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Quaternion geometries on the twistor space of the six-sphere

Francisco Martín Cabrera · Andrew Swann

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Abstract We explicitly describe all $SO(7)$-invariant almost quaternion-Hermitian structures on the twistor space of the six-sphere and determine the types of their intrinsic torsion.

Keywords Twistor space · Almost quaternion-Hermitian structure · Homogeneous space · Canonical variation · Intrinsic torsion

Mathematics Subject Classification (2000) Primary 53C26; Secondary 53C10 · 53C30

1 Introduction

Recently, Moroianu, Pilca and Semmelmann [5] found that the twistor space $M = SO(7)/U(3)$ of the six-sphere S^6 admits a homogeneous almost quaternion-Hermitian structure. This arose as part of their striking result that M is the only such homogeneous space with non-zero Euler characteristic that is neither quaternionic Kähler (the quaternionic symmetric spaces of Wolf [9]) nor $S^2 \times S^2$.

In this paper, we show that there is exactly a one-dimensional family of invariant almost quaternion-Hermitian structures on M, with fixed volume, and determine the types of their intrinsic torsion. We will see that the family contains inequivalent structures and includes

F. M. Cabrera
Department of Fundamental Mathematics,
University of La Laguna, 38200 La Laguna, Tenerife, Spain
e-mail: fmartin@ull.es

A. F. Swann (✉)
Department of Mathematics,
Aarhus University, Ny Munkegade 118, Bldg 1530, 8000 Aarhus C, Denmark
e-mail: swann@imf.au.dk

A. F. Swann
CP3-Origins Centre of Excellence for Cosmology and Particle Physics Phenomenology,
University of Southern Denmark, Campusvej 55, 5230 Odense M, Denmark

the symmetric Kähler metric of the quadric $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^6) = SO(8)/SO(2)SO(6)$. Each member of the family will be shown to have almost quaternion-Hermitian type $\Lambda_0^3 E(S^3H + H)$ with the first component non-zero, confirming that they are not quaternionic Kähler; one member of the family has pure type $\Lambda_0^3 ES^3H$, and this is the first known example of such a geometry. However, the structure singled out by this almost quaternionic-Hermitian intrinsic torsion is not the Kähler metric of the quadric nor the squashed Einstein metric in the canonical variation.

2 Invariant forms

The subgroup $U(3)$ of $SO(7)$ arises from a choice of identification of $\mathbb{R}^7$ as $\mathbb{R}\oplus\mathbb{C}^3$. Regarding $U(3)$ as $U(1)SU(3)$, we may write $\mathbb{C}^3 = \mathbb{R}^6 = [\![L\lambda^{1,0}]\!]$, meaning that $\mathbb{R}^6 \otimes \mathbb{C} = L\lambda^{1,0} + \overline{L\lambda^{1,0}} \cong L\lambda^{1,0} + L^{-1}\lambda^{0,1}$, where $L = \mathbb{C}$ and $\lambda^{1,0} = \mathbb{C}^3$ as the standard representations of $U(1)$ and $SU(3)$, respectively. We thus have $U(3) \leqslant SO(6) \leqslant SO(7)$, so $M = SO(7)/U(3)$ fibres over $S^6 = SO(7)/SO(6)$ with fibre $SO(6)/U(3)$, the almost complex structures on T_xS^6. Thus, M is the (Riemannian) twistor space of S^6.

Since $\lambda^{3,0} = \Lambda^3\lambda^{1,0} = \mathbb{C}$ is trivial, we have $\lambda^{2,0} \cong \lambda^{0,1}$ as $SU(3)$-modules. The Lie algebra of $SO(7)$ now decomposes as

$$\begin{aligned}\mathfrak{so}(7) &= \Lambda^2\mathbb{R}^7 = \Lambda^2(\mathbb{R} + [\![L\lambda^{1,0}]\!]) = [\![L\lambda^{1,0}]\!] + [\![L^2\lambda^{2,0}]\!] + [\lambda^{1,1}]\\ &\cong [\![L\lambda^{1,0}]\!] + [\![L^2\lambda^{0,1}]\!] + \mathfrak{u}(1) + \mathfrak{su}(3).\end{aligned}$$

Here, $[\lambda^{1,1}]$ is the real module whose complexification is $\lambda^{1,1} = \lambda^{1,0}\otimes\lambda^{0,1}$; it splits in to two irreducible modules $[\lambda_0^{1,1}] \cong \mathfrak{su}(3)$ and $\mathbb{R} = \mathfrak{u}(1)$.

