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Oscillation of second-order Emden–Fowler neutral delay differential equations

Ravi P. Agarwal · Martin Bohner · Tongxing Li · Chenghui Zhang

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Abstract We establish some new criteria for the oscillation of second-order Emden–Fowler neutral delay differential equations. We study the case of superlinear and the case of sublinear equations subject to various conditions. The results obtained show that the presence of a neutral term in a differential equation can cause or destroy oscillatory properties. Several examples are provided to illustrate the relevance of new theorems.

Keywords Oscillation · Neutral differential equation · Emden–Fowler equation · Second-order

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1 Introduction

Emden–Fowler-type differential equations have some applications in the real world. For instance, equation

$$x''(t) + \frac{a}{t}x'(t) + bt^{m-1}x^n(t) = 0, \tag{1.1}$$

R. P. Agarwal
Department of Mathematics, Texas A&M University–Kingsville,
700 University Blvd., Kingsville, TX 78363-8202, USA
e-mail: agarwal@tamuk.edu

M. Bohner
Department of Mathematics and Statistics, Missouri S&T, Rolla, MO 65409-0020, USA
e-mail: bohner@mst.edu

T. Li · C. Zhang (✉)
School of Control Science and Engineering, Shandong University,
Jinan 250061, Shandong, People's Republic of China
e-mail: zchui@sdu.edu.cn

T. Li
e-mail: litongx2007@163.com

where $n \neq 0, n \neq 1, a, b, m$ are parameters, is used in mathematical physics, theoretical physics, and chemical physics, etc., see [9,31]. This paper is concerned with the oscillatory behavior of second-order Emden–Fowler neutral delay differential equations of the form

$$\left(r(t)\left((x(t)+p(t)x(\tau(t)))'\right)^{\alpha}\right)' + q(t)x^{\gamma}(\sigma(t)) = 0, \quad t \geq t_0 \tag{1.2}$$

subject to the following hypotheses:

(H_1) $\alpha, \gamma \in \mathfrak{R}$, where $\mathfrak{R}$ is the set of all ratios of odd positive integers;
(H_2) $r \in \mathrm{C}([t_0,\infty),(0,\infty))$, $p, q \in \mathrm{C}([t_0,\infty),\mathbb{R})$, $0 \leq p(t) < 1$, $q(t) \geq 0$, and q is not identically zero for large t;
(H_3) $\tau, \sigma \in \mathrm{C}([t_0,\infty),\mathbb{R})$, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\lim_{t\to\infty}\tau(t) = \infty$, and $\lim_{t\to\infty}\sigma(t) = \infty$.

By a solution of (1.2), we mean a nontrivial function x satisfying (1.2) for $t \geq t_x \geq t_0$. In the sequel, we assume that solutions of (1.2) exist and can be continued indefinitely to the right. A solution of (1.2) is called oscillatory if it has arbitrarily large zeros on $[t_x,\infty)$; otherwise, it is called nonoscillatory. Equation (1.2) is said to be oscillatory if all its solutions are oscillatory.

During the past few years, there has been constant interest in obtaining sufficient conditions for oscillatory or nonoscillatory behavior of different classes of differential and functional differential equations; see, e.g., [1–8,10–38]. Very recently, Baculíková and Džurina [7] established several oscillation theorems for equation

$$\left(r(t)\left(x(t)+p(t)x(\tau(t))\right)'\right)' + q(t)x(\sigma(t)) = 0$$

via a comparison with associated first-order delay differential equations in the case when, in addition to

$$\tau \in \mathrm{C}^1([t_0,\infty),\mathbb{R}), \quad \tau'(t) \geq \tau_0 > 0, \quad \text{and} \quad \tau \circ \sigma = \sigma \circ \tau, \tag{1.3}$$

condition

$$\int_{t_0}^{\infty} r^{-1}(t)\mathrm{d}t = \infty$$

is satisfied. Assuming

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)\mathrm{d}t = \infty,$$

Baculíková and Džurina [8] extended results of [7] to Eq. (1.2). Under conditions

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)\mathrm{d}t < \infty \tag{1.4}$$

and

$$p'(t) \geq 0, \quad \lim_{t\to\infty} p(t) = A, \tag{1.5}$$

Xu and Meng [34, Theorem 2.3] obtained sufficient conditions for oscillation and asymptotic behavior of a nonlinear neutral differential equation of the form

$$\left(r(t)\left((x(t)+p(t)x(t-\tau))'\right)^{\alpha}\right)' + q(t)x^{\alpha}(\sigma(t)) = 0, \tag{1.6}$$

where $\tau \geq 0$ is a constant. Further results in that direction were obtained by Ye and Xu [35] under the assumptions that

$$p'(t) \geq 0, \quad \sigma(t) \leq t - \tau, \tag{1.7}$$

see also the paper by Han et al. [13] where inaccuracies in [35] have been corrected and new oscillation criteria for (1.6) were established [13, Theorem 2.1 and Theorem 2.2]. Developing further ideas from the paper by Hasanbulli and Rogovchenko [14] concerned with a particular case of Eq. (1.6) with $\alpha = 1$, Li et al. [24] studied the oscillation of (1.6) in the case where (1.4) holds and $\alpha \geq 1$. Li et al. [22] considered the Emden–Fowler neutral delay differential equation

$$\left(r(t)\left(x(t) + p(t)x(t-\tau)\right)'\right)' + q(t)x^{\gamma}(\sigma(t)) = 0, \tag{1.8}$$

where $\tau \geq 0$ is a constant, $\gamma \in \mathfrak{R}, \gamma \geq 1, \sigma \in \mathrm{C}^1([t_0, \infty), \mathbb{R}), \sigma' > 0, \sigma(t) \leq t$, and $\lim_{t\to\infty} \sigma(t) = \infty$, and they presented the following result.

