Annali di Matematica (2015) 194:903–918
DOI 10.1007/s10231-014-0403-9

Superquadratic or asymptotically quadratic Hamiltonian systems: ground state homoclinic orbits

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Received: 30 October 2013 / Accepted: 22 January 2014 / Published online: 8 February 2014

Abstract Under super or asymptotically quadratic assumptions at infinity, we obtain the existence of nontrivial *ground state* homoclinic orbits for a class of second-order Hamiltonian systems with general potentials by a variant generalized weak linking theorem. For the asymptotically quadratic case, a *necessary and sufficient condition* is obtained for the existence of nontrivial homoclinic orbits. For the superquadratic case, we use general superquadratic conditions to replace the Ambrosetti–Rabinowitz condition.

Keywords Second-order Hamiltonian systems · Ground state homoclinic orbits · Superquadratic · Asymptotically quadratic · Strongly indefinite functionals

Mathematics Subject Classification (2010) 37J45 · 37K05 · 58E05

1 Introduction and main results

In this paper, we consider the following second-order Hamiltonian system

$$-\ddot{u}(t) + A(t)u(t) = \nabla W(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $A(t)$ is continuous T-periodic $N \times N$ symmetric matrix, $W(t, u) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is continuous for each $u \in \mathbb{R}^N$ and T-periodic in t, and $\nabla W(t, u)$ denotes its gradient with respect to the u variable. We say that a solution $u(t)$ of (1.1) is homoclinic (with 0) if $u(t) \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$. If $u(t) \not\equiv 0$, then $u(t)$ is called a nontrivial homoclinic solution.

Research supported by the Tianyuan Fund for Mathematics of NSFC (Grant No. 11326113) and the Key Project of Natural Science Foundation of Educational Committee of Henan Province of China (Grant No. 13A110015).

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In recent decades, many authors are devoted to the existence and multiplicity of homoclinic orbits for second-order Hamiltonian systems with super or asymptotically linear terms, see [1–10,12,14–17,21,24] and the references therein. If the matrix $A(t)$ is positive definite uniformly in t, some authors [6–8,16,17] obtained the existence of homoclinic orbits for (1.1). **However**, for some mathematical physics, the global positive definiteness of $A(t)$ is not satisfied; thus, it is necessary for us to study the case that the matrix $A(t)$ is not uniformly positively definite for $t \in \mathbb{R}$.

We notice that, in these works (except for [8,21]), it was always assumed that $W(t,u)$ satisfies the following **superquadratic** condition (see [2]): there exists a constant $\mu > 2$ such that

$$0 < \mu W(t,u) \le (\nabla W(t,u), u), \quad u \in \mathbb{R}^N \backslash \{0\}, \tag{1.2}$$

where $(\cdot,\cdot)$ denotes the standard inner product in $\mathbb{R}^N$, and the associated norm is denoted by $|\cdot|$. **However**, we are interested in the case where $W(t,u)$ satisfies conditions that are more general than (1.2). Also, there are some authors who have considered (1.1) with $W(t,u)$ satisfying **asymptotically quadratic** growth at infinity, see [8,21] and so on. **However**, to the best of our knowledge, there is no result published concerning *necessary and sufficient conditions* for the existence of nontrivial homoclinic orbits of (1.1) with W satisfying asymptotically quadratic growth at infinity.

We assume 0 lies in a gap of $\sigma(B)$, the spectrum of $B := -\frac{d^2}{dt^2} + A(t)$, that is,

(A₁) $\underline{\Lambda} := \sup(\sigma(B) \cap (-\infty, 0)) < 0 < \overline{\Lambda} := \inf(\sigma(B) \cap (0,\infty))$.

Let $\widetilde{W}(t,u) := \frac{1}{2}(\nabla W(t,u), u) - W(t,u)$. Firstly, we consider the *superquadratic* situation. We shall use a general assumption to replace the superquadratic condition (1.2) and assume

(W_1) $|\nabla W(t,u)| = o(|u|)$ as $|u| \to 0$ uniformly in $t \in \mathbb{R}$.
(W_2) $W(t,u) \ge 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$ and $\widetilde{W}(t,u) > 0$ if $u \in \mathbb{R}^N \backslash \{0\}$.
(W_3) $\frac{W(t,u)}{|u|^2} \to +\infty$ as $|u| \to +\infty$ uniformly in $t \in \mathbb{R}$.
(W_4) There exist $c_0, r_0 > 0$, and $\sigma > 1$ such that

$$\frac{|\nabla W(t,u)|^\sigma}{|u|^\sigma} \le c_0 \widetilde{W}(t,u) \quad \text{if } |u| \ge r_0.$$

Next we consider the *asymptotically quadratic* situation. We still need (W_2) and assume

(W'_1) $|\nabla W(t,u)| \le c|u|^{\mu-1}$ if $|u| \le R$ for some $c, R > 0$ and $\mu > 2$, $\forall t \in \mathbb{R}$.
(W'_3) $W(t,u) = \frac{1}{2}V|u|^2 + F(t,u)$, where

$$|\nabla F(t,u)| = o(|u|) \text{ as } |u| \to +\infty \text{ uniformly in } t, \quad 0 < V < \infty.$$

(W'_4) There exist $c_1, c_2, R_1, R_2 > 0$ and $1 < \alpha < 2$ such that

$$\widetilde{W}(t,u) \ge c_1|u|^\mu \quad \text{if } |u| \le R_1, \qquad \widetilde{W}(t,u) \ge c_2|u|^\alpha \quad \text{if } |u| \ge R_2, \quad \forall t \in \mathbb{R}.$$

Now, our main results read as follows:

Theorem 1.1 *Assume that (A_1) and (W_2) hold. If either $((W_1)$, (W_3) and $(W_4))$ or $((W'_1)$, (W'_4), and (W'_3) with $V > \overline{\Lambda})$ hold, then (1.1) has at least one nontrivial ground state homoclinic orbit.*

For the asymptotically quadratic situation, by Theorem 1.1, we obtain a **necessary and sufficient** condition for the existence of nontrivial homoclinic orbit of (1.1).

Corollary 1.1 *Assume that* (A_1), (W_1'), (W_2), (W_3'), *and* (W_4') *hold. If* $\frac{|\nabla W(t,u)|}{|u|} \leq V$ *for all* $(t,u) \in \mathbb{R} \times \mathbb{R}^N$ *and*

$$\underline{\Lambda} + V \leq \min\left\{0, \overline{\Lambda} - V\right\}, \tag{1.3}$$

then (1.1) admits a nontrivial homoclinic orbit if and only if $V > \overline{\Lambda}$.

Remark 1.1 Notice that the inequality (1.3) always holds if $0 < V < \infty$ is small enough. Therefore, Corollary 1.1 shows that $V > \overline{\Lambda}$ is a sharp condition for the existence of nontrivial homoclinic orbit for (1.1). To the best of our knowledge, there is no result published concerning *necessary and sufficient conditions* for the existence of nontrivial homoclinic orbits of (1.1).