We thus have that the complexified tangent space of $M = SO(7)/U(3)$ is the bundle associated with

$$\begin{aligned}T\otimes\mathbb{C} &= \big([\![L\lambda^{1,0}]\!] + [\![L^2\lambda^{0,1}]\!]\big)\otimes\mathbb{C}\\ &= L\lambda^{1,0} + L^{-1}\lambda^{0,1} + L^2\lambda^{0,1} + L^{-2}\lambda^{1,0}\\ &= (L^{1/2}\lambda^{0,1} + L^{-1/2}\lambda^{1,0})(L^{3/2} + L^{-3/2}). \qquad (2.1)\end{aligned}$$

This allows us to write $T\otimes\mathbb{C} = EH$, where $E = L^{1/2}\lambda^{0,1} + L^{-1/2}\lambda^{1,0}$ and $H = L^{3/2} + L^{-3/2}$ are representations of $U(1)_2 \times SU(3)$ as a subgroup of $U(1)_L SU(3) \times U(1)_R \leqslant Sp(3)\times Sp(1)$. Here, $U(1)_2$ is a double cover of $U(1)$ and is included in $U(1)_L \times U(1)_R$ via the map $e^{i\theta} \mapsto (e^{-i\theta}, e^{3i\theta})$. In this way, we see that $M = SO(7)/U(3)$ carries an invariant $Sp(3)Sp(1)$-structure, where $Sp(3)Sp(1) = (Sp(3)\times Sp(1))/\{\pm(1,1)\}$. This is the G-structure description of an almost quaternion-Hermitian structure.

Geometrically, an almost quaternion-Hermitian structure is specified by a Riemannian metric g and a three-dimensional subbundle $\mathcal{G}$ of End(TM) which locally has a basis $I,\ J,\ K$ satisfying the quaternion identities

$$I^2 = -1 = J^2, \qquad IJ = K = -JI$$

and the compatibility conditions

$$g(I\cdot, I\cdot) = g(\cdot,\cdot) = g(J\cdot, J\cdot).$$

There are then local two-forms

$$\begin{aligned}\omega_I(X,Y) = g(X, IY), \quad \omega_J(X,Y) = g(X, JY),\\ \omega_K(X,Y) = g(X, KY)\end{aligned}$$

and with the local form $\omega_c = \omega_J + i\omega_K$ of type (2,0) with respect to I. Since they are non-degenerate, the local forms $\omega_I,\ \omega_J,\ \omega_K$ are sufficient to determine the local almost complex structures $I,\ J$ and K and the metric g.

Equation (2.1) show us that T has two inequivalent irreducible summands $[\![L\lambda^{1,0}]\!]$ and $[\![L^2\lambda^{0,1}]\!]$ and so there are two invariant forms ω_0 and $\tilde{\omega}_0$ spanning $\Omega^2(M)^{SO(7)}$. However, we have that

$$\begin{aligned}\Lambda^2 T &= \Lambda^2[\![L\lambda^{1,0}]\!] + \Lambda^2[\![L^2\lambda^{0,1}]\!] + [\![L\lambda^{1,0}]\!] \wedge [\![L^2\lambda^{0,1}]\!]\\ &= (\mathbb{R}\omega_0 + [\lambda_0^{1,1}] + [\![L^2\lambda^{0,1}]\!]) + (\mathbb{R}\tilde{\omega}_0 + [\lambda_0^{1,1}] + [\![L^4\lambda^{1,0}]\!])\\ &\quad +([\![L^3]\!] + [\![L^3]\!][\lambda_0^{1,1}] + [\![L\lambda^{1,0}]\!] + [\![L\sigma^{0,2}]\!]), \end{aligned} \tag{2.2}$$

