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Summability for solutions to some quasilinear elliptic systems

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Abstract We prove local regularity in Lebesgue spaces for weak solutions u of quasilinear elliptic systems whose off-diagonal coefficients are small when $|u|$ is large: the faster off-diagonal coefficients decay, the higher integrability of u becomes.

Keywords Local · Integrability · Solution · Quasilinear · Elliptic · System

Mathematics Subject Classification 35J62 · 35J47 · 35D30

1 Introduction

We study integrability properties of mappings $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ solving nonlinear elliptic systems

$$-\sum_{i=1}^{n} D_i \left(A_i^\alpha(x, u(x), Du(x)) \right) = 0, \quad x \in \Omega, \ \alpha = 1, \dots, N. \tag{1.1}$$

We assume coercivity:

$$\sum_{\alpha=1}^{N} \sum_{i=1}^{n} A_i^\alpha(x, y, z) z_i^\alpha \geq |z|^2. \tag{1.2}$$

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We require also growth condition: for some constant $M \in (0, +\infty)$, we have

$$\left|A_i^\alpha(x, y, z)\right| \le M|z|. \tag{1.3}$$

In addition, we suppose that $x \to A_i^\alpha(x, y, z)$ is measurable and $(y, z) \to A_i^\alpha(x, y, z)$ is continuous. Inequalities (1.2) and (1.3) are assumed to hold for almost every $x \in \Omega$, for any $y \in \mathbb{R}^N$ and every $z \in \mathbb{R}^{N\times n}$. If $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a weak solution to system (1.1), then

$$u \in L_{loc}^{2^*(1+t)}(\Omega, \mathbb{R}^N) \tag{1.4}$$

for every $t \in (0, t_1)$, for some $t_1 = t_1(n, N, M) > 0$, according to Remark (a) after theorem 4 in [15], see also [6,8,16] and chapter 6 in [7]; 2^* is the Sobolev exponent: $2^* = 2n/(n-2)$. In general, t_1 cannot be very large because of counterexamples [4,5,14]; see also [17,18]. If we add the following restriction

$$\sum_{\alpha,\beta=1}^{N} \sum_{i=1}^{n} A_i^\alpha(x, y, z) \frac{y^\alpha}{|y|} z_i^\beta \frac{y^\beta}{|y|} \ge -\delta|z|^2 \tag{1.5}$$

for some constant $\delta > 0$, then (1.4) holds true for every $t \in (0, 1/(2\delta))$, see theorem 4 in [15]. Note that (1.5) with small δ gives very high degree of integrability. Now let us assume that system (1.1) is quasilinear, that is,

$$A_i^\alpha(x, y, z) = \sum_{j=1}^{n} \sum_{\beta=1}^{N} a_{ij}^{\alpha\beta}(x, y) z_j^\beta. \tag{1.6}$$

In such a case, structure condition (1.5) is satisfied with any $\delta > 0$ when

$$a_{ij}^{\alpha\beta}(x, y) = 0 \quad \text{if } \alpha \ne \beta, \tag{1.7}$$

$$a_{ij}^{\alpha\alpha}(x, y) = b_{ij}(x, y) \tag{1.8}$$

and

$$\sum_{i,j=1}^{n} b_{ij}(x, y)\xi_j\xi_i \ge 0. \tag{1.9}$$

Note that (1.7) says that off-diagonal coefficients are zero; (1.8) requires that diagonal coefficients are taken from the same matrix. If diagonal coefficients are taken from different bounded and elliptic matrices, then structure condition (1.5) is satisfied only for $\delta \ge \delta_0 > 0$ (see the "Appendix" of the present paper) thus theorem 4 in [15] guarantees integrability only up to a certain degree. On the other hand, when off-diagonal coefficients vanish, the system is decoupled and we can apply standard regularity theory for a single elliptic equation thus every component u^α of the solution u is locally bounded, thus it is integrable with any exponent. In the present paper, we deal with the quasilinear case (1.6); so we consider weak solutions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of quasilinear systems

$$-\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} \sum_{\beta=1}^{N} a_{ij}^{\alpha\beta}(x, u(x)) D_j u^\beta(x) \right) = 0, \quad x \in \Omega, \ \alpha = 1, \dots, N. \tag{1.10}$$

