsilon} s^{2\delta}\right). \end{aligned}$$

When $N \geq 11$, then $2(N-2) < \frac{(N-2)^2}{4}$. Therefore, if $\varepsilon > 0$ is small, then there is $\rho_1 \in (0, \rho_0)$ such that $f'hatu^*) \leq (N-2)^2/(4s^2)$ $(0 < s < \rho_1)$. We define

$$\chi_0(s) := \begin{cases} 1 & (0 < s < \rho_1/2), \\ 0 & (\rho_1 < s), \end{cases}$$

where $0 \leq \chi_0(s) \leq 1$ and $\chi_0(s) \in C^1$. Let $\chi_1(s) := 1 - \chi_0(s)$. Then we have

$$\begin{aligned} \int_0^{\sqrt{\lambda^*}} \left(|\nabla \hat{\phi}|^2 - f'(\hat{u}^*)\hat{\phi}^2\right) s^{N-1} \mathrm{d}s &= \int \left(|\nabla \hat{\phi}|^2 - (\chi_0 + \chi_1) f'hatu^*)\hat{\phi}^2\right) s^{N-1} \mathrm{d}s \\ &= \int \left(|\nabla \hat{\phi}|^2 - \chi_0 f'(\hat{u}^*)\hat{\phi}^2\right) s^{N-1} \mathrm{d}s + \int \left(|\nabla \hat{\phi}|^2 - \chi_1 f'(\hat{u}^*)\hat{\phi}^2\right) s^{N-1} \mathrm{d}s \\ &\geq \int \left(|\nabla \hat{\phi}|^2 - \frac{(N-2)^2}{4s^2}\hat{\phi}^2\right) s^{N-1} \mathrm{d}s + \int \left(|\nabla \hat{\phi}|^2 - \chi_1 f'(\hat{u}^*)\hat{\phi}^2\right) s^{N-1} \mathrm{d}s \\ &\geq \int \left(|\nabla \hat{\phi}|^2 - \chi_1 f'(\hat{u}^*)\hat{\phi}^2\right) s^{N-1} \mathrm{d}s. \end{aligned} \tag{8.7}$$

Let $\mathcal{R}$ be a function space of radial functions on B. Let

$$X := \left\{ \hat{\phi} \in H_0^1(B) \cap \mathcal{R};\ \int_0^{\sqrt{\lambda^*}} \left(|\nabla \hat{\phi}|^2 - \chi_1 f'(\hat{u}^*)\hat{\phi}^2\right) s^{N-1} \mathrm{d}s \leq 0 \right\}.$$

Since $|\chi_1 f'(\hat{u}^*)|$ is bounded on B, the operator $\Delta + \chi_1 f'(\hat{u}^*)$ with the Dirichlet boundary condition has at most finitely many negative eigenvalues, i.e., $\dim X < \infty$. This inequality indicates that

$$\dim \left\{ \hat{\phi} \in H_0^1(B) \cap \mathcal{R};\ \int_0^{\sqrt{\lambda^*}} \left(|\nabla \hat{\phi}|^2 - f'(\hat{u}^*)\hat{\phi}^2 \right) s^{N-1} \mathrm{d}s \le 0 \right\} < \infty,$$

otherwise (8.7) indicates that $\dim X = \infty$. Thus, $m(\hat{u}^*) < \infty$, which means that $m(u^*) < \infty$. □

Lemmas 8 and 9 prove Theorem C.

9 Examples

9.1 First example

Let $f(u) := e^u$. Then (f1) holds. (1.3) has the singular solution $(\lambda^*, u^*) = (\lambda_N^*, u_N^*)$. Let $g(u) := f(u) - e^u = 0$. Then (f2) clearly holds. Theorems A and C are applicable. Hence, if $3 \le N \le 9$, then the branch of the positive solutions is of Type I and $m(u^*) = \infty$.