where $\sigma^{0,2} = S^2\lambda^{0,1}$. There is thus an addition two-dimensional subspace $[\![L^3]\!]$ preserved by the $SU(3)$-action. This space is spanned by local $SU(3)$-invariant forms ω_J and ω_K that are mixed under the $U(1)$-action, so that $\omega_c = \omega_J + i\omega_K$ is a basis element of L^3. We may now consider the triple of forms

$$\omega_I = \lambda\omega_0 + \mu\tilde{\omega}_0, \quad \omega_J \quad \text{and} \quad \omega_K \tag{2.3}$$

which will be seen to result in an almost quaternion-Hermitian structure when

$$20\lambda^3\mu^3(\omega_0)^3(\tilde{\omega}_0)^3 = (\omega_J)^6. \tag{2.4}$$

This equation is necessary, as each two-form in the triple must define the same volume element.

We note that for an almost quaternion-Hermitian structure the four-form $\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2$ is globally defined. For an invariant structure, this form must lie in $\Omega^4(M)^{SO(7)}$ which in our particular case is four-dimensional. Indeed, the complete decomposition of $\Lambda^4 T$ in to irreducible $U(3)$-modules is

$$\begin{aligned}\Lambda^4 T = {} & [\![L^6]\!] + 2[\![L^3]\!] + 4\mathbb{R} + [\![L^7\lambda^{1,0}]\!] + 3[\![L^4\lambda^{1,0}]\!] + 5[\![L\lambda^{1,0}]\!]\\ & +4[\![L^2\lambda^{0,1}]\!] + 2[\![L^5\lambda^{0,1}]\!] + 2[\![L^2\sigma^{2,0}]\!] + 2[\![L\sigma^{0,2}]\!] + [\![L^4\sigma^{0,2}]\!]\\ & +[\![L^3\sigma^{3,0}]\!] + [\![\sigma^{3,0}]\!] + [\![L^3\sigma^{0,3}]\!] + [\![L^6\lambda_0^{1,1}]\!] + 4[\![L^3\lambda_0^{1,1}]\!] + 6[\lambda_0^{1,1}]\\ & +[\![L^4\sigma_0^{2,1}]\!] + 2[\![L^2\sigma_0^{2,1}]\!] + [\![L^2\sigma_0^{1,2}]\!] + [\![\sigma_0^{2,2}]\!]. \end{aligned}$$

Now, the four-forms $\omega_0^2,\ \tilde{\omega}_0^2,\ \omega_0 \wedge \tilde{\omega}_0$ and $\omega_J^2 + \omega_K^2$ are invariant and linearly independent, so they provide a basis for $\Omega^4(M)^{SO(7)}$. It follows, Lemma 4.1 below, that any invariant almost hyperHermitian structure on M is described via the forms of (2.3).

3 Intrinsic torsion

Given an invariant almost Hermitian structure on M, there is a unique $Sp(3)Sp(1)$-connection ∇ characterised by the condition that the pointwise norm of its torsion is the least possible. More precisely, ∇ is related to the Levi-Civita connection by

$$\nabla = \nabla^{\mathrm{LC}} + \xi,$$

where ξ is the intrinsic torsion given [4] by

$$\xi_X Y = -\tfrac{1}{4} \sum_{A=I,J,K} A(\nabla_X^{\mathrm{LC}} A)Y + \tfrac{1}{2} \sum_{A=I,J,K} \lambda_A(X)AY,$$

with

$$6\lambda_I(X) = g(\nabla^{\mathrm{LC}}_X \omega_J, \omega_K),$$

etc. The tensor ξ takes values in

$$\mathcal{Q} = T^* \otimes (\mathfrak{sp}(3) + \mathfrak{sp}(1))^{\perp} \subset T^* \otimes \Lambda^2 T^*$$

where $\mathfrak{sp}(3) = [S^2 E]$ and $\mathfrak{sp}(1) = [S^2 H]$ are the Lie algebras of $Sp(3)$ and $Sp(1)$. Under the action of $Sp(3)Sp(1)$, the space $\mathcal{Q} \otimes \mathbb{C}$ decomposes as

$$\mathcal{Q} \otimes \mathbb{C} = (\Lambda^3_0 E + K + E)(S^3 H + H)$$

with $\Lambda^3_0 E$ and K irreducible $Sp(3)$-modules satisfying $\Lambda^3 E = \Lambda^3_0 E + E$ and $E \otimes S^2 E = S^3 E + K + E$. The space $\mathcal{Q}$ thus has six irreducible summands under $Sp(3)Sp(1)$.