Now we no longer assume that off-diagonal coefficients vanish; we only know that they are small when $|u^\gamma|$ is large:

$$|a_{ij}^{\gamma\beta}(x, u)| \le \frac{c}{(1 + |u^\gamma|)^q} \quad \text{for} \quad \beta \ne \gamma \tag{1.11}$$

for some constants $c, q \in (0, +\infty)$. We assume ellipticity only for diagonal coefficients $a_{ij}^{\gamma\gamma}(x, u)$ and only for large values of $|u^\gamma|$:

$$\theta \leq |u^\gamma| \implies \nu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x, u)\xi_j\xi_i \tag{1.12}$$

for some constants $\theta \in [0, +\infty)$ and $\nu \in (0, +\infty)$. Also diagonal coefficients are assumed to be bounded:

$$|a_{ij}^{\gamma\gamma}(x, u)| \leq \tilde{c} \tag{1.13}$$

for some constant $\tilde{c} \in (0, +\infty)$. In this paper, we prove that every weak solution $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, with $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, of quasilinear system (1.10) enjoys the following higher integrability

$$u \in L_{loc}^{2^*(1+q)}(\Omega, \mathbb{R}^N) \tag{1.14}$$

where q is the exponent in the right-hand side of (1.11), 2^* is the Sobolev exponent $\frac{2n}{n-2}$ and $n \geq 3$, see theorem 2.1 in Sect. 2. The proof, in Sect. 3, is based on a finite version of Moser's iteration [19]. To apply such an iteration, it is usual to test (1.1) with

$$v = |u|^p u, \tag{1.15}$$

in the scalar case $N = 1$ [1,13] as well in the vectorial one $N \geq 2$ [2,3,15]. In the vectorial case, we have

$$D_i v^\alpha = p|u|^{p-1} \sum_{\gamma=1}^{N} \frac{u^\gamma}{|u|}(D_i u^\gamma)u^\alpha + |u|^p D_i u^\alpha; \tag{1.16}$$

the second piece is "good" and the first one gives us

$$\sum_{i=1}^{n} \sum_{\alpha=1}^{N} A_i^\alpha(x, u, Du) \sum_{\gamma=1}^{N} \frac{u^\gamma}{|u|}(D_i u^\gamma)\frac{u^\alpha}{|u|}; \tag{1.17}$$

such a sum might be negative (see the "Appendix" at the end of the paper) and it has to be controlled by means of assumption (1.5). In the scalar case, the derivative of the test function (1.15) is

$$D_i v = p|u|^{p-1} \frac{u}{|u|}(D_i u)u + |u|^p D_i u; \tag{1.18}$$

now the first piece is

$$p|u|^{p-1} \frac{u}{|u|}(D_i u)u = p|u|^{p-1} \frac{u^2}{|u|}(D_i u) = p|u|^p (D_i u) \tag{1.19}$$

so it is like the second piece in (1.18) and it is "good". In the present paper, we fix one component u^γ and we test the equation by means of

$$v = (0, \ldots, 0, |u^\gamma|^p u^\gamma, 0, \ldots, 0). \tag{1.20}$$

Due to the quasilinear structure and to small off-diagonal coefficients, we are able to argue as in the scalar case and we get higher integrability of the selected component u^γ; we cannot arrive up to infinity since off-diagonal coefficients have to be controlled in a way that the exponent p cannot be too high with respect to exponent q of the assumption (1.11). Please, note that the above discussion gives only the main idea of the proofs: they actually need the test v to be multiplied by a cut off function and a suitable approximation of the power $|u|^p$. Our result (1.14) improves on theorem 2.1 of [11]: in both cases quasilinear systems with small off-diagonal coefficients are considered; in theorem 2.1 of [11], the integrability of the solution u does not reach the degree of (1.14) since the proof of theorem 2.1 in [11] uses $|u|^p$ in the test function. We thank the referee for usuful suggestions.