Theorem 1.1 (See [22, Theorem 2.1]) *Assume* (H_2) *and let* (1.4) *hold with* $\alpha = 1$. *Assume further that there exists a function* $\rho \in \mathrm{C}^1([t_0, \infty), \mathbb{R})$ *with* $\rho(t) \geq t, \rho'(t) > 0$, *and* $\sigma(t) \leq \rho(t) - \tau$ *such that, for all sufficiently large* t_1 *and for all positive constants* M *and* L

$$\int^{\infty} \left[q(t)(1 - p(\sigma(t)))^{\gamma} R^{\gamma}(\sigma(t)) - \frac{\gamma M^{1-\gamma} \sigma'(t) R^{\gamma-1}(\sigma(t))}{r(\sigma(t)) \int_{t_1}^{t} \frac{\sigma'(s)}{r(\sigma(s))} \mathrm{d}s} \right] \mathrm{d}t = \infty$$

and

$$\int^{\infty} \left[q(t) \left(\frac{1}{1 + p(\rho(t))} \right)^{\gamma} \delta^{\gamma}(t) - \frac{\gamma \rho'(t)}{L^{\gamma-1} \delta(t) r(\rho(t))} \right] \mathrm{d}t = \infty,$$

where $R(t) := \int_{t_0}^{t} r^{-1}(s)\mathrm{d}s$ *and* $\delta(t) := \int_{\rho(t)}^{\infty} r^{-1}(s)\mathrm{d}s$. *Then,* (1.8) *is oscillatory.*

As a special case of Eq. (1.2), Zhang et al. [37] employed Riccati transformation to study oscillation of a nonlinear differential equation

$$\left(r(t)\left(x'(t)\right)^{\alpha}\right)' + q(t)x^{\gamma}(\sigma(t)) = 0, \tag{1.9}$$

where $\alpha, \gamma \in \mathfrak{R}, r, q \in \mathrm{C}([t_0, \infty), (0, \infty)), \sigma \in \mathrm{C}([t_0, \infty), \mathbb{R}), \sigma(t) < t$, and $\lim_{t\to\infty} \sigma(t) = \infty$. They established several new results, one of which we present below for the convenience of the reader.

Theorem 1.2 (See [37, Theorem 2.1]) *Assume* (1.4), $\gamma \leq \alpha$, *and let the differential equation*

$$y'(t) + q(t) \left(\frac{\lambda_0 \sigma(t)}{r^{1/\alpha}(\sigma(t))} \right)^{\gamma} y^{\gamma/\alpha}(\sigma(t)) = 0 \tag{1.10}$$

be oscillatory for some constant $\lambda_0 \in (0, 1)$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[M^{\gamma-\alpha} q(s) \xi^{\alpha}(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{1}{\xi(s) r^{1/\alpha}(s)} \right] \mathrm{d}s = \infty \tag{1.11}$$

holds for every constant $M > 0$*, where*

$$\xi(t) := \int_t^{\infty} r^{-1/\alpha}(s)\mathrm{d}s, \tag{1.12}$$

then (1.9) *is oscillatory.*

On the basis of conditions (1.3), $0 \le p(t) \le p_0 < \infty$, $\tau(t) \le t$, $\sigma(t) \le t$, and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\delta(s)Q(s) - \frac{1 + \frac{p_0}{\tau_0}}{4\delta(s)r(s)} \right] \mathrm{d}s = \infty, \tag{1.13}$$

where $Q(t) := \min\{q(t), q(\tau(t))\}$ and $\delta(t) := \int_t^{\infty} r^{-1}(s)\mathrm{d}s$, Han et al. [13] established some oscillation criteria [13, Theorem 3.1 and Theorem 3.2] for the second-order neutral delay differential equation

$$\left(r(t)\left(x(t) + p(t)x(\tau(t))\right)'\right)' + q(t)x(\sigma(t)) = 0.$$

Sun et al. [30] investigated nonlinear differential equation

$$\left(r(t)\left(\left(x(t) + p(t)x(\tau(t))\right)'\right)^{\alpha}\right)' + q(t)x^{\alpha}(\sigma(t)) = 0, \tag{1.14}$$

where $\alpha \in \Re, \alpha \ge 1, r \in \mathrm{C}([t_0,\infty),(0,\infty)), p, q \in \mathrm{C}([t_0,\infty),\mathbb{R}), 0 \le p(t) \le p_0 < \infty, q(t) \ge 0$, q is not identically zero for large t, $\tau, \sigma \in \mathrm{C}^1([t_0,\infty),\mathbb{R}), \sigma' > 0, \sigma(t) \le \tau(t) \le t$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$. They obtained the following oscillation criterion.