When $N \ge 10$, Brezis and Vázquez [3] studied this case by a method different from Joseph and Lundgren [14]. They obtained

$$\lambda^* f'(u^*) = \frac{2(N-2)}{r^2}.$$

They have shown that if $N \ge 10$, then

$$\lambda^* f'(u^*) = \frac{2(N-2)}{r^2} \le \frac{(N-2)^2}{4r^2} \quad \text{for } r \in (0, 1].$$

Hence, if $N \ge 10$, then

$$\int_B \left(|\nabla \phi|^2 - \lambda^* f'(u^*)\phi^2 \right) \mathrm{d}x \ge \int_B \left(|\nabla \phi|^2 - \frac{(N-2)^2}{4r^2}\phi^2 \right) \mathrm{d}x \ge 0$$

for all $\phi \in C_0^1(B)$, where we use Hardy's inequality. This inequality indicates that $m(u^*) = 0$. Proposition 2 says that the branch is of Type II. Hence,

$$\text{the branch the positive solutions is of } \begin{cases} \text{Type I and } m(u^*) = \infty \text{ if } 3 \le N \le 9, \\ \text{Type II and } m(u^*) = 0 \text{ if } N \ge 10. \end{cases}$$

9.2 Second example

Let f be defined by (1.7). (f1) clearly holds. Let $g(u) := f(u) - e^u$. Then

$$g(u) = \frac{3N-5}{N-2} e^{\frac{u}{2}} + \frac{3N-3}{N-2} + \frac{N+1}{N-2} e^{-\frac{u}{2}} + \frac{1}{N-2} e^{-u}.$$

Since

$$g'(u) = \frac{3N-5}{2N-4} e^{\frac{u}{2}} - \frac{N+1}{2N-4} e^{-\frac{u}{2}} - \frac{1}{N-2} e^{-u},$$

(f2) holds. Thus, Theorem A is applicable and the branch of the positive solutions is of Type I if $3 \leq N \leq 9$. By direct calculation, we see that (1.3) has the singular solution (λ^*, u^*) defined by (1.8). Theorem C tells us that $m(u^*) = \infty$ if $3 \leq N \leq 9$.

Next we consider the case $N \geq 10$. By direct calculation, we can check (1.5). We have

$$\lambda^* f'(u^*) = \frac{2(N-2)}{r^2} - \frac{1}{r^2}\left(\frac{(N-5)r}{2-r} + \frac{4r}{(2-r)^2}\right).$$

We easily see that $u^* \in H^1(B)$ if $N \geq 3$. When $N \geq 10$,

$$\frac{2(N-2)}{r^2} - \frac{1}{r^2}\left(\frac{(N-5)r}{2-r} + \frac{4r}{(2-r)^2}\right) \leq \frac{(N-2)^2}{4r^2} \text{ for } 0 < r \leq 1.$$

Using Hardy's inequality, we have

$$\int_B \left(|\nabla\phi|^2 - \lambda^* f'(u^*)\phi^2\right) dx \geq \int_B \left(|\nabla\phi|^2 - \frac{(N-2)^2}{4r^2}\phi^2\right) dx \geq 0 \qquad (9.1)$$

for all $\phi \in C_0^1(B)$ if $N \geq 10$. Thus, (1.6) holds. Proposition 2 says that the branch of the positive solutions is of Type II. Moreover, (9.1) indicates that $m(u^*) = 0$ if $N \geq 10$. We have obtained Corollary D.

When $N = 10$, (f1′) does not hold, but the bifurcation diagram is of Type II. Hence, (f1′) is not a necessary condition for the bifurcation diagram to be of Type II.

9.3 Other examples

We study the case where the exact expression of the singular solution is not known.

We consider the case $g(u) := \frac{1}{u+1}$. Then (f1) and (f2) are satisfied. If $N \geq 10$, then (f1′) holds. Hence, (1.9) holds.

We consider the case $g(u) := \frac{1}{4}e^{\frac{u}{2}}$. Then (f1) and (f2) are satisfied. When $N \geq 11$, then (f1′) holds. However, if $N = 10$, then (f1′) does not hold. The bifurcation diagram is of Type I (resp. Type II) if $3 \leq N \leq 9$ (resp. $N \geq 11$).