2 Assumptions and results

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$. For $N \geq 2$, let $a_{ij}^{\alpha\beta} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be Carathéodory functions, that is, $a_{ij}^{\alpha\beta}(x, y)$ are measurable with respect to x and continuous with respect to y. We assume that diagonal coefficients $a_{ij}^{\gamma\gamma}$ are bounded: there exists $c_1 \in (0, +\infty)$ such that

$$|a_{ij}^{\gamma\gamma}(x, y)| \leq c_1 \tag{2.1}$$

for almost every $x \in \Omega$, for every $y \in \mathbb{R}^N$, for all $i, j \in \{1, \dots, n\}$, for any $\gamma \in \{1, \dots, N\}$. Now we assume ellipticity of diagonal coefficients $a_{ij}^{\gamma\gamma}$ for large values of y^γ: there exist $\theta \in [0, +\infty)$ and $\nu \in (0, +\infty)$ such that

$$\theta \leq |y^\gamma| \implies \nu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x, y)\xi_j\xi_i \tag{2.2}$$

for almost every $x \in \Omega$, for any $\xi \in \mathbb{R}^n$ and for any $\gamma \in \{1, \dots, N\}$. Now we assume that off-diagonal coefficients $a_{ij}^{\gamma\beta}(x, y)$ do not vanish any more, but they are small when y^γ is large: there exist $q \in (0, +\infty)$ and $c_2 \in (0, +\infty)$ such that

$$|a_{ij}^{\gamma\beta}(x, y)| \leq \frac{c_2}{(1 + |y^\gamma|)^q} \quad \text{for} \quad \beta \neq \gamma. \tag{2.3}$$

Note that both diagonal and off-diagonal coefficients are bounded.

Theorem 2.1 *Under the previous assumptions* (2.1), (2.2), (2.3) *let* $u = (u^1, \dots, u^N)$ *be a weak solution of the system* (1.10)*, that is,* $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ *and*

$$\int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}(x, u(x)) D_j u^\beta(x) D_i v^\alpha(x) dx = 0 \qquad \forall v \in W_0^{1,2}(\Omega, \mathbb{R}^N). \tag{2.4}$$

Then

$$u \in L_{loc}^{2^*(q+1)}(\Omega, \mathbb{R}^N). \tag{2.5}$$

A global integrability result is contained in [10]. Let us note that, when off-diagonal coefficients are zero for large values of $|y^\gamma|$, regularity has been studied in [9,12,20].

Remark Assumptions of theorem 2.1 in [11] are more restrictive than the present ones; moreover, in the present paper, the degree of integrability (2.5) is better than the one in theorem 2.1 of [11]: this is due to the smarter choice of the test function. In the present paper, we fix one component u^γ and we (basically) test the equation by means of

$$v = (0, \dots, 0, |u^\gamma|^p u^\gamma, 0, \dots, 0); \tag{2.6}$$

in the proof of theorem 2.1 in [11] it was (basically) used

$$v = (0, \dots, 0, |u|^p u^\gamma, 0, \dots, 0) \tag{2.7}$$

so all of u appears in $|u|^p$ and this gives trouble in the calculations, thus resulting in a worse result. After changing the test function, the present proof closely follows the one given in [11]: for this reason, we only give the required modification with respect to [11].

3 Proof of Theorem 2.1

We start as in the proof of theorem 2.1 in [11]. Let $\phi : [0, +\infty) \to [0, +\infty)$ be increasing and $C^1([0, +\infty))$; moreover, we assume that there exists a constant $\tilde{c} \in [1, +\infty)$ such that

$$0 \le \phi(t) \le \tilde{c} \quad \forall t \in [0, +\infty) \tag{3.1}$$

$$0 \le \phi'(t) \le \tilde{c} \quad \forall t \in [0, +\infty) \tag{3.2}$$

$$0 \le \phi'(t)t \le \tilde{c} \quad \forall t \in [0, +\infty). \tag{3.3}$$

Let $B_\rho = B(x_0, \rho)$ and $B_R = B(x_0, R)$ be open balls with the same center x_0 and radii $0 < \rho < R \le 1$, with $\overline{B_R} \subset \Omega$. We assume that $\eta : \mathbb{R}^n \to \mathbb{R}$, $\eta \in C_0^1(B_R)$ with $0 \le \eta \le 1$ in $\mathbb{R}^n$, $\eta = 1$ on B_ρ, $|D\eta| \le 4/(R-\rho)$ in $\mathbb{R}^n$. Note that $0 < R-\rho < R \le 1$ so $4/(R-\rho) > 4$. We fix $\gamma \in \{1, \dots, N\}$; we consider the test function $v = (v^1, \dots, v^N)$ defined as follows

$$v^\alpha = \begin{cases} 0 & \text{if } \alpha \ne \gamma, \\ \phi(|u^\alpha|)u^\alpha \eta^2 & \text{if } \alpha = \gamma; \end{cases} \tag{3.4}$$