Theorem 1.3 (See [30, Theorem 4.1]) *Assume* (1.3)*,* (1.4)*, and there exists a function* $\rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{\left(1 + \frac{p_0{}^{\alpha}}{\tau_0}\right) r(\sigma(s))(\rho_+'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\rho(s)\sigma'(s))^{\alpha}} \right] \mathrm{d}s = \infty.$$

If there exists a function $\eta \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$ *such that* $\eta(t) \ge t$, $\eta'(t) > 0$*, and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{\pi^{\alpha}(s)Q(s)}{2^{\alpha-1}} - \frac{\alpha^{\alpha+1}\left(1 + \frac{p_0{}^{\alpha}}{\tau_0}\right)\eta'(s)}{(\alpha+1)^{\alpha+1}\pi(s)r^{1/\alpha}(\eta(s))} \right] \mathrm{d}s = \infty,$$

where $Q(t) := \min\{q(t), q(\tau(t))\}$, $\rho_+'(t) := \max\{0, \rho'(t)\}$*, and* $\pi(t) := \int_{\eta(t)}^{\infty} r^{-1/\alpha}(s)\mathrm{d}s$*, then* (1.14) *is oscillatory.*

The objective of this paper is to improve the results in [13,22,30,34,35,37]. This paper is organized as follows: In the next section, we give some lemmas. In Sect. 3, four new oscillation criteria are obtained. In Sect. 4, we present some conclusions to summarize the contents of this paper.

2 Some lemmas

We begin with the following lemma.

Lemma 2.1 (See [28, Theorem 1]) *Suppose* $\lambda \in \Re$, $g, h \in \mathrm{C}[t_0, \infty)$, $g(t) \geq 0$, $h(t) < t$, *and* $\lim_{t\to\infty} h(t) = \infty$. *If the first-order delay differential inequality*

$$z'(t) + g(t)z^{\lambda}(h(t)) \leq 0$$

has an eventually positive solution, so does the delay differential equation

$$z'(t) + g(t)z^{\lambda}(h(t)) = 0. \tag{2.1}$$

Lemma 2.2 (See [16, Theorem 2]) *Assume* $\lambda \in \Re$, $g, h \in \mathrm{C}[t_0, \infty)$, $g(t) \geq 0$, $h(t) < t$, *and* $\lim_{t\to\infty} h(t) = \infty$. *Then, Eq.* (2.1) *with* $\lambda \in (0, 1)$ *is oscillatory if*

$$\int_{t_0}^{\infty} g(t)\mathrm{d}t = \infty. \tag{2.2}$$

Lemma 2.3 (See [6, Lemma 2.3]) *Suppose* $\lambda \in \Re$, $g, \eta \in \mathrm{C}[t_0, \infty)$, $g(t) \geq 0$, *and* $\eta(t) > t$. *If the first-order advanced differential inequality*

$$z'(t) - g(t)z^{\lambda}(\eta(t)) \geq 0$$

has an eventually positive solution, so does the advanced differential equation

$$z'(t) - g(t)z^{\lambda}(\eta(t)) = 0. \tag{2.3}$$

Lemma 2.4 (See [16, Theorem 1]) *Assume* $\lambda \in \Re$, $g, \eta \in \mathrm{C}[t_0, \infty)$, $g(t) \geq 0$, *and* $\eta(t) > t$. *Then, Eq.* (2.3) *with* $\lambda \in (1, \infty)$ *is oscillatory if* (2.2) *holds.*

3 Oscillation criteria

In what follows, all functional inequalities are assumed to hold eventually, that is, for all t large enough. We also use the notation

$$z(t) := x(t) + p(t)x(\tau(t)) \quad \text{and} \quad f(t) := \left(1 - p(\sigma(t))\frac{\xi(\tau(\sigma(t)))}{\xi(\sigma(t))}\right)^{\gamma} > 0,$$

where ξ is as in (1.12).

Theorem 3.1 *Let* (H_1)–(H_3), (1.4), *and* $\gamma \geq \alpha$ *hold. Assume that*

$$\int_{t_2}^{\infty} q(t)\,(1 - p(\sigma(t)))^{\gamma} \left(\int_{t_1}^{\sigma_1(t)} r^{-1/\alpha}(s)\mathrm{d}s\right)^{\beta} \mathrm{d}t = \infty \tag{3.1}$$

holds for some $\beta \in \Re$ *with* $\beta < \alpha$, *for all sufficiently large* $t_1 \geq t_0$, *for some* $t_2 > t_1$, *and for some function* $\sigma_1 \in \mathrm{C}([t_0, \infty), \mathbb{R})$ *with* $\sigma_1(t) \leq \sigma(t)$, $\sigma_1(t) < t$, *and* $\lim_{t\to\infty} \sigma_1(t) = \infty$. *Suppose further that*

$$\int_{t_0}^{\infty} q(t)f(t)\xi^{\theta}(\sigma_2(t))\mathrm{d}t = \infty \tag{3.2}$$

holds for some $\theta \in \mathfrak{R}$ with $\theta \geq \gamma$ and $\theta > \alpha$, and for some function $\sigma_2 \in \mathrm{C}([t_0, \infty), \mathbb{R})$ with $\sigma_2(t) > t$. Then, (1.2) is oscillatory.

