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Harmonic maps from bounded symmetric domains to Finsler manifolds

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Abstract In this paper, we consider the harmonic maps from a Riemannian manifold with non-positive pinching curvature to any Finsler manifold, and we can prove that there is no non-degenerate harmonic maps from a classical bounded symmetric domain to any Finsler manifold with moderate divergent energy. In particular, we obtain that any harmonic map from a classical bounded symmetric domain to any Riemannian manifold with finite energy has to be constant, which improves the Xin's result in Acta Math Sinica 15:277–292, 1999.

Keywords Harmonic maps · Bounded symmetric domains · Finsler manifolds

Mathematics Subject Classification (2010) 58E20 · 53C60

1 Introduction

Bounded symmetric domains were introduced by Cartan [1,2] and were systematically studied by Hua, Look, Siegel and Xin [4–11]. Those are important Cartan–Hadamard manifolds whose further geometrical and analytical properties should be explored and discovered. By the work done by Wong [8], a bounded symmetric domain can be viewed as a pseudo-Grassmannian manifold of all the spacelike subspaces of dimension m in pseudo-Euclidean space R_n^{m+n} of index n. Let C_m^{n+m} be an $(n+m)$-dimensional complex vector space. For any $u = (x, y) = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \in C_m^{n+m}$, define a pseudo-Euclidean inner product

$$\langle u, u\rangle = |x_1|^2 + \cdots + |x_n|^2 - |x_{n+1}|^2 - \cdots - |x_{n+m}|^2.$$

Let A be an n-subspace. If the induced inner product on A is positive definite, A is called a spacelike n-subspace. The set of all the n-dimensional spacelike subspaces forms a pseudo-

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Grassmannian manifold $G_{n,m}^m(C)$. $\Re_I(n,m)$, the first type of bounded symmetric domains, can be identified with $G_{n,m}^m(C)$. The Bergman metric corresponds to the canonical metric in $G_{n,m}^m(C)$. The second and third types of bounded symmetric domains $\Re_{II}(n)$, $\Re_{III}(n)$ are totally geodesic submanifolds of $G_{n,m}^m(C)$. The fourth type of bounded symmetric domains $\Re_{IV}(n)$ is identified with $G_{n,2}^2$ (see [8]).

Harmonic maps between Riemannian manifolds are defined as the critical points of energy functionals. They are important in both classical and modern differential geometry. As is well known, any harmonic map from a space form of non-positive sectional curvature to any Riemannian manifold with finite energy has to be constant [6]. Xin considered the case of Cartan–Hadamard manifolds with some pinching condition and obtained that

Theorem A ([9]). *Let M be an $n(n \geq 3)$-dimensional Cartan–Hadamard manifold with sectional curvature $K : -a^2 \leq K \leq 0$ and Ricci curvature bounded from above by $-b^2$. Let ϕ be a harmonic map from M to any Riemannian manifold with moderate divergent energy. If $b \geq 2a$, then ϕ is constant.*

For the bounded symmetric domains, this implies that

Theorem B ([9]). *Let*

$$M = \begin{cases} \Re_I(n,m), & when \quad n+m \geq 8, \\ \Re_{II}(n), & when \quad n \geq 7, \\ \Re_{III}(n), & when \quad n \geq 5, \\ \Re_{IV}(n), & when \quad n \geq 8, \end{cases}$$

Then any harmonic map from M to any Riemannian manifold with moderate divergent energy has to be constant.

In [10], by studying the harmonic maps from certain k$\ddot{a}$hler manidolds and the geometry of classical bounded symmetric domains, Xin proved that

Theorem C ([10]). *A harmonic map of finite energy from a classical bounded symmetric domain (except $\Re_{IV}(2) = \mathbb{H}^2 \times \mathbb{H}^2$) to any Riemannian manifold has to be constant.*

Xin [10] raised the question as follows:

Question. Is there any harmonic map of finite energy from $\Re_{IV}(2)$ to any Riemannian manifold?

The main purpose of this paper is to find a way to answer this question. In this paper, we consider the harmonic maps from a Riemannian manifold with non-positive pinching curvature to any Finsler manifold. We can prove the following

Main Theorem. *There is no non-degenerate harmonic map from a classical bounded symmetric domain to any Finsler manifold with moderate divergent energy.*

Corollary 1 *Any harmonic map from a classical bounded symmetric domain to any Riemannian manifold with finite energy has to be constant.*

2 Preliminaries

We shall use the following convention of index ranges unless otherwise stated:

$$1 \leq i, j, \ldots \leq n; \quad 1 \leq \alpha, \beta, \ldots \leq m; \quad 1 \leq \lambda, \mu, \ldots \leq n-1.$$

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle. Let (x, Y) be a point of TM with $x \in M$, $Y \in T_x M$ and let (x^i, Y^i) be the local coordinates on TM with $Y = Y^i \frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) Regularity: $F(x, Y)$ is smooth in $TM \backslash 0$;
(ii) Positive homogeneity: $F(x, \lambda Y) = \lambda F(x, Y)$ for $\lambda > 0$;
(iii) Strong convexity: The fundamental quadratic form $g = g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$ is positive definite, where $g_{ij} = \frac{\partial^2(F^2)}{2\partial Y^i \partial Y^j}$.