For the superquadratic situation, we give the following example. As is shown in the next example, assumptions $(W_1)-(W_4)$ are reasonable, and there are cases in which the condition (1.2) is not satisfied.

Example 1.1 (Superquadratic). Let

$$W(t,u) = g(t)\left(|u|^p + (p-2)|u|^{p-\varepsilon}\sin^2(|u|^\varepsilon/\varepsilon)\right),$$

where $g(t) > 0$ is T-periodic in t, $0 < \varepsilon < p-2$ and $p > 2$. It is not hard to check that $W(t,u)$ satisfies $(W_1)-(W_4)$. However, similar to Remark 1.2 of [23], let $u_m := (\varepsilon(m\pi + \frac{3\pi}{4}))^{\frac{1}{\varepsilon}} L_N$, where $L_N = (1, 0, \ldots, 0)$. Then, for any $\gamma > 2$, one has

$$\begin{aligned}
&(\nabla W(t,u_m), u_m) - \gamma W(t,u_m)\\
&= g(t)\left[(p-\gamma)|u_m|^p + (p-2)(p-\varepsilon-\gamma)|u_m|^{p-\varepsilon}\sin^2\left(\frac{|u_m|^\varepsilon}{\varepsilon}\right)\right.\\
&\quad \left. + (p-2)|u_m|^p \sin 2(|u_m|^\varepsilon/\varepsilon)\right]\\
&= g(t)|u_m|^p\left[2-\gamma + \frac{(p-2)(p-\varepsilon-\gamma)\sin^2\left(\frac{|u_m|^\varepsilon}{\varepsilon}\right)}{|u_m|^\varepsilon}\right] \to -\infty \text{ as } m \to \infty,
\end{aligned}$$

that is, the condition (1.2) cannot be satisfied for $\gamma > 2$.

For the asymptotically quadratic situation, we give the following example.

Example 1.2 (Asymptotically quadratic). Let

$$W(t,u) = \begin{cases} \left(\frac{1}{2}V - d(t)\right)|u|^\mu & \text{if } |u| \leq 1,\\ \frac{1}{2}V|u|^2 - d(t)|u|^\alpha & \text{if } |u| \geq 1,\end{cases}$$

where $0 < \inf_{t\in\mathbb{R}} d(t) \leq \sup_{t\in\mathbb{R}} d(t) < \frac{1}{2}V$ and $\mu > 2 > \alpha > 1$. It is not hard to check that the above function satisfies (W_1'), (W_2), (W_3'), and (W_4').

The rest of this paper is organized as follows. In Sect. 2, we firstly establish the variational framework of (1.1), and then, we give some preliminary lemmas, which are useful in the proofs of our main results. In Sect. 3, we give the detailed proofs of our main results.

2 Variational framework and preliminary lemmas

Throughout this paper, we denote by $\|\cdot\|_{L^q}$ the usual $L^q(\mathbb{R}, \mathbb{R}^N)$-norm, and we set $B_r(s) := [s-r, s+r]$.

Under assumption (A_1), $B := -\frac{d^2}{dt^2} + A(t)$ is a self-adjoint operator acting on $L^2 := L^2(\mathbb{R}, \mathbb{R}^N)$ with domain $\mathcal{D}(B) = H^2(\mathbb{R}, \mathbb{R}^N)$, and we have the orthogonal decomposition

$$L^2 = L^- \oplus L^+, \quad u = u^- + u^+$$

such that B is negative (respectively, positive) in L^- (respectively, in L^+). Let $E := \mathcal{D}(|B|^{1/2})$ be equipped, respectively, with the inner product and norm

$$\langle u, v\rangle := \left(|B|^{1/2}u, |B|^{1/2}v\right)_{L^2}, \quad \|u\| := \||B|^{1/2}u\|_{L^2}, \tag{2.1}$$

where $(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(\mathbb{R}, \mathbb{R}^N)$. Then, we have the decompostion

$$E = E^- \oplus E^+, \quad E^\pm = E \cap L^\pm,$$

orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $\langle\cdot,\cdot\rangle$. By (A_1), $E = H^1(\mathbb{R}, \mathbb{R}^N)$ with equivalent norms. Then, E is a Hilbert space and it is not difficult to show that $E \subset C^0(\mathbb{R}, \mathbb{R}^N)$, the space of continuous functions u on $\mathbb{R}$ such that $u(t) \to 0$ as $|t| \to \infty$ (see, e.g., [16]).

Now, the corresponding functional with (1.1) can be rewritten as:

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}} \left(|\dot{u}|^2 + (A(t)u, u)\right) dt - \int_{\mathbb{R}} W(t, u)dt \\ &= \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \int_{\mathbb{R}} W(t, u)dt. \end{aligned} \tag{2.2}$$

The hypotheses on W imply that $I \in C^1(E, \mathbb{R})$. Moreover, critical points of I are classical solutions of (1.1) satisfying $\dot{u}(t) \to 0$ as $|t| \to \infty$. Thus u is a homoclinic solution of (1.1).

The following abstract critical point theorem plays an important role in proving our main result. Let E be a Hilbert space with norm $\|\cdot\|$ and have an orthogonal decomposition $E = N \oplus N^\perp$, $N \subset E$ is a closed and separable subspace. There exists norm $|v|_\omega$ satisfies $|v|_\omega \le \|v\|$ for all $v \in N$ and induces an topology equivalent to the weak topology of N on bounded subset of N. For $u = v + w \in E = N \oplus N^\perp$ with $v \in N$, $w \in N^\perp$, we define $|u|_\omega^2 = |v|_\omega^2 + \|w\|^2$. Particularly, if $(u_n = v_n + w_n)$ is $\|\cdot\|$-bounded and $u_n \xrightarrow{|\cdot|_\omega} u$, then $v_n \rightharpoonup v$ weakly in N, $w_n \to w$ strongly in $N^\perp$, $u_n \rightharpoonup v + w$ weakly in E (cf. [18]).

Let $E = E^- \oplus E^+$, $z_0 \in E^+$ with $\|z_0\| = 1$. Let $N := E^- \oplus \mathbb{R}z_0$ and $E_1^+ := N^\perp = (E^- \oplus \mathbb{R}z_0)^\perp$. For $R > 0$, let

$$Q := \left\{u := u^- + sz_0 : \ s \in \mathbb{R}^+, \ u^- \in E^-, |u\| < R\right\}$$

with $p_0 = s_0 z_0 \in Q$, $s_0 > 0$. We define

$$D := \left\{u := sz_0 + w^+ : \ s \in \mathbb{R}, \ w^+ \in E_1^+, |sz_0 + w^+\| = s_0\right\}.$$

For $I \in C^1(E, \mathbb{R})$, define $\Gamma := \{h|h : [0,1] \times \bar{Q} \mapsto E$ is $|\cdot|_\omega$-continuous, $h(0,u) = u$, $I(h(s,u)) \le I(u)$, $\forall u \in \bar{Q}$. For any $(s_0, u_0) \in [0,1] \times \bar{Q}$, there is a $|\cdot|_\omega$-neighborhood $U_{(s_0,u_0)}$, such that $\{u - h(t,u) : (t,u) \in U_{(s_0,u_0)} \cap ([0,1] \times \bar{Q})\} \subset E_{fin}.\}$, where E_{fin} denotes various finite-dimensional subspaces of E, $\Gamma \ne 0$ since $id \in \Gamma$.