Proof Suppose to the contrary that x is a nonoscillatory solution of (1.2). Without loss of generality, we may assume that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all large t. From (1.2), one can easily obtain that there exists a $t_1 \geq t_0$ such that either

$$z(t) > 0, \quad z'(t) > 0, \quad (r(z')^\alpha)'(t) \leq 0, \tag{3.3}$$

or

$$z(t) > 0, \quad z'(t) < 0, \quad (r(z')^\alpha)'(t) \leq 0 \tag{3.4}$$

for $t \geq t_1$.

Suppose first (3.3). Then, we have

$$x(t) = z(t) - p(t)x(\tau(t)) \geq z(t) - p(t)z(\tau(t)) \geq (1 - p(t))\, z(t) \tag{3.5}$$

and

$$z(t) \geq (r^{1/\alpha} z')(t) \int_{t_1}^{t} r^{-1/\alpha}(s)\mathrm{d}s. \tag{3.6}$$

It follows from (1.2) and (3.5) that

$$(r(z')^\alpha)'(t) + q(t)\,(1 - p(\sigma(t)))^\gamma\, z^\beta(\sigma(t)) z^{\gamma-\beta}(\sigma(t)) \leq 0.$$

By virtue of $z' > 0$ and $\beta < \alpha$, there exists a constant $c_1 > 0$ such that

$$(r(z')^\alpha)'(t) + c_1 q(t)\,(1 - p(\sigma(t)))^\gamma\, z^\beta(\sigma_1(t)) \leq 0. \tag{3.7}$$

Letting $y := r(z')^\alpha$ and using (3.6) and (3.7), we have

$$y'(t) + c_1 q(t)\,(1 - p(\sigma(t)))^\gamma \left(\int_{t_1}^{\sigma_1(t)} r^{-1/\alpha}(s)\mathrm{d}s \right)^\beta y^{\beta/\alpha}(\sigma_1(t)) \leq 0.$$

By Lemma 2.1, we obtain that the delay differential equation

$$y'(t) + c_1 q(t)\,(1 - p(\sigma(t)))^\gamma \left(\int_{t_1}^{\sigma_1(t)} r^{-1/\alpha}(s)\mathrm{d}s \right)^\beta y^{\beta/\alpha}(\sigma_1(t)) = 0$$

also has positive solutions. Using Lemma 2.2 and condition (3.1), one can obtain that the above equation is oscillatory, which is a contradiction.

Suppose now (3.4). From $(r(z')^\alpha)' \leq 0$, we obtain that $r(z')^\alpha$ is nonincreasing. Hence, we have

$$r^{1/\alpha}(s)z'(s) \leq r^{1/\alpha}(t)z'(t), \quad s \geq t. \tag{3.8}$$

Dividing (3.8) by $r^{1/\alpha}(s)$ and integrating the resulting inequality from t to l, we obtain

$$z(l) \leq z(t) + r^{1/\alpha}(t)z'(t) \int_{t}^{l} r^{-1/\alpha}(s)\mathrm{d}s, \quad l \geq t.$$

Letting $l \to \infty$ in the above inequality, we obtain

$$0 \leq z(t) + r^{1/\alpha}(t)z'(t)\xi(t),$$

i.e.,

$$z(t) \geq -\xi(t)r^{1/\alpha}(t)z'(t). \tag{3.9}$$

From (3.9), we have

$$\left(\frac{z}{\xi}\right)' \geq \frac{z'\xi + \xi r^{1/\alpha}z'\xi'}{\xi^2} = \frac{z'(1+\xi' r^{1/\alpha})}{\xi} = 0. \tag{3.10}$$

Thus, we get by (3.10) that

$$x(t) = z(t) - p(t)x(\tau(t)) \geq z(t) - p(t)z(\tau(t)) \geq \left(1 - p(t)\frac{\xi(\tau(t))}{\xi(t)}\right)z(t). \tag{3.11}$$

It follows from (1.2) that

$$(r(z')^\alpha)'(t) + q(t)f(t)z^\gamma(\sigma(t)) \leq 0, \tag{3.12}$$

which yields

$$(r(z')^\alpha)'(t) + q(t)f(t)z^\gamma(\sigma_2(t)) \leq 0.$$

Writing the latter inequality in the form

$$(r(z')^\alpha)'(t) + q(t)f(t)z^\theta(\sigma_2(t))z^{\gamma-\theta}(\sigma_2(t)) \leq 0.$$

By virtue of $z' < 0$ and $\theta \geq \gamma$, there exists a constant $c_2 > 0$ such that

$$(r(z')^\alpha)'(t) + c_2 q(t)f(t)z^\theta(\sigma_2(t)) \leq 0. \tag{3.13}$$

Letting $u := r(z')^\alpha$ and using (3.9) and (3.13), we obtain

$$u'(t) - c_2 q(t)f(t)\xi^\theta(\sigma_2(t))u^{\theta/\alpha}(\sigma_2(t)) \leq 0.$$