For an invariant structure on $M = SO(7)/U(3)$, the intrinsic torsion lies in a $U(3)$-invariant submodule of $\mathcal{Q}$. As $\mathfrak{sp}(3) = [S^2(L^{1/2}\lambda^{0,1})] = [\![L\sigma^{0,2}]\!] + [\lambda^{1,1}_0] + \mathbb{R}$ and $\mathfrak{sp}(1) = [S^2(L^{3/2})] = [\![L^3]\!] + \mathbb{R}$, Eq. (2.2) implies that

$$(\mathfrak{sp}(3) + \mathfrak{sp}(1))^{\perp} \cong [\lambda^{1,1}_0] + [\![L^2\lambda^{0,1}]\!] + [\![L^4\lambda^{1,0}]\!] + [\![L^3]\!][\lambda^{1,1}_0] + [\![L\lambda^{1,0}]\!].$$

Comparing with Eq. (2.1), we see that $(\mathfrak{sp}(3)+\mathfrak{sp}(1))^{\perp}$ contains a unique copy of each of the irreducible summands of T, so $\mathcal{Q}^{U(3)}$ is two-dimensional. As $\Lambda^3(A + B) \cong \Lambda^3 A + \Lambda^2 A \otimes B + A \otimes \Lambda^2 B + \Lambda^3 B$, we find that

$$\Lambda^3_0 E = (L^{3/2} + L^{-3/2}) + (L^{1/2}\sigma^{2,0} + L^{-1/2}\sigma^{0,2}).$$

The first summand is a copy of H and is also a submodule of $S^3 H = L^{9/2} + L^{3/2} + L^{-3/2} + L^{-9/2}$. This shows that $[\Lambda^3_0 E S^3 H]^{U(3)}$ and $[\Lambda^3_0 E H]^{U(3)}$ are each one-dimensional, and so we have

$$\xi \in \mathcal{Q}^{U(3)} \subset [\Lambda^3_0\, E S^3 H] + [\Lambda^3_0\, E H]. \tag{3.1}$$

4 Explicit structures

We now wish to determine the components of ξ in each of the summands of (3.1). An invariant almost Hermitian structure on M may be described by two-forms as in (2.3). As ω_J and ω_K are only invariant under $SU(3)$, they do not define global forms on M. However, we do get two such invariant forms on the total space of the circle bundle $N = SO(7)/SU(3) \to M = SO(7)/U(3)$.

Let $0, 1, 2, 3, 1', 2', 3'$ be an orthonormal basis for $\mathbb{R}^7 = \mathbb{R} + \mathbb{C}^3$, with $0 \in \mathbb{R}$ and $i1 = 1'$, etc. Writing 12 for $1 \wedge 2$, a standard basis for $[\![L\lambda^{1,0}]\!] \subset \mathfrak{so}(7)$ is given by

$$A = 01, \quad B = 02, \quad C = 03, \quad A' = 01', \quad B' = 02', \quad C' = 03'$$

and a corresponding basis for $[\![L^2\lambda^{0,1}]\!]$ is

$$\begin{aligned} P &= 23 - 2'3', \quad & Q &= 31 - 3'1', \quad & R &= 12 - 1'2', \\ P' &= 23' - 32', \quad & Q' &= 31' - 13', \quad & R' &= 12' - 21'. \end{aligned}$$