Then (M, F) is called an n-dimensional Finsler manifold. F determines the *Hilbert form* and *Cartan tensor* as follows:

$$\omega = \frac{\partial F}{\partial Y^i} \mathrm{d}x^i, \quad A = A_{ijk} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k, \quad A_{ijk} = F \frac{\partial g_{ij}}{2\partial Y^k}.$$

The natural dual of the $Hilbert$ form ω is $\ell = \frac{Y^i}{F} \frac{\partial}{\partial x^i}$ which is called the *distinguished section*.

Let π be the canonical projection $TM \backslash 0 \to M$. It is well known that there is a unique connection the Chern connection ∇ on $\pi^* TM$ with $\nabla \frac{\partial}{\partial x^i} = \omega_i^j \frac{\partial}{\partial x^j}$ and $\omega_i^j = \Gamma_{ik}^j h x^k$ satisfying that

$$dg_{ij} - g_{ik}\omega_j^k - g_{jk}\omega_i^k = 2A_{ijk} \frac{\delta Y^k}{F}, \tag{2.1}$$

where $\delta Y^i = dY^i + N_j^i dx^j$, $N_j^i = \gamma_{jk}^i Y^k - \frac{1}{F} A_{jk}^i \gamma_{st}^k Y^s Y^t$ and γ_{jk}^i are the formal Christoffel symbols of the second kind for g_{ij}.

The curvature 2-forms of the Chern connection ∇ are

$$d\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i = \frac{1}{2} R_{jkl}^i \mathrm{d}x^k \wedge \mathrm{d}x^l + \frac{1}{F} P_{jkl}^i \mathrm{d}x^k \wedge \delta Y^l, \tag{2.2}$$

where R_{jkl}^i and P_{jkl}^i are the components of the hh-curvature tensor and hv-curvature tensor of the Chern connection, respectively.

Take a g-orthonormal frame $\{e_i = u_i^j \frac{\partial}{\partial x^j}\}$ with $e_n = \ell = \frac{Y^i}{F} \frac{\partial}{\partial x^i}$ for each fiber of $\pi^* TM$ and $\{\omega^i\}$ with $\omega^n = \omega$ is its dual coframe. The collection $\{\omega^i, \omega_n^i\}$ forms an orthonormal basis for $T^*(TM \backslash \{0\})$ with respect to the Sasaki type metric $g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ij} \delta Y^i \otimes \delta Y^j$. The pullback of the Sasaki metric from $TM \backslash \{0\}$ to the sphere bundle SM is a Riemannian metric

$$\widehat{g} = g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j + \delta_{\lambda\mu} \omega_n^\lambda \otimes \omega_n^\mu. \tag{2.3}$$

Let $\psi = \psi_i \omega^i \in \Gamma(\pi^* T^* M)$. With respect to the Chern connection ∇, the covariant derivatives of ψ_i are defined by

$$d\psi_i - \psi_j \omega_i^j = \psi_{i|j} \omega^j + \psi_{i;\lambda} \omega_n^\lambda.$$

where "|" and ";" denote the horizontal and the vertical covariant differentials with respect to the Chern connection, respectively.

Then we have

Lemma 2.1 ([3]). *For $\psi = \psi_i \omega^i \in \Gamma(\pi^* T^* M)$, we have*

$$div_{\widehat{g}} \psi = \sum_i \psi_{i|i} + \sum_{\lambda,\mu} \psi_\lambda P_{\mu\mu\lambda} = (\nabla_{e_i^H} \psi) e_i + \sum_{\lambda,\mu} \psi_\lambda P_{\mu\mu\lambda},$$

where $e_i^H = u_i^j \frac{\delta}{\delta x^j} = u_i^j(\frac{\partial}{\partial x^j} - N_j^k \frac{\partial}{\partial Y^k})$ *denotes the horizontal part of* e_i *and* $P_{\mu\mu\lambda} = P^n_{\mu\mu\lambda}$.

Lemma 2.2 ([3]). *Let* (M, F) *be a Finsler manifold. Then any function* f *on the sphere bundle* SM *satisfies*

$$\int_{S_xM} g^{ij}(F^2 f)_{Y^iY^j}\Omega d\tau = 2n \int_{S_xM} f\Omega d\tau,$$

where $(F^2 f)_{Y^iY^j} = \frac{\partial}{\partial Y^j}\frac{\partial}{\partial Y^i}(F^2 f), d\tau = \sum_i (-1)^{i-1} Y^i dY^1 \wedge \cdots \wedge \widehat{dY^i} \wedge \cdots \wedge dY^n$ *and* $\Omega = det(\frac{g_{ij}}{F})$.

For any fixed $x \in M$, $S_xM = \{Y \in T_xM | F(Y) = 1\}$ has a natural Riemannian metric

$$\widehat{r}_x = \sum_\lambda \theta_n^\lambda \otimes \theta_n^\lambda, \qquad where \ \ \theta_n^\lambda = \omega_n^\lambda|_{S_xM}.$$

On the Riemannian manifold $(S_xM, \widehat{r}_x)$, considering an 1-form $\Psi = \nu\Psi_i dY^i$, where $\nu = \sqrt{det(g_{ij})}$, we have [3]

$$div_{\widehat{r}_x}\Psi = r^{ij}[\nu\Psi_i]_{Y^j} - (n-1)\nu\Psi_i Y^i - Fg^{ij}\nu\Psi_i\eta_j, \tag{2.4}$$

where $r^{ij} = F^2 g^{ij} - Y^iY^j$ and $\eta_i = Fg^{jk}C_{jki}$.