please, note that such a test function is different from the one in [11]. It results that

$$v \in W_0^{1,2}(B_R; \mathbb{R}^N) \subset W_0^{1,2}(\Omega; \mathbb{R}^N) \tag{3.5}$$

and

$$D_i v^\gamma = \left[\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)\right](D_i u^\gamma)\eta^2 + \left[\phi(|u^\gamma|)u^\gamma\right] D_i(\eta^2) : \tag{3.6}$$

this derivatives is better than the corresponding derivative in [11]. We insert such a test function v into (2.4) and we get:

$$\begin{aligned}
&\int_{\{\theta \le |u^\gamma|\}} \sum_{i,j=1}^n a_{ij}^{\gamma\gamma}(x,u) D_j u^\gamma \left[\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)\right](D_i u^\gamma)\eta^2 \\
&= - \int_{\{\theta > |u^\gamma|\}} \sum_{i,j=1}^n a_{ij}^{\gamma\gamma}(x,u) D_j u^\gamma \left[\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)\right](D_i u^\gamma)\eta^2 \\
&\quad - \int_\Omega \sum_{i,j=1}^n \sum_{\beta \ne \gamma} a_{ij}^{\gamma\beta}(x,u) D_j u^\beta \left[\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)\right](D_i u^\gamma)\eta^2 \\
&\quad - \int_\Omega \sum_{i,j=1}^n a_{ij}^{\gamma\gamma}(x,u) D_j u^\gamma \phi(|u^\gamma|)u^\gamma D_i(\eta^2) \\
&\quad - \int_\Omega \sum_{i,j=1}^n \sum_{\beta \ne \gamma} a_{ij}^{\gamma\beta}(x,u) D_j u^\beta \phi(|u^\gamma|)u^\gamma D_i(\eta^2).
\end{aligned} \tag{3.7}$$

Now we use ellipticity (2.2) on the left-hand side and decay for off-diagonal coefficients (2.3) on the right-hand side: we get

$$\begin{aligned} &\nu \int_{\{\theta \le |u^\gamma|\}} [\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)]|Du^\gamma|^2\eta^2 \\ &\le nc_1 \int_{\{\theta > |u^\gamma|\}} [\phi'(|u^\gamma|) \quad |u^\gamma| + \phi(|u^\gamma|)]|Du^\gamma|^2\eta^2 \\ &\quad + \int_\Omega \frac{n^2Nc_2}{(1+|u^\gamma|)^q}|Du|[\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)]|Du^\gamma|\eta^2 \\ &\quad + \int_\Omega \frac{n^2Nc_2}{(1+|u^\gamma|)^q}|Du|\phi(|u^\gamma|)|u^\gamma|2\eta|D\eta| + nc_1 \int_\Omega 2\eta|Du^\gamma|\phi(|u^\gamma|)|u^\gamma||D\eta|. \end{aligned} \tag{3.8}$$

We add to both sides

$$\nu \int_{\{\theta > |u^\gamma|\}} [\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)]|Du^\gamma|^2\eta^2 \tag{3.9}$$

and we get

$$\begin{aligned} &\nu \int_\Omega [\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)]|Du^\gamma|^2\eta^2 \\ &\le (\nu + nc_1) \int_{\{\theta > |u^\gamma|\}} [\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)]|Du^\gamma|^2\eta^2 \\ &\quad + \int_\Omega \frac{n^2Nc_2}{(1+|u^\gamma|)^q}|Du|[\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)]|Du^\gamma|\eta^2 \\ &\quad + \int_\Omega \frac{n^2Nc_2}{(1+|u^\gamma|)^q}|Du|\phi(|u^\gamma|)|u^\gamma|2\eta|D\eta| + nc_1 \int_\Omega 2\eta|Du^\gamma|\phi(|u^\gamma|)|u^\gamma||D\eta|. \end{aligned} \tag{3.10}$$

We use the inequality $2AB \le \epsilon A^2 + B^2/\epsilon$, twice with $\epsilon = \nu/4$ and once with $\epsilon = 1$; then we get

$$\begin{aligned} &\frac{\nu}{2} \int_\Omega [\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)]|Du^\gamma|^2\eta^2 \\ &\le (\nu + nc_1) \int_{\{\theta > |u^\gamma|\}} [\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)]|Du^\gamma|^2\eta^2 \\ &\quad + \left(1 + \frac{4n^2c_1^2}{\nu}\right) \int_\Omega \phi(|u^\gamma|)|u^\gamma|^2|D\eta|^2 \\ &\quad + \int_\Omega \left(1 + \frac{4}{\nu}\right) \frac{n^4N^2c_2^2}{(1+|u^\gamma|)^{2q}}[\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)]|Du|^2\eta^2. \end{aligned} \tag{3.11}$$