We put $E = 11' + 22' + 33'$, and note that, this is a generator of the central $\mathfrak{u}(1)$ in $\mathfrak{u}(3)$. Then $\{E, A, \ldots, R'\}$ is a basis for $\mathfrak{n} = T_{\mathrm{Id}\,SU(3)}N$ and $\{A, \ldots, R'\}$ is a basis for $\mathfrak{m} = T_{\mathrm{Id}\,U(3)}M$. We use lower case letters to denote the corresponding dual bases of $\mathfrak{n}^*$ and $\mathfrak{m}^*$. These give left-invariant one-forms on $SO(7)$, with $da(X, Y) = -a([X, Y])$ for $X, Y \in \mathfrak{so}(7)$, etc. We write

$$d_N a = (\mathrm{da})|_{\Lambda^2\mathfrak{n}} \quad \text{and} \quad d_M a = (\mathrm{da})|_{\Lambda^2\mathfrak{m}}$$

at Id $\in SO(7)$. For a left-invariant form $\alpha \in \Omega^k(SO(7))$, we have at Id $\in SO(7)$ that $d\alpha = d_N\alpha$ if α is right $SU(3)$-invariant and $d\alpha = d_M\alpha$ if α is right $U(3)$-invariant. For our choice of bases, we have

$$\begin{aligned} d_M a &= -b \wedge r + c \wedge q - b' \wedge r' + c' \wedge q', \quad d_M p = -\tfrac{1}{2}(b \wedge c - b' \wedge c'), \\ d_M a' &= -b \wedge r' + c \wedge q' + b' \wedge r - c' \wedge q, \quad d_M p' = -\tfrac{1}{2}(b \wedge c' + b' \wedge c) \end{aligned}$$

with the other derivatives obtained by applying the cyclic permutation $(a, a', p, p') \to (b, b', q, q') \to (c, c', r, r') \to (a, a', p, p')$. We use $\mathfrak{S}$ to denote sums over this group of permutations.

The two-form ω_I of (2.3) is

$$\begin{aligned} \omega_I &= \lambda(a' \wedge a + b' \wedge b + c' \wedge c) + \mu(p' \wedge p + q' \wedge q + r' \wedge r) \\ &= \mathfrak{S}(\lambda a' \wedge a + \mu p' \wedge p). \end{aligned}$$

On N, we have the forms $\hat{\omega}_J$ and $\hat{\omega}_K$ given by

$$\hat{\omega}_J + i\hat{\omega}_K = \mathfrak{S}\left((p + ip') \wedge (a + ia')\right).$$

Choosing a local section s of $\pi : N \to M$ such that $s(\mathrm{Id}\, U(3)) = \mathrm{Id}\, SU(3)$ and $s^*e = 0$, we then obtain local two-forms

$$\omega_J = s^*\hat{\omega}_J, \quad \omega_K = s^*\hat{\omega}_K$$

completing the triple of (2.3). The corresponding metric on M is

$$g = \mathfrak{S}(\lambda(a^2 + a'^2) + \mu(p^2 + p'^2)) \tag{4.1}$$

and condition (2.4) is simply

$$\lambda\mu = 1. \tag{4.2}$$

These are the only invariant metrics on M with normalised volume form, since TM (2.1) has exactly two irreducible summands.

At Id $U(3)$, the almost complex structures satisfy

$$\begin{gathered} IA = A', \quad IP = P', \quad J\tfrac{1}{\sqrt{\lambda}}A = \tfrac{1}{\sqrt{\mu}}P, \quad J\tfrac{1}{\sqrt{\lambda}}A' = -\tfrac{1}{\sqrt{\mu}}P', \\ K\tfrac{1}{\sqrt{\lambda}}A = \tfrac{1}{\sqrt{\mu}}P', \quad K\tfrac{1}{\sqrt{\lambda}}A' = \tfrac{1}{\sqrt{\mu}}P. \end{gathered}$$

These act on forms via $Ia = -a(I\cdot)$, so with the normalisation condition (4.2), we have $Ja = \mu p$, $Jp = -\lambda a$, etc.

Lemma 4.1 *These describe all invariant almost quaternion-Hermitian structures on M with normalised volume form.*

Proof We have noted above that (4.1) gives all the invariant metrics. Now, the local almost complex structures, or equivalently their Hermitian two-forms, associated with the almost quaternion-Hermitian structure span a $U(3)$-invariant subspace V of $\Lambda^2 T$ of dimension 3. Counting dimensions in the decomposition (2.2) shows that V is a subspace of $\mathbb{R}\omega_0 + \mathbb{R}\tilde{\omega}_0 + [\![L^3]\!]$. In particular, $V \cap [\![L^3]\!]$ is at least one-dimensional; $U(3)$-invariance implies that $[\![L^3]\!] \leqslant V$. As ω_J and ω_K are g-orthogonal of the same length for each normalised g in (4.1), we see that J and K are local almost complex structures belonging to the almost quaternion-Hermitian geometry. Finally, $I = JK$ is specified too.