The following variant generalized weak linking theorem due to Schechter and Zou [18], see also [20,23], where the authors developed the idea of monotonicity trick for strongly indefinite problems, the original idea is due to [11,19].

Lemma 2.1 [18] *The family of C^1-functional $\{I_\lambda\}$ has the form*

$$I_\lambda(u) := J(u) - \lambda K(u), \quad \forall\lambda \in [1,2].$$

Assume that

(a) $K(u) \geq 0, \ \forall u \in E, \ I_1 = I$;
(b) $J(u) \to \infty$ *or* $K(u) \to \infty$ *as* $\|u\| \to \infty$;
(c) I_λ *is* $|\cdot|_\omega$*-upper semicontinuous, and* I'_λ *is weakly sequentially continuous on* E*. Moreover,* I_λ *maps bounded sets to bounded sets;*
(d) $\sup_{\partial Q} I_\lambda < \inf_D I_\lambda, \forall \lambda \in [1, 2]$.

Then, for almost all $\lambda \in [1, 2]$*, there exists a sequence* $\{u_n\}$ *such that*

$$\sup_n \|u_n\| < \infty, \quad I'_\lambda(u_n) \to 0, \quad I_\lambda(u_n) \to c_\lambda,$$

where $c_\lambda := \inf_{h \in \Gamma} \sup_{u \in Q} I_\lambda(h(1, u)) \in [\inf_D I_\lambda, \ \sup_{\bar{Q}} I]$.

In order to apply Lemma 2.1, we consider

$$I_\lambda(u) := \frac{1}{2}\|u^+\|^2 - \lambda \left(\frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} W(t, u)dt \right).$$

It is easy to see that I_λ satisfies conditions (a), (b) in Lemma 2.1. To see (c), if $u_n \overset{|\cdot|_\omega}{\to} u$ and $I_\lambda(u_n) \geq a$, then $u_n^+ \to u^+$ and $u_n^- \rightharpoonup u^-$ in E, going to a subsequence if necessary, $u_n \to u$ a.e. on $\mathbb{R}$. Next, we prove $I_\lambda(u) \geq a$, which means that I_λ is $|\cdot|_\omega$-upper semicontinuous. Since

$$I_\lambda(u_n) = \frac{1}{2}\|u_n^+\|^2 - \lambda \left(\frac{1}{2}\|u_n^-\|^2 + \int_{\mathbb{R}} W(t, u_n)dt \right) \geq a,$$

it follows from $u_n^+ \to u^+$ and $u_n^- \rightharpoonup u^-$ in E, the weak lower semicontinuity of the norm, $W(t, u_n) \geq 0$ and the Fatou's lemma that

$$\begin{aligned} a \leq \limsup_{n\to\infty} I_\lambda(u_n) &= \limsup_{n\to\infty} \left(\frac{1}{2}\|u_n^+\|^2 - \lambda \left(\frac{1}{2}\|u_n^-\|^2 + \int_{\mathbb{R}} W(t, u_n)dt \right) \right) \\ &\leq \frac{1}{2}\|u^+\|^2 - \liminf_{n\to\infty} \lambda \left(\frac{1}{2}\|u_n^-\|^2 + \int_{\mathbb{R}} W(t, u_n)dt \right) \\ &\leq \frac{1}{2}\|u^+\|^2 - \lambda \left(\frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} W(t, u)dt \right) = I_\lambda(u). \end{aligned}$$

Thus, we get $I_\lambda(u) \geq a$. I'_λ is weakly sequentially continuous on E is due to [22]. To continue the discussion, we still need to verify condition (d). Indeed, we have:

Lemma 2.2 *Under assumptions of Theorem 1.1, the following facts hold true:*

(i) *There exists* $\rho > 0$ *independent of* $\lambda \in [1, 2]$ *such that* $\kappa := \inf I_\lambda(S_\rho E^+) > 0$*, where*

$$S_\rho E^+ := \left\{ z \in E^+ : \ \|z\| = \rho \right\}.$$

(ii) *For fixed* $z_0 \in E^+$ *with* $\|z_0\| = 1$ *and any* $\lambda \in [1, 2]$*, there is* $R > \rho > 0$ *such that* $\sup I_\lambda(\partial Q) \leq 0$*, where* $Q := \left\{ u := v + s z_0 : \ s \geq 0, \ v \in E^-, \ \|u\| < R \right\}$.

Proof (i) By $((W_1)$ and $(W_4))$ or $((W_1')$ and $(W_3'))$, we know for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|\nabla W(t,u)| \le \varepsilon|u| + C_\varepsilon |u|^{p-1} \tag{2.3}$$

and

$$|W(t,u)| \le \varepsilon|u|^2 + C_\varepsilon |u|^p, \tag{2.4}$$

where $p > 2$ in case (W_3') and $p \ge \frac{2\sigma}{\sigma-1}$ with $\sigma > 1$ in case (W_4). Hence, by the Sobolev embedding theorem, for any $u \in E^+$, we have

$$I_\lambda(u) \ge \frac{1}{2}\|u\|^2 - \lambda\varepsilon\|u\|^2 - C_\varepsilon'\|u\|^p,$$

which implies the conclusion.

(ii) **Case 1** (Superquadratic case). That is, if (W_3) holds.

Part 1. Suppose by contradiction that there exist $u_n \in E^- \oplus \mathbb{R}^+ z_0$ such that $I_\lambda(u_n) > 0$ for all n and $\|u_n\| \to \infty$ as $n \to \infty$. Set $w_n = \frac{u_n}{\|u_n\|} = s_n z_0 + w_n^-$, then

$$0 < \frac{I_\lambda(u_n)}{\|u_n\|^2} = \frac{1}{2}\left(s_n^2 - \lambda\|w_n^-\|^2\right) - \lambda \int_{\mathbb{R}} \frac{W(t,u_n)}{|u_n|^2}|w_n|^2 dt. \tag{2.5}$$

From (W_2), we know $W(t,u) \ge 0$ and have

$$\|w_n^-\|^2 \le \lambda\|w_n^-\|^2 < s_n^2 = 1 - \|w_n^-\|^2,$$

therefore, $\|w_n^-\| \le \frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}} \le s_n \le 1$. So $s_n \to s \ne 0$ after passing to a subsequence, $w_n \rightharpoonup w$ and $w_n \to w$ a.e. in $\mathbb{R}$. Hence, $w = sz_0 + w^- \ne 0$, and thus,

$$|u_n| = |w_n| \cdot \|u_n\| \to +\infty.$$

Part 2. By (W_3), the fact $|u_n| \to \infty$ and the Fatou's lemma, we have

$$\int_{\mathbb{R}} \frac{W(t,u_n)}{u_n^2} w_n^2 dt \to +\infty,$$

which contradicts with (2.5).