Using the fact $\frac{\partial \nu}{\partial Y^i} = \frac{\nu\eta_i}{F}$, (2.4) implies the following

Lemma 2.3 *Let* (M, F) *be a Finsler manifold and* $\Psi = \nu\Psi_i dY^i$ *be a global section on* $T^*(S_xM)$, *where* $\nu = \sqrt{det(g_{ij})}$. *Then we have*

$$div_{\widehat{r}_x}\Psi = F^2\nu g^{ij}(\Psi_i)_{Y^j} - (n-2)\nu\Psi_i Y^i.$$

Let $\phi : (M^n, F) \to (\overline{M}^m, \overline{F})$ be a non-degenerate smooth map between Finsler manifolds, i.e., $ker(\mathrm{d}\phi) = \emptyset$, and $\widetilde{\nabla}$ be the pullback Chern connection on $\pi^*(\phi^{-1}T\overline{M})$. We have

Lemma 2.4

$$\begin{aligned} X\langle d\phi U, d\phi V\rangle = &\langle \widetilde{\nabla}_X(d\phi U), d\phi V\rangle + \langle d\phi U, \widetilde{\nabla}_X(d\phi V)\rangle \\ &+ 2\overline{C}(d\phi U, d\phi V, (\widetilde{\nabla}_X(d\phi F\ell)), \end{aligned}$$

where $\overline{C}_{ijk} = \frac{1}{F}\overline{A}_{ijk}$ *and* $X, U, V \in \Gamma(\pi^*TM)$.

The energy density of ϕ is the function $e(\phi) : SM \to R$ defined by

$$e(\phi)(x, Y) = \frac{1}{2}|\mathrm{d}\phi|^2 = \frac{1}{2}g^{ij}(x, Y)\phi_i^\alpha\phi_j^\beta\overline{g}_{\alpha\beta}(\overline{x}, \overline{Y}), \tag{2.5}$$

where $\mathrm{d}\phi(\frac{\partial}{\partial x^i}) = \phi_i^\alpha\frac{\partial}{\partial \overline{x}^\alpha}$ and $\overline{Y} = \overline{Y}^\alpha\frac{\partial}{\partial \overline{x}^\alpha} = Y^i\phi_i^\alpha\frac{\partial}{\partial \overline{x}^\alpha}$.

We define the energy functional $E(\phi)$ by

$$E(\phi) = \frac{1}{C_{n-1}}\int_{SM} e(\phi)\mathrm{d}V_{SM}, \tag{2.6}$$

where $\mathrm{d}V_{SM} = \Omega\mathrm{d}\tau \wedge \mathrm{d}x$, $\mathrm{d}x = dx^1 \wedge \cdots \wedge dx^n$ and C_{n-1} denotes the volume of the unit Euclidean sphere S^{n-1}.

We call ϕ a harmonic map if it is a critical point of the energy functional.

Theorem 2.5 ([3]). *ϕ is harmonic map if and only if*

$$\int_{SM} \langle V, (\widetilde{\nabla}_{\ell^H} d\phi)\ell\rangle dV_{SM} = 0,$$

for any vector $V \in \Gamma(\phi^{-1}T\overline{M})$.

We need the following

Theorem 2.6 ([9]). *Let M and $\widetilde{M}$ be n-dimensional complete Riemannian manifolds, where M is simply connected without focal points. Let r and $\widetilde{r}$ be the distance functions from $x_0 \in M$ and $\widetilde{x}_0 \in \widetilde{M}$, respectively. Suppose for any $x \in M, \widetilde{x} \in \widetilde{M}, r(x) = \widetilde{r}(\widetilde{x}) \neq 0$*

$$Ric_{(x)}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \leq \frac{1}{n-1}\widetilde{Ric}_{(\widetilde{x})}\left(\frac{\partial}{\partial \widetilde{r}}, \frac{\partial}{\partial r}\right),$$

where Ric and $\widetilde{Ric}$ denote the Ricci curvatures of M and $\widetilde{M}$, respectively. Then at any differentiable point $\widetilde{x}$ of $\widetilde{r}$

$$\triangle r \geq \frac{1}{n-1}\widetilde{\triangle}\widetilde{r}.$$

In particular, when $Ric \leq -b^2$, where b is a positive constant, then

$$\triangle r \geq b\coth(br). \tag{2.7}$$

3 The proof of main theorem

Let $\phi : M^n \to \overline{M}$ be a smooth map form a Riemannian manifold to a Finsler manifold. For any vector field $\mathrm{d}\phi X \in \Gamma(\phi^{-1}T\overline{M})$, by Lemmas 2.1, 2.4, $\nabla_{X^H}\ell = 0$ and $\overline{C}(\mathrm{d}\phi\ell, \bullet, \bullet) = 0$ we have that