Lemma 4.2 *For the choices of ω_I, ω_J and ω_K above normalised by* (4.2) *we have at the base point* Id $U(3) \in M$ *that*

$$Id\omega_I = Id_M\omega_I = (\tfrac{1}{2}\mu - 2\lambda)\Phi,$$
$$Jd\omega_J = 2\lambda\Phi - \tfrac{1}{2}\mu^3\Psi, \quad Kd\omega_K = 2\lambda\Phi + \tfrac{1}{2}\mu^3\Psi,$$

where

$$\Phi = \mathfrak{S}(a \wedge b \wedge r - a' \wedge b' \wedge r + a \wedge b' \wedge r' + a' \wedge b \wedge r'),$$
$$\Psi = \mathfrak{S}(p \wedge q \wedge r - 3p \wedge q' \wedge r')$$

and $Ad\omega_A(\cdot,\cdot,\cdot) = -d\omega_A(A\cdot, A\cdot, A\cdot)$, for $A = I, J, K$.

Proof As ω_I is $U(3)$-invariant, we have $Id\omega_I = Id_M\omega_I$ which equals

$$(2\lambda - \tfrac{1}{2}\mu)I\mathfrak{S}(a \wedge b' \wedge r + a' \wedge b \wedge r - a \wedge b \wedge r' + a' \wedge b' \wedge r')$$

and gives the first claimed formula valid at any point of M.

For our choice of section s, we have at Id $U(3)$ that $Jd\,\omega_J = Js^*d_N\tilde{\omega}_J = Jd_M\tilde{\omega}_J$ which is

$$J\mathfrak{S}\left(-\tfrac{1}{2}a \wedge b \wedge c + \tfrac{3}{2}a \wedge b' \wedge c' + 2(a \wedge q \wedge r - a \wedge q' \wedge r' + a' \wedge q \wedge r' + a' \wedge q' \wedge r)\right).$$

Combined with the description of J, we thus get the claimed formula. The computation for $Kd\omega_K$ is similar.

To compute the intrinsic torsion, we use the "minimal description" of [4] which relies on computing the forms $\beta_I = Jd\omega_J + Kd\omega_K$, etc., and the contractions $\Lambda_A\beta_B$ of β_B with ω_A. For our structures, we have at the base point

$$\beta_I = 4\lambda\Phi, \quad \beta_J = \tfrac{1}{2}(\mu\Phi + \mu^3\Psi), \quad \beta_K = \tfrac{1}{2}(\mu\Phi - \mu^3\Psi)$$

and all contractions $\Lambda_A\beta_B = 0$. This confirms that the intrinsic torsion ξ has no components in $[E(S^3H + H)]$.

Theorem 4.3 *The component of ξ in $[\Lambda_0^3 ES^3H]$ is always non-zero, so the almost quaternion-Hermitian is never quaternionic. The component of ξ in $[\Lambda_0^3 EH]$ is zero if and only if $2\lambda = \mu$.*

Proof Since we have shown in §3 that ξ has no component in $[K(S^3H + H)]$ and we saw above that each one form $\Lambda_A\beta_B$ is zero, at the base point, the results of [4] show that the $\Lambda_0^3 ES^3H$-component of ξ corresponds to

$$\psi^{(3)} := \tfrac{1}{12}(\beta_I + \beta_J + \beta_K) = \tfrac{1}{12}(4\lambda + \mu)\Phi$$

which is always non-zero under condition (4.2). The component in $\Lambda_0^3 EH$ is determined by

$$\psi_I^{(3)} := \tfrac{1}{8}(-\beta_I + 2(3 + \mathcal{L}_I)\psi^{(3)}),$$

where $\mathcal{L}_I = I_{(12)} + I_{(13)} + I_{(23)}$, with $I_{(12)}\alpha = \alpha(I\cdot, I\cdot, \cdot)$, etc. Now $\mathcal{L}_I\Phi = \Phi$, so

$$\psi_I^{(3)} = \tfrac{1}{12}(\mu - 2\lambda)\Phi$$

and the result follows.