Case 2 (Asymptotically quadratic case). That is, if (W_3') with $V > \overline{\Lambda}$ holds.

Since $V > \overline{\Lambda}$, we can choose $\varepsilon_0 > 0$ such that

$$V \ge \overline{\Lambda} + 2\varepsilon_0. \tag{2.6}$$

Since $\sigma(B)$ is absolutely continuous, we can choose $z_0 \in E^+$ with $\|z_0\| = 1$ such that

$$\|z_0\|^2 \le (\overline{\Lambda} + \varepsilon_0)\|z_0\|_{L^2}^2. \tag{2.7}$$

Next, we use z_0 in Case 2 to replace the z_0 in the Part 1 of the Case 1. Then, the Part 1 is still true. By (W_2), (W_3'), (2.5)–(2.7), the facts $|u_n| \to \infty$ and $\|z_0\| = 1$, the Fatou's lemma and the weak lower semicontinuity of the norm, we have

$$0 \leq \limsup_{n\to\infty} \frac{I_\lambda(u_n)}{\|u_n\|^2} = \limsup_{n\to\infty} \left(\frac{1}{2}\left(s_n^2 - \lambda\|w_n^-\|^2\right) - \lambda \int_{\mathbb{R}} \frac{W(t,u_n)}{|u_n|^2}|w_n|^2 dt \right)$$
$$\leq \frac{1}{2}\left(s^2\|z_0\|^2 - \|w^-\|^2\right) - \frac{1}{2}\int_{\mathbb{R}} V w^2 dt$$
$$\leq \frac{1}{2}s^2\|z_0\|^2 - \frac{1}{2}Vs^2\|z_0\|_{L^2}^2$$
$$\leq \frac{1}{2}s^2(\overline{\Lambda}+\varepsilon_0)\|z_0\|_{L^2}^2 - \frac{1}{2}(\overline{\Lambda}+2\varepsilon_0)s^2\|z_0\|_{L^2}^2$$
$$= -\frac{1}{2}\varepsilon_0 s^2\|z_0\|_{L^2}^2 < 0,$$

which is a contradiction.

Therefore, the proof is finished. □

Lemma 2.3 *Under assumptions of Theorem 1.1, for almost all* $\lambda \in [1,2]$, *there exists a* u_λ *such that* $I'_\lambda(u_\lambda) = 0$ *and* $I_\lambda(u_\lambda) \leq \sup_{\bar{Q}} I$.

Proof By Lemmas 2.1 and 2.2, for almost all $\lambda \in [1,2]$, there exists a sequence $\{u_n\}$ such that

$$\sup_n \|u_n\| < \infty, \quad I'_\lambda(u_n) \to 0, \quad I_\lambda(u_n) \to c_\lambda \in [\kappa, \sup_{\bar{Q}} I],$$

where κ is defined in Lemma 2.2. We write $u_n = u_n^- + u_n^+$ with $u_n^\pm \in E^\pm$. Since $\{u_n^+\}$ is bounded, by a Lion's concentration compactness principle [13], either $\{u_n^+\}$ is vanishing, i.e., for each $l > 0$,

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \int_{B_l(s)} |u_n^+|^2 dt = 0$$

(in this case $u_n^+ \to 0$ in $L^q(\mathbb{R},\mathbb{R}^N)$ for all $q \in (2,\infty)$), or it is nonvanishing, i.e., there exist $r, \delta > 0$ and a sequence $s_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_{B_r(s_n)} |u_n^+|^2 dt \geq \delta.$$

If $\{u_n^+\}$ is vanishing, then $u_n^+ \to 0$ in $L^q(\mathbb{R},\mathbb{R}^N)$ for all $q \in (2,\infty)$, it follows from (2.3), the boundedness of $\{u_n\}$ and the Hölder's inequality that

$$\int_{\mathbb{R}} |(\nabla W(t,u_n), u_n^+)| dt \leq \varepsilon \int_{\mathbb{R}} |u_n|\cdot|u_n^+| dt + C_\varepsilon \int_{\mathbb{R}} |u_n|^{p-1}|u_n^+| dt$$
$$\leq \varepsilon\|u_n\|_{L^2}\|u_n^+\|_{L^2} + C_\varepsilon\|u_n\|_{L^p}^{p-1}\|u_n^+\|_{L^p} \to 0 \quad \text{as } n\to\infty.$$

Therefore,

$$I_\lambda(u_n) \leq \|u_n^+\|^2 = I'_\lambda(u_n)u_n^+ + \lambda \int_{\mathbb{R}} (\nabla W(t,u_n), u_n^+) dt \to 0,$$

which contradicts with the fact that $I_\lambda(u_n) \geq \kappa$. Hence, $\{u_n^+\}$ must be nonvanishing. Let us define $v_n = u_n(\cdot - s_n)$, then

$$\lim_{n\to\infty} \int_{B_r(0)} |v_n^+|^2 dt \geq \frac{\delta}{2}. \tag{2.8}$$

Since I_λ and I'_λ are both invariant under translation, we know

$$I'_\lambda(v_n) \to 0, \quad I_\lambda(v_n) \to c_\lambda.$$

Since $\{v_n\}$ is still bounded, we may assume $v_n^+ \rightharpoonup u_\lambda^+$, $v_n^- \rightharpoonup u_\lambda^-$ in E, $v_n \rightharpoonup u_\lambda$ a.e. on $\mathbb{R}$ and $v_n^+ \to u_\lambda^+$ in $L^2_{loc}(\mathbb{R}, \mathbb{R}^N)$, which together with (2.9) implies that $u_\lambda = u_\lambda^+ + u_\lambda^- \neq 0$ and

$$I'_\lambda(u_\lambda)\varphi = \lim_{n\to\infty} I'_\lambda(v_n)\varphi = 0, \ \forall \varphi \in C_0^\infty(\mathbb{R}). \tag{2.9}$$

By (W_2) and the Fatou's lemma, we have

$$\begin{aligned}
\sup_{\bar{Q}} I \geq c_\lambda &= \lim_{n\to\infty} \left(I_\lambda(v_n) - \frac{1}{2} I'_\lambda(v_n)v_n \right) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} \left(\frac{1}{2} (\nabla W(t, v_n), v_n) - W(t, v_n) \right) dt \\
&\geq \int_{\mathbb{R}} \left(\frac{1}{2} (\nabla W(t, u_\lambda), u_\lambda) - W(t, u_\lambda) \right) dt = I_\lambda(u_\lambda).
\end{aligned}$$

Thus, we get $I_\lambda(u_\lambda) \leq \sup_{\bar{Q}} I$. □

Lemma 2.4 *Under assumptions of Theorem 1.1, there exist* $\{\lambda_n\} \subset [1, 2]$ *with* $\lambda_n \to 1$ *and sequence* $\{u_{\lambda_n}\}$ *such that* $I'_{\lambda_n}(u_{\lambda_n}) = 0$ *and* $I_{\lambda_n}(u_{\lambda_n}) \leq \sup_{\bar{Q}} I$*; moreover,* $\{u_{\lambda_n}\}$ *is bounded.*

Proof The existence of $\{\lambda_n\} \subset [1, 2]$ with $\lambda_n \to 1$ and $\{u_{\lambda_n}\}$ such that

$$I'_{\lambda_n}(u_{\lambda_n}) = 0 \quad \text{and} \quad I_{\lambda_n}(u_{\lambda_n}) \leq \sup_{\bar{Q}} I$$

is the direct consequence of Lemma 2.3. Next, we divide our proof into two parts according to super and asymptotically quadratic case, i.e., the following Part 1 and Part 2.