Let us consider $p \in (0, +\infty)$ and let us assume that

$$|u^\gamma|^{2(p+1)} \in L^1(B_R). \tag{3.12}$$

For $t \in [0, +\infty)$ we set

$$\psi(t) = (p+1)^2 t^{2p}; \tag{3.13}$$

we would like to take $\phi = \psi$ in (3.11) but we cannot do that, since ϕ must satisfy (3.1), (3.2) and (3.3). So we approximate ψ in this way: for every $k \in \mathbb{N}$, when $p < 1/2$ we take

$$\theta_k(t) = \begin{cases} \psi'(\frac{1}{k}) & \text{if} \quad t \in [0, \frac{1}{k}) \\ \psi'(t) & \text{if} \quad t \in [\frac{1}{k}, k] \\ \psi'(k)(k+1-t) & \text{if} \quad t \in (k, k+1) \\ 0 & \text{if} \quad t \in [k+1, +\infty); \end{cases} \tag{3.14}$$

when $1/2 \le p$ we take

$$\theta_k(t) = \begin{cases} \psi'(t) & \text{if} \quad t \in [0, k] \\ \psi'(k)(k+1-t) & \text{if} \quad t \in (k, k+1) \\ 0 & \text{if} \quad t \in [k+1, +\infty); \end{cases} \tag{3.15}$$

in both cases we consider

$$\psi_k(s) = \int_0^s \theta_k(t)\mathrm{d}t. \tag{3.16}$$

Then $\psi_k : [0, +\infty) \to [0, +\infty)$ is increasing, $C^1([0, +\infty))$ and

$$0 \le \psi_k'(t) \le \psi'(t) \quad \forall t \in (0, +\infty), \tag{3.17}$$
$$0 \le \psi_k(t) \le \psi(t) \quad \forall t \in [0, +\infty); \tag{3.18}$$

in addition, there exists $c_k \in [0, +\infty)$ such that

$$0 \le \psi_k(t) \le c_k \quad \forall t \in [0, +\infty), \tag{3.19}$$
$$0 \le \psi_k'(t) \le c_k \quad \forall t \in [0, +\infty), \tag{3.20}$$
$$0 \le \psi_k'(t)t \le c_k \quad \forall t \in [0, +\infty). \tag{3.21}$$

Moreover

$$\lim_{k \to +\infty} \psi_k(t) = \psi(t) \quad \forall t \in [0, +\infty). \tag{3.22}$$

Please, note that the present approximation is not the same as in [11]. Now we can take $\phi = \psi_k$ in (3.11); we remark that

$$\psi_k(t) \le \psi_k'(t)t + \psi_k(t) \le (p+1)^2[2p+1]t^{2p} \tag{3.23}$$

and we get

$$\frac{\nu}{2}\int_{\Omega}\psi_k(|u^\gamma|)|Du^\gamma|^2\eta^2 \le (\nu+nc_1)(p+1)^2[2p+1]|\theta|^{2p}\int_{\Omega}|Du^\gamma|^2\eta^2$$
$$+\left(1+\frac{4n^2c_1^2}{\nu}\right)\int_{\Omega}(p+1)^2[2p+1]|u^\gamma|^{2(p+1)}|D\eta|^2$$
$$+\int_{\Omega}\left(1+\frac{4}{\nu}\right)\frac{n^4N^2c_2^2}{(1+|u^\gamma|)^{2q}}(p+1)^2[2p+1]|u^\gamma|^{2p}|Du|^2\eta^2. \tag{3.24}$$

Now we require that

$$p \le q \tag{3.25}$$

in order to have

$$\frac{|u^\gamma|^{2p}}{(1+|u^\gamma|)^{2q}} \le 1; \tag{3.26}$$

positivity of ψ_k and pointwise convergence (3.22) allow us to use Fatou lemma so that

$$\frac{\nu}{2}\int_{\Omega}|u^\gamma|^{2p}|Du^\gamma|^2\eta^2 \le \left((\nu+nc_1)\theta^{2p}+\left(1+\frac{4}{\nu}\right)(n^4N^2c_2^2)\right)(2p+1)||Du||^2_{L^2(\Omega)}$$
$$+\left(1+\frac{4n^2c_1^2}{\nu}\right)[2p+1]\frac{16}{(R-\rho)^2}\int_{B_R}|u^\gamma|^{2(p+1)}. \tag{3.27}$$