$$\begin{aligned}
&div_{\widehat{g}}\left(\frac{1}{2}\langle \mathrm{d}\phi\ell, \mathrm{d}\phi\ell\rangle X\right)\\
&= \nabla_{X^H}\left(\frac{1}{2}\langle \mathrm{d}\phi\ell, \mathrm{d}\phi\ell\rangle\right) + \frac{1}{2}\langle \mathrm{d}\phi\ell, \mathrm{d}\phi\ell\rangle\langle \nabla_{e_i^H} X, e_i\rangle\\
&= \langle (\widetilde{\nabla}_{X^H}\mathrm{d}\phi)\ell, \mathrm{d}\phi\ell\rangle + \frac{1}{2}\langle \mathrm{d}\phi\ell, \mathrm{d}\phi\ell\rangle\langle \nabla_{e_i^H} X, e_i\rangle\\
&= \langle \widetilde{\nabla}_{\ell^H}\mathrm{d}\phi(X), \mathrm{d}\phi\ell\rangle - \langle \mathrm{d}\phi(\nabla_\ell X), \mathrm{d}\phi\ell\rangle + \frac{1}{2}\langle \mathrm{d}\phi\ell, \mathrm{d}\phi\ell\rangle\langle \nabla_{e_i} X, e_i\rangle.
\end{aligned} \tag{3.1}$$

Let $\psi = \langle \mathrm{d}\phi X, \mathrm{d}\phi\ell\rangle\omega^n$ which is a global section on π^*T^*M. By Lemma 2.1 and $P_{aan} = 0$, we know that

$$\begin{aligned}
div_{\widehat{g}}\psi &= \widetilde{\nabla}_{\ell^H}\langle \mathrm{d}\phi X, \mathrm{d}\phi\ell\rangle\\
&= \langle \widetilde{\nabla}_{\ell^H}\mathrm{d}\phi(X), \mathrm{d}\phi\ell\rangle + \langle \mathrm{d}\phi(X), (\widetilde{\nabla}_{\ell^H}\mathrm{d}\phi)\ell\rangle.
\end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1), we obtain that

$$\begin{aligned}
&div_{\widehat{g}}(\frac{1}{2}\langle \mathrm{d}\phi\ell, \mathrm{d}\phi\ell X) - div_{\widehat{g}}(\langle \mathrm{d}\phi X, \mathrm{d}\phi\ell\rangle\omega^n)\\
&= \langle \mathrm{d}\phi(X), (\widetilde{\nabla}_{\ell^H}\mathrm{d}\phi)\ell\rangle - \langle \mathrm{d}\phi(\nabla_\ell X), \mathrm{d}\phi\ell\rangle + \frac{1}{2}\langle \mathrm{d}\phi\ell, \mathrm{d}\phi\ell\rangle\langle \nabla_{e_i} X, e_i\rangle.
\end{aligned} \tag{3.3}$$

Since M is a Riemannian manifold, we have that $d\phi X$ only depends on the local coordinates (x^i) for $\mathrm{d}\phi X \in \Gamma(\phi^{-1}T\overline{M})$. When ϕ is a harmonic map. Integrating (3.3) implies that

$$\begin{aligned}&\int\limits_{\partial(SM)} \Big\{\frac{1}{2}\langle \mathrm{d}\phi\ell, \mathrm{d}\phi\ell\rangle\langle X, \mathbf{n}\rangle - \langle \mathrm{d}\phi X, \mathrm{d}\phi\ell\rangle\langle \ell, \mathbf{n}\rangle\Big\}\mathrm{d}V_{\partial SM}\\ &= \int\limits_{SM}\Big\{-\langle \mathrm{d}\phi(\nabla_\ell X), \mathrm{d}\phi\ell\rangle + \frac{1}{2}\langle \mathrm{d}\phi\ell, \mathrm{d}\phi\ell\rangle\langle \nabla_{e_i}X, e_i\rangle\Big\}\mathrm{d}V_{SM},\end{aligned} \tag{3.4}$$

where $\mathbf{n}$ is the unit normal vector of the boundary $\partial(SM)$ in SM and ℓ is the natural dual of ω^n.

From (3.4), we obtain the following immediately

Proposition 3.1 *Let ϕ be a harmonic map from a Riemannian manifold M to any Finsler manifold. If $e(\phi)|_{\partial(SM)} = 0$, then for any vector field $d\phi X \in \Gamma(\phi^{-1}T\overline{M})$*

$$\int\limits_{SM}\Big\{-\langle d\phi(\nabla_\ell X), d\phi\ell\rangle + \frac{1}{2}\langle d\phi\ell, d\phi\ell\rangle tr\nabla X\Big\}dV_{SM} = 0, \tag{3.5}$$

where $tr\nabla X = \langle \nabla_{e_i}X, e_i\rangle_g$.

Theorem 3.2 *Let ϕ be a harmonic map from a Riemannian manifold M to any Finsler manifold. If $e(\phi)|_{\partial(SM)} = 0$, then for any vector field $d\phi X \in \Gamma(\phi^{-1}T\overline{M})$*

$$\int\limits_{SM}\Big\{e(\phi)tr\nabla X - tr\langle d\phi(\nabla X), d\phi\rangle\Big\}dV_{SM} = 0, \tag{3.6}$$

where $tr\langle d\phi(\nabla X), d\phi\rangle = \langle d\phi(\nabla_{e_i}X), \phi_*(e_i)\rangle$.