That is, $y := -u$ is a positive solution of inequality

$$y'(t) - c_2 q(t)f(t)\xi^\theta(\sigma_2(t))y^{\theta/\alpha}(\sigma_2(t)) \geq 0.$$

Then, we obtain by Lemma 2.3 that the advanced differential equation

$$y'(t) - c_2 q(t)f(t)\xi^\theta(\sigma_2(t))y^{\theta/\alpha}(\sigma_2(t)) = 0$$

also has positive solutions. Applications of Lemma 2.4 and condition (3.2) yield a contradiction. The proof is complete. □

Example 3.2 For $t \geq 4$, consider the second-order superlinear Emden–Fowler neutral delay differential equation

$$\left(t^2\left(x(t) + \frac{1}{2}x(t-1)\right)'\right)' + t^2x^3(t-2) = 0. \tag{3.14}$$

Let $\sigma_1(t) = t - 2$, $\beta = 1/3$, $\sigma_2(t) = t + 1$, $\theta = 3$. Then, condition (3.1) is satisfied. Note that $\xi(t) = t^{-1}$ and

$$\int_{t_0}^{\infty} q(t)f(t)\xi^{\theta}(\sigma_2(t))\mathrm{d}t = \int_4^{\infty} t^2\left(1 - \frac{t-2}{2(t-3)}\right)^3 (t+1)^{-3}\mathrm{d}t = \infty.$$

An application of Theorem 3.1 yields oscillation of Eq. (3.14). Theorem 1.1 fails to apply in (3.14) because, for any $L \in (0, 9\sqrt{2}/4)$,

$$\begin{aligned}
&\int_4^{\infty}\left[q(t)\left(\frac{1}{1+p(\rho(t))}\right)^{\gamma}\delta^{\gamma}(t) - \frac{\gamma\rho'(t)}{L^{\gamma-1}\delta(t)r(\rho(t))}\right]\mathrm{d}t \\
&= \int_4^{\infty}\left[\left(\frac{2}{3}\right)^3 \rho^{-3}(t)t^2 - \frac{3\rho'(t)}{L^2\rho(t)}\right]\mathrm{d}t \\
&\le \int_4^{\infty}\left[\left(\frac{2}{3}\right)^3 t^{-1} - \frac{3\rho'(t)}{L^2\rho(t)}\right]\mathrm{d}t \\
&= \lim_{s\to\infty}\int_4^{s}\left[\left(\frac{2}{3}\right)^3 t^{-1} - \frac{3\rho'(t)}{L^2\rho(t)}\right]\mathrm{d}t \\
&= \lim_{s\to\infty}\left[\left(\frac{2}{3}\right)^3 \ln s - \frac{3}{L^2}\ln\rho(s) - \left(\frac{2}{3}\right)^3\ln 4 + \frac{3}{L^2}\ln\rho(4)\right] \\
&\le \lim_{s\to\infty}\left[\left(\left(\frac{2}{3}\right)^3 - \frac{3}{L^2}\right)\ln s - \left(\frac{2}{3}\right)^3\ln 4 + \frac{3}{L^2}\ln\rho(4)\right] = -\infty.
\end{aligned}$$

It may well happen that condition (3.2) of Theorem 3.1 is not satisfied, in which case the following result proves to be useful.

Theorem 3.3 *Assume* (H_1)–(H_3), (1.4), (3.1), *and* $\gamma \ge \alpha$. *If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[(M\xi(\sigma(s)))^{\gamma-\alpha}q(s)f(s)\xi^{\alpha}(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{\xi(s)r^{1/\alpha}(s)}\right]\mathrm{d}s = \infty \tag{3.15}$$

holds for all constants $M > 0$, *then* (1.2) *is oscillatory.*

Proof We proceed as in the proof of Theorem 3.1, assuming, without loss of generality, that there exists a solution x of (1.2) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all large t. Then, there exists a $t_1 \ge t_0$ such that either (3.3) or (3.4) holds for all $t \ge t_1$. One can obtain a contradiction to (3.1) when (3.3) holds. Assume now (3.4). Define the function w by

$$w(t) := \frac{r(t)(z')^{\alpha}(t)}{z^{\alpha}(t)}, \quad t \ge t_1. \tag{3.16}$$

Then, $w(t) < 0$ for $t \geq t_1$. From the proof of Theorem 3.1, we get (3.9), (3.10), and (3.12). Hence, by (3.9) and (3.16), we have

$$-w(t)\xi^{\alpha}(t) \leq 1. \tag{3.17}$$

Differentiating (3.16), we have

$$w'(t) = \frac{(r(z')^{\alpha})'(t)}{z^{\alpha}(t)} - \alpha\frac{r(t)(z')^{\alpha+1}(t)}{z^{\alpha+1}(t)}.$$

It follows from (3.12) and (3.16) that

$$w'(t) \leq -q(t)f(t)\frac{z^{\gamma}(\sigma(t))}{z^{\alpha}(t)} - \alpha\frac{w^{(\alpha+1)/\alpha}(t)}{r^{1/\alpha}(t)}. \tag{3.18}$$