Part 1 (superquadratic case). If $(W_1) - (W_4)$ hold. To prove the boundedness of $\{u_{\lambda_n}\}$, arguing by contradiction, suppose that $\|u_{\lambda_n}\| \to \infty$. Let $v_{\lambda_n} := \frac{u_{\lambda_n}}{\|u_{\lambda_n}\|}$. Then, $\|v_{\lambda_n}\| = 1$, $v_{\lambda_n} \rightharpoonup v$ in E and $v_{\lambda_n} \to v$ a.e. in $\mathbb{R}$, after passing to a subsequence.

Recall that $I'_{\lambda_n}(u_{\lambda_n}) = 0$. Thus, for any $\varphi \in E$, we have

$$\langle u_{\lambda_n}^+, \varphi \rangle - \lambda_n \langle u_{\lambda_n}^-, \varphi \rangle = \lambda_n \int_{\mathbb{R}} \left(\nabla W(t, u_{\lambda_n}), \varphi \right) dt. \tag{2.10}$$

Consequently, $\{v_{\lambda_n}\}$ satisfies

$$\langle v_{\lambda_n}^+, \varphi \rangle - \lambda_n \langle v_{\lambda_n}^-, \varphi \rangle = \lambda_n \int_{\mathbb{R}} \frac{\left(\nabla W(t, u_{\lambda_n}), \varphi \right)}{\|u_{\lambda_n}\|} dt. \tag{2.11}$$

Let $\varphi = v^{\pm}_{\lambda_n}$ in (2.11), respectively. Then, we have

$$\langle v^+_{\lambda_n}, v^+_{\lambda_n}\rangle = \lambda_n \int_{\mathbb{R}} \frac{\left(\nabla W(t, u_{\lambda_n}), v^+_{\lambda_n}\right)}{\|u_{\lambda_n}\|} dt$$

and

$$-\lambda_n \langle v^-_{\lambda_n}, v^-_{\lambda_n}\rangle = \lambda_n \int_{\mathbb{R}} \frac{\left(\nabla W(t, u_{\lambda_n}), v^-_{\lambda_n}\right)}{\|u_{\lambda_n}\|} dt.$$

Since $1 = \|v_{\lambda_n}\|^2 = \|v^+_{\lambda_n}\|^2 + \|v^-_{\lambda_n}\|^2$, we have

$$1 = \int_{\mathbb{R}} \frac{\left(\nabla W(t, u_{\lambda_n}), \lambda_n v^+_{\lambda_n} - v^-_{\lambda_n}\right)}{\|u_{\lambda_n}\|} dt. \tag{2.12}$$

For $r \geq 0$, let

$$h(r) := \inf \left\{ \widetilde{W}(t, u) : \ t \in \mathbb{R} \text{ and } u \in \mathbb{R}^N \text{ with } |u| \geq r \right\}.$$

By (W_2), we have $h(r) > 0$ for all $r > 0$. By (W_2) and (W_4), for $|u| \geq r_0$,

$$c_0 \widetilde{W}(t, u) \geq \frac{|\nabla W(t, u)|^\sigma}{|u|^\sigma} = \left(\frac{|\nabla W(t, u)||u|}{|u|^2}\right)^\sigma \geq \left(\frac{(\nabla W(t, u), u)}{|u|^2}\right)^\sigma \geq \left(\frac{2W(t, u)}{|u|^2}\right)^\sigma,$$

it follows form (W_3) and the definition of $h(r)$ that

$$h(r) \to \infty \quad \text{as } r \to \infty.$$

For $0 \leq a < b$, let

$$\Omega_n(a, b) := \left\{ t \in \mathbb{R} : \ a \leq |u_{\lambda_n}(t)| < b \right\}$$

and

$$C_a^b := \inf \left\{ \frac{\widetilde{W}(t, u)}{|u|^2} : \ t \in \mathbb{R} \text{ and } u \in \mathbb{R}^N \text{ with } a \leq |u| \leq b \right\}.$$

Since $W(t, u)$ depends periodically on t and $\widetilde{W}(t, u) > 0$ if $u \in \mathbb{R}^N \setminus \{0\}$, one has $C_a^b > 0$ and

$$\widetilde{W}(t, u_{\lambda_n}) \geq C_a^b |u_{\lambda_n}|^2 \quad \text{for all } t \in \Omega_n(a, b).$$

Since

$$I'_{\lambda_n}(u_{\lambda_n}) = 0, \quad I_{\lambda_n}(u_{\lambda_n}) \leq \sup_{\bar{Q}} I,$$

we have there exists a constant $C_0 > 0$ such that for all n

$$C_0 \geq I_{\lambda_n}(u_{\lambda_n}) - \frac{1}{2} I'_{\lambda_n}(u_{\lambda_n}) u_{\lambda_n} = \int_{\mathbb{R}} \widetilde{W}(t, u_{\lambda_n}) dt, \tag{2.13}$$

from which we have

$$C_0 \ge \int_{\Omega_n(0,a)} \widetilde{W}(t, u_{\lambda_n})dt + \int_{\Omega_n(a,b)} \widetilde{W}(t, u_{\lambda_n})dt + \int_{\Omega_n(b,\infty)} \widetilde{W}(t, u_{\lambda_n})dt$$
$$\ge \int_{\Omega_n(0,a)} \widetilde{W}(t, u_{\lambda_n})dt + C_a^b \int_{\Omega_n(a,b)} |u_{\lambda_n}|^2 dt + h(b)|\Omega_n(b,\infty)|. \tag{2.14}$$

Invoking (W_4), set $\tau := 2\sigma/(\sigma-1)$ and $\sigma' = \tau/2$. Since $\sigma > 1$, one sees $\tau \in (2,\infty)$. Fix arbitrarily $\hat{\tau} \in (\tau,\infty)$. Using (2.14), we have

$$|\Omega_n(b,\infty)| \le \frac{C_0}{h(b)} \to 0$$

as $b \to \infty$ uniformly in n, which implies by the Hölder inequality and the Sobolev embedding theorem that

$$\int_{\Omega_n(b,\infty)} |v_{\lambda_n}|^\tau dt \le C|\Omega_n(b,\infty)|^{1-\frac{\tau}{\hat{\tau}}} \to 0 \tag{2.15}$$

as $b \to \infty$ uniformly in n. Using (2.14) again, for any fixed $0 < a < b$,

$$\int_{\Omega_n(a,b)} |v_{\lambda_n}|^2 dt = \frac{1}{\|u_{\lambda_n}\|^2} \int_{\Omega_n(a,b)} |u_{\lambda_n}|^2 dt \le \frac{C_0}{C_a^b \|u_{\lambda_n}\|^2} \to 0 \quad \text{as } n \to \infty. \tag{2.16}$$