Proof Since M is a Riemannian manifold, we have that $tr\nabla X = g^{ij}\langle \nabla_{\frac{\partial}{\partial x^i}}X, \frac{\partial}{\partial x^j}\rangle_g$ only depends on the local coordinates (x^i) for $\mathrm{d}\phi X \in \Gamma(\phi^{-1}T\overline{M})$. Denote $f = \frac{1}{2}\langle \mathrm{d}\phi\ell, \mathrm{d}\phi\ell\rangle tr\nabla X \in C^\infty(SM)$. Using the fact $\overline{C}(\mathrm{d}\phi\ell, \bullet, \bullet) = 0$ and Lemma 2.4, we obtain that

$$\begin{aligned}g^{ij}(F^2 f)_{Y^iY^j} &= g^{ij}\left[\frac{1}{2}Y^kY^l\left\langle \mathrm{d}\phi\frac{\partial}{\partial x^k}, \mathrm{d}\phi\frac{\partial}{\partial x^k}\right\rangle tr\nabla X\right]_{Y^iY^j}\\ &= \sum_i\langle \mathrm{d}\phi e_i, \mathrm{d}\phi e_i\rangle tr\nabla X.\end{aligned} \tag{3.7}$$

Substituting (3.7) into Lemma 2.2 yields that

$$\int\limits_{S_xM}\frac{n}{2}|\mathrm{d}\phi\ell|^2 tr\nabla X\Omega\mathrm{d}\tau = \int\limits_{S_xM}\frac{1}{2}|\mathrm{d}\phi|^2 tr\nabla X\Omega\mathrm{d}\tau. \tag{3.8}$$

Considering $\Psi = \frac{1}{F}\langle \mathrm{d}\phi(\nabla_{\frac{\partial F\ell}{\partial Y^i}}X), \mathrm{d}\phi\ell\rangle dY^i = \frac{1}{F}\langle \mathrm{d}\phi(\nabla_{\frac{\partial}{\partial x^i}}X), \mathrm{d}\phi\ell\rangle dY^i$. Since $\mathrm{d}\phi(\nabla_{\frac{\partial}{\partial x^i}}X)$ only depends on (x^i) for $\mathrm{d}\phi X \in \Gamma(\phi^{-1}T\overline{M})$, by Lemma 2.3 we have that

$$\begin{aligned} div_{\widehat{r}_x}\psi &= F^2 \nu g^{ij}\left[\frac{1}{F^2}\left\langle \mathrm{d}\phi(\nabla_{\frac{\partial}{\partial x^i}}X), \mathrm{d}\phi F\ell\right\rangle\right]_{Y^j} - (n-2)\nu\frac{1}{F}\left\langle \mathrm{d}\phi(\nabla_{\frac{\partial}{\partial x^i}}X), \mathrm{d}\phi\ell\right\rangle Y^i \\ &= F^2\nu g^{ij}\left[-\frac{2Y^k g_{jk}}{F^4}\langle \mathrm{d}\phi(\nabla_{\frac{\partial}{\partial x^i}}X), \mathrm{d}\phi F\ell\rangle + \frac{1}{F^2}\langle \mathrm{d}\phi(\nabla_{\frac{\partial}{\partial x^i}}X), \mathrm{d}\phi\frac{\partial}{\partial x^j}\rangle\right] \\ &\quad -(n-2)\nu\frac{1}{F}\langle \mathrm{d}\phi(\nabla_{\frac{\partial}{\partial x^i}}X), \mathrm{d}\phi\ell\rangle Y^i \\ &= \nu\left[\langle \mathrm{d}\phi(\nabla_{e_i}X), \mathrm{d}\phi e_i\rangle - n\langle \mathrm{d}\phi(\nabla_\ell X), \mathrm{d}\phi\ell\rangle\right]. \end{aligned} \tag{3.9}$$

Integrating (3.9) implies that

$$n\int_{S_xM}\langle \mathrm{d}\phi(\nabla_\ell X), \mathrm{d}\phi\ell\rangle\Omega \mathrm{d}\tau = \int_{S_xM} tr\langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi\rangle\Omega\mathrm{d}\tau, \tag{3.10}$$

where $tr\langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi\rangle = \langle \mathrm{d}\phi(\nabla_{e_i}X), \mathrm{d}\phi e_i\rangle$.

It can be seen from (3.8), (3.10) and Proposition 3.1 that

$$\int_{SM}\left\{e(\phi)tr\nabla X - tr\langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi\rangle\right\}\mathrm{d}V_{SM} = 0.$$

This completes the proof of Theorem 3.2. □

For a harmonic map $\phi : M^n \to \overline{M}$ between Riemannian manifolds, We also have that

Theorem 3.2′. Let ϕ be a harmonic map from a Riemannian manifold M into any Riemannian manifold. If $e(\phi)|_{\partial M} = 0$, then for any vector field $\mathrm{d}\phi X \in \Gamma(\phi^{-1}T\overline{M})$

$$\int_{SM}\left\{e(\phi)tr\nabla X - tr\langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi\rangle\right\}\mathrm{d}V_{SM} = 0, \quad (3.6')$$

i.e.,

$$\int_{M}\left\{e(\phi)tr\nabla X - tr\langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi\rangle\right\}\mathrm{d}V_{M} = 0. \tag{3.11}$$