Then, we obtain by (3.10) and (3.18) that there exists a constant $M > 0$ such that

$$\begin{aligned} w'(t) &\leq -q(t)f(t)z^{\gamma-\alpha}(\sigma(t))\frac{z^{\alpha}(\sigma(t))}{z^{\alpha}(t)} - \alpha\frac{w^{(\alpha+1)/\alpha}(t)}{r^{1/\alpha}(t)} \\ &\leq -(M\xi(\sigma(t)))^{\gamma-\alpha}q(t)f(t) - \alpha\frac{w^{(\alpha+1)/\alpha}(t)}{r^{1/\alpha}(t)}. \end{aligned} \tag{3.19}$$

Multiplying (3.19) by $\xi^{\alpha}(t)$ and integrating the resulting inequality from t_1 to t, we have

$$\xi^{\alpha}(t)w(t) - \xi^{\alpha}(t_1)w(t_1) + \alpha\int_{t_1}^{t} r^{-1/\alpha}(s)\xi^{\alpha-1}(s)w(s)\mathrm{d}s$$
$$+\int_{t_1}^{t}(M\xi(\sigma(s)))^{\gamma-\alpha}q(s)f(s)\xi^{\alpha}(s)\mathrm{d}s + \alpha\int_{t_1}^{t}\frac{w^{(\alpha+1)/\alpha}(s)}{r^{1/\alpha}(s)}\xi^{\alpha}(s)\mathrm{d}s \leq 0.$$

Set $B := r^{-1/\alpha}(s)\xi^{\alpha-1}(s)$, $A := \xi^{\alpha}(s)/r^{1/\alpha}(s)$, and $v := -w(s)$. Using (3.17) and the inequality (see [37,38])

$$Av^{(\alpha+1)/\alpha} - Bv \geq -\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \quad A > 0,$$

we have

$$\int_{t_1}^{t}\left[(M\xi(\sigma(s)))^{\gamma-\alpha}q(s)f(s)\xi^{\alpha}(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{\xi(s)r^{1/\alpha}(s)}\right]\mathrm{d}s \leq \xi^{\alpha}(t_1)w(t_1) + 1,$$

which contradicts (3.15). This completes the proof. □

Example 3.4 For $t \geq 1$, consider the second-order superlinear Emden–Fowler neutral delay differential equation

$$\left(\mathrm{e}^{t}\left(x(t) + \frac{1}{2\mathrm{e}}x(t-1)\right)'\right)' + t\mathrm{e}^{2t}x^{3}\left(\frac{t}{2}\right) = 0. \tag{3.20}$$

Let $\sigma_1(t) = t/2$ and $\beta = 1/3$. Then, condition (3.1) is satisfied. Further, $\xi(t) = \mathrm{e}^{-t}$ and for all constants $M > 0$,

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[(M\xi(\sigma(s)))^{\gamma-\alpha} q(s) f(s) \xi^\alpha(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{1}{\xi(s) r^{1/\alpha}(s)} \right] \mathrm{d}s$$

$$= \limsup_{t\to\infty} \int_{1}^{t} \left[\frac{M^2}{8} s - \frac{1}{4} \right] \mathrm{d}s = \infty.$$

An application of Theorem 3.3 yields oscillation of Eq. (3.20). Theorem 1.1 cannot be applied to (3.20) because, for any $L > 0$,

$$\int_{1}^{\infty} \left[q(t) \left(\frac{1}{1+p(\rho(t))} \right)^\gamma \delta^\gamma(t) - \frac{\gamma\rho'(t)}{L^{\gamma-1}\delta(t) r(\rho(t))} \right] \mathrm{d}t$$

$$= \int_{1}^{\infty} \left[\left(\frac{2\mathrm{e}}{2\mathrm{e}+1} \right)^3 \mathrm{e}^{-3\rho(t)} t \mathrm{e}^{2t} - \frac{3\rho'(t)}{L^2} \right] \mathrm{d}t$$

$$\le \int_{1}^{\infty} \left[\left(\frac{2\mathrm{e}}{2\mathrm{e}+1} \right)^3 t\mathrm{e}^{-t} - \frac{3\rho'(t)}{L^2} \right] \mathrm{d}t < \infty.$$

Theorem 3.1 also fails to apply in (3.20) since

$$\int_{t_0}^{\infty} q(t) f(t) \xi^\theta(\sigma_2(t)) \mathrm{d}t = \int_{1}^{\infty} \frac{t}{8} \mathrm{e}^{2t} \mathrm{e}^{-\theta\sigma_2(t)} \mathrm{d}t \le \int_{1}^{\infty} \frac{t}{8} \mathrm{e}^{2t} \mathrm{e}^{-3t} \mathrm{d}t < \infty.$$

Example 3.5 For $t \ge 1$, consider the second-order half-linear neutral delay differential equation

$$\left(t^6 \left(\left(x(t) + \frac{1}{4} x\left(\frac{t}{2}\right) \right)' \right)^3 \right)' + q_0 t^2 x^3 \left(\frac{t}{3} \right) = 0, \tag{3.21}$$

where $q_0 > 0$ is a constant. Let $\sigma_1(t) = t/4$ and $\beta = 1/3$. Then, condition (3.1) holds. Moreover, $\xi(t) = t^{-1}$ and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[(M\xi(\sigma(s)))^{\gamma-\alpha} q(s) f(s) \xi^\alpha(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{1}{\xi(s) r^{1/\alpha}(s)} \right] \mathrm{d}s$$

$$= \left[\frac{q_0}{8} - \left(\frac{3}{4}\right)^4 \right] \limsup_{t\to\infty} \int_{1}^{t} \frac{\mathrm{d}s}{s} = \infty,$$

if $q_0 > 81/32$. An application of Theorem 3.3 yields oscillation of Eq. (3.21) when $q_0 > 81/32$. Using Theorem 1.3, it is not difficult to see that Eq. (3.21) is oscillatory if $q_0 > 2673/512$. Hence, Theorem 3.3 improves Theorem 1.3 sufficiently.