Let $0 < \varepsilon < \frac{1}{3}$. Note that the Sobolev embedding theorem implies $\|v_{\lambda_n}\|_{L^2}^2 \le C\|v_{\lambda_n}\|^2 = C$ and $|\lambda_n| \le C_1$. By (W_1) there is $a_\varepsilon > 0$ such that $|\nabla W(t,u)| < \frac{\varepsilon}{C_1 C}|u|$ for all $|u| \le a_\varepsilon$, consequently,

$$\begin{aligned}
\int_{\Omega_n(0,a_\varepsilon)} \frac{\left(\nabla W(t,u_{\lambda_n}), \lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right)}{\|u_{\lambda_n}\|} dt &\le \int_{\Omega_n(0,a_\varepsilon)} \frac{|\nabla W(t,u_{\lambda_n})|}{|u_{\lambda_n}|} |v_{\lambda_n}| \cdot |\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-| dt \\
&\le \frac{\varepsilon}{C_1 C} \int_{\Omega_n(0,a_\varepsilon)} |v_{\lambda_n}| \cdot |\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-| dt \\
&\le \frac{\varepsilon}{C_1 C} \left(\int_{\mathbb{R}} v_{\lambda_n}^2 dt \right)^{1/2} \left(\int_{\mathbb{R}} (\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-)^2 dt \right)^{1/2} \\
&\le \frac{\varepsilon}{C} \|v_{\lambda_n}\|_{L^2}^2 \le \varepsilon
\end{aligned} \tag{2.17}$$

for all n. By (W_4), (2.13), (2.15), and the Sobolev embedding theorem, we can take $b_\varepsilon \ge r_0$ large so that

$$\int_{\Omega_n(b_\varepsilon,\infty)} \frac{\left(\nabla W(t,u_{\lambda_n}), \lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right)}{\|u_{\lambda_n}\|} dt \le \int_{\Omega_n(b_\varepsilon,\infty)} \frac{|\nabla W(t,u_{\lambda_n})|}{|u_{\lambda_n}|} |v_{\lambda_n}| \cdot |\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-| dt$$
$$\le \left(\int_{\Omega_n(b_\varepsilon,\infty)} \frac{|\nabla W(t,u_{\lambda_n})|^\sigma}{|u_{\lambda_n}|^\sigma} dt \right)^{1/\sigma} \left(\int_{\Omega_n(b_\varepsilon,\infty)} \left(|v_{\lambda_n}| \cdot |\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-|\right)^{\sigma'} dt \right)^{1/\sigma'}$$

$$\leq \left(\int_{\mathbb{R}} c_0 \widetilde{W}(t, u_{\lambda_n}) dt \right)^{1/\sigma} \left(\int_{\mathbb{R}} |\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-|^\tau dt \right)^{1/\tau} \left(\int_{\Omega_n(b_\varepsilon, \infty)} |v_{\lambda_n}|^\tau dt \right)^{1/\tau} < \varepsilon \tag{2.18}$$

for all n. Note that there is $\gamma = \gamma(\varepsilon) > 0$ independent of n such that $|\nabla W(t, u_{\lambda_n})| \leq \gamma |u_{\lambda_n}|$ for $t \in \Omega_n(a_\varepsilon, b_\varepsilon)$. By (2.16), there is n_0 such that

$$\begin{aligned}
&\int_{\Omega_n(a_\varepsilon, b_\varepsilon)} \frac{\left(\nabla W(t, u_{\lambda_n}), \lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right)}{\|u_{\lambda_n}\|} dt \\
&\leq \int_{\Omega_n(a_\varepsilon, b_\varepsilon)} \frac{|\nabla W(t, u_{\lambda_n})|}{|u_{\lambda_n}|} |v_{\lambda_n}| \cdot |\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-| dt \\
&\leq \gamma \int_{\Omega_n(a_\varepsilon, b_\varepsilon)} |v_{\lambda_n}| \cdot |\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-| dt \\
&\leq \gamma \left(\int_{\mathbb{R}} v_{\lambda_n}^2 dt \right)^{1/2} \left(\int_{\Omega_n(a_\varepsilon, b_\varepsilon)} (\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-)^2 dt \right)^{1/2} \\
&\leq \gamma \lambda_n \|v_{\lambda_n}\|_{L^2} \left(\int_{\Omega_n(a_\varepsilon, b_\varepsilon)} |v_{\lambda_n}|^2 dt \right)^{1/2} < \varepsilon
\end{aligned} \tag{2.19}$$

for all $n \geq n_0$. Therefore, the combination of (2.17)–(2.19) implies that for $n \geq n_0$, we have

$$\int_{\mathbb{R}} \frac{\left(\nabla W(t, u_{\lambda_n}), \lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right)}{\|u_{\lambda_n}\|} dt < 3\varepsilon < 1,$$

which contradicts with (2.12). Thus, $\{u_{\lambda_n}\}$ is bounded.

Part 2 (asymptotically quadratic case). If (W_1'), (W_2), (W_4'), and (W_3') with $V > \overline{\Lambda}$ hold. Note that (W_3') implies that there exists $c_3, R_3 > 0$ such that

$$|\nabla W(t, u)| \leq c_3 |u|, \quad |u| \geq R_3. \tag{2.20}$$

Let $R_0 := \min\{1, R, R_1, R_2, R_3\}$, where R, R_1, and R_2 are defined, respectively, in (W_1') and (W_4'). Note that $I'_{\lambda_n}(u_{\lambda_n}) = 0$ and $I_{\lambda_n}(u_{\lambda_n}) \leq \sup_{\bar{Q}} I$, thus

$$\frac{I_{\lambda_n}(u_{\lambda_n}) - \frac{1}{2} I'_{\lambda_n}(u_{\lambda_n}) u_{\lambda_n}}{\lambda_n} \leq C.$$

It follows from (W_2), (W_4') and the definition of $\widetilde{W}$ that

$$\begin{aligned}
C &\geq \frac{I_{\lambda_n}(u_{\lambda_n}) - \frac{1}{2} I'_{\lambda_n}(u_{\lambda_n}) u_{\lambda_n}}{\lambda_n} \\
&= \int_{\mathbb{R}} \widetilde{W}(t, u_{\lambda_n}) dt \\
&= \int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\leq R_0\}} \widetilde{W}(t, u_{\lambda_n}) dt + \int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\geq R_0\}} \widetilde{W}(t, u_{\lambda_n}) dt \\
&\geq c_1 \int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\leq R_0\}} |u_{\lambda_n}|^{\mu} dt + c_2' \int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\geq R_0\}} |u_{\lambda_n}|^{\alpha} dt. \qquad (2.21)
\end{aligned}$$