Proof For any vector field $\mathrm{d}\phi X \in \Gamma(\phi^{-1}T\overline{M})$, we have that

$$\begin{aligned} \mathrm{div}(e(\phi)X) = \nabla_X e(\phi) + e(\phi)\langle\nabla_{e_i}X, e_i\rangle \\ = \mathrm{div}(\langle\phi_* X, \phi_*(e_i)\rangle e_i) - \langle\phi_* X, \tau(\phi)\rangle \\ -\langle \mathrm{d}\phi(\nabla_{e_i}X), \phi_*(e_i)\rangle + e(\phi)\langle\nabla_{e_i}X, e_i\rangle, \end{aligned} \tag{3.12}$$

where $\{e_i\}$ is a local orthonormal frame field on M.

When ϕ is a harmonic map. Integrating (3.12) implies that

$$\begin{aligned} &\int_{\partial M}\left\{e(\phi)\langle X, \mathbf{n}\rangle - \langle\phi_* X, \phi_*(\mathbf{n})\rangle\right\}\mathrm{d}V_M \\ &= \int_M\left\{e(\phi)\langle\nabla_{e_i}X, e_i\rangle - \langle \mathrm{d}\phi(\nabla_{e_i}X), \phi_*(e_i)\rangle\right\}\mathrm{d}V_M, \end{aligned} \tag{3.13}$$

where $\mathbf{n}$ is the unit normal vector of ∂M in M,

By (3.13) and our assumption condition $e(\phi)|_{\partial M} = 0$, we obtain that

$$\int\limits_M \Big\{ e(\phi) tr \nabla X - tr\langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi \rangle \Big\} \mathrm{d}V_M = 0. \tag{3.14}$$

For a harmonic map ϕ between Riemannian manifolds, we have that $e(\phi) tr \nabla X - tr\langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi\rangle$ only depends on the local coordinates (x^i) for $\mathrm{d}\phi X \in \Gamma(\phi^{-1}T\overline{M})$. Then

$$\begin{aligned} &\frac{1}{C_{n-1}} \int\limits_{SM} \Big\{ e(\phi) tr \nabla X - tr\langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi \rangle \Big\} \mathrm{d}V_{SM} \\ &= \frac{1}{C_{n-1}} \int\limits_M \Big\{ \int\limits_{S_x M} [e(\phi) tr \nabla X - tr\langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi \rangle] \Omega \mathrm{d}\tau \Big\} \mathrm{d}x \\ &= \int\limits_M \Big\{ e(\phi) tr \nabla X - tr\langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi \rangle \Big\} \mathrm{d}V_M. \end{aligned} \tag{3.15}$$

We complete the proof of Theorem 3.2′. □

Definition 1 The energy of a map ϕ from a Riemannian manifold M to any Finsler manifold is called *moderate divergent energy* if there exists a positive function $\psi(r)$ satisfying

$$\int\limits_{R_0}^{\infty} \frac{\mathrm{d}r}{r\psi(r)} = \infty,$$

such that

$$\lim_{R\to\infty} \int\limits_{SB_R(x_0)} \frac{e(\phi)}{\psi(r)} * 1 < \infty.$$

where $SB_R(x_0) \subseteq SM$ is a geodesic ball of radius R and centered at x_0 in SM whose boundary is the geodesic sphere $\partial(SB_R(x_0))$.

Remark For a map ϕ between Riemannian manifolds, the $\lim_{R\to\infty} \int_{SB_R(x_0)} \frac{e(\phi)}{\psi(r)} * 1 < \infty$ is equivalent to $\lim_{R\to\infty} \int_{B_R(x_0)} \frac{e(\phi)}{\psi(r)} * 1 < \infty$, where $B_R(x_0) \subseteq M$ is a geodesic ball of radius R and centered at x_0 in M.

Theorem 3.3 *Let ϕ be a harmonic map from a Riemannian manifold M to any Finsler manifold. If the energy of ϕ is moderate divergent energy, then $e(\phi)|_{\partial(SM)} = 0$.*

Proof We assume that $\int_{\partial(SM)} e(\phi) * 1 \neq 0$. Let $\int_{\partial(SM)} e(\phi) * 1 = \delta > 0$, i.e.,

$$\lim_{R\to\infty} \int\limits_{\partial(SB_R(x_0))} e(\phi) * 1 = \delta > 0. \tag{3.16}$$

Then there is a $R_0 > 1$, for $R > R_0$ we have that

$$\int\limits_{\partial(SB_R(x_0))} e(\phi) * 1 \geq \varepsilon, \tag{3.17}$$

where ε is a sufficiently small constant.