Theorem 3.6 *Let (H_1)–(H_3), (1.4), and $\gamma < \alpha$ hold. Assume that*

$$\int_{t_2}^{\infty} q(t)\,(1 - p(\sigma(t)))^{\gamma} \left(\int_{t_1}^{\sigma_1(t)} r^{-1/\alpha}(s)\mathrm{d}s \right)^{\gamma} \mathrm{d}t = \infty \tag{3.22}$$

holds for all sufficiently large $t_1 \geq t_0$, for some $t_2 > t_1$, and for some function $\sigma_1 \in \mathrm{C}([t_0, \infty), \mathbb{R})$ with $\sigma_1(t) \leq \sigma(t)$, $\sigma_1(t) < t$, and $\lim_{t\to\infty} \sigma_1(t) = \infty$. Suppose also that

$$\int_{t_0}^{\infty} q(t) f(t) \xi^{\beta}(\sigma_2(t))\mathrm{d}t = \infty \tag{3.23}$$

holds for some $\beta \in \Re$ with $\beta > \alpha$ and for some function $\sigma_2 \in \mathrm{C}([t_0, \infty), \mathbb{R})$ with $\sigma_2(t) > t$. Then, (1.2) is oscillatory.

Proof Suppose to the contrary that x is a nonoscillatory solution of (1.2). Without loss of generality, we may assume that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all large t. From (1.2), we can easily obtain that there exists a $t_1 \geq t_0$ such that either (3.3) or (3.4) holds for all $t \geq t_1$.

Suppose first (3.3). Similar as in the proof of Theorem 3.1, we obtain that the delay differential equation

$$y'(t) + q(t)\,(1 - p(\sigma(t)))^{\gamma} \left(\int_{t_1}^{\sigma_1(t)} r^{-1/\alpha}(s)\mathrm{d}s \right)^{\gamma} y^{\gamma/\alpha}(\sigma_1(t)) = 0$$

has positive solutions. Using Lemma 2.2 and condition (3.22), one can obtain a contradiction.

Suppose now (3.4). Proceeding as in the proof of Theorem 3.1, we have (3.12). That is,

$$(r(z')^{\alpha})'(t) + q(t) f(t) z^{\beta}(\sigma(t)) z^{\gamma-\beta}(\sigma(t)) \leq 0. \tag{3.24}$$

Since $z' < 0$, there exist a $t_2 \geq t_1$ and a constant $k > 0$ such that $z(t) \leq k$ for $t \geq t_2$. Hence, by (3.24) and $\sigma_2(t) > t$, we find

$$(r(z')^{\alpha})'(t) + k^{\gamma-\beta} q(t) f(t) z^{\beta}(\sigma_2(t)) \leq 0.$$

Similar as in the proof of Theorem 3.1, we see that the advanced differential equation

$$y'(t) - k^{\gamma-\beta} q(t) f(t) \xi^{\beta}(\sigma_2(t)) y^{\beta/\alpha}(\sigma_2(t)) = 0$$

also has positive solutions. Applications of Lemma 2.4 and condition (3.23) yield a contradiction. This completes the proof. □

It may well happen that condition (3.23) of Theorem 3.6 is not satisfied, in which case the following result proves to be useful.

Theorem 3.7 *Assume (H_1)–(H_3), (1.4), (3.22), and $\gamma < \alpha = 1$. If*

$$\int_{t_1}^{\infty} \frac{\int_{t_1}^{s} q(v) f(v)\mathrm{d}v}{r(s)} \mathrm{d}s = \infty \tag{3.25}$$

holds for all sufficiently large $t_1 \geq t_0$, then (1.2) is oscillatory.

Proof We proceed as in the proof of Theorem 3.6, assuming, without loss of generality, that there exists a solution x of (1.2) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all large t. Then, there exists a $t_1 \geq t_0$ such that either (3.3) or (3.4) holds for all $t \geq t_1$. One can obtain a contradiction to (3.22) when (3.3) holds. Assume now (3.4). Then, we have (3.11) when using the proof of Theorem 3.1. Writing (1.2) in the form

$$(rz')'(t) + q(t)x^\gamma(\sigma(t)) = 0.$$

Integrating this equation from t_1 to s, we get

$$r(s)z'(s) - r(t_1)z'(t_1) + \int_{t_1}^{s} q(v)x^\gamma(\sigma(v))\mathrm{d}v = 0.$$