Take $s \in (0, \frac{\alpha}{2})$, then by (2.21), the Hölder's inequality, and the Sobolev imbedding theorem, we have

$$\begin{aligned}
&\int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\geq R_0\}} |u_{\lambda_n}|^2 dt \\
&= \int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\geq R_0\}} |u_{\lambda_n}|^{2s} |u_{\lambda_n}|^{2(1-s)} dt \\
&\leq \left(\int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\geq R_0\}} |u_{\lambda_n}|^{\alpha} \right)^{\frac{2s}{\alpha}} \left(\int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\geq R_0\}} |u_{\lambda_n}|^{\frac{2\alpha(1-s)}{\alpha-2s}} \right)^{\frac{\alpha-2s}{\alpha}} \leq C_1 \|u_{\lambda_n}\|^{2(1-s)}
\end{aligned} \qquad (2.22)$$

for some constant $C_1 > 0$, where $\frac{2\alpha(1-s)}{\alpha-2s} > 2$. Note that $I'_{\lambda_n}(u_{\lambda_n}) u^+_{\lambda_n} = 0$, it follows from (W_1'), (2.20)–(2.22), the Hölder's inequality, and the Sobolev imbedding theorem that

$$\begin{aligned}
\|u^+_{\lambda_n}\|^2 &= \lambda_n \int_{\mathbb{R}} (\nabla W(t, u_{\lambda_n}), u^+_{\lambda_n}) dt \\
&\leq C_2 \int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\leq R_0\}} |\nabla W(t, u_{\lambda_n})| \cdot |u^+_{\lambda_n}| dt + C_2 \int_{\{t\in\mathbb{R}:|u_{\lambda_n}|\geq R_0\}} |\nabla W(t, u_{\lambda_n})| \cdot |u^+_{\lambda_n}| dt \\
&\leq C_3 \int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\leq R_0\}} |u_{\lambda_n}|^{\mu-1} \cdot |u^+_{\lambda_n}| dt + C_3 \int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\geq R_0\}} |u_{\lambda_n}| \cdot |u^+_{\lambda_n}| dt \\
&\leq C_3 \left(\int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\leq R_0\}} |u_{\lambda_n}|^{\mu} dt \right)^{\frac{\mu-1}{\mu}} \left(\int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\leq R_0\}} |u^+_{\lambda_n}|^{\mu} dt \right)^{\frac{1}{\mu}} \\
&\quad + C_3 \left(\int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\geq R_0\}} |u_{\lambda_n}|^2 dt \right)^{\frac{1}{2}} \left(\int_{\{t\in\mathbb{R}:\ |u_{\lambda_n}|\geq R_0\}} |u^+_{\lambda_n}|^2 dt \right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq C_4\|u^+_{\lambda_n}\| + C_4\|u^+_{\lambda_n}\| \cdot \|u_{\lambda_n}\|^{1-s}$$
$$= C_4\|u^+_{\lambda_n}\| + C_4\|u^+_{\lambda_n}\| \cdot \left(\|u^+_{\lambda_n}\|^2 + \|u^-_{\lambda_n}\|^2\right)^{\frac{1-s}{2}} \tag{2.23}$$

for some constants $C_2, C_3, C_4 > 0$, where $\frac{1-s}{2} < \frac{1}{2}$. By $I'_{\lambda_n}(u_{\lambda_n})u_{\lambda_n} = 0$, we have

$$\|u^+_{\lambda_n}\|^2 - \lambda_n\|u^-_{\lambda_n}\|^2 = \lambda_n \int_{\mathbb{R}} (\nabla W(t, u_{\lambda_n}), u_{\lambda_n})dt \geq 0,$$

that is,

$$\|u^+_{\lambda_n}\|^2 \geq \lambda_n\|u^-_{\lambda_n}\|^2 \geq \|u^-_{\lambda_n}\|^2. \tag{2.24}$$

By (2.23) and (2.24), we have $\{u_{\lambda_n}\}$ is bounded.

Therefore, the proof is finished by Part 1 and Part 2. □

Lemma 2.5 *If* $\{u_{\lambda_n}\}$ *is the sequence obtained in Lemma 2.4, then it is also a* (PS) *sequence for* I *satisfying* $\lim_{n\to\infty} I'(u_{\lambda_n}) = 0$ *and* $\lim_{n\to\infty} I(u_{\lambda_n}) \leq \sup_{\bar{Q}} I$.

Proof Note that u_{λ_n} is bounded. From

$$\lim_{n\to\infty} I(u_{\lambda_n}) = \lim_{n\to\infty} \left(I_{\lambda_n}(u_{\lambda_n}) + (\lambda_n - 1)\left(\frac{1}{2}\|u^-_{\lambda_n}\|^2 + \int_{\mathbb{R}} W(t, u_{\lambda_n})dt\right)\right)$$

and note that

$$\lim_{n\to\infty} I'(u_{\lambda_n})\varphi = \lim_{n\to\infty} \left(I'_{\lambda_n}(u_{\lambda_n})\varphi + (\lambda_n - 1)\left(\langle u^-_{\lambda_n}, \varphi^-\rangle + \int_{\mathbb{R}} \left(\nabla W(t, u_{\lambda_n}), \varphi\right) dt\right)\right)$$

for any $\varphi \in E$, we obtain the conclusion. □

3 Proofs of main results

In this section, we are in a position to prove our main results.

Proof of Theorem 1.1 From Lemma 2.4, we know $\{u_{\lambda_n}\}$ is bounded, we have $\{u_{\lambda_n}\}$ is either vanishing, that is, for each $l > 0$,

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \int_{B_l(s)} |u_{\lambda_n}|^2 dt = 0 \tag{3.1}$$

or nonvanishing, i.e., there exist $r, \delta > 0$ and a sequence $\{s_n\} \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_{B_r(s_n)} |u_{\lambda_n}|^2 dt \geq \delta. \tag{3.2}$$

If $\{u_{\lambda_n}\}$ is vanishing, by the Lion's concentration compactness principl, we have that $u_{\lambda_n} \to 0$ in $L^p(\mathbb{R}, \mathbb{R}^N)$ for all $p \in (2, \infty)$. However, by (2.3), the Hölder's inequality, the Sobolev embedding theorem, and the fact that $I'_{\lambda_n}(u_{\lambda_n})u^+_{\lambda_n} = 0$, we have

$$\begin{aligned}\|u_{\lambda_n}^+\|^2 &= \lambda_n \int\limits_{\mathbb{R}} \left(\nabla W(t, u_{\lambda_n}), u_{\lambda_n}^+\right) dt \\ &\leq \varepsilon \int\limits_{\mathbb{R}} |u_{\lambda_n}| \cdot |u_{\lambda_n}^+| dt + C_\varepsilon \int\limits_{\mathbb{R}} |u_{\lambda_n}|^{p-1} |u_{\lambda_n}^+| dt \\ &\leq \varepsilon \|u_{\lambda_n}\| \cdot \|u_{\lambda_n}^+\| + C'_\varepsilon \|u_{\lambda_n}\|_{L^p}^{p-1} \|u_{\lambda_n}^+\| \\ &\leq \varepsilon \|u_{\lambda_n}\| \cdot \|u_{\lambda_n}^+\| + C''_\varepsilon \|u_{\lambda_n}\|_{L^p}^{p-2} \|u_{\lambda_n}\| \cdot \|u_{\lambda_n}^+\| \\ &\leq \varepsilon \|u_{\lambda_n}\|^2 + C''_\varepsilon \|u_{\lambda_n}\|_{L^p}^{p-2} \|u_{\lambda_n}\|^2. \end{aligned} \tag{3.3}$$