10 Singular solution

We briefly show that (1.3) has the singular solution. We consider the problem

$$\begin{cases} y'' + (N-2)y' + 2(N-2)(e^y - 1) + e^{2t}g(y - 2t + \kappa) = 0, \\ y(t) \to 0 \ \ (t \to -\infty). \end{cases} \qquad (10.1)$$

Let $\tau := -t$ and $\eta(\tau) := y(t)$. Then the equation in (10.1) becomes

$$\eta'' - (N-2)\eta' + 2(N-2)(e^\eta - 1) + e^{-2\tau}g(\eta + 2\tau + \kappa) = 0. \qquad (10.2)$$

Lemma 10 *Let $\eta(\tau)$ be a solution of (10.2) such that $\eta(\tau) \to 0$ ($\tau \to \infty$). Then $\eta(\tau) = O(e^{-2\delta\tau})$ ($\tau \to \infty$).*

Proof Let $h(\eta, \tau) := -2(N-2)(e^\eta - 1 - \eta) - 2e^{-2\tau}g(\eta + 2\tau + \kappa)$. Then η satisfies

$$\eta'' - (N-2)\eta' + 2(N-2)\eta = h(\eta, \tau).$$

We divide the possibilities into three cases:

$$\text{(a) } 3 \le N \le 9, \qquad \text{(b) } N = 10, \qquad \text{(c) } N \ge 11.$$

We prove only the case (a). The other cases can be proved similarly.

Let $\mu := \sqrt{|(N-2)(N-10)|}/2$. The function η satisfies

$$\eta(\tau) = \frac{1}{\mu} e^{\frac{N-2}{2}\tau} \int_\tau^\infty e^{-\frac{N-2}{2}\sigma} \sin(\mu(\sigma-\tau)) h(\eta,\sigma) \mathrm{d}\sigma. \tag{10.3}$$

Since $\eta(\tau) \to 0$ $(\tau \to \infty)$, there exists $\varepsilon > 0$ such that

$$|h(\eta,\tau)| \le \varepsilon|\eta(\tau)| + Ce^{-2\delta\tau} \ \text{ for large } \ \tau > 0. \tag{10.4}$$

Using (10.3), we have

$$|\eta(\tau)| \le Ce^{-2\delta\tau} + \varepsilon \int_\tau^\infty |\eta(\sigma)| \mathrm{d}\sigma \ \text{ for large } \ \tau > 0.$$

By Gronwall's inequality, we have

$$|\eta(\tau)| \le Ce^{-2\delta\tau} \ \text{ for large } \ \tau > 0. \tag{10.5}$$

□

Proof of Proposition 1 In the proof, we consider only the case (a). Let

$$\mathcal{F}(\eta)(\tau) := \frac{1}{\mu} e^{\frac{N-2}{2}\tau} \int_\tau^\infty e^{-\frac{N-2}{2}\sigma} \sin(\mu(\sigma-\tau)) h(\eta,\sigma) \mathrm{d}\sigma.$$

For large $T > 0$ and let X be the space of continuous functions on (T,∞) equipped with the norm $\|\xi\| := \sup\{\xi(\tau);\ \tau > T\}$. Let $\mathcal{B} := \{\xi \in X;\ \|\xi\| < \varepsilon\}$. Then if T is large, then $\mathcal{F}$ is a contraction mapping from $\mathcal{B}$ into itself. Hence, we see that the integral equation $\eta(\tau) = \mathcal{F}(\eta)(\tau)$ has a unique solution in $\mathcal{B}$.

Let $\hat{u}(s) := -2\log s + \kappa + y(t)$. Then $\hat{u}(s)$ satisfies the equation in (1.12). Therefore, we can extend the domain of $\hat{u}(s)$ in the positive direction of s. We show by contradiction that there is $\lambda^* > 0$ such that

$$\begin{cases} \hat{u}(s) > 0, & 0 < s < \sqrt{\lambda^*}, \\ \hat{u}(\sqrt{\lambda^*}) = 0. \end{cases}$$

Suppose the contrary, i.e., we assume that $\hat{u}(s) > 0$ $(0 < s < \infty)$. Since $\hat{u}_{ss} + \frac{N-1}{s}\hat{u}_s = -f(\hat{u}) < 0$, $\hat{u}$ does not have a local minimum point, hence $\hat{u}(s)$ is decreasing. Because $\hat{u}(s) > 0$, $\hat{u}_s \to 0$ $(s \to \infty)$ and there is $c \ge 0$ such that $\hat{u} \to c$ $(s \to \infty)$. Since $\hat{u}_{ss} = -\frac{N-1}{s}\hat{u}_s - f(\hat{u}) \to -f(c) < 0$ $(s \to \infty)$, this contradicts that $\hat{u} \to c$ $(s \to \infty)$.