For any positive function $\psi(r)$ satisfying $\int_{R_0}^{\infty} \frac{\mathrm{d}r}{r\psi(r)} = \infty$, we have that

$$\begin{aligned}
\lim_{R\to\infty} \int\limits_{SB_R(x_0)} \frac{e(\phi)}{\psi(r)} * 1 &= \int\limits_0^{\infty} \left[\frac{1}{\psi(R)} \int\limits_{\partial(SB_R(x_0))} e(\phi) * 1 \right] \mathrm{d}R \\
&\geq \varepsilon \int\limits_{R_0}^{\infty} \frac{1}{\psi(R)} \mathrm{d}R \\
&\geq \varepsilon \int\limits_{R_0}^{\infty} \frac{1}{R\psi(R)} \mathrm{d}R = \infty,
\end{aligned}$$

which is a contradiction to the assumption that the energy of ϕ is moderate divergent energy. This completes the proof of Theorem 3.3. □

Corollary 3.4 *Let ϕ be a harmonic map from a Riemannian manifold M to any Riemannian manifold. If the energy of ϕ is moderate divergent energy, then $e(\phi)|_{\partial M} = 0$.*

Theorem 3.5 *Let M be an $n(n \geq 3)$-dimensional Cartan–Hadamard Riemannian manifold with sectional curvature $K : -a^2 \leq K \leq 0$ and Ricci curvature bounded from above by $-b^2$. If $b \geq a$, then there is no non-degenerate harmonic maps from M to any Finsler manifold with moderate divergent energy.*

Remark This theorem improves the Xin's result in [10].

Proof For the harmonic map ϕ. It can be seen from Theorems 3.2 and 3.3 that

$$\lim_{R\to\infty} \int\limits_{B_R(x_0)} \left[\int\limits_{S_x B_R(x_0)} \Big\{ e(\phi) tr \nabla X - tr \langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi \rangle \Big\} \Omega \mathrm{d}\tau \right] \mathrm{d}x = 0, \tag{3.18}$$

where $B_R(x_0) \subseteq M$ be a geodesic ball of radius R and centered at x_0 whose boundary is the geodesic sphere $S_R(x_0)$.

The square of the distance function r^2 from x_0 in Riemannian manifold M is a smooth function. Set $X = sh(Cr)\frac{\partial}{\partial r} = \frac{e^{Cr} - e^{-Cr}}{2} \frac{\partial}{\partial r}$, where C is constant and $\frac{\partial}{\partial r}$ denotes the unit radial vector. Obviously, the unit normal vector to $S_R(x_0)$ is $\frac{\partial}{\partial r}$. Let $\{e_\lambda, \frac{\partial}{\partial r}\}$ is an orthonormal frame field on $B_R(x_0)$, where $\lambda = 1, \ldots, n-1$. Then we have that

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial r}} X = Cch(Cr)\tfrac{\partial}{\partial r}, \\
&\nabla_{e_\lambda} X = sh(Cr) \textstyle\sum_\mu h_{\lambda\mu} e_\mu = sh(Cr) \sum_\mu Hess(r)(e_\lambda, e_\mu) e_\mu,
\end{aligned} \tag{3.19}$$

where $Hess(r)$ is the Hessian of the distance function r and $-h_{\lambda\mu}$ is the components of the second fundamental form of $S_R(x_0)$ in $B_R(x_0)$.

Then we obtain that by (3.19)

$$\begin{aligned}
&e(\phi) tr \nabla X - tr \langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi \rangle \\
&\quad = e(\phi) \Big\{ Cch(Cr) + sh(Cr) \sum_\lambda h_{\lambda\lambda} \Big\} \\
&\qquad - Cch(Cr) |\mathrm{d}\phi \left(\frac{\partial}{\partial r} \right)|^2 - sh(Cr) h_{\lambda\mu} \big\langle \mathrm{d}\phi(e_\lambda), \mathrm{d}\phi(e_\mu) \big\rangle.
\end{aligned} \tag{3.20}$$

We can choose an orthonormal frame field $\{e_\lambda\}$ on $S_R(x_0)$ such that $h_{\lambda\mu} = \delta_{\lambda\mu}h_{\lambda\lambda}$. It follows from (3.20) that

$$\begin{aligned} &e(\phi)tr\nabla X - tr\langle \mathrm{d}\phi(\nabla X), \mathrm{d}\phi\rangle \\ &= \frac{1}{2}\Big\{sh(Cr)\sum_\lambda h_{\lambda\lambda} - Cch(Cr)\Big\}|\mathrm{d}\phi\left(\frac{\partial}{\partial r}\right)|^2 \\ &\quad + \frac{1}{2}\sum_\lambda \Big\{Cch(Cr) + \sum_\mu sh(Cr)h_{\mu\mu} - 2sh(Cr)h_{\lambda\lambda}\Big\}|\mathrm{d}\phi(e_\lambda)|^2. \end{aligned} \tag{3.21}$$

Substituting (3.21) into (3.19), we have that

$$\begin{aligned} \lim_{R\to\infty}\int_{B_R(x_0)}\Bigg[\int_{S_x B_R(x_0)} &\Big\{[sh(Cr)\sum_\lambda h_{\lambda\lambda} - Cch(Cr)]|\mathrm{d}\phi(\frac{\partial}{\partial r})|^2 \\ &+ \sum_\lambda [Cch(Cr) + \sum_\mu sh(Cr)h_{\mu\mu} - 2sh(Cr)h_{\lambda\lambda}]|\mathrm{d}\phi(e_\lambda)|^2\Big\}\Omega \mathrm{d}\tau\Bigg]\mathrm{d}x = 0. \end{aligned} \tag{3.22}$$