Let us set

$$w = |u^\gamma|^{p+1}\eta; \tag{3.28}$$

then

$$w \in W_0^{1,2}(B_R) \tag{3.29}$$

and

$$|Dw|^2 \le 2(p+1)^2|u^\gamma|^{2p}|Du^\gamma|^2\eta^2 + 2n|u^\gamma|^{2(p+1)}\left(\frac{4}{R-\rho}\right)^2. \tag{3.30}$$

The previous inequality and (3.27) give

$$\int_{B_R}|Dw|^2 \le \frac{4}{\nu}(p+1)^2\left((\nu+nc_1)\theta^{2p}+\left(1+\frac{4}{\nu}\right)(n^4N^2c_2^2)\right)(2p+1)||Du||^2_{L^2(\Omega)}$$
$$+\left(\frac{4}{\nu}(p+1)^2\left(1+\frac{4n^2c_1^2}{\nu}\right)[2p+1]+2n\right)\frac{16}{(R-\rho)^2}\int_{B_R}|u^\gamma|^{2(p+1)}. \tag{3.31}$$

Now we use Sobolev embedding and the properties of η in order to get

$$\int_{B_\rho} |u^\gamma|^{(p+1)2^*} \le \int_{B_R} ||u^\gamma|^{p+1}\eta|^{2^*} = \int_{B_R} |w|^{2^*} \le \left[\frac{2(n-1)}{n-2} \int_{B_R} |Dw|^2 \right]^{2^*/2}$$
$$\le \left[\frac{4}{\nu}(p+1)^2 \left((\nu + nc_1)\theta^{2p} + \left(1+\frac{4}{\nu}\right)(n^4N^2c_2^2) \right)(2p+1)||Du||^2_{L^2(\Omega)} \right.$$
$$\left. + \left(\frac{4}{\nu}(p+1)^2 \left(1 + \frac{4n^2c_1^2}{\nu}\right)[2p+1] + 2n \right) \frac{16}{(R-\rho)^2} \int_{B_R} |u^\gamma|^{2(p+1)} \right]^{2^*/2}$$
$$\times \left(\frac{2(n-1)}{n-2} \right)^{2^*/2}. \tag{3.32}$$

Let us summarize as follows: if for some $p \in (0, +\infty)$ with

$$p \le q \tag{3.33}$$

and for some $0 < \rho < R \le 1$ with $\overline{B_R} \subset \Omega$ we have

$$|u^\gamma|^{2(p+1)} \in L^1(B_R) \tag{3.34}$$

then it results that

$$|u^\gamma|^{2^*(p+1)} \in L^1(B_\rho). \tag{3.35}$$

Since $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ and $\overline{B_R} \subset \Omega$, Sobolev embedding gives us

$$|u^\gamma|^{2\frac{n}{n-2}} \in L^1(B_R) \tag{3.36}$$

thus (3.34) and (3.33) are fulfilled with $p = \min\{2/(n-2); q\}$; this improves the integrability accordingly to (3.35); the procedure can be iterated and, following [11] page 129, after a finite number of steps, we reach the desired integrability.

This ends the proof of Theorem 2.1. □

4 Appendix

For $N = 2$, let us consider the following $n \times n$ matrices $a^{11}, a^{12}, a^{21}, a^{22}$:

$$a^{11}_{ij}(x, y) = \begin{cases} 0 & \text{if } i \ne j, \\ \sigma_1(x) & \text{if } i = j; \end{cases} \tag{4.1}$$
$$a^{12}_{ij}(x, y) = a^{21}_{ij}(x, y) = 0; \tag{4.2}$$
$$a^{22}_{ij}(x, y) = \begin{cases} 0 & \text{if } i \ne j, \\ \sigma_2(x) & \text{if } i = j. \end{cases} \tag{4.3}$$