By Lemma 10, we see that $y(t) = O(e^{2\delta t})$, hence $\hat{u}(s) = -2\log s + \kappa + O(s^{2\delta})$. Therefore, (1.4) is obtained.

Next, we show that there is a small $R > 0$ such that

$$|u_r^*(r)| \le \frac{2}{r}(1 + o(1)) \text{ for } 0 < r < R. \tag{10.6}$$

Differentiating (10.3) with respect to τ, we have

$$\eta'(\tau) = \frac{N-2}{2\mu} e^{\frac{N-2}{2}\tau} \int_\tau^\infty e^{-\frac{N-2}{2}\sigma} \sin(\mu(\sigma-\tau)) h(\eta,\sigma) \mathrm{d}\sigma$$
$$- e^{\frac{N-2}{2}\tau} \int_\tau^\infty e^{-\frac{N-2}{2}\sigma} \cos(\mu(\sigma-\tau)) h(\eta,\sigma) \mathrm{d}\sigma.$$

Because of (10.4) and (10.5),

$$\begin{aligned} |h(\eta,\sigma)| &\le \varepsilon |\eta(\tau)| + Ce^{-2\delta\tau} \\ &\le \varepsilon C e^{-2\delta\tau} + Ce^{-2\delta\tau} \\ &= Ce^{-2\delta\tau}. \end{aligned}$$

Hence,

$$\begin{aligned} |\eta'(\tau)| &\le \frac{N-2}{2\mu} e^{\frac{N-2}{2}\tau} \int_\tau^\infty e^{-\frac{N-2}{2}\sigma} Ce^{-2\delta\sigma} \mathrm{d}\sigma + e^{\frac{N-2}{2}\tau} \int_\tau^\infty e^{-\frac{N-2}{2}\sigma} Ce^{-2\delta\sigma} \mathrm{d}\sigma. \\ &\le Ce^{-2\delta\tau}. \end{aligned}$$

Since $t = -\tau$ and $\eta(\tau) = y(t)$,

$$|y'(t)| = |\eta'(\tau)| \le Ce^{-2\delta\tau} = Ce^{2\delta t}. \tag{10.7}$$

Since $u^*(r) = y(t) - 2t + \kappa$ and $t = \log \sqrt{\lambda^*} r$, we have $u_r^*(r) = y'(\log \sqrt{\lambda} r)\frac{1}{r} - \frac{2}{r}$. Using (10.7), we have

$$\begin{aligned} |u_r^*(r)| &\le |y'(\log \sqrt{\lambda} r)| \frac{1}{r} + \frac{2}{r} \\ &\le \frac{C\lambda^\delta r^{2\delta}}{r} + \frac{2}{r}. \end{aligned}$$

This inequality proves (10.6).

We prove that $u^* \in H^1(B)$. Because $u^*(r)$ has a singularity only at the origin, it is enough to show that $u^* \in H^1(B_R)$ for small $R > 0$. Using (1.4) and (10.6), we have

$$\begin{aligned} \int_{B_R} \left(|\nabla u^*|^2 + |u^*|^2\right) \mathrm{d}x &= \int_0^R \left(|u_r^*(r)|^2 + |u^*(r)|^2\right) r^{N-1} \mathrm{d}r \\ &\le \int_0^R \left(9r^{N-3} + (-2\log r + c)^2 r^{N-1}\right) \mathrm{d}r < \infty, \end{aligned}$$

if $N \ge 3$. Thus, $u^* \in H^1(B_R)$. The proof is complete. □

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