That is,

$$z'(s) - \frac{r(t_1)z'(t_1)}{r(s)} + \frac{\int_{t_1}^{s} q(v)x^\gamma(\sigma(v))\mathrm{d}v}{r(s)} = 0.$$

Integrating again from t_1 to t, we find

$$z(t) - z(t_1) - r(t_1)z'(t_1)\int_{t_1}^{t} r^{-1}(s)\mathrm{d}s + \int_{t_1}^{t} \frac{\int_{t_1}^{s} q(v)x^\gamma(\sigma(v))\mathrm{d}v}{r(s)}\mathrm{d}s = 0.$$

The latter equality and (3.11) yield

$$\begin{aligned} z'(t) &= \frac{r(t_1)z'(t_1)}{r(t)} - \frac{\int_{t_1}^{t} q(v)x^\gamma(\sigma(v))\mathrm{d}v}{r(t)} \\ &\leq -\frac{\int_{t_1}^{t} q(v)f(v)z^\gamma(\sigma(v))\mathrm{d}v}{r(t)} \leq -\frac{\int_{t_1}^{t} q(v)f(v)\mathrm{d}v}{r(t)}z^\gamma(t), \end{aligned}$$

which implies that

$$\frac{\int_{t_1}^{t} q(v)f(v)\mathrm{d}v}{r(t)} \leq -\frac{z'(t)}{z^\gamma(t)} = -\frac{(z^{1-\gamma})'(t)}{1-\gamma}.$$

Integrating the last inequality from t_1 to t, we obtain

$$\int_{t_1}^{t} \frac{\int_{t_1}^{s} q(v)f(v)\mathrm{d}v}{r(s)}\mathrm{d}s \leq -\frac{z^{1-\gamma}(t)}{1-\gamma} + \frac{z^{1-\gamma}(t_1)}{1-\gamma} \leq \frac{z^{1-\gamma}(t_1)}{1-\gamma},$$

which contradicts (3.25). The proof is complete. □

Example 3.8 For $t \geq 4$, consider the second-order sublinear Emden–Fowler neutral delay differential equation

$$\left(t^2\left(x(t) + \frac{1}{2}x(t-1)\right)'\right)' + q_0 x^{1/3}(t-2) = 0, \tag{3.26}$$

where $q_0 > 0$ is a constant. It is not difficult to verify that all conditions of Theorem 3.7 are satisfied. Hence, Eq. (3.26) is oscillatory. Note that Theorem 3.6 cannot be applied to (3.26) since condition (3.23) does not hold for this equation (due to $\int_{t_0}^{\infty} s^{-\beta}\mathrm{d}s < \infty$ in the case $\beta > 1$).

Example 3.9 For $t \geq 1$, consider the second-order sublinear Emden–Fowler delay differential equation

$$\left(t^2 x'(t)\right)' + q_0 x^{1/3}(\sigma(t)) = 0, \tag{3.27}$$

where $q_0 > 0$ is a constant. It is easy to see that all conditions of Theorem 3.7 are satisfied. Hence, Eq. (3.27) is oscillatory. Note that Theorem 1.2 cannot be applied to (3.27) since condition (1.11) does not hold for this equation (due to the arbitrariness in the choice of M).

4 Conclusions

In this paper, we suggest four new oscillation criteria for the neutral differential equation (1.2) without requiring conditions (1.3), (1.5), and (1.7). These results are of independent interest (note that Theorem 3.3 cannot be applied to Eq. (3.14) due to the arbitrary choice of M). Example 3.9 and Example 3.8 show that the Emden–Fowler delay differential equation

$$x''(t) + \frac{2}{t}x'(t) + q_0 t^{-2} x^{1/3}(t-2) = 0, \quad q_0 > 0$$

and the Emden–Fowler neutral delay differential equation

$$x''(t) + \frac{1}{2}x''(t-1) + \frac{2}{t}x'(t) + \frac{x'(t-1)}{t} + q_0 t^{-2} x^{1/3}(t-2) = 0, \quad q_0 > 0$$

are oscillatory, respectively. Note that [26, Theorem 11.3 and Theorem 11.4] fail to apply in these equations due to the existence of deviating arguments and neutral term.

It is well known [10] that the presence of a neutral term in a differential equation can cause oscillation, but it can also destroy oscillatory properties of a differential equation. For example, using Theorem 3.3, the second-order ordinary differential equation

$$\left(e^{2t} x'(t)\right)' + \left(e^{2t} + \frac{e^{2t+2}}{2}\right) x(t) = 0$$

and the second-order neutral delay differential equation

$$\left(e^{2t}\left(x(t) + \frac{1}{2e^4}x(t-2)\right)'\right)' + \left(e^{2t} + \frac{e^{2t+2}}{2}\right) x(t) = 0$$

are oscillatory (note that results in [13] and [30] cannot be applied to this neutral equation due to restrictive conditions (1.7) and (1.13)). However, the second-order neutral delay differential equation

$$\left(e^{2t}\left(x(t) + \frac{1}{2}x(t-2)\right)'\right)' + \left(e^{2t} + \frac{e^{2t+2}}{2}\right) x(t) = 0$$

has a nonoscillatory solution $x(t) = e^{-t}$. This phenomenon is caused by the different choices of neutral term.

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