Now we assume the quasilinear structure (1.6), and we compute the left-hand side of (1.5) with the previous choice of $a_{ij}^{\alpha\beta}$:

$$\begin{aligned}
\sum_{\alpha,\beta=1}^{N}\sum_{i=1}^{n} A_i^\alpha(x,y,z)\frac{y^\alpha}{|y|}z_i^\beta\frac{y^\beta}{|y|} &= \sum_{\alpha,\beta=1}^{N}\sum_{i=1}^{n}\sum_{\gamma=1}^{N}\sum_{j=1}^{n} a_{ij}^{\alpha\gamma}(x,y)z_j^\gamma\frac{y^\alpha}{|y|}z_i^\beta\frac{y^\beta}{|y|}\\
&= (\text{since } a_{ij}^{\alpha\gamma}=0 \text{ when } \gamma\neq\alpha)\sum_{\alpha,\beta=1}^{N}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{\alpha\alpha}(x,y)z_j^\alpha\frac{y^\alpha}{|y|}z_i^\beta\frac{y^\beta}{|y|}\\
&= (\text{since } a_{ij}^{\alpha\alpha}=0 \text{ when } i\neq j)\sum_{\alpha,\beta=1}^{N}\sum_{i=1}^{n} a_{ii}^{\alpha\alpha}(x,y)z_i^\alpha\frac{y^\alpha}{|y|}z_i^\beta\frac{y^\beta}{|y|}\\
&= (\text{since } a_{ii}^{\alpha\alpha}(x,y)=\sigma_\alpha(x))\sum_{\alpha,\beta=1}^{N}\sum_{i=1}^{n}\sigma_\alpha(x)z_i^\alpha\frac{y^\alpha}{|y|}z_i^\beta\frac{y^\beta}{|y|}\\
&= \sum_{i=1}^{n}\sum_{\alpha=1}^{N}\sigma_\alpha(x)z_i^\alpha\frac{y^\alpha}{|y|}\sum_{\beta=1}^{N}z_i^\beta\frac{y^\beta}{|y|} = (*). \qquad (4.4)
\end{aligned}$$

Now we choose

$$y^1 = y^2 > 0 \tag{4.5}$$

so that

$$\frac{y^\alpha}{|y|} = \frac{1}{\sqrt{2}}. \tag{4.6}$$

With such a choice of y, we have

$$(*) = \frac{1}{2}\sum_{i=1}^{n}\sum_{\alpha=1}^{N}\sigma_\alpha(x)z_i^\alpha\sum_{\beta=1}^{N}z_i^\beta = (**). \tag{4.7}$$

Now we choose z as follows:

$$z_i^\alpha = 0 \quad \text{if} \quad i\geq 2 \tag{4.8}$$

and

$$z_1^\alpha = \begin{cases} -1 & \text{if} \quad \alpha=1,\\ 2 & \text{if} \quad \alpha=2.\end{cases} \tag{4.9}$$

Thus

$$|z|^2 = 5 \tag{4.10}$$

and, keeping in mind that $N=2$,

$$\begin{aligned}
(**) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{\alpha=1}^{2}\sigma_\alpha(x)z_i^\alpha\sum_{\beta=1}^{2}z_i^\beta = \frac{1}{2}\sum_{\alpha=1}^{2}\sigma_\alpha(x)z_1^\alpha\sum_{\beta=1}^{2}z_1^\beta\\
&= \frac{1}{2}(-\sigma_1(x)+2\sigma_2(x))(-1+2) = \frac{-\sigma_1(x)+2\sigma_2(x)}{10}|z|^2. \qquad (4.11)
\end{aligned}$$

Eventually

$$\sum_{\alpha,\beta=1}^{N}\sum_{i=1}^{n} A_i^{\alpha}(x,y,z)\frac{y^{\alpha}}{|y|}z_i^{\beta}\frac{y^{\beta}}{|y|} = \frac{-\sigma_1(x)+2\sigma_2(x)}{10}|z|^2. \tag{4.12}$$

If we insert this equality into (1.5), we get

$$\frac{-\sigma_1(x)+2\sigma_2(x)}{10}|z|^2 \geq -\delta|z|^2; \tag{4.13}$$

this means that

$$\frac{\sigma_1(x)-2\sigma_2(x)}{10} \leq \delta. \tag{4.14}$$

Let us take

$$\sigma_1(x) = 14 + 2\sin(|x|^2), \quad \sigma_2(x) = 2 + \sin(|x|^2); \tag{4.15}$$

then (4.14) becomes

$$1 \leq \delta. \tag{4.16}$$

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