Similarly, we have

$$\|u_{\lambda_n}^-\|^2 \leq \varepsilon \|u_{\lambda_n}\|^2 + C''_\varepsilon \|u_{\lambda_n}\|_{L^p}^{p-2} \|u_{\lambda_n}\|^2. \tag{3.4}$$

From (3.3) and (3.4), we get

$$\|u_{\lambda_n}\|^2 \leq 2\varepsilon \|u_{\lambda_n}\|^2 + 2C''_\varepsilon \|u_{\lambda_n}\|_{L^p}^{p-2} \|u_{\lambda_n}\|^2,$$

which means $\|u_{\lambda_n}\|_{L^p} \geq C$ for some constant C, hence (3.1) does not hold. Let us define $v_{\lambda_n} = u_{\lambda_n}(\cdot - s_n)$, from (3.2), we have

$$\lim_{n\to\infty} \int\limits_{B_r(0)} |v_{\lambda_n}|^2 dt \geq \frac{\delta}{2}. \tag{3.5}$$

I and I' are both invariant under translation, we know $I'(v_{\lambda_n}) \to 0$. Since $\{v_{\lambda_n}\}$ is still bounded, we may assume $v_{\lambda_n} \rightharpoonup u$ in E and $v_{\lambda_n} \to u$ in $L^2_{loc}(\mathbb{R}, \mathbb{R}^N)$, which together with (3.5) implies that $u \neq 0$ with $I'(u) = 0$.

Let $K := \left\{u \in E :\ I'(u) = 0,\ u \neq 0\right\}$ be the critical set of I and

$$c := \inf\left\{I(z) :\ z \in K \backslash \{0\}\right\}.$$

For any critical point u of I, assumption (W_2) implies that

$$I(u) = I(u) - \frac{1}{2} I'(u)u = \int\limits_{\mathbb{R}} \left(\frac{1}{2}(\nabla W(t, u), u) - W(t, u)\right) dt > 0 \quad \text{if } u \neq 0. \tag{3.6}$$

Therefore, we have $c \geq 0$. We prove that $c > 0$ and there is $u \in K$ such that $I(u) = c$. Let $u_j \in K\backslash\{0\}$ be such that $I(u_j) \to c$. Then, the proof in Lemma 2.4 shows that $\{u_j\}$ is bounded; then, by the concentration compactness principle discussion above, we know $u_j \rightharpoonup u \in K\backslash\{0\}$. Thus,

$$\begin{aligned} c = \lim_{j\to\infty} I(u_j) &= \lim_{j\to\infty} \left(I(u_j) - \frac{1}{2} I'(u_j)u_j\right) \\ &= \lim_{j\to\infty} \int\limits_{\mathbb{R}} \left(\frac{1}{2}\left(\nabla W(t, u_j), u_j\right) - W(t, u_j)\right) dt \\ &\geq \int\limits_{\mathbb{R}} \left(\frac{1}{2}(\nabla W(t, u), u) - W(t, u)\right) dt = I(u) \geq c, \end{aligned}$$

where the first inequality dues to the Fatou's lemma. So $I(u) = c$ and $c > 0$ because $u \neq 0$.
□

Proof of Corollary 1.1 By virtue of Theorem 1.1, it suffices to show that (1.1) has no nontrivial homoclinic orbit if (A_1), (W_1'), (W_2), (W_3'), and (W_4') hold, $\frac{|\nabla W(t,u)|}{|u|} \leq V$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$ and

$$V \leq \Lambda_0 := \min\{-\underline{\Lambda}, \overline{\Lambda}\}.$$

By way of contradiction, assume that (1.1) has a nontrivial homoclinic orbit $u \in E$, then for any small $\varepsilon > 0$ there exists $R > 0$ such that

$$|u(t)| < \varepsilon \quad \text{if } |t| \geq R.$$

It follows from (W_1') that

$$\frac{|\nabla W(t,u)|}{|u|} < V \quad \text{if } |t| \geq R. \tag{3.7}$$

Since u is a nonzero critical point of I, we get $I'(u)(u^+ - u^-) = 0$, it follows from (2.1), (2.2), and (3.7), $\frac{|\nabla W(t,u)|}{|u|} \leq V$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$ and $V \leq \Lambda_0 := \min\{-\underline{\Lambda}, \overline{\Lambda}\}$ that

$$\begin{aligned}
(Bu^+, u^+)_{L^2} - (Bu^-, u^-)_{L^2} &= (Bu, u^+ - u^-)_{L^2} = \int_{\mathbb{R}} (\nabla W(t,u), u^+ - u^-)dt \\
&\leq \int_{\mathbb{R}} |\nabla W(t,u)| \cdot |u^+ - u^-|dt \\
&= \int_{\{t\in\mathbb{R}:\ |t|\leq R\}} \frac{|\nabla W(t,u)|}{|u|}|u| \cdot |u^+ - u^-|dt + \int_{\{t\in\mathbb{R}:\ |t|\geq R\}} \frac{|\nabla W(t,u)|}{|u|}|u| \cdot |u^+ - u^-|dt \\
&< \int_{\mathbb{R}} V|u| \cdot |u^+ - u^-|dt \\
&= \int_{\mathbb{R}} \sqrt{V}|u| \cdot \sqrt{V}|u^+ - u^-|dt \\
&\leq \left(\int_{\mathbb{R}} Vu^2 dt\right)^{1/2} \left(\int_{\mathbb{R}} V(u^+ - u^-)^2 dt\right)^{1/2} \\
&\leq \Lambda_0 \left(\int_{\mathbb{R}} u^2 dt\right)^{1/2} \left(\int_{\mathbb{R}} (u^+ - u^-)^2 dt\right)^{1/2} \\
&= \Lambda_0\|u^+\|_{L^2}^2 + \Lambda_0\|u^-\|_{L^2}^2 \leq \overline{\Lambda}\|u^+\|_{L^2}^2 - \underline{\Lambda}\|u^-\|_{L^2}^2.
\end{aligned}$$

That is,

$$(Bu^+, u^+)_{L^2} - (Bu^-, u^-)_{L^2} < \overline{\Lambda}\|u^+\|_{L^2}^2 - \underline{\Lambda}\|u^-\|_{L^2}^2. \tag{3.8}$$

However, we know

$$(Bu^+, u^+)_{L^2} - (Bu^-, u^-)_{L^2} \geq \overline{\Lambda}\|u^+\|_{L^2}^2 - \underline{\Lambda}\|u^-\|_{L^2}^2. \tag{3.9}$$

Therefore, by (3.8) and (3.9), we get a contradiction. This contradiction completes the proof. □

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