By $Ric \le -b^2$ and Theorem 2.6, we get that

$$\sum_\mu h_{\mu\mu} \ge bcoth(br).$$

Set $C = a$ in (3.22). Since $g(x) = xcoth(x)$ is non-decreasing function and $b \ge a$, we obtain that

$$sh(Cr)\sum_\lambda h_{\lambda\lambda} - Cch(Cr) \ge sh(ar)[bcoth(br) - acoth(ar)] \ge 0. \tag{3.23}$$

When $-a^2 \le K \le 0$, the Hessian comparison theorem implies that

$$\frac{1}{r} \le h_{\lambda\lambda} \le acoth(ar), \quad \forall\lambda. \tag{3.24}$$

Then

$$\begin{aligned} &Cch(Cr) + \sum_\mu sh(Cr)h_{\mu\mu} - 2sh(Cr)h_{\lambda\lambda} \\ &= ach(ar) + \sum_{\mu\ne\lambda} sh(ar)h_{\mu\mu} - sh(ar)h_{\lambda\lambda} \\ &\ge \sum_{\mu\ne\lambda} sh(ar)h_{\mu\mu} \ge \delta, \end{aligned} \tag{3.25}$$

where $\delta > 0$ is a constant.

By (3.22), (3.23) and (3.25), we have that

$$\mathrm{d}\phi(e_\lambda) = 0, \forall\lambda.$$

Put $C = 0$ in (3.22). Then (3.22), together with $\mathrm{d}\phi(e_\lambda) = 0, \forall\lambda$, yields that $\mathrm{d}\phi(\frac{\partial}{\partial r}) = 0$. We get that $e(\phi) = 0$ which is a contradiction to the assumption that ϕ is a non-degenerate harmonic maps. □

Corollary 3.6 *Let M be an $n(n \geq 3)$-dimensional Cartan–Hadamard Riemannian manifold with sectional curvature $K : -a^2 \leq K \leq 0$ and Ricci curvature bounded from above by $-b^2$. If $b \geq a$, then there is no non-degenerate harmonic maps from M to any Riemannian manifold with moderate divergent energy.*

From [7] we know the Ricci curvatures of bounded symmetric domains and estimate the bounds of the sectional curvatures of bounded symmetric domains as follows

	Dim.	Sec. curvature	Ric. curvature
$\Re_I(n, m)(min(n, m) \geq 2)$	$2nm$	$-4 \leq k \leq 0$	$-2(n+m)$
$\Re_{II}(n)(n \geq 2)$	$n(n+1)$	$-4 \leq k \leq 0$	$-2(n+1)$
$\Re_{III}(n)(n \geq 4)$	$n(n-1)$	$-2 \leq k \leq 0$	$-2(n-1)$
$\Re_{IV}(n)(n \geq 2)$	$2n$	$-2 \leq k \leq 0$	$-n$

This table together with Theorem 3.5 yields immediately.

Main Theorem. *There is no non-degenerate harmonic map from a classical bounded symmetric domain to any Finsler manifold with moderate divergent energy.*

Corollary 3.7 *Any harmonic map from a classical bounded symmetric domain to any Riemannian manifold with moderate divergent energy has to be constant.*

Using the fact that any harmonic map of finite energy must be moderate divergent energy, we have immediately

Corollary 3.8 *Any harmonic map from a classical bounded symmetric domain to any Riemannian manifold with finite energy has to be constant.*

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References

1. Cartan, E.: Sur certaines formes riemanniennes remarquables des geometric a groupe fondamental simple. Ann Sci. Ecole Norm. Sup. **44**, 345–367 (1927)
2. Cartan, E.: Sur les domaines bornrs homogenes de l'espace de n variables complexes. Abh. Math. Sem. Univ. Hamburg **11**, 116–162 (1935)
3. He, Q., Shen, Y.B.: Some results on harmonic maps for Finsler manifolds. Int. J. Math. **16**, 1017–1031 (2005)
4. Hua, L.K.: On the theory of automorphic functions of a matrix variables I: geometrical basis. Am. J. Math. **66**, 470–488 (1944)
5. Liu, X.S.: Curvature estimates for irreducible symmetric spaces. Chin. Ann. Math. **27B**, 287–302 (2006)
6. Sealey, H.C.J.: Some conditions ensuring the vanishing of harmonic differential forms with applications to harmonic maps and Yang-Mills theory. Math. Prec. Camb. Phil. Soc. **91**, 441–452 (1982)
7. Siegle, C.L.: Symplectic geometry. Am. J. Math. **55**, 1–86 (1943)
8. Wong, Y.C.: Euclidean n-planes in pseudo-Euclidean space and differential geometry of Cartan domains. Bull. Am. Math. Soc. **75**, 409–414 (1969)
9. Xin, Y.L.: Harmonic maps of bounded symmetric domains. Math. Ann. **303**, 417–433 (1995)
10. Xin, Y.L.: harmonic maps from Kähler manifolds. Acta Math. Sinica **15**, 277–292 (1999)
11. Yang, Y.H.: Harmonic functions on product of certain Kahler manifolds with a pole. Chin. Ann. Math. **22B**, 381–384 (2001)