Corollary 4.4 *The invariant almost quaternion-Hermitian structures on M are not quaternionic integrable, and their quaternionic twistor spaces are not complex.*

Proof This follows directly from the following two facts [7]: (1) The underlying quaternionic structure is integrable if and only if the intrinsic torsion ξ has no S^3H component, i.e., it lies in $(\Lambda_0^3 E + K + E)H$. (2) The quaternionic twistor space is complex if and only if the underlying quaternionic structure is integrable. But, we have shown the $\Lambda_0^3\, ES^3H$-component of ξ is non-zero, so the result follows.

The almost Hermitian structure (g, ω_I) is easily seen to be integrable: $d_M(a + ia') = -(b - ib') \wedge (r + ir') + (c - ic') \wedge (q + iq') \in \Lambda_I^{1,1}$, $d_M(p + ip') = -\frac{1}{2}(b + ib') \wedge (c + ic') \in \Lambda_I^{2,0}$. In addition, from Lemma 4.2, we see that $d\omega_I$ is orthogonal to $\omega_I \wedge \Lambda^1$. It follows that $d\omega_I$ is primitive.

Now, recall that Gray and Hervella [3] showed that the intrinsic torsion of an almost Hermitian structure (g, ω) lies in

$$\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4 = [\![\Lambda^{3,0}]\!] + [\![U^{3,0}]\!] + [\![\Lambda_0^{2,1}]\!] + [\![\Lambda^{1,0}]\!],$$

with $U^{3,0}$ irreducible: the $\mathcal{W}_1 + \mathcal{W}_2$-part is determined by the Nijenhuis tensor; the $\mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_4$-part by $d\omega$. We now have from Lemma 4.2:

Proposition 4.5 *The Hermitian structure (g, ω_I, I) is of Gray-Hervella type $\mathcal{W}_3$, except when $4\lambda = \mu$, when it is Kähler. Furthermore, the Kähler metric is symmetric.*

Note that the Kähler parameters do not correspond to the parameters in Theorem 4.3 that give $\xi \in [\Lambda_0^3\, ES^3H]$.

Proof It remains to prove the last assertion. As in [8], note that $SO(7)/U(3) \cong SO(8)/U(6) \cong SO(8)/SO(2)SO(6)$, which is the quadric. The latter is isotropy irreducible and carries a unique $SO(8)$-invariant metric with fixed volume, which is Hermitian symmetric so Kähler. However, we have seen that there is a unique Kähler metric with the same volume invariant under the smaller group $SO(7)$, so these Kähler metrics must agree.

Remark 4.6 Each $SO(7)$-invariant metric g on M is given by (4.1) and so is a Riemannian submersion over $\mathbb{C}\mathrm{P}(3)$ with fibre S^6. The standard theory of the canonical variation [2] tell us that precisely two of these metrics are Einstein. One is the symmetric case $4\lambda = \mu$. The other is when $8\lambda = 3\mu$, as verified by Musso [6] in slightly different notation. Again these particular parameters are not those for which ξ is special.

Remark 4.7 It can be shown that the local almost Hermitian structures (g, ω_J, J) and (g, ω_K, K) above are each of strict Gray-Hervella type $\mathcal{W}_1 + \mathcal{W}_3$ at the base point, unless $4\lambda = 3\mu$, when they have type $\mathcal{W}_1$. In particular, the Nijenhuis tensors N_J and N_K are skew-symmetric at the base point and equal to $\frac{1}{6}(4\lambda + \mu)(3\Phi \mp \mu^2\Psi)$ at Id $U(3)$. In [4] we showed how N_I is determined by $\mathrm{J}d\omega_J - \mathrm{K}d\omega_K$. In this case, we have the interesting situation that this latter tensor is non-zero, even though N_I vanishes. Using [1], one can prove that the obstruction to quaternionic integrability is proportional to $N_I + N_J + N_K = (4\lambda + \mu)\Phi$, confirming that this is non-zero and the results of Corollary 4.4.

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