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Multiple existence results of solutions for quasilinear elliptic equations with a nonlinearity depending on a parameter

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Abstract We provide existence results of multiple solutions for quasilinear elliptic equations depending on a parameter under the Neumann and Dirichlet boundary condition. Our main result shows the existence of two opposite constant sign solutions and a sign changing solution in the case where we do not impose the subcritical growth condition to the nonlinear term not including derivatives of the solution. The studied equations contain the p-Laplacian problems as a special case. Our approach is based on variational methods combining super- and sub-solution and the existence of critical points via descending flow.

Keywords Quasilinear elliptic equations · Nonhomogeneous operator · Super-solution and sub-solution · Critical point · Invariant sets of descending flow

Mathematics Subject Classification (2000) 35J20 · 58E05 · 35J62

1 Introduction and statements of main results

In this paper, we consider the existence of nontrivial multiple solutions for the following quasilinear elliptic equation

$$\begin{cases} -\operatorname{div} A(x, \nabla u) = \mu f(x, u) & \text{in} \quad \Omega, \\ Bu = 0 & \text{on} \quad \partial\Omega, \end{cases} \qquad (\mathrm{P})_\mu$$

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where $\mu > 0$ is a parameter and $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$. Here, $Bu = 0$ denotes the Dirichlet or Neumann boundary condition, namely $Bu := u$ or $Bu := \partial u/\partial\nu$, respectively, where ν denotes the outward unit normal vector on $\partial\Omega$. Moreover, $A\colon \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a map which is strictly monotone in the second variable and satisfies certain regularity conditions (see the following assumption (A)).

Throughout this paper, we assume that the map A and the nonlinear term f satisfy the following assumptions (A) *and* (f), *respectively*:

(A) $A(x, y) = a(x, |y|)y$, where $a(x, t) > 0$ for all $(x, t) \in \overline{\Omega} \times (0, +\infty)$, $1 < p < \infty$ and

(i) $A \in C^0(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$;

(ii) there exists $C_1 > 0$ such that

$$|D_y A(x, y)| \le C_1|y|^{p-2} \quad \text{for every } x \in \overline{\Omega}, \text{ and } y \in \mathbb{R}^N \setminus \{0\};$$

(iii) there exists $C_0 > 0$ such that

$$D_y A(x, y)\xi \cdot \xi \ge C_0|y|^{p-2}|\xi|^2 \quad \text{for every } \xi \in \overline{\Omega},\ y \in \mathbb{R}^N \setminus \{0\} \text{ and } \xi \in \mathbb{R}^N;$$

(iv) there exists $C_2 > 0$ such that

$$|D_x A(x, y)| \le C_2(1 + |y|^{p-1}) \quad \text{for every } x \in \overline{\Omega},\ y \in \mathbb{R}^N \setminus \{0\};$$

(v) there exist $C_3 > 0$ and $1 \ge t_0 > 0$ such that

$$|D_x A(x, y)| \le C_3|y|^{p-1}\ (-\log|y|)$$

for every $x \in \overline{\Omega}$, $y \in \mathbb{R}^N$ with $0 < |y| < t_0$.

(f) f is a Carathéodory function on $\Omega \times \mathbb{R}$ with $f(x, 0) = 0$ for a.e. $x \in \Omega$ and f is bounded on bounded sets.

In this paper, we say that $u \in W^{1,p}(\Omega)$ (resp. $W_0^{1,p}(\Omega)$) is a (weak) solution of $(P)_\mu$ under the Neumann boundary condition (resp. the Dirichlet boundary condition) if

$$\int_\Omega A(x, \nabla u)\nabla\varphi \,\mathrm{d}x = \mu \int_\Omega f(x, u)\varphi \,\mathrm{d}x$$

for all $\varphi \in W^{1,p}(\Omega)$ (resp. $W_0^{1,p}(\Omega)$) provided the integral on the right-hand side exists. We say that u is a positive (resp. negative) solution of $(P)_\mu$ if $u \in W^{1,p}(\Omega)$ is a solution in the above sense and $u(x) > 0$ (resp. $u(x) < 0$) for a.e. $x \in \Omega$.

A similar hypothesis to (A) is considered in the study of quasilinear elliptic problems (cf. [28, Example 2.2.] and see [14,26,27,32] too). We also refer to [19,29,31] for the generalized p-Laplace operators. In particular, for $A(x, y) = |y|^{p-2}y$, that is, $\operatorname{div} A(x, \nabla u)$ stands for the usual p-Laplacian $\Delta_p u$, we can take $C_0 = C_1 = p - 1$ in (A). Conversely, in the case where $C_0 = C_1 = p - 1$ holds in (A), by the inequalities in Remark 6 (ii) and (iii) in Sect. 2, we see that $a(x, t) = |t|^{p-2}$ whence $A(x, y) = |y|^{p-2}y$. Hence, our equation $(\mathrm{P})_\mu$ contains the corresponding p-Laplacian problem as a special case.

The main purpose of this paper is to show the existence of at least three nontrivial solutions for $(P)_\mu$, provided μ is sufficiently large, without assuming the subcritical growth condition for the term f. This is achieved through a variational approach that encompasses both the Dirichlet and Neumann problems. It is well known that for the *Dirichlet problems*, due to the Poincaré inequality, we can construct a coercive functional corresponding to the equation simply by the truncation of the nonlinearity. This argument does not work for the Neumann problems because for them the Poincaré inequality does not hold. However, in [26], the present authors overcome this difficulty by introducing new functionals (see also Sects. 4, 5) by means of which we can prove under the Neumann boundary condition, an existence result as in the Dirichlet case via super- and sub-solution, that is, the existence of a solution within the ordered interval determined by a sub-solution and a super-solution.

Here, for such a coercive functional under several hypotheses, we show that it has at least three critical points via the descending flow argument which is done in our main abstract result stated as Theorem 11. This result is developed from the one in [5] which deals with the p-Laplace operator in place of our generalized operator and a super-linear nonlinearity under the *Dirichlet boundary condition* (so, a functional as in [5] is not coercive). Furthermore, we point out that we drop the hypothesis regarding N and p imposed in Bartsch and Liu [5] and removed in [7] in the case of *Dirichlet boundary condition*. We overcome this difficulty by a different way from the one in [7], roughly speaking by constructing a suitable descending flow on $C_0^1(\Omega)$ and $C^1(\overline{\Omega})$ in place of the Sobolev space $W_0^{1,p}(\Omega)$. As a result, although the result in [7] covers only the case of $f(x,u) = o(|u|^{p-1})$ as $|u| \to 0$, our abstract theorem can provide the result to the cases of $f(x,u) = O(|u|^{p-1})$ as $|u| \to 0$ (see Sect. 1.1) and concave near zero (see Sect. 1.2) in Neumann problem too.

Moreover, even without assuming the subcritical growth condition, in [11] (for the Dirichlet problems with the p-Laplacian) and in [26] (for the Neumann problems with a general operator in the principal part), the existence of multiple solutions can be established. However, the statements in [11] and [26] did not depend on parameters and the nonlinearities treated there were essentially different from those considered here because now we focus on a nonlinearity $f(x,u)$ without the local sign-condition and whose growth condition near $u = 0$ matters only when the parameter μ is sufficiently large. In addition, since we do not impose the subcritical growth condition, we can handle nonlinearities $f(x,u)$ containing terms like $|u|^{q-2}u$ $(1 < q < \infty)$ and e^u. Based on Theorem 11, we are able to establish the existence of multiple solutions with complete sign information for both Dirichlet and Neumann nonlinear elliptic equations whose principal part is much more general than the p-Laplacian.

Let us recall some relevant results for the p-Laplacian problems under the Dirichlet boundary condition. Consider

$$-\Delta_p u = f(x,u,\mu) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and μ is a parameter. When the parameter μ is large, there are few results of multiple existences containing a sign-changing solution (cf. [6,12]). In [6], Bartsch and Liu treated the nonlinearity $f(x,u,\mu) = \mu f(x,u)$ satisfying $f(x,u) = o(|u|^{p-1})$ as $|u| \to \infty$ under additional hypotheses. In [12], Carl and the first author consider the nonlinearity f such that $f(x,u,\mu) = \mu|u|^{p-2}u - g(x,u)$ with $g(x,u) = o(|u|^{p-1})$ as $u \to 0$ and $g(x,u)/|u|^{p-2}u \to +\infty$ as $|u| \to \infty$. In this paper, we give a general result which admits $f(x,u)$ to behave like $m(x)|u|^{p-2}u$ near $u = 0$ with a bounded sign-changing function m [see Corollary 2 (ii)].

We also mention that many authors studied a positive (or a nonnegative) solution of Eq. (1) (cf. [1,10,17,20,21,30]). This occurred, in particular, when the nonlinearity f is concave–convex, that is,

$$f(x,u,\mu) = \mu|u|^{q-2}u + |u|^{r-2}u$$

with $1 < q < p < r \le p^*$. In the semilinear case ($p = 2$), the study of existence or nonexistence of a positive solution is well known from Ambrosetti et al. [2]. Later, it has been developed by many authors for the p-Laplace problem with concave–convex nonlinearity (cf. [4,18,20]). One of our purposes is to provide a *sign-changing* solution when a positive solution exists (see Corollary 4 and Example 1).

For the generalized operator under the Dirichlet boundary condition, we can see the existence of a nontrivial or a positive radially symmetric solution in [29] or [19], respectively. For the Neumann problems with p-Laplacian, we refer to [8,9]. However, there are no results regarding a sign-changing solution. In the present paper, we prove the existence of three solutions of problem $(P)_\mu$: one positive, one negative and one changing its sign, provided the parameter μ is sufficiently large. We emphasize that in problem $(P)_\mu$ the operator $\operatorname{div} A(x, \nabla u)$ is much more general than the p-Laplacian, in particular it is not required to be $(p-1)$-homogeneous. Our main result in this direction is Theorem 1 that applies to both Dirichlet and Neumann boundary value problems. Various corollaries of it provide verifiable conditions for the nonlinearity $f(x, u)$ in order to guarantee the conclusion of Theorem 1. Our approach relies on the analysis with respect to the positive and negative cones of a descending flow related to problem $(P)_\mu$.

The contents of the paper: Section 1.1 contains the statements of our main results without assuming the (local) sign-condition for f and the growth condition for f near zero [see $(H3)$ and $(H6)$].

In Sect. 1.2, we present existence results in the case where the nonlinearity f contains a concave term near zero [see $(\widetilde{H3})$].

Section 2 is devoted to the properties of the general map A in problem $(P)_\mu$.

In Sect. 3, we prove the main abstract theorem (Theorem 11). In Sect. 4, we give the proofs of our results in the case where the parameter is sufficiently large. In Sect. 5, we prove our results in special cases where the parameter μ is arbitrary.

1.1 Statements of main results

To simplify the notation, we introduce the following spaces:

$$
\begin{aligned}
W_B &:= W^{1,p}(\Omega) \quad \text{or} \quad W_B := W_0^{1,p}(\Omega), \\
X_B &:= C^1(\overline{\Omega}) \quad \text{or} \quad X_B := C_0^1(\overline{\Omega}), \\
C_B^{1,\alpha}(\overline{\Omega}) &:= C^{1,\alpha}(\overline{\Omega}) \quad \text{or} \quad C_B^{1,\alpha}(\overline{\Omega}) := C_0^{1,\alpha}(\overline{\Omega}) \quad \text{for } \alpha \in (0,1)
\end{aligned}
$$

in the case of $Bu := \partial u/\partial \nu$ or $Bu := u$, respectively. Denote the positive cone $P_B := \{u \in X_B\,;\ u(x) \geq 0 \text{ for every } x \in \Omega\}$ and the closure of P_B in W_B by $\widetilde{P}_B$. For simplicity, we denote the positive cone in $C^1(\overline{\Omega})$ by P, and so

$$
\operatorname{int} P := \{u \in C^1(\overline{\Omega})\,;\ u(x) > 0 \text{ for every } x \in \overline{\Omega}\}.
$$

In this paper, we set $t_\pm := \max\{\pm t, 0\}$ and so u_+ and u_- denote the positive and the negative part of a function u, respectively (that is, $u = u_+ - u_-$).

Next, we formulate the following hypothesis: there exists $\mu_0 \geq 0$ such that

(H1) for each $\mu > \mu_0$, there exist a super-solution $u_\mu \in W^{1,p}(\Omega) \cap L^\infty(\Omega)_+ \setminus \{0\}$ and a sub-solution $v_\mu \in W^{1,p}(\Omega) \cap (-L^\infty(\Omega)_+) \setminus \{0\}$ of $(P)_\mu$;

Here, we say that $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a super-solution (resp. sub-solution) of $(P)_\mu$ if u satisfies

$$\int_\Omega A(x, \nabla u)\nabla\varphi \, dx \geq \mu \int_\Omega f(x, u)\varphi \, dx$$

$$\left(\text{resp.} \int_\Omega A(x, \nabla u)\nabla\varphi \, dx \leq \mu \int_\Omega f(x, u)\varphi \, dx\right)$$

for every $\varphi \in W_B$ with $\varphi \geq 0$. In addition, in the case of $W_B = W_0^{1,p}(\Omega)$ (that is, Dirichlet boundary problem), we impose $u \geq 0$ resp. $u \leq 0$) on $\partial\Omega$ in the sense of trace operator.

We introduce several conditions for f which are not necessarily simultaneously assumed in our results.

(H2) there exist $D_1 > 0$ and $\delta_0 > 0$ such that $f(x, u)u \geq -D_1|u|^p$ for every $|u| < \delta_0$ and a.e. $x \in \Omega$;

(H3) there exist open subsets Ω_1, Ω_2 of Ω, positive constants δ_0, d_1 and d_2 such that

$$\inf_{x\in\Omega_1} u_\mu(x) \geq d_1, \qquad \sup_{x\in\Omega_2} v_\mu(x) \leq -d_2 \quad \text{for every } \mu \geq \mu_0,$$
$$f(x, u) > 0 \quad \text{for every } 0 < u < \delta_0, \text{ a.e. } x \in \Omega_1,$$
$$f(x, u) < 0 \quad \text{for every } 0 > u > -\delta_0, \text{ a.e. } x \in \Omega_2,$$

where $u_\mu \geq 0$ and $v_\mu \leq 0$ denote a super- or sub-solution as in $(H1)$, respectively;

(H4) there exist $m \in L^\infty(\Omega)$ and $\delta_1 > 0$ such that $|\{x \in \Omega\,;\, m(x) > 0\}| > 0$ and

$$f(x, u)u \geq m(x)|u|^p \quad \text{for every } |u| \leq \delta_1, \text{ a.e. } x \in \Omega;$$

(H5) there exist $T^- < 0 < T^+$ such that $f(x, T^+) \leq 0 \leq f(x, T^-)$ for a.e. $x \in \Omega$;

(H6) there exist $\delta_2 > 0$ and an open subset Ω_3 of Ω such that $f(x, u)u > 0$ for every $0 < |u| < \delta_2$, a.e. $x \in \Omega_3$.

Theorem 1 *Assume* $(H1)$, $(H2)$ *and* $(H3)$ *or* $(H4)$. *In addition, in the case of* $(H4)$, *we also suppose that* $u_\mu \in \text{int}\, P_B \cup \text{int}\, P$ *and* $v_\mu \in -\text{int}\, P_B \cup -\text{int}\, P$ *for a super-solution* u_μ *and a sub-solution* v_μ *as in* $(H1)$. *Then, for sufficiently large* μ, $(P)_\mu$ *has a positive solution* $w_{\mu,1} \in \text{int} P_B$, *a negative solution* $w_{\mu,2} \in -\text{int}\, P_B$ *and a sign-changing solution* $w_{\mu,3} \in X_B \setminus (P_B \cup -P_B)$ *with* $w_{\mu,i} \in [v_\mu, u_\mu]$ *for* $i = 1, 2, 3$, *where*

$$\text{int}\, P_B = \left\{u \in C_0^1(\overline{\Omega})\,;\, u > 0 \textit{ in } \Omega \quad \textit{and} \quad \frac{\partial u}{\partial \nu} < 0 \textit{ on } \partial\Omega\right\}$$
$$\textit{and} \qquad \text{int}\, P_B = \{u \in C^1(\overline{\Omega})\,;\, u > 0 \textit{ on } \overline{\Omega}\}$$

in the Dirichlet case $(Bu = u)$ *or the Neumann case* $(Bu = \partial u/\partial\nu)$, *respectively.*

By applying the above theorem, we have the following result.

Corollary 2 *If one of the following conditions holds*

(i) $(H2)$, $(H4)$ *and* $(H5)$;
(ii) $(H2)$, $(H5)$ *and* $(H6)$;
(iii) $Bu = u$, $(H4)$ *and* $\operatorname{ess\,sup}_{x\in\Omega} \limsup_{|u|\to\infty} \frac{f(x,u)}{|u|^{p-2}u} \leq 0$,

then for sufficiently large μ, $(P)_\mu$ *has a positive solution* $w_{\mu,1} \in \operatorname{int} P_B$, *a negative solution* $w_{\mu,2} \in -\operatorname{int} P_B$ *and a sign-changing solution* $w_{\mu,3} \in X_B \setminus (P_B \cup -P_B)$.

1.2 The existence result in special cases

In this subsection, we state the existence result in the case where the nonlinearity contains a concave term near zero. Precisely, we consider the problem

$$\begin{cases} -\operatorname{div} A(x, \nabla u) = \psi_1(\mu) f_1(x, u) + \psi_2(\mu) f_2(x, u) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\widetilde{P})_\mu$$

where ψ_1 and ψ_2 are functions defined on some subset $\mathscr{M} \subset \mathbb{R}$ (ψ_1 can be identically zero). Here, we suppose that f_1 and f_2 satisfy (f).

Theorem 3 *Assume that*

- $(\widetilde{H1})$ *for each* $\mu \in \mathscr{M}$, *there exist a super-solution* $u_\mu \in \operatorname{int} P_B \cup \operatorname{int} P$ *and a sub-solution* $v_\mu \in -\operatorname{int} P_B \cup -\operatorname{int} P$ *of* $(\widetilde{P})_\mu$*;*
- $(\widetilde{H2})$ *there exist* $D_1 > 0$ *and* $\delta_0 > 0$ *such that* $f_1(x, u)u \geq -D_1|u|^p$ *for every* $|u| < \delta_0$ *and a.e.* $x \in \Omega$*;*
- $(\widetilde{H3})$ *there exists* $1 < \beta < p$ *such that*

$$\operatorname*{ess\,inf}_{x\in\Omega} \liminf_{u\to 0} \frac{f_2(x, u)}{|u|^{\beta-2}u} > 0;$$

- $(\widetilde{H4})$ $\psi_1(\mu) \geq 0$ *and* $\psi_2(\mu) > 0$ *for every* $\mu \in \mathscr{M}$.

Then, for every $\mu \in \mathscr{M}$, $(\widetilde{P})_\mu$ *has a positive solution* $w_{\mu,1} \in \operatorname{int} P_B$, *a negative solution* $w_{\mu,2} \in -\operatorname{int} P_B$ *and a sign-changing solution* $w_{\mu,3} \in X_B \setminus (P_B \cup -P_B)$ *with* $w_{\mu,i} \in [v_\mu, u_\mu]$ *for* $i = 1, 2, 3$.

Since any solutions are super- and sub-solutions, the following result follows from Theorem 3.

Corollary 4 *Let* $\mathscr{M} = \{\mu\}$ *for some* $\mu \in \mathbb{R}$. *Assume* $(\widetilde{H2})$, $(\widetilde{H3})$ *and* $(\widetilde{H4})$. *If* $(\widetilde{P})_\mu$ *has two solutions* $u \in \operatorname{int} P_B$ *and* $v \in -\operatorname{int} P_B$, *then* $(\widetilde{P})_\mu$ *has at least one sign-changing solution within the order interval* $[v, u]$.

Moreover, if we suppose the additional hypothesis that f_1 *and* f_2 *are odd in the second variable, then the existence of a solution belonging to* $\operatorname{int} P_B$ *ensures a pair of sign-changing solutions.*

Example 1 *In the following cases 1–3, it is known that there exists a positive solution (in* $\operatorname{int} P_B$*) for sufficiently small* $\mu > 0$ *of the* p*-Laplace equation*

$$-\Delta_p u = \psi_1(\mu) f_1(x, u) + \psi_2(\mu) f_2(x, u) \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \partial\Omega.$$

Thus, according to Corollary 4, there exists a sign-changing solution in each case for sufficiently small $\mu > 0$ since f_1 and f_2 are odd in the second variable.

Dirichlet problem:

1. $\psi_1(\mu) \equiv 1$, $\psi_2(\mu) = \mu$, $f_1(x, u) = |u|^{q-2}u$, $f_2(x, u) = |u|^{\beta-2}u$ with $1 < \beta < p < q \leq p^*$ (that is, ABC problem for the p-Laplacian);
2. $\psi_1(\mu) = \mu$, $\psi_2(\mu) \equiv 1$, $f_1(x, u) = a(x)|u|^{q-2}u$, $f_2(x, u) = |u|^{\beta-2}u - m(x)|u|^{p-2}u$, where $0 \neq m \in L^\infty(\Omega)_+$, $a \in C(\overline{\Omega})$ and $2 < \beta < p < q < p^*$.
   Refer to [20] and [1], respectively.

Neumann problem:

3. $\psi_1(\mu) = \mu$, $\psi_2(\mu) \equiv 1$, $f_1(x,u) = m(x)|u|^{p-2}u$, $f_2(x,u) = |u|^{\beta-2}u$ with $1 < \beta < p$, where $m \in L^\infty(\Omega)$ is a sign-changing function satisfying $\int_\Omega m(x)\,\mathrm{d}x \neq 0$ (see [8]).

Corollary 5 *Assume* $(\widetilde{H2})$, $(\widetilde{H}3)$ *and* $(\widetilde{H4})$. *Set*

$$\widetilde{f}(x,u,\mu) = \psi_1(\mu) f_1(x,u) + \psi_2(\mu) f_2(x,u).$$

Suppose that

(i) for every $\mu \in \mathscr{M}$ *there exist* $T(\mu)_- < 0 < T(\mu)_+$ *such that* $\widetilde{f}(x, T(\mu)_+, \mu) \le 0 \le \widetilde{f}(x, T(\mu)_-, \mu)$ *for a.e.* $x \in \Omega$*;*

or

(ii) $Bu = u$ *and for every* $\mu \in \mathscr{M}$,

$$\operatorname*{ess\,sup}_{x\in\Omega} \limsup_{|u|\to\infty} \frac{\widetilde{f}(x,u,\mu)}{|u|^{p-2}u} < \frac{C_0\lambda_1}{p-1},$$

where $\lambda_1 > 0$ *denotes the first eigenvalue of* $-\Delta_p$ *under the Dirichlet boundary condition. Then, for every* $\mu \in \mathscr{M}$, $(\widetilde{P})_\mu$ *has a positive solution* $w_{\mu,1} \in \operatorname{int} P_B$, *a negative solution* $w_{\mu,2} \in -\operatorname{int} P_B$ *and a sign-changing solution* $w_{\mu,3} \in X_B \setminus (P_B \cup -P_B)$.

2 The properties of the map A

In what follows, the norm on W_B is given by $\|u\|^p := \|\nabla u\|_p^p + \|u\|_p^p$, where $\|u\|_q$ denotes the usual norm of $L^q(\Omega)$ for $u \in L^q(\Omega)$ $(1 \le q \le \infty)$. Setting

$$G(x,y) := \int_0^{|y|} a(x,t)t\,\mathrm{d}t, \tag{2}$$

we can easily see that

$$\nabla_y G(x,y) = A(x,y) \quad \text{and} \quad G(x,0) = 0$$

for every $x \in \overline{\Omega}$ (see [27] for details).

Remark 6 the following assertions hold under condition (A):

(i) for all $x \in \overline{\Omega}$, $A(x,y)$ is maximal monotone and strictly monotone in y;
(ii) $|A(x,y)| \le \frac{C_1}{p-1}|y|^{p-1}$ for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$;
(iii) $A(x,y)y \ge \frac{C_0}{p-1}|y|^p$ for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$;
(iv) $G(x,y)$ is convex in y for all x and satisfies the following inequalities:

$$A(x,y)y \ge G(x,y) \ge \frac{C_0}{p(p-1)}|y|^p \quad \text{and} \quad G(x,y) \le \frac{C_1}{p(p-1)}|y|^p \tag{3}$$

for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$,

where C_0 and C_1 are the positive constants in (A).

Lemma 7 [14, Lemma 2.1.] *The map A satisfies the following inequalities:*

(i) $\left|A(x,y)-A(x,y')\right| \le c_1(|y|+|y'|)^{p-2}|y-y'|$;
(ii) $\left(A(x,y)-A(x,y')\right)\cdot(y-y') \ge c_2(|y|+|y'|)^{p-2}|y-y'|^2$ if $|y|+|y|'>0$;
(iii) $\left|A(x,y)-A(x,y')\right| \le c_3|y-y'|^{p-1}$ if $1<p\le 2$;
(iv) $\left(A(x,y)-A(x,y')\right)\cdot(y-y') \ge c_4|y-y'|^p$ if $p\ge 2$

for every $y,\ y'\in\mathbb{R}^N$ and $x\in\Omega$, where c_i is a positive constant ($i=1,2,3,4$).

The following result is important for the proof of the Palais–Smale condition for the functionals related to our problem.

Proposition 8 [27, Proposition 1] *Let $V: W_B\to W_B^*$ be the map defined by*

$$\langle V(u),v\rangle=\int_\Omega A(x,\nabla u)\nabla v\,\mathrm{d}x$$

for $u,\ v\in W_B$. Then, V has the $(S)_+$ property, that is, any sequence $\{u_m\}$ weakly convergent to u strongly converges to u provided $\limsup_{m\to\infty}\langle V(u_m),u_m-u\rangle\le 0$.

Proposition 9 *For $\lambda>0$, we define a map $T_\lambda: W_B\to W_B^*$ by*

$$\langle T_\lambda(u),v\rangle=\int_\Omega A(x,\nabla u)\nabla v\,\mathrm{d}x+\lambda\int_\Omega |u|^{p-2}uv\,\mathrm{d}x \tag{4}$$

for $u,\ v\in W_B$. Then, the inverse $T_\lambda^{-1}: W_B^*\to W_B$ of T_λ exists and it is continuous.

Proof We have that T_λ is injective due to the monotonicity of A and $|u|^{p-2}u$. Furthermore, T_λ is continuous being the potential operator of a C^1-function and is coercive on the basis of Remark 6 (iii). Since T_λ is monotone, hemicontinuous and coercive, we infer that T_λ is surjective (see [32, p. 557]). So, there exists the inverse operator $T_\lambda^{-1}: W_B^*\to W_B$, which is known to be strictly monotone, semicontinuous and bounded (see [32, p. 557]).

We show that $T_\lambda^{-1}: W_B^*\to W_B$ is continuous. Let $\xi_n\to\xi$ in W_B^*. There exists a unique $u_n\in W_B$ such that $T_\lambda(u_n)=\xi_n$. This ensures that

$$\begin{aligned}\min\left\{\frac{C_0}{p-1},\lambda\right\}\|u_n\|^p &\le \int_\Omega A(x,\nabla u_n)\nabla u_n\,\mathrm{d}x+\lambda\int_\Omega |u_n|^p\,\mathrm{d}x\\ &=\langle\xi_n,u_n\rangle\le\|\xi_n\|_{W_B^*}\|u_n\|,\end{aligned}$$

thereby the sequence $\{u_n\}$ is bounded in W_B. Hence, along a relabeled subsequence, we may suppose that $u_n\rightharpoonup u$ in W_B and $u_n\to u$ in $L^p(\Omega)$ as $n\to\infty$, with some $u\in W_B$. Passing to the limit in the equality

$$\int_\Omega A(x,\nabla u_n)\nabla(u_n-u)\,\mathrm{d}x+\lambda\int_\Omega |u_n|^{p-2}u_n(u_n-u)\,\mathrm{d}x=\langle\xi_n,u_n-u\rangle,$$

we find that

$$\lim_{n\to\infty}\int_\Omega A(x,\nabla u_n)\nabla(u_n-u)\,\mathrm{d}x=0.$$

Recalling that the operator $V: W_B\to W_B^*$ satisfies the $(S)_+$-property (refer to Proposition 8), it turns out that $u_n\to u$ strongly in W_B (for the whole sequence $\{u_n\}$). Hence, $\xi_n=T_\lambda(u_n)\to T_\lambda(u)$, which leads to $\xi=T_\lambda(u)$. Therefore, we have that $T_\lambda^{-1}(\xi_n)\to T_\lambda^{-1}(\xi)$. We conclude that T_λ^{-1} is continuous. □

The following result is proved by an argument similar to [22, Lemma 4.6.]. We give the proof in "Appendix".

Proposition 10 *Assume $\lambda > 0$, $1 < p \le N$ and $r > N/p$. Let T_λ be the map defined by* (4)*. Then, there exists a $D_2 > 0$ such that*

$$\|T_\lambda^{-1}(u)\|_\infty \le D_2 \|u\|_r^{1/(p-1)} \quad \text{for every } u \in L^r(\Omega),$$

so, T_λ^{-1} maps a bounded set of $L^r(\Omega)$ into a bounded set of $L^\infty(\Omega)$.

3 The existence of critical points via descending flow

Throughout this section, we suppose that $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $h(x,0) = 0$ a.e. $x \in \Omega$ and there exist $C > 0$ such that

$$|h(x,t)| \le C(1 + |t|^{p-1}) \quad \text{for every } t \in \mathbb{R}, \text{ a.e. } x \in \Omega, \tag{5}$$

Under (5), we define a C^1 functional J on W_B by

$$J(u) := \int_\Omega G(x, \nabla u)\, \mathrm{d}x - \int_\Omega \int_0^{u(x)} h(x,t)\, \mathrm{d}t \mathrm{d}x.$$

$$\text{There holds} \quad \langle J'(u), v\rangle = \int_\Omega A(x, \nabla u)\nabla v\, \mathrm{d}x - \int_\Omega h(x,u) v\, \mathrm{d}x$$

for $u, v \in W_B$.

Theorem 11 *Assume the following conditions:*

(A1) there exists $\lambda_0 > 0$ such that

$$h(x,u)u + \lambda_0 |u|^p \ge 0 \quad \text{for every } u \in \mathbb{R}, \text{ a.e. } x \in \Omega;$$

(A2) there exists $\gamma \in C([0,1], X_B)$ such that $\gamma(0) \in P_B$, $\gamma(1) \in -P_B$ and $\max_{t\in[0,1]} J(\gamma(t)) < 0$.

If J is coercive on W_B, then J has at least three critical points $w_1 \in \operatorname{int} P_B$, $w_2 \in -\operatorname{int} P_B$ and $w_3 \in X_B \setminus (P_B \cup -P_B)$.

Set

$$\varphi_p(u) := |u|^{p-2}u \quad \text{and} \quad B_\lambda(u) := T_\lambda^{-1}(h(\cdot, u) + \lambda \varphi_p(u)) \tag{6}$$

for $u \in W_B$ and $\lambda > 0$, where T_λ^{-1} is the inverse of T_λ (see Proposition 9 for the existence of the inverse). Note that under $N > p$, $h(\cdot, u) + \varphi_p(u) \in L^{p^*/(p^*-1)}(\Omega)$ provided $u \in L^{p^*(p-1)/(p^*-1)}(\Omega)$ and so $B_\lambda\colon L^{p^*(p-1)/(p^*-1)}(\Omega) \to W_B$ is well defined. Also, we note that $B_\lambda\colon W_B \to W_B$ is continuous according to Proposition 9 and (5).

Throughout this section, we denote the critical set of J by K, that is, $K := \{u \in W_B\, ;\ J'(u) = 0\}$.

Remark 12 If $u \in W_B \cap L^\infty(\Omega)$, then $v = B_\lambda(u) \in C^{1,\alpha}(\overline{\Omega})$ (some $\alpha \in (0,1)$) and v is a solution of

$$-\operatorname{div} A(x, \nabla v) + \lambda |v|^{p-2} v = h(x,u) + \lambda |u|^{p-2} u \quad \text{in } \Omega, \quad Bv = 0 \text{ on } \partial\Omega.$$

Indeed, $v = B_\lambda(u)$ satisfies

$$\int_\Omega A(x, \nabla v)\nabla w \,\mathrm{d}x + \lambda \int_\Omega |v|^{p-2} v w \,\mathrm{d}x = \int_\Omega h(x, u) w \,\mathrm{d}x + \lambda \int_\Omega |u|^{p-2} u w \,\mathrm{d}x$$

for every $w \in W_B$. Because of $u \in L^\infty(\Omega)$, we have $v \in L^\infty(\Omega)$ by the Moser iteration process or Lemma 13 in the next subsection. Therefore, we see that $v \in C^{1,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ by the regularity result in [23].

In the case of the Neumann problem, by [13, Theorem 3], v satisfies the boundary condition

$$0 = \frac{\partial v}{\partial \nu_A} = A(\cdot, \nabla v)\,\nu = a(\cdot, |\nabla v|)\frac{\partial v}{\partial \nu} \quad \text{in } W^{-1/q,q}(\partial\Omega)$$

for every $1 < q < \infty$ (see [13] for the definition of $W^{-1/q,q}(\partial\Omega)$). Since $v \in C^{1,\alpha}(\overline{\Omega})$ and $a(x, t) > 0$ for every $t \neq 0$, v satisfies the Neumann boundary condition, that is, $\frac{\partial v}{\partial \nu}(x) = 0$ for every $x \in \partial\Omega$.

Consequently, the critical points of J correspond to the fixed points of B_λ.

Moreover, we remark that $K \subset X_B$. In fact, if u is a critical point of J, then we have $u \in L^\infty(\Omega)$ by the Moser iteration process (refer to Theorem C in [26]). Thus, from the above argument, $u \in X_B$ follows.

3.1 Constructing a descending flow

Lemma 13 *For $\lambda > 0$, B_λ satisfies the following properties:*

(i) there exists a $D_3 = D_3(\lambda) > 0$ such that

$$\|B_\lambda(u)\|_\infty \le D_3(\|u\|_\infty + 1) \quad \text{for every } u \in L^\infty(\Omega);$$

(ii) for every $R > 0$, there exist $\alpha = \alpha(R, \lambda) \in (0, 1)$ and $M = M(R, \lambda) > 0$ such that $\|B_\lambda(u)\|_{C_B^{1,\alpha}(\overline{\Omega})} \le M$ for all $u \in L^\infty(\Omega)$ with $\|u\|_\infty \le R$;

(iii) If $N \ge p$ and $r > \max\{N/p, 1/(p-1)\}$, then there exists a $D_4 = D_4(\lambda) > 0$ such that

$$\|B_\lambda(u)\|_\infty \le D_4(\|u\|_{r(p-1)} + 1) \quad \text{for every } u \in L^{r(p-1)}(\Omega).$$

Proof (i) Let $u \in L^\infty(\Omega)$ and $v = B_\lambda(u) (\in W_B)$.

First, we consider $N < p$. Because $W_B \hookrightarrow L^\infty(\Omega)$ is continuous, there exists $D > 0$ such that $D\|w\|_\infty \le \|w\|$ for every $w \in W_B$. Hence, by Hölder's inequality and (5), we obtain

$$\begin{aligned}
&\min\left\{\frac{C_0}{p-1}, \lambda\right\} D^{p-1}\|v\|_\infty^{p-1}\|v\| \le \min\left\{\frac{C_0}{p-1}, \lambda\right\}\|v\|^p \\
&\le \int_\Omega A(x, \nabla v)\nabla v \,\mathrm{d}x + \lambda \int_\Omega |v|^p \,\mathrm{d}x = \langle T_\lambda(v), v\rangle = \langle h(\cdot, u) + \lambda\varphi_p(u), v\rangle \\
&\le \|v\|_p |\Omega|^{p/(p-1)}\left(C + (C+\lambda)\|u\|_\infty^{p-1}\right) \le \|v\||\Omega|^{p/(p-1)}\left(C + (C+\lambda)\|u\|_\infty^{p-1}\right).
\end{aligned}$$

This proves our conclusion.

In the case of $N \ge p$, we choose $r > N/p$. Then, by Proposition 10 and (5), we have

$$\|v\|_\infty \le D_2\|h(\cdot, u) + \lambda\varphi_p(u)\|_r^{1/(p-1)} \le 2^{\frac{1}{p-1}} D_2\left(C^{\frac{1}{p-1}} + (\lambda+C)^{\frac{1}{p-1}}\|u\|_\infty\right)|\Omega|^{1/(r(p-1))},$$

which establishes (i).

(ii) This assertion follows from (i) and the argument in Remark 12. Note that α and M depend on $\|u\|_\infty$ generally (refer to [23]).

(iii) By Proposition 10 and (5), we obtain

$$\|B_\lambda(u)\|_\infty \le D_2\|h(\cdot,u)+\lambda\varphi_p(u)\|_r^{1/(p-1)} \le D_2C'(1+(\lambda+1)\|u\|_{r(p-1)})$$

for every $u \in L^{r(p-1)}(\Omega)$, where $C' > 0$ is a constant independent of u.

□

Lemma 14 *Let $p < N$. Define inductively*

$$q_0 := p^* \quad \text{and} \quad q_{n+1} := p^* q_n/p = (p^*/p)^{n+1}p^*. \tag{7}$$

For every $n \in \mathbb{N}\cup\{0\}$, there exists a positive constant C^*_{n+1} such that

$$\|B_\lambda(u)\|_{q_{n+1}} \le C^*_{n+1}(1+\|u\|_{q_n}) \quad \text{for every } u \in L^{q_n}(\Omega).$$

Proof Let $u \in L^{q_n}(\Omega)$ and $v = B_\lambda(u) \in W_B$. Now, for $\varphi = v_+$ or v_- and $M > 0$, we put $\varphi_M(x) := \min\{\varphi(x), M\}$. By taking φ_M^{q+1} (if $\varphi = u_+$) or $-\varphi_M^{q+1}$ (if $\varphi = u_-$) with $q > 0$ as test function in $T_\lambda(v) = h(\cdot,u)+\lambda|u|^{p-2}u$ in W_B^*, (note that $\nabla(\varphi_M^{q+1}) = (q+1)\varphi_M^q\nabla\varphi_M$), we have

$$\begin{aligned}
&\frac{C_0(q+1)}{p-1}\int_\Omega |\nabla\varphi_M|^p\varphi_M^q\,\mathrm{d}x + \lambda\|\varphi_M\|_{p+q}^{p+q}\\
&\le (q+1)\int_\Omega \varphi_M^q\,A(x,\pm\nabla\varphi_M)(\pm\nabla\varphi_M)\,\mathrm{d}x + \lambda\|\varphi_M\|_{p+q}^{p+q}\\
&\le \int_\Omega (C+(C+\lambda)|u|^{p-1})|\varphi_M|^{q+1}\,\mathrm{d}x\\
&\le C'(1+(1+\lambda)\|u\|_{q_n}^{p-1})\|\varphi_M\|_{(q+1)q_n/(q_n-p+1)}^{q+1},
\end{aligned} \tag{8}$$

where we use (5), Hölder's inequality and Remark 6 (iii). On the other hand, the embedding of W_B into $L^{p^*}(\Omega)$ guarantees the existence of $C_* > 0$ satisfying

$$\begin{aligned}
C_*\|\varphi_M\|_{p^*(p+q)/p}^{p+q} &= C_*\|(\varphi_M)^{1+q/p}\|_{p^*}^p \le \|(\varphi_M)^{1+q/p}\|^p\\
&= (1+q/p)^p\int_\Omega |\nabla\varphi_M|^p\,(\varphi_M)^q\,\mathrm{d}x + \|\varphi_M\|_{p+q}^{p+q}.
\end{aligned} \tag{9}$$

Combining (8) and (9), it follows that

$$C_*\|\varphi_M\|_{p^*(p+q)/p}^{p+q} \le D(1+\|u\|_{q_n}^{p-1})\|\varphi_M\|_{(q+1)q_n/(q_n-p+1)}^{q+1},$$

where $D = D(\lambda,p,q,|\Omega|)$ is a positive constant independent of u and v. Here, we choose $q = q_n - p$. Then, we obtain

$$C_*\|\varphi_M\|_{q_{n+1}}^{p-1} = C_*\|\varphi_M\|_{p^*q_n/p}^{p-1} \le D(1+\|u\|_{q_n}^{p-1})|\Omega|^{(p^*-p)(q_n-p+1)/p^*q_n}$$

because of $q_n < q_np^*/p = q_{n+1}$ and using Hölder's inequality. Therefore, by letting $M \to +\infty$, we see that $v_\pm \in L^{q_{n+1}}(\Omega)$ and $\|v_\pm\|_{q_{n+1}} \le D'(1+\|u\|_{q_n})$ for some positive constant $D' = D'(\lambda,p,q_n,|\Omega|,C_*)$. □

Lemma 15 *Assume (A1) and let $\lambda_0 > 0$ be a constant as in (A1). Then, for each $\lambda > \lambda_0$, we have $B_\lambda(u) \in \pm\,\mathrm{int}\,P_B$ provided $u \in \pm\widetilde{P}_B \cap L^\infty(\Omega)\setminus\{0\}$, respectively.*

Proof We may assume that $u \in \widetilde{P}_B \cap L^\infty(\Omega) \setminus \{0\}$ by considering $-u$ in the other case. Let $u \in \widetilde{P}_B \cap L^\infty(\Omega) \setminus \{0\}$ and $v = B_\lambda(u)$. Then, $v \in C^{1,\alpha}(\overline{\Omega})$ (some $\alpha \in (0,1)$) and v is a solution of

$$-\mathrm{div}\, A(x, \nabla v) + \lambda |v|^{p-2} v = h(x, u) + \lambda u^{p-1} \quad \text{in } \Omega, \quad Bv = 0 \text{ on } \partial\Omega$$

(see Remark 12). By taking $-v_-$ as a test function, we obtain

$$\min\left\{\frac{C_0}{p-1}, \lambda\right\} \|v_-\|^p \le \int_\Omega A(x, \nabla v)(-\nabla v_-)\, \mathrm{d}x + \lambda \int_\Omega v_-^p\, \mathrm{d}x$$
$$= -\int_\Omega (h(x,u) + \lambda u^{p-1})\, v_-\, \mathrm{d}x \le 0$$

since $h(x,u) + \lambda u^{p-1} \ge 0$ holds by $(A1)$, whence $v_- = 0$ a.e. Ω. Furthermore, we note that $h(\cdot, u) + \lambda u^{p-1} \ne 0$ in W_B^* by the inequality

$$\langle h(\cdot, u) + \lambda u^{p-1}, u\rangle = \int_\Omega (h(x,u) + \lambda_0 u^{p-1}) u\, \mathrm{d}x + (\lambda - \lambda_0)\int_\Omega u^p\, \mathrm{d}x$$
$$\ge (\lambda - \lambda_0)\int_\Omega u^p\, \mathrm{d}x > 0$$

(note $u \ne 0$). This yields $v \ne 0$ because of $v = B_\lambda(u) = T_\lambda^{-1}(h(\cdot, u) + \lambda \varphi_p(u))$.

By noting that $v \in C^{1,\alpha}(\overline{\Omega})$ and $v \ne 0$, we have $v(x) > 0$ for every $x \in \Omega$ by Theorem B in [26]. In addition, due to the strong maximum principle (see Theorem A in [26]), we easily see that $\partial v(x)/\partial \nu < 0$ for every $x \in \partial\Omega$ provided $v(x) = 0$. Hence, under the Neumann boundary condition (that is, $Bv = \partial v/\partial \nu = 0$), $v(x) > 0$ for every $x \in \overline{\Omega}$. This means that $v \in \mathrm{int}\, P_B$. □

The proof of the following lemma can be shown by the argument in [5, Lemma 3.7 and 3.8] and Lemma 7. Thus, we omit the proof.

Lemma 16 *Let $\lambda > 0$. Then, there exist $d_i = d_i(\lambda) > 0$ $(1 \le i \le 4)$ such that*

(i) $\langle J'(u), u - B_\lambda(u)\rangle \ge d_1 \|u - B_\lambda(u)\|^2 (\|u\| + \|B_\lambda(u)\|)^{p-2}$ if $1 < p \le 2$*;*
(ii) $\langle J'(u), u - B_\lambda(u)\rangle \ge d_2 \|u - B_\lambda(u)\|^p$ if $p \ge 2$*;*
(iii) $\|J'(u)\|_{W_B^*} \le d_3 \|u - B_\lambda(u)\|^{p-1}$ if $1 < p \le 2$*;*
(iv) $\|J'(u)\|_{W_B^*} \le d_4 \|u - B_\lambda(u)\| (\|u\| + \|B_\lambda(u)\|)^{p-2}$ if $p \ge 2$

for every $u \in W_B$, where B_λ is the operator defined by (6)*.*

The next result follows from a similar argument as in [5, Lemma 4.1.] using the properties of B_λ described in Lemmas 16, 15, 14 and 13.

Lemma 17 *Let $\lambda > \lambda_0$ (λ_0 being the positive constant as in $(A1)$). Then, there exists a locally Lipschitz continuous operator V_λ from $X_B \setminus K$ into X_B such that*

(i) For $u \in X_B \setminus K$, we have

$$\langle J'(u), u - V_\lambda(u)\rangle \ge \frac{d_1}{2}\|u - B_\lambda(u)\|^2 (\|u\| + \|B_\lambda(u)\|)^{p-2} \quad \text{if } 1 < p \le 2;$$
$$\langle J'(u), u - V_\lambda(u)\rangle \ge \frac{d_2}{2}\|u - B_\lambda(u)\|^p \quad \text{if } p \ge 2;$$
$$\frac{1}{2}\|u - B_\lambda(u)\| \le \|u - V_\lambda(u)\| \le 2\|u - B_\lambda(u)\|;$$

where d_1 and d_2 are the positive constants in Lemma 16.

(ii) *$V_\lambda(u) \in \pm \operatorname{int} P_B$ for every $u \in \pm P_B \setminus K$, respectively;*

(iii) *For the sequence $\{q_n\}$ defined by (7), V_λ satisfies*

$$\|V_\lambda(u)\|_{q_{n+1}} \le C^*_{n+1}(2 + |\Omega| + \|u\|_{q_n}) \quad \text{for every } u \in X_B \setminus K,$$

where C^*_{n+1} is the positive constant obtained in Lemma 14.

(iv) *If $N \ge p$ and $r > \max\{N/p, 1/(p-1)\}$, then V_λ satisfies*

$$\|V_\lambda(u)\|_\infty \le D_4(\|u\|_{r(p-1)} + 2 + |\Omega|) \quad \text{for every } u \in X_B \setminus K,$$

where D_4 is the positive constant obtained in Lemma 13 (iii) for r above;

(v) *there holds*

$$\|V_\lambda(u)\|_\infty \le D_3(2 + \|u\|_\infty) \quad \text{for every } u \in X_B \setminus K,$$

where D_3 is the positive constant obtained in Lemma 13 (i);

(vi) *for every $R > 0$, there exist $\alpha \in (0, 1)$ and $M > 0$ such that $\|V_\lambda(u)\|_{C^{1,\alpha}_B(\overline{\Omega})} \le M$ for every $u \in X_B \setminus K$ with $\|u\|_\infty \le R$.*

Proof For $u \in W_B \setminus K$, we define

$$\begin{aligned}
\delta_1(u) &:= \frac{1}{2}\|u - B_\lambda(u)\|,\\
\delta_2(u) &:= \frac{d_1}{2d_3}\|u - B_\lambda(u)\|^{3-p}(\|u\| + \|B_\lambda(u)\|)^{p-2} \quad \text{if } 1 < p \le 2,\\
\delta_2(u) &:= \frac{d_2}{2d_4}\|u - B_\lambda(u)\|^{p-1}(\|u\| + \|B_\lambda(u)\|)^{2-p} \quad \text{if } p > 2,
\end{aligned}$$

where $d_i > 0$ $(i = 1, 2, 3, 4)$ denotes a constant as in Lemma 16. Note that $\delta_1(u) > 0$ and $\delta_2(u) > 0$ by $u \notin K$. We denote the usual X_B norm by $\|\cdot\|_X$. Choose an open neighborhood $N(u)$ of u in X_B such that

$$N(u) := \left\{ \begin{array}{l} v \in X_B \setminus K \ ; \ \|u - v\|_X < 1/2,\ \delta_1(v) > \delta_1(u)/2,\ \delta_2(v) > \delta_1(u)/2, \\ \qquad\qquad\qquad \|B_\lambda(u) - B_\lambda(v)\| < \min\{\delta_1(u), \delta_2(u)\}/4 \end{array} \right\}.$$

Note that

$$\|v - w\|_X < 1 \quad \text{and} \quad \|B_\lambda(v) - B_\lambda(w)\| < \min\{\delta_1(v), \delta_1(w), \delta_2(v), \delta_2(w)\} \tag{10}$$

for every v, $w \in N(u)$. It is obvious that $\{N(u)\,;\, u \in X_B \setminus K\}$ is an open covering of $X_B \setminus K$. According to the paracompactness of $X_B \setminus K$, $X_B \setminus K$ has a locally finite open refinement $\{U_\xi\}_{\xi \in I}$ of $\{N(u)\,;\, u \in X_B \setminus K\}$. If there exists a U_ξ such that $U_\xi \cap P_B \ne \emptyset$ and $U_\xi \cap -P_B \ne \emptyset$, then we replace U_ξ with two open sets $U_\xi \setminus P_B$ and $U_\xi \setminus -P_B$. In this way, we may assume that U_ξ satisfies one of the following conditions:

(a) $U_\xi \cap P_B = \emptyset$ and $U_\xi \cap -P_B = \emptyset$; (b) $U_\xi \cap P_B \ne \emptyset$ and $U_\xi \cap -P_B = \emptyset$; (c) $U_\xi \cap P_B = \emptyset$ and $U_\xi \cap -P_B \ne \emptyset$.

For each case above, we choose a point u_ξ satisfying $u_\xi \in U_\xi$, $u_\xi \in U_\xi \cap P_B$ and $u_\xi \in U_\xi \cap -P_B$ in the case of (a), (b) and (c), respectively. Let $\{\psi_\xi\}_{\xi \in I}$ be a C^1 partition of the unity with respect to $\{U_\xi\}_{\xi \in I}$.

Define

$$V_\lambda(u) := \sum_{\xi \in I} \psi_\xi(u) B_\lambda(u_\xi)$$

for every $u \in X_B \setminus K$. It is easily shown that V_λ is locally Lipschitz continuous since $\{U_\xi\}_{\xi \in I}$ is locally finite.

(i) For $\xi \in I$ such that $\psi_\xi(u) \neq 0$, we have $\|B_\lambda(u_\xi) - B_\lambda(u)\| < \min\{\delta_1(u), \delta_2(u)\}$ by (10). Therefore, we get $\|B_\lambda(u) - V_\lambda(u)\| < \delta_1(u) = \|u - B_\lambda(u)\|/2$, whence $\|u - B_\lambda(u)\|/2 \le \|u - V_\lambda(u)\| \le 3\|u - B_\lambda(u)\|/2$ holds. The other inequalities follow from $\|B_\lambda(u_\xi) - B_\lambda(u)\| < \delta_2(u)$, the inequality for $\|J'(\cdot)\|_{W_B^*}$ in Lemma 16, (10) and $\langle J'(u), u - V_\lambda(u)\rangle \ge \langle J'(u), u - B_\lambda(u)\rangle - \|J'(u)\|_{W_B^*}\|B_\lambda(u) - V_\lambda(u)\|$.

(ii) Let $u \in P_B \setminus K$. Then, for $\xi \in I$ such that $\psi_\xi(u) \neq 0$, we see that $u_\xi \in P_B$ because U_ξ satisfies (b). Thus, $B_\lambda(u_\xi) \in \operatorname{int} P_B$ holds by Lemma 15. Since $\operatorname{int} P_B$ is convex, our assertion is proved. Similarly, we can show that $V_\lambda(u) \in -\operatorname{int} P_B$ provided $u \in (-P_B) \setminus K$.

(iii) According to Lemma 14, we obtain

$$\begin{aligned} \|V_\lambda(u)\|_{q_{n+1}} &\le C_{n+1}^* \sum_{\xi \in I} \psi_\xi(u)(1 + \|u_\xi\|_{q_n}) \\ &\le C_{n+1}^* \sum_{\xi \in I} \psi_\xi(u)(1 + \|u\|_{q_n} + \|u_\xi - u\|_{q_n}) \le C_{n+1}^*(2 + |\Omega| + \|u\|_{q_n}) \end{aligned}$$

by noting (10), $\|u_\xi - u\|_{q_n} \le \|u_\xi - u\|_\infty |\Omega|^{1/q_n}$ and $|\Omega|^{1/q_n} \le 1 + |\Omega|$.

(iv) By a similar argument to (iii) and Lemma 13 (iii), our conclusion holds.

(v) By a similar argument to (iii) and Lemma 13 (i), our conclusion holds.

(vi) For $\xi \in I$ such that $\psi_\xi(u) \neq 0$, $\|u_\xi\|_\infty \le \|u_\xi - u\|_\infty + \|u\|_\infty \le 1 + \|u\|_\infty$ holds. Thus, if $\|u\|_\infty \le R$, then by Lemma 13 (ii), there exist $\alpha = \alpha(R+1, \lambda) \in (0, 1)$ and $M = M(R+1, \lambda) > 0$ with $\|B_\lambda(u_\xi)\|_{C_B^{1,\alpha}(\overline{\Omega})} \le M$ for $\xi \in I$ such that $\psi_\xi(u) \neq 0$. Hence, our conclusion follows.

□

Fix $\lambda > \lambda_0$, where $\lambda_0 > 0$ is the constant as in $(A1)$. For $u \in X_B \setminus K$, we consider the following initial value problem in X_B:

$$\begin{cases} \dfrac{d\eta}{dt}(t) = -\eta(t) + V_\lambda(\eta(t)) \\ \eta(0) = u, \end{cases}$$

where V_λ is the locally Lipschitz continuous map from $X_B \setminus K$ into X_B constructed in Lemma 17. Let $\eta(t, u)$ be the unique solution of the above problem considered in X_B. Moreover, we denote by $[0, \tau(u))$ the right maximal interval of existence of $\eta(t, u)$.

Lemma 18 *The following assertions hold:*

(i) $J(\eta(t, u)) \le J(u)$ for every $u \in X_B \setminus K$ and $t \in [0, \tau(u))$;

(ii) If $\tau(u) < +\infty$ and $\eta(t, u)$ weakly converges to some w in W_B as $t \to \tau(u) - 0$, then $w \in K$ and $\eta(t, u)$ converges to w in X_B as $t \to \tau(u) - 0$.

(iii) if $\eta(t, u)$ converges to some w in W_B as $t \to \tau(u) - 0$, then $w \in X_B$ and $\eta(t, u)$ converges to w in X_B as $t \to \tau(u) - 0$.

Proof (i) Let $u \in X_B \setminus K$ and $t \in [0, \tau(u))$. Then, by the property of V_λ in Lemma 17 (i), $d\,J(\eta(t, u))/dt \le 0$. Thus, our conclusion holds.

(ii) Let $\tau(u) < +\infty$. Note that $\eta(t, u)$ satisfies

$$\eta(t, u) = e^{-t}u + \int_0^t e^{-t+s} V_\lambda(\eta(s, u))\, ds \quad \text{for } 0 \le t < \tau(u)$$

in X_B. Then, we obtain

$$\|\eta(t,u)\|_\infty \le e^{-t}\|u\|_\infty + 2D_3 + D_3\int_0^t e^{-t+s}\|\eta(s,u)\|_\infty\, ds \quad \text{for } 0 \le t < \tau(u)$$

by Lemma 17 (v). Hence, due to Gronwall's inequality, we have

$$\|\eta(t,u)\|_\infty \le e^{D_3\tau(u)}(\|u\|_\infty + 2D_3) \quad \text{for every } t \in [0,\tau(u)).$$

Therefore, it follows from Lemma 17 (vi) that there exists $\alpha \in (0,1)$ such that $\{\int_0^t e^{-t+s}V_\lambda(\eta(s,u))\, ds\,;\ t \in [0,\tau(u))\}$ is bounded in $C_B^{1,\alpha}(\overline{\Omega})$. Since the embedding of $C_B^{1,\alpha}(\overline{\Omega})$ into X_B is compact and $\eta(t,u) \rightharpoonup w$ in W_B as $t \to \tau(u)-0$, $\eta(t,u)$ converges to w in X_B as $t \to \tau(u)-0$. By the definition of $\tau(u)$ and $\tau(u) < +\infty$, we see that $w \in K$ holds.

(iii) For the proof, it suffices to show that $\{\int_0^t e^{-t+s}V_\lambda(\eta(s,u))\, ds\,;\ t \in [0,\tau(u))\}$ is bounded in $L^\infty(\Omega)$ proceeding as in (ii). In the case of $p > N$, it is obvious because $W_B \hookrightarrow L^\infty(\Omega)$ is continuous and $\eta(t,u)$ converges to some w in W_B.

In the case of $N = p$, then by taking an $r > \max\{1, 1/(p-1)\}$, we obtain

$$\int_0^t e^{-t+s}\|V_\lambda(\eta(s,u))\|_\infty\, ds \le D_4(2 + |\Omega| + \sup_{t\in[0,\tau(u))}\|\eta(t,u)\|_{r(p-1)})$$

due to Lemma 17 (iv) and the continuity of $W_B \hookrightarrow L^{r(p-1)}(\Omega)$ (note that $\{\eta(t,u)\,;\ t \in [0,\tau(u))\}$ is bounded in W_B).

Now, we consider the case of $N > p$. Note that we can choose $n_0 \in \mathbb{N}$ such that $q_{n_0} > N(p-1)/p$ since $q_n \to \infty$ as $n \to \infty$, where $\{q_n\}$ is the increasing sequence defined by (7). By considering

$$W_B \xrightarrow{V_\lambda} L^{q_0}(\Omega) \xrightarrow{V_\lambda} L^{q_1}(\Omega) \xrightarrow{V_\lambda} \cdots \xrightarrow{V_\lambda} L^{q_{n_0}}(\Omega) \xrightarrow{V_\lambda} L^\infty(\Omega)$$

(that is, a bootstrap argument), we can show that $\sup\{\|V_\lambda(\eta(t,u))\|_\infty\,;\ t \in [0,\tau(u))\} < \infty$ holds by $\sup\{\|\eta(t,u)\|\,;\ t \in [0,\tau(u))\} < \infty$ because V_λ is bounded from $L^{q_n}(\Omega)$ to $L^{q_{n+1}}(\Omega)$ and also from $L^{q_{n_0}}(\Omega)$ to $L^\infty(\Omega)$ due to Lemma 17 (iii) and (iv) , respectively. This boundedness leads to our claim. □

3.2 Proof of Theorem 11

The next result follows from Lemma 17 (ii) and the argument in [24, Lemma 3.2.]. We omit the proof.

Lemma 19 *If $u \in \pm P_B \setminus K$, then $\eta(t,u) \in \pm \operatorname{int} P_B$ for every $0 < t < \tau(u)$.*

The following result is well known (see [24, Lemma 2.3.]).

Lemma 20 *Set*

$$Q_\pm := \{u \in X_B \setminus K\,;\ \eta(t,u) \in \pm \operatorname{int} P_B \text{ for some } t \in [0,\tau(u))\} \cup (\pm \operatorname{int} P_B). \quad (11)$$

Then, Q_+ and Q_- are open subsets of X_B and they are invariant for the descending flow η, that is, $\eta(t,u) \in Q_\pm$ for every $t \in [0,\tau(u))$ provided $u \in Q_\pm \setminus K$, respectively. In addition, $\partial Q_\pm$ are invariant closed subsets of X_B for the descending flow η, where $\partial Q_\pm$ denotes the boundary of $Q_\pm$ in X_B.

An additional preliminary result is needed.

Lemma 21 *If J is coercive on W_B, then for every $\beta \in \mathbb{R}$, there exists an $R = R(\beta) > 0$ such that $\|u\| \le R$ and $\|B_\lambda(u)\| \le R$ for every $u \in J^{-1}((-\infty, \beta])$.*

Proof By the coercivity of J, we have an $R_1 > 0$ satisfying $\|u\| \le R_1$ for every $u \in J^{-1}((-\infty, \beta])$. Let $u \in J^{-1}((-\infty, \beta])$ and $v = B_\lambda(u)$, namely $T_\lambda(v) = h(\cdot, u) + \lambda\varphi_p(u)$ in W_B^*. By taking v as a test function, we obtain

$$\min\left\{\frac{C_0}{p-1}, \lambda\right\}\|v\|^p \le \int_\Omega A(x, \nabla v)\nabla v \,\mathrm{d}x + \lambda\|v\|_p^p$$

$$\le (C+\lambda)\int_\Omega |u|^{p-1}|v|\,\mathrm{d}x + C\|v\|_1 \le (C+\lambda)\|u\|_p^{p-1}\|v\|_p + C|\Omega|^{(p-1)/p}\|v\|_p$$

$$\le (C+\lambda)R_1^{p-1}\|v\| + C|\Omega|^{(p-1)/p}\|v\|$$

[we use the Hölder's inequality and Remark 6 (iii)], where C is a positive constant as in (5). Because $p > 1$, this yields the desired conclusion. □

Proof of Theorem 11 First, we note that the boundary of $\pm P_B$ in X_B includes no nontrivial critical points of J by combining Lemma 15 and the fact that the critical points of J are exactly the fixed points of B_λ. Due to the $(S)_+$ property of V (see Proposition 8) and the compactness of $W_B \hookrightarrow L^p(\Omega)$, it is clear that J satisfies the *bounded* Palais–Smale condition.

Choose a constant β satisfying $\max_{t\in[0,1]} J(\gamma(t)) < \beta < 0$, where γ is the continuous path in $(A2)$. Since it follows from Lemmas 19 and 20 that $\gamma(0) \in Q_+$, $\gamma(1) \in Q_-$ and $Q_\pm$ are open in X_B, there exist $0 < t_+ \le t_- < 1$ such that $\gamma(t_+) \in \partial Q_+$ and $\gamma(t_-) \in \partial Q_-$ (it may happen that $t_+ = t_-$ because it is not known whether $\partial Q_+ \ne \partial Q_-$). Set $u_1 := \gamma(0)$, $u_2 := \gamma(1)$ and $u_3 := \gamma(t_+)$. Note that there exists an $R > 0$ such that

$$\|\eta(t, u_i)\| \le R \quad \text{and} \quad \|B_\lambda(\eta(t, u_i))\| \le R \quad \text{for every } t \in [0, \tau(u_i)) \tag{12}$$

by Lemma 21 and $\inf_{W_B} J \le J(\eta(t, u_i)) \le \beta$ for every $t \in [0, \tau(u_i))$. Therefore, if $\tau(u_i) < \infty$ holds $(i = 1, 2, 3)$, then we have for every $0 < t_1 < t_2 < \tau(u_i) < \infty$

$$\|\eta(t_1, u_i) - \eta(t_2, u_i)\| \le \int_{t_1}^{t_2} \|\eta(s, u_i) - V_\lambda(\eta(s, u_i))\|\,\mathrm{d}s$$

$$\le 2\int_{t_1}^{t_2} \|\eta(s, u_i) - B_\lambda(\eta(s, u_i))\|\,\mathrm{d}s \le 4R(t_2 - t_1)$$

by Lemma 17 (i) and (12). Hence, $\eta(t, u_i)$ converges to some w_i in W_B as $t \to \tau(u_i) - 0$ provided $\tau(u_i) < \infty$. Moreover, according to Lemmas 18, 19 and 20, $w_i \in K$, $\eta(t, u_i)$ converges to w_i in X_B as $t \to \tau(u_i) - 0$, $J(w_i) \le J(u_i) \le \beta < 0$ $(i = 1, 2, 3)$ and $w_i \in \operatorname{int} P_B$ if $i = 1$, $w_i \in -\operatorname{int} P_B$ if $i = 2$ and $w_i \in \partial Q_+$ if $i = 3$. Because of $\partial Q_+ \cap (\pm P_B \setminus \{0\}) = \emptyset$ (note that $\pm P_B \setminus \{0\} \subset Q_\pm$), our conclusion is shown when $\tau(u_i) < \infty$ for every $i = 1, 2, 3$. Thus, we suppose that $\tau(u_i) = \infty$ for some $i \in \{1, 2, 3\}$. We claim that there exists a sequence $\{t_n\} \subset \mathbb{R}^+$ such that

$$t_n \to +\infty \quad \text{and} \quad J'(\eta(t_n, u_i)) \to 0 \quad \text{in } W_B^* \quad \text{as } n \to \infty.$$

If our claim is shown, then it provides the existence of a Palais–Smale sequence of J which is bounded because of (12). Thus, there exists $w_i \in W_B \cap K$ such that $\lim_{n\to\infty} \eta(t_n, u_i) = w_i$

in W_B by choosing a subsequence if necessary. Furthermore, by the argument in Lemma 18 (iii) and (12), we see that $\{\eta(t, u_i)\,;\ t \geq 0\}$ is bounded in $C_B^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. This yields that $\lim_{n\to\infty} \eta(t_n, u_i) = w_i$ in X_B due to the compactness of $C_B^{1,\alpha}(\overline{\Omega}) \hookrightarrow X_B$ and $\lim_{n\to\infty} \eta(t_n, u_i) = w_i$ in W_B. Therefore, there holds $w_i \in \operatorname{int} P_B$ if $i = 1$, $w_i \in -\operatorname{int} P_B$ if $i = 2$ and $w_i \in X_B \setminus (P_B \cup -P_B)$ if $i = 3$.

Finally, we prove our claim. Note that there exists a sequence $\{t_n\} \subset \mathbb{R}^+$ such that $t_n \to +\infty$ and $\frac{\mathrm{d}}{\mathrm{d}t} J(\eta(t_n, u_i)) \to 0$ because $-\infty < \inf_{W_B} J \leq J(\eta(t, u_i)) \leq \beta$ for every $t \geq 0$ and $J(\eta(t, u_i))$ is nondecreasing in t.

In the case of $1 < p \leq 2$, the following inequality follows from Lemma 16 (iii), Lemma 17 (i) and (12):

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t} J(\eta(t, u_i)) &\geq \frac{d_1}{2} \|\eta(t, u_i) - B_\lambda(\eta(t, u_i))\|^2 (\|\eta(t, u_i)\| + \|B_\lambda(\eta(t, u_i))\|)^{p-2} \\
&\geq d_1 d_3^{-2/(p-1)} 2^{p-3} R^{p-2} \|J'(\eta(t, u_i))\|_{W_B^*}^{2/(p-1)}
\end{aligned}
$$

for every $t > 0$. Similarly, in the case of $p > 2$, we obtain

$$
-\frac{\mathrm{d}}{\mathrm{d}t} J(\eta(t, u_i)) \geq 2^{2p-p^2-1} R^{2p-p^2} d_2 d_4^{-p} \|J'(\eta(t, u_i))\|_{W_B^*}^{p}
$$

for every $t > 0$. These inequalities applied for the sequence $\{t_n\}$ imply the proof of our claim. □

4 Proof of Theorem 1 and Corollary 2

Throughout this section, we denote a super-solution and a sub-solution of $(P)_\mu$ in $(H1)$ by u_μ and v_μ, respectively. We define

$$
f_{[v_\mu, u_\mu]}(x, u) := \begin{cases} f(x, u_\mu(x)) & \text{if} \quad u \geq u_\mu(x), \\ f(x, u) & \text{if} \quad v_\mu(x) < u < u_\mu(x), \\ f(x, v_\mu(x)) & \text{if} \quad u \leq v_\mu(x), \end{cases} \tag{13}
$$

Moreover, we set

$$
h_\mu(x, u) := \mu f_{[v_\mu, u_\mu]}(x, u) - p(u - u_\mu(x))_+^{p-1} + p(u - v_\mu(x))_-^{p-1}. \tag{14}
$$

Lemma 22 *Assume $(H1)$ and $(H2)$. Then, for every $\mu > \mu_0$, there exists a $\lambda = \lambda(\mu) > 0$ such that*

$$
h_\mu(x, u)u + \lambda |u|^p \geq 0 \quad \textit{for every } u \in \mathbb{R},\ \textit{a.e. } x \in \Omega.
$$

Proof Because f is bounded on each bounded set and by $(H2)$, there exists $\lambda_0 > 0$ such that

$$
f(x, t)t + \lambda_0 |t|^p \geq 0 \quad \text{for every } |t| \leq \max\{\|v_\mu\|_\infty, \|u_\mu\|_\infty\},\ \text{a.e. } x \in \Omega. \tag{15}
$$

Set $\lambda := \mu\lambda_0 + p$. If $u_\mu(x) \geq t \geq v_\mu(x)$ holds, then $h_\mu(x, t)t + \lambda|t|^p = \mu(f(x, t)t + \lambda_0|t|^p) + p|t|^p \geq 0$ follows from (15). In the case of $t > u_\mu(x) (\geq 0)$, we have

$$
\begin{aligned}
h_\mu(x, t)t + \lambda|t|^p = \mu t(f(x, u_\mu(x)) + \lambda_0 u_\mu(x)^{p-1}) + \lambda_0(t^{p-1} - u_\mu(x)^{p-1})\mu t \\
+ p(t^{p-1} - (t - u_\mu(x))^{p-1})t \geq 0
\end{aligned}
$$

since s^{p-1} is nondecreasing on $\mathbb{R}^+$ and (15). Similarly, we can show that $h_\mu(x, t)t + \lambda|t|^p \geq 0$ for $t < v_\mu(x)$. □

Now, we introduce the functional I_μ on W_B by

$$I_\mu(u) := \int_\Omega G(x, \nabla u)\,\mathrm{d}x - \int_\Omega \int_0^{u(x)} h_\mu(x,t)\,\mathrm{d}t\mathrm{d}x \tag{16}$$

$$= \int_\Omega G(x, \nabla u)\,\mathrm{d}x - \mu \int_\Omega \int_0^{u(x)} f_{[v_\mu,u_\mu]}(x,t)\,\mathrm{d}t\mathrm{d}x + \|(u-u_\mu)_+\|_p^p + \|(u-v_\mu)_-\|_p^p \tag{17}$$

for $u \in W_B$ [see (2), (14) and (13) for the definitions of G, h_μ and $f_{[v_\mu,u_\mu]}$].

Because $f_{[v_\mu,u_\mu]}(\cdot,u) \in L^\infty(\Omega)$ for every $u \in W_B$ by $u_\mu,\ v_\mu \in L^\infty(\Omega)$, we easily prove the following result due to the last two terms in (17) [that is, we use that $\|(u-w)_\pm\|_p/\|u_\pm\|_p \to 1$ as $\|u_\pm\|_p \to \infty$ for $w \in L^\infty(\Omega)$]. See Lemma 11 in [26] for the proof.

Lemma 23 *Assume* $(H1)$. *Then, for every* $\mu > \mu_0$, I_μ *is coercive on* W_B.

Moreover, we state the following important fact.

Lemma 24 *Assume* $(H1)$ *and let* $\mu > \mu_0$. *If* $u \in W_B$ *is a critical point of* I_μ, *then* u *satisfies* $v_\mu(x) \le u(x) \le u_\mu(x)$ *for a.e.* $x \in \Omega$. *Therefore,* u *is a solution of* $(P)_\mu$ *with* $u \in [v_\mu, u_\mu] = \{w \in W_B\,;\ v_\mu \le w(x) \le u_\mu \text{ a.e. } x \in \Omega\}$.

Proof This proof has been essentially done in [26, Lemma 14.]. For the readers' convenience, we give it.

Let $u \in W_B$ be a critical point of I_μ. Because v_μ and u_μ are a sub-solution and a super-solution of $(P)_\mu$, we have

$$\int_\Omega A(x,\nabla v_\mu)\nabla w\,\mathrm{d}x \le \mu \int_\Omega f(x, v_\mu(x))w\,\mathrm{d}x = \mu \int_\Omega f_{[v_\mu,u_\mu]}(x, v_\mu(x))w\,\mathrm{d}x$$

$$\int_\Omega A(x,\nabla u_\mu)\nabla w\,\mathrm{d}x \ge \mu \int_\Omega f(x, u_\mu(x))w\,\mathrm{d}x = \mu \int_\Omega f_{[v_\mu,u_\mu]}(x, u_\mu(x))w\,\mathrm{d}x \tag{18}$$

for every $w \in W_B$ with $w \ge 0$. Because of $(u-u_\mu)_+ \in W_B$ (note that in the definition of a super-solution, we assume that $u_\mu \ge 0$ on $\partial\Omega$ in the Dirichlet problem), by taking $(u-u_\mu)_+$ as a test function in $I'_\mu(u) = 0$ and (18), we have

$$\begin{aligned} 0 &\ge \langle I'_\mu(u), (u-u_\mu)_+\rangle - \int_\Omega A(x,\nabla u_\mu)\nabla(u-u_\mu)_+\,\mathrm{d}x \\ &\quad + \mu \int_\Omega f_{[v_\mu,u_\mu]}(x, u_\mu(x))(u-u_\mu)_+\,\mathrm{d}x \\ &= \int_{u \ge u_\mu} (A(x,\nabla u) - A(x,\nabla u_\mu))(\nabla u - \nabla u_\mu)\,\mathrm{d}x + p\|(u-u_\mu)_+\|_p^p \ge 0 \end{aligned}$$

(note that $u_\mu \ge v_\mu$ and the map A is strictly monotone in the second variable). This leads to $u(x) \le u_\mu(x)$ for a.e. $x \in \Omega$. Similarly, we obtain $u(x) \ge v_\mu(x)$ for a.e. $x \in \Omega$ by replacing $(u-u_\mu)_+$ and u_μ with $-(u-v_\mu)_-$ and v_μ, respectively. Consequently, u is a solution of $(P)_\mu$ with $v_\mu \le u \le u_\mu$ because of $h_\mu(x,u) = \mu f_{[v_\mu,u_\mu]}(x,u) = \mu f(x,u)$ (see Remark 12 for the boundary condition). □

Lemma 25 *Assume* $(H1)$ *and* $(H3)$. *Then, there exist* $\mu_1 > \mu_0$, $u_0 \in P_B$ *and* $v_0 \in -P_B$ *such that*

$$\max_{t\in[0,1]} I_\mu(tu_0 + (1-t)v_0) < 0 \quad \textit{for every } \mu \geq \mu_1.$$

Proof Let Ω_1 and Ω_2 be open subsets as in $(H3)$. By taking new open subsets of Ω_1 and Ω_2 if necessary, we may assume that $\Omega_1 \cap \Omega_2 = \emptyset$.

Choose a nonnegative function $u_0 \in C_0^\infty(\overline{\Omega}_1)$ and a nonpositive function $v_0 \in C_0^\infty(\overline{\Omega}_2)$ such that $\|u_0\|_\infty < \min\{d_1, \delta_0\}$, $u_0 > 0$ somewhere, $\|v_0\|_\infty < \min\{d_2, \delta_0\}$ and $v_0 < 0$ somewhere, where δ_0, d_1 and d_2 are positive constants as in $(H3)$.

Then, by the sign-condition for f on Ω_1 and Ω_2 as in $(H3)$, we have $\int_\Omega F(x, tu_0)\,\mathrm{d}x > 0$ and $\int_\Omega F(x, tv_0)\,\mathrm{d}x > 0$ for every $0 < t \leq 1$, where $F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s$. Define

$$d_+ := \min_{1\geq t\geq 1/2} \int_\Omega F(x, tu_0)\,\mathrm{d}x > 0, \quad d_- := \min_{1\geq t\geq 1/2} \int_\Omega F(x, tv_0)\,\mathrm{d}x > 0,$$

and choose a positive number μ_1 satisfying $\mu_1 > \mu_0$ and

$$\mu_1 > \frac{C_1}{p(p-1)\min\{d_+, d_-\}} \left(\|\nabla u_0\|_p^p + \|\nabla v_0\|_p^p \right), \tag{19}$$

where $C_1 > 0$ is the positive constant in (A) (ii). Because $\operatorname{supp} u_0 \cap \operatorname{supp} v_0 = \emptyset$, $0 \leq u_0 \leq d_1 \leq u_\mu$ in Ω_1 and $0 \geq v_0 \geq -d_2 \geq v_\mu$ in Ω_2 hold, it is clear that

$$\int_\Omega H_\mu(x, tu_0 + (1-t)v_0)\,\mathrm{d}x = \mu \int_\Omega F(x, tu_0)\,\mathrm{d}x + \mu \int_\Omega F(x, (1-t)v_0)\,\mathrm{d}x$$

for every $t \in [0,1]$, where $H_\mu(x,s) := \int_0^s h_\mu(x,\tau)\,\mathrm{d}\tau$. Therefore, we have

$$\begin{aligned} I_\mu(tu_0 + (1-t)v_0) &= \int_{\Omega_1} G(x, t\nabla u_0)\,\mathrm{d}x + \int_{\Omega_2} G(x, (1-t)\nabla v_0)\,\mathrm{d}x \\ &\quad - \mu \int_\Omega F(x, tu_0)\,\mathrm{d}x - \mu \int_\Omega F(x, (1-t)v_0)\,\mathrm{d}x \\ &\leq \frac{C_1}{p(p-1)} \left(t^p \|\nabla u_0\|_p^p + (1-t)^p \|\nabla v_0\|_p^p \right) \\ &\quad - \mu \int_\Omega F(x, tu_0)\,\mathrm{d}x - \mu \int_\Omega F(x, (1-t)v_0)\,\mathrm{d}x \end{aligned} \tag{20}$$

for every $t \in [0,1]$, where we use (3) in the last inequality. If $\mu \geq \mu_1$ and $0 \leq t \leq 1/2$, then from (19), (20) and the definition of d_-,

$$I_\mu(tu_0 + (1-t)v_0) \leq \frac{C_1}{p(p-1)} \left(\|\nabla u_0\|_p^p + \|\nabla v_0\|_p^p \right) - \mu d_- < 0$$

follows. Similarly, in the case of $\mu \geq \mu_1$ and $1 \geq t \geq 1/2$, we easily obtain the inequality $I_\mu(tu_0 + (1-t)v_0) \leq C_1(\|\nabla u_0\|_p^p + \|\nabla v_0\|_p^p)/(p(p-1)) - \mu d_+ < 0$. Hence, our conclusion holds since $u_0 \in P_B$ and $v_0 \in -P_B$. □

Lemma 26 *Assume* $(H1)$, $(H2)$ *and* $(H4)$. *In addition, we suppose that* $u_\mu \in \operatorname{int} P_B \cup \operatorname{int} P$ *and* $v_\mu \in -\operatorname{int} P_B \cup -\operatorname{int} P$ *for every* $\mu > \mu_0$. *Then, there exist* $\mu_1 > \mu_0$, $u_0 \in P_B$ *and*

$v_0 \in -P_B$ *such that for every* $\mu \geq \mu_1$ *we can choose an* $r_\mu > 0$ *satisfying*

$$\max_{t\in[0,1]} I_\mu(tr_\mu u_0 + (1-t)r_\mu v_0) < 0.$$

Proof Let $m \in L^\infty(\Omega)$ and $\delta_1 > 0$ be as in $(H4)$. Then, since $|\{x \in \Omega\,;\, m(x) > 0\}| > 0$, we can take two open balls B_1 and B_2 such that $B_1 \cap B_2 = \emptyset$ and $|\{x \in B_i\,;\, m(x) > 0\}| > 0$ $(i = 1, 2)$ (refer to [15, Corollary 2, p 28] for the existence). It is well known that the first eigenvalue of the following weighted eigenvalue (Dirichlet) problem for the p-Laplacian on B_i is positive and simple and that it has a positive eigenfunction belonging to $C_0^1(\overline{B}_i)$ (refer to [3] and [16, section 6.2]):

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in } B_i, \quad u = 0 \quad \text{on } \partial B_i, \quad (i = 1, 2), \tag{21}$$

where Δ_p denotes the p-Laplacian. Therefore, since the above eigenvalue problem is $(p-1)$ homogeneous, for each $i = 1, 2$, there exist a positive solution $\psi_i \in C_0^1(\overline{B}_i)$ with $\|\psi_i\|_{C_0^1(\overline{B}_i)} = 1$ of

$$-\Delta_p u = \lambda_1(m, B_i)m(x)|u|^{p-2}u \quad \text{in } B_i, \quad u = 0 \quad \text{on } \partial B_i, \tag{22}$$

where $\lambda_1(m, B_i) > 0$ denotes the first eigenvalue of (21). By taking $r^p\psi_i$ as a test function in (22), we have

$$r^p\lambda_1(m, B_i)\int_{B_i} m\psi_i^p\,dx = \int_{B_i} |r\nabla\psi_i|^p\,dx \geq \frac{p(p-1)}{C_1}\int_{B_i} G(x, r\nabla\psi_i)\,dx \tag{23}$$

for every $r > 0$, where we use (3) in the last inequality. Take $\mu_1 > 0$ satisfying $\mu_1 > \mu_0$ and

$$\mu_1 > \max\left\{\frac{C_1\lambda_1(m, B_i)}{p(p-1)}\,;\, i = 1, 2\right\}. \tag{24}$$

Since $u_\mu \in \operatorname{int} P_B \cup \operatorname{int} P$ and $v_\mu \in -\operatorname{int} P_B \cup -\operatorname{int} P$ for each $\mu > \mu_0$, there exists an $0 < r_\mu < \delta_1$ such that $u_\mu - r_\mu\psi_i \in P_B \cup P$ and $v_\mu + r_\mu\psi_i \in -P_B \cup -P$ for $i = 1, 2$. As a result, it is easily shown that $v_\mu \leq -r_\mu\psi_i < 0 < r_\mu\psi_i \leq u_\mu$ $(i = 1, 2)$ and

$$\begin{aligned}
&\int_\Omega H_\mu(x, tr_\mu\psi_1 - (1-t)r_\mu\psi_2)\,dx \\
&\quad = \mu\int_\Omega F(x, tr_\mu\psi_1)\,dx + \mu\int_\Omega F(x, -(1-t)r_\mu\psi_2)\,dx \\
&\quad \geq \mu(tr_\mu)^p\int_\Omega m\psi_1^p\,dx + \mu(1-t)^p r_\mu^p\int_\Omega m\psi_2^p\,dx
\end{aligned} \tag{25}$$

for every $t \in [0, 1]$ by $(H4)$ (note $\|r_\mu\psi_i\|_\infty < \delta_1$).

Therefore, for every $\mu \geq \mu_1$ and $t \in [0, 1]$, we obtain

$$\begin{aligned}
I_\mu(tr_\mu\psi_1 - (1-t)r_\mu\psi_2) &\leq t^p r_\mu^p\left(\frac{C_1}{p(p-1)} - \frac{\mu}{\lambda_1(m, B_1)}\right)\|\nabla\psi_1\|_p^p \\
&\quad + (1-t)^p r_\mu^p\left(\frac{C_1}{p(p-1)} - \frac{\mu}{\lambda_1(m, B_2)}\right)\|\nabla\psi_2\|_p^p < 0
\end{aligned}$$

by (24), (25) and (23) with $tr_\mu\psi_1$ and $(1-t)r_\mu\psi_2$ in the place of $r\psi_i$. □

Proof of Theorem 1 First, we note that it suffices to find critical points of I_μ in int P_B, −int P_B and $X_B \setminus (P_B \cup -P_B)$ for sufficiently large μ according to Lemma 24. We already know that Lemmas 22, 25 or 26 imply $(A1)$ and $(A2)$ for h_μ and I_μ if $\mu > \max\{\mu_0, \mu_1\}$. Moreover, Lemma 23 ensures that I_μ is coercive on W_B for every $\mu > \mu_0$. Therefore, by applying Theorem 11 to I_μ for each $\mu > \max\{\mu_0, \mu_1\}$, we obtain three critical points $w_{\mu,1} \in \text{int}\, P_B$, $w_{\mu,2} \in -\text{int}\, P_B$ and $w_{\mu,3} \in X_B \setminus (P_B \cup -P_B)$ of I_μ. □

Proof of Corollary 2 (i) and (ii): By $(H5)$, we easily see that $T_+ > 0$ and $T_- < 0$ are a super-solution and a sub-solution of $(P)_\mu$ for every $\mu > 0$, respectively. Consequently, our conclusion follows from Theorem 1. (iii): Note that we are assuming $Bu = u = 0$ (that is, Dirichlet boundary condition) in this case. Moreover, we also note that $(H2)$ follows from $(H4)$ (with $D_1 = \|m\|_\infty$ and $\delta_0 = \delta_1$).

According to Theorem 1, it is sufficient to obtain a super-solution in int P_B and a sub-solution in −int P_B of $(P)_\mu$ for each $\mu > 0$. Here, we fix any $\mu > 0$ and choose $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{\lambda_1 C_0}{\mu(p-1)}, \tag{26}$$

where λ_1 denotes the first eigenvalue of $-\Delta_p$ in Ω under the Dirichlet boundary condition. By the condition $\text{ess sup}_{x\in\Omega} \limsup_{|u|\to\infty} f(x,u)/|u|^{p-2}u \le 0$, there exists an $R > 0$ such that

$$\begin{aligned} f(x,u) &\le \varepsilon u^{p-1} && \text{for } u \ge R, \text{ a.e. } x \in \Omega \\ \text{and} \quad f(x,u) &\ge -\varepsilon |u|^{p-1} && \text{for } u \le -R, \text{ a.e. } x \in \Omega. \end{aligned} \tag{27}$$

We set $M_+ := \sup\{f(x,u)\,;\, x \in \Omega,\ R \ge u \ge 0\} + 1 \ge 1$ and $M_- := \inf\{f(x,u)\,;\, x \in \Omega,\ -R \le u \le 0\} - 1 \le -1$ where the inequalities hold because f is bounded on a bounded set and $f(x,0) = 0$ for a.e. $x \in \Omega$. We define two functionals $I_\mu^\pm$ on W_B by

$$I_\mu^\pm(u) := \int_\Omega G(x,\nabla u)\,\mathrm{d}x - \mu M_\pm \int_\Omega u\,\mathrm{d}x - \frac{\varepsilon\mu}{p}\int_\Omega u_\pm^p\,\mathrm{d}x$$

for $u \in W_B$. Then, it is easily shown that $I_\mu^\pm$ is coercive and bounded from below on W_B by Poincaré's inequality, (3) and (26). Furthermore, $\int_\Omega G(x,\nabla u)\,\mathrm{d}x$ is weakly lower semicontinuous (w.l.s.c.) on W_B because G is convex in the second variable (see Remark 6) and $\int_\Omega G(x,\nabla u)\,\mathrm{d}x$ is continuous on W_B (see [25, Theorem 1.2.]). Thus, $I_\mu^\pm$ is also w.l.s.c. on $W^{1,p}(\Omega)$ since the inclusion of W_B into $L^p(\Omega)$ is compact. As a result, I_μ^+ and I_μ^- have a global minimizer u_μ and v_μ, respectively. Let us prove that $u_\mu \ne 0$ and $v_\mu \ne 0$. Indeed, by taking a positive eigenfunction φ_1 of $-\Delta_p$ corresponding to λ_1, we obtain

$$I_\mu^+(u_\mu) = \min_{W_B} I_\mu^+ \le I_\mu^+(t\varphi_1) \le \frac{t^p C_1}{p(p-1)}\|\nabla\varphi_1\|_p^p - t\mu M_+\|\varphi_1\|_1 - \frac{\varepsilon\mu t^p}{p}\|\varphi_1\|_p^p < 0$$

for sufficiently small $t > 0$ because of $p > 1$ and $M_+ > 0$. Hence, $u_\mu \ne 0$. Similarly, by considering $-t\varphi_1$, we have $v_\mu \ne 0$.

We point out that $u_\mu \in X_B$ and $v_\mu \in X_B$ due to the regularity theorem in [23] because $u_\mu \in L^\infty(\Omega)$ and $v_\mu \in L^\infty(\Omega)$ by Moser's iteration process (refer to Theorem C in [26] by noting that the nonlinearity satisfies the subcritical growth condition).

Next, we see that u_μ and v_μ are a super-solution and a sub-solution of $(P)_\mu$, respectively. Indeed, by the definition of $M_\pm$ and (27), we obtain

$$-\text{div}\, A(x, \nabla u_\mu) = \mu M_+ + \mu\varepsilon (u_\mu)_+^{p-1} \geq \mu f(x, u_\mu)$$
$$-\text{div}\, A(x, \nabla v_\mu) = \mu M_- - \mu\varepsilon (v_\mu)_-^{p-1} \leq \mu f(x, v_\mu)$$

in Ω. This shows our claim [note that $u_\mu, v_\mu \in X_B = C_0^1(\overline{\Omega})$].

Finally, we prove that $u_\mu \in \text{int}\, P_B$ and $v_\mu \in -\text{int}\, P_B$. In fact, by taking $-(u_\mu)_-$ as a test function, we have

$$\begin{aligned} 0 = \langle (I_\mu^+)'(u_\mu), -(u_\mu)_- \rangle &= \int_\Omega A(x, \nabla u_\mu)(-\nabla (u_\mu)_-)\, \mathrm{d}x + \mu M_+ \int_\Omega (u_\mu)_-\, \mathrm{d}x \\ &\geq \frac{C_0}{p-1} \|\nabla (u_\mu)_-\|_p^p \geq \frac{\lambda_1 C_0}{p-1} \|(u_\mu)_-\|_p^p \geq 0 \end{aligned}$$

by Remark 6 (iii), $M_+ > 0$ and Poincaré's inequality. Thus, $(u_\mu)_-(x) = 0$ for every $x \in \Omega$, whence $u_\mu \geq 0$. Since $-\text{div}\, A(x, \nabla u_\mu) = \mu M_+ + \mu\varepsilon u_\mu^{p-1} \geq 0$ in Ω, we have $u_\mu(x) > 0$ for every $x \in \Omega$ by Theorem B in [26] (note that $u_\mu \in X_B$ and $u_\mu \not\equiv 0$). In addition, due to the strong maximum principle (see Theorem A in [26]), we see that $\partial u_\mu(x)/\partial \nu < 0$ for every $x \in \partial\Omega$. This implies $u_\mu \in \text{int}\, P_B$.

Concerning v_μ, by replacing u_μ with $-v_\mu$ in the above argument, we see that $-v_\mu \in \text{int}\, P_B$ (note that A is odd in the second variable). □

5 Proofs in the special cases

First, in a similar way to I_μ as in Sect. 4, we define a functional $\widetilde{I}_\mu$ on W_B as follows:

$$\widetilde{I}_\mu(u) := \int_\Omega G(x, \nabla u)\, \mathrm{d}x - \int_\Omega H(x, u, \mu)\, \mathrm{d}x$$

for $u \in W_B$ and $\mu \in \mathscr{M}$, where $H(x, u, \mu) := \int_0^u h(x, t, \mu)\, \mathrm{d}x$ and

$$\begin{aligned} h(x, u, \mu) := \psi_1(\mu) f_{1,[v_\mu, u_\mu]}(x, u) + \psi_2(\mu) f_{2,[v_\mu, u_\mu]}(x, u) \\ - p(u - u_\mu(x))_+^{p-1} + p(u - v_\mu(x))_-^{p-1} \end{aligned}$$

with a super-solution u_μ and a sub-solution v_μ [see (13) for the definition of $f_{i,[v_\mu, u_\mu]}$].

Throughout this section, we denote a super-solution and a sub-solution of $(\widetilde{P})_\mu$ in $(\widetilde{H1})$ by u_μ and v_μ, respectively.

By the same argument as in Sect. 4, we can prove the following two lemmas. Here, we omit the proofs.

Lemma 27 *Assume $(\widetilde{H1})$. Then, for every $\mu \in \mathscr{M}$, $\widetilde{I}_\mu$ is coercive on W_B.*

Lemma 28 *Assume $(\widetilde{H1})$. If $u \in W_B$ is a critical point of $\widetilde{I}_\mu$, then u satisfies $v_\mu(x) \leq u(x) \leq u_\mu(x)$ for a.e. $x \in \Omega$. Therefore, u is a solution of $(\widetilde{P})_\mu$ within the order interval $[v_\mu, u_\mu]$.*

Lemma 29 *Assume $(\widetilde{H1})$, $(\widetilde{H2})$, $(\widetilde{H3})$ and $(\widetilde{H4})$. Then, for every $\mu \in \mathscr{M}$, there exists a $\lambda = \lambda(\mu) > 0$ such that*

$$h(x, u, \mu)u + \lambda |u|^p \geq 0 \quad \textit{for every } u \in \mathbb{R},\ \textit{a.e. } x \in \Omega.$$

Proof Fix any $\mu \in \mathscr{M}$. Because f_1 is bounded on each bounded set and by $(\widetilde{H2})$, there exists $\lambda_1 > 0$ such that

$$f_1(x,t)t + \lambda_1|t|^p \geq 0 \quad \text{for every } |t| \leq \max\{\|v_\mu\|_\infty, \|u_\mu\|_\infty\}, \text{ a.e. } x \in \Omega.$$

Moreover, there exists $\lambda_2 > 0$ such that

$$f_2(x,t)t + \lambda_2|t|^p \geq 0 \quad \text{for every } |t| \leq \max\{\|v_\mu\|_\infty, \|u_\mu\|_\infty\}, \text{ a.e. } x \in \Omega.$$

since $f_2(x,t)t$ is positive for sufficiently small $|t| > 0$ by $(\widetilde{H3})$ and f_2 is also bounded on each bounded set. Set $\lambda = \psi_1(\mu)\lambda_1 + \psi_2(\mu)\lambda_2 + p > 0$. By the same argument as in Lemma 22, we can reach our conclusion. □

Lemma 30 *Assume* $(\widetilde{H1})$, $(\widetilde{H2})$, $(\widetilde{H3})$ *and* $(\widetilde{H4})$. *Then, for every* $\mu \in \mathscr{M}$, *there exist* $w_\mu^1 \in P_B$ *and* $w_\mu^2 \in -P_B$ *such that*

$$\max_{t\in[0,1]} \widetilde{I}_\mu\left(tw_\mu^1 + (1-t)w_\mu^2\right) < 0.$$

Proof First, we choose smooth functions $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$ satisfying $\operatorname{supp} u_0 \cap \operatorname{supp} v_0 = \emptyset$ and $u_0 \geq 0 \geq v_0$ in $\overline{\Omega}$. Fix any $\mu \in \mathscr{M}$. Since $u_\mu \in \operatorname{int} P_B \cup \operatorname{int} P$ and $v_\mu \in -\operatorname{int} P_B \cup -\operatorname{int} P$ [see $(\widetilde{H1})$], there exists an $r_\mu > 0$ such that $u_\mu \pm ru_0 \in P_B \cup P$ and $v_\mu \pm rv_0 \in -P_B \cup -P$ for every $0 < r \leq r_\mu$. Because of $\operatorname{supp} u_0 \cap \operatorname{supp} v_0 = \emptyset$, this implies that $u_\mu \geq tru_0 \geq 0 \geq (1-t)rv_0 \geq v_\mu$ and

$$\int_\Omega H(x, tru_0 + (1-t)rv_0, \mu)\,\mathrm{d}x = \sum_{i=1}^{2}\psi_i(\mu)\int_\Omega F_i(x, tru_0)\,\mathrm{d}x + \sum_{i=1}^{2}\psi_i(\mu)\int_\Omega F_i(x,(1-t)rv_0)\,\mathrm{d}x \tag{28}$$

for every $0 < r \leq r_\mu$ and $t \in [0,1]$, where $F_i(x,s) := \int_0^s f_i(x,\tau)\,\mathrm{d}\tau$ $(i = 1, 2)$. By the hypothesis $(\widetilde{H3})$, there exist $\delta_1 > 0$ and $\rho_1 > 0$ such that

$$f_2(x,u)u \geq \beta\rho_1|u|^\beta \quad \text{for every } |u| < \delta_1.$$

Thus, we have

$$\int_\Omega F_2(x,w)\,\mathrm{d}x \geq \rho_1\|u\|_\beta^\beta \quad \text{for every } w \in L^\infty(\Omega) \text{ with } \|w\|_\infty < \delta_1. \tag{29}$$

Moreover, it follows from $(\widetilde{H2})$ that

$$\int_\Omega F_1(x,w)\,\mathrm{d}x \geq -\frac{D_1}{p}\|w\|_p^p \quad \text{for every } w \in L^\infty(\Omega) \text{ with } \|w\|_\infty < \delta_0, \tag{30}$$

where D_1 and δ_0 are the positive constants in $(\widetilde{H2})$. Here, to simplify the notation, we set

$$\Phi(r,w) := C_1r^{p-\beta}\|\nabla w\|_p^p + D_1(p-1)r^{p-\beta}\psi_1(\mu)\|w\|_p^p - \rho_1 p(p-1)\psi_2(\mu)\|w\|_\beta^\beta$$

for $w \in W_B$. Then, by $p > \beta > 1$, $\psi_2(\mu) > 0$, $\|u_0\|_\beta > 0$ and $\|v_0\|_\beta > 0$, we can choose an $r > 0$ such that

$$r < \min\left\{r_\mu, \frac{\delta_0}{\|u_0\|_\infty}, \frac{\delta_0}{\|v_0\|_\infty}, \frac{\delta_1}{\|u_0\|_\infty}, \frac{\delta_1}{\|v_0\|_\infty}\right\}, \Phi(r,u_0) < 0 \text{ and } \Phi(r,v_0) < 0. \tag{31}$$

Therefore, for every $t \in [0, 1]$ and such $r > 0$, we obtain

$$\begin{aligned}
&\widetilde{I}_\mu(tru_0 + (1-t)rv_0) \\
&= \int_\Omega G(x, tr\nabla u_0)\,\mathrm{d}x + \int_\Omega G(x, (1-t)r\nabla v_0)\,\mathrm{d}x \\
&\quad - \sum_{i=1}^{2} \psi_i(\mu) \int_\Omega F_i(x, tru_0)\,\mathrm{d}x - \sum_{i=1}^{2} \psi_i(\mu) \int_\Omega F_i(x, (1-t)rv_0)\,\mathrm{d}x \\
&\le \frac{C_1\left((tr)^p \|\nabla u_0\|_p^p + (1-t)^p r^p \|\nabla v_0\|_p^p\right)}{p(p-1)} + \frac{D_1\psi_1(\mu)\left(\|tru_0\|_p^p + \|(1-t)rv_0\|_p^p\right)}{p} \\
&\quad -\rho_1\psi_2(\mu)\left(\|tru_0\|_\beta^\beta + \|(1-t)rv_0\|_\beta^\beta\right) \\
&\le \frac{(tr)^\beta}{p(p-1)}\Phi(r, u_0) + \frac{(1-t)^\beta r^\beta}{p(p-1)}\Phi(r, v_0) < 0
\end{aligned}$$

by (3), (28), (29), (30) and (31). □

Proof of Theorem 3 Fix any $\mu \in \mathscr{M}$. It follows from Lemma 27 that $\widetilde{I}_\mu$ is coercive on W_B. According to Lemma 29 and Lemma 30, $h(x, u, \mu)$ satisfies $(A1)$ and $(A2)$ holds for $J = \widetilde{I}_\mu$. Thus, by applying Theorem 11, $\widetilde{I}_\mu$ has three critical points $w_1 \in \operatorname{int} P_B$, $w_2 \in -\operatorname{int} P_B$ and $w_3 \in X_B \setminus (P_B \cup -P_B)$. Moreover, Lemma 28 guarantees that they are solutions of $(\widetilde{P})_\mu$ with $w_i \in [u_\mu, v_\mu]$ $(i = 1, 2, 3)$. The proof is complete. □

Proof of Corollary 5 (i) Since the constant function $T(\mu)_+ > 0$ is a super-solution and $T(\mu)_- < 0$ is a sub-solution of $(\widetilde{P})_\mu$, our conclusion follows from Theorem 3.

(ii) First, we recall that in this case we assume the Dirichlet boundary condition. According to Theorem 3, it suffices to prove that for every $\mu \in \mathscr{M}$, there exists a super-solution $u_\mu \in \operatorname{int} P_B$ and a sub-solution $v_\mu \in -\operatorname{int} P_B$ of $(\widetilde{P})_\mu$. Fix $\mu \in \mathscr{M}$. Because $\operatorname{ess\,sup}_{x\in\Omega} \limsup_{|u|\to\infty} \tilde{f}(x, u, \mu)/|u|^{p-2}u < C_0\lambda_1/(p-1)$ there exist $\varepsilon > 0$ and $R > 0$ such that

$$\tilde{f}(x, u, \mu) \le \frac{C_0(\lambda_1 - \varepsilon)}{p-1} u^{p-1} \quad \text{for } u \ge R, \text{ a.e. } x \in \Omega,$$
$$\text{and} \quad \tilde{f}(x, u, \mu) \ge -\frac{C_0(\lambda_1 - \varepsilon)}{p-1} |u|^{p-1} \quad \text{for } u \le -R, \text{ a.e. } x \in \Omega.$$

Set $M_+ := \sup\{\tilde{f}(x, u, \mu)\,;\ 0 \le u \le R,\ x \in \Omega\} + 1 > 0$ and $M_- := \inf\{\tilde{f}(x, u, \mu)\,;\ 0 \ge u \ge -R,\ x \in \Omega\} - 1 < 0$. Define the functionals $\widetilde{I}_\mu^\pm$ on $W_0^{1,p}(\Omega)$ by

$$\widetilde{I}_\mu^\pm(u) := \int_\Omega G(x, \nabla u)\,\mathrm{d}x - M_\pm \int_\Omega u\,\mathrm{d}x - \frac{C_0(\lambda_1 - \varepsilon)}{p(p-1)} \int_\Omega u_\pm^p\,\mathrm{d}x$$

for $u \in W_0^{1,p}(\Omega)$. Due to the same argument as in Corollary 2, these functionals have a global minimizer with $\min \widetilde{I}_\mu^\pm < 0$. Moreover, this guarantees the existence of a super-solution in $\operatorname{int} P_B$ and a sub-solution in $-\operatorname{int} P_B$ (refer to Corollary 2 for details). Thus, our conclusion holds. □

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Appendix

For readers' convenience, we give the proof of Proposition 10. We remark that we do not need inequality (35) in the following proof under the Dirichlet boundary condition. However, it is necessary for our approach because we consider the Neumann problem too.

Proof of Proposition 10 Let $u \in L^r(\Omega)$ and $v = T_\lambda^{-1}(u)(\in W_B)$ with $r > N/p$. Consider $\bar{p}^*$ such that $\bar{p}^* := p^*$ if $N > p$ and $\bar{p}^* > pr/(r-1)$ if $N = p$.

Set $k = \|u\|_r^{1/(p-1)}$ and for $\varphi = v_+$ or v_-, we put $\varphi_M(x) := \min\{\varphi(x), M\}$ for $M > 0$ to simplify the notation. For $q \geq 1$, we define

$$H(z) := \begin{cases} 0 & \text{if } z < k, \\ z^q - k^q & \text{if } k \leq z. \end{cases}$$

Let $w := \varphi + k$ and $w_M := \varphi_M + k$. Define

$$K(w_M) := \int_k^{w_M} H'(s)^p \, \mathrm{d}s.$$

Note that $H'(s) \geq 0$ for every $s \neq 0$ and that if $\varphi \in W_0^{1,p}(\Omega)$, then $w_M = k$ on $\partial\Omega$ in the sense of the trace operator, and hence $K(w_M) \in W_B$.

First, we consider $\varphi = v_+$. By taking $K(w_M)$ as a test function in $T_\lambda(v) = u$, we obtain

$$\int_\Omega A(x, \nabla v)\nabla K(w_M)\, \mathrm{d}x + \lambda \int_\Omega |v|^{p-2} v K(w_M)\, \mathrm{d}x \tag{32}$$

$$= \int_\Omega u K(w_M)\, \mathrm{d}x \leq \int_\Omega |u| \left(\int_k^{w_M} H'(s)^p \, \mathrm{d}s \right) \mathrm{d}x$$

$$\leq \int_\Omega |u| w_M H'(w_M)^p \, \mathrm{d}x \leq k^{-p+1} \int_\Omega |u| w_M^p H'(w_M)^p \, \mathrm{d}x$$

$$\leq k^{-p+1} \|u\|_r \|w_M H'(w_M)\|_{pr/(r-1)}^p = \|w_M H'(w_M)\|_{pr/(r-1)}^p, \tag{33}$$

by Hölder's inequality and $k = \|u\|_r^{1/(p-1)}$, where we use $w_M/k \geq 1$ and $H'(s) \leq H'(w_M)$ for every $s \leq w_M$. On the other hand, the inequality

$$\int_\Omega A(x, \nabla v)\nabla K(w_M)\, \mathrm{d}x = \int_\Omega A(x, \nabla v)\nabla \varphi_M H'(w_M)^p \, \mathrm{d}x$$

$$\geq \frac{C_0}{p-1} \int_\Omega |\nabla \varphi_M|^p H'(w_M)^p \, \mathrm{d}x = \frac{C_0}{p-1} \|\nabla H(w_M)\|_p^p \tag{34}$$

follows from Remark 6 (iii), $w_M = \varphi_M + k$ and $v_+ = \varphi$. Concerning the second term in the left-hand side of (32), we have

$$\begin{aligned}
\int_\Omega |v|^{p-2} v K(w_M)\,\mathrm{d}x &= \int_{v\geq 0} v^{p-1} \int_k^{w_M} H'(s)^p\,\mathrm{d}s\mathrm{d}x \\
&\geq \int_{v\geq 0} \int_k^{w_M} H'(s)\left((w_M - k)H'(s)\right)^{p-1}\,\mathrm{d}s\mathrm{d}x \\
&= \int_{v\geq 0} \int_{2k}^{w_M+k} H'(s-k)\big((w_M - k)H'(s-k)\big)^{p-1}\,\mathrm{d}s\mathrm{d}x \\
&\geq \int_{v\geq 0} \int_{2k}^{w_M+k} H'(s-k)(H(s-k))^{p-1}\,\mathrm{d}s\mathrm{d}x = \frac{1}{p}\int_\Omega H(w_M)^p\,\mathrm{d}x
\end{aligned} \tag{35}$$

by $(v_+ =)\varphi \geq \varphi_M = w_M - k$ and $(w_M - k)H'(s-k) \geq q(s-k)^q - kq(s-k)^{q-1} \geq H(s-k)$ for $2k \leq s \leq w_M + k$. Because $W_B \hookrightarrow L^{\bar{p}^*}(\Omega)$ is continuous, according to (33), (34) and (35), there exists a $C_* > 0$ such that

$$C_* \min\left\{\frac{C_0}{p-1}, \frac{\lambda}{p}\right\} \|H(w_M)\|_{\bar{p}^*}^p \leq \|w_M H'(w_M)\|_{pr/(r-1)}^p = q^p \|w_M\|_{pqr/(r-1)}^{pq}. \tag{36}$$

Similarly, by taking $-K(w_M)$ as a test function in $T_\lambda(v) = u$ for the case of $\varphi = v_-$, we have the same inequality as above. Consequently, we have

$$\begin{aligned}
C'_* \|w_M\|_{\bar{p}^* q}^q &\leq q\|w_M\|_{pqr/(r-1)}^q + C'_* k^q |\Omega|^{1/\bar{p}^*} \\
&\leq q(1 + C'_* |\Omega|^{1/\bar{p}^* + (1-r)/(pr)})\|w_M\|_{pqr/(r-1)}^q
\end{aligned}$$

for every $q \geq 1$, where $C'_* = C_*^{1/p} \min\{C_0/(p-1), \lambda/p\}^{1/p}$ and we use Holder's inequality, $k^q = \|k\|_q^q / |\Omega|$ and $k \leq w_M$ in the last inequality. This yields

$$\|w_M\|_{\bar{p}^* q} \leq (q D_*)^{1/q} \|w_M\|_{\alpha q} \tag{37}$$

for every $q \geq 1$ with $\alpha := pr/(r-1)$, where $D_* > 0$ is a positive constant independent of M, k, u, v. Note that $\alpha < \bar{p}^*$ by our assumption $r > N/p$ and the definition of $\bar{p}^*$. At this point, we define a sequence $\{q_n\}$ by $q_0 := \bar{p}^*/\alpha (> 1)$ and $q_{n+1} := q_n \bar{p}^*/\alpha = (\bar{p}^*/\alpha)^{n+2}$. Then, by taking $M \to \infty$ in (37), we see that if $w \in L^{\alpha q_n}(\Omega)$, then $w \in L^{\alpha q_{n+1}}(\Omega)$ and it satisfies $\|w\|_{\alpha q_{n+1}} \leq (q_n D_*)^{1/q_n} \|w\|_{\alpha q_n}$. This leads to $\|w\|_\infty \leq C\|w\|_{\bar{p}^*}$ by a standard argument, where C is a positive constant independent of w and n. Therefore, we have

$$\|\varphi\|_\infty \leq C\|\varphi\|_{\bar{p}^*} + k(1 + C|\Omega|^{1/\bar{p}^*}) = C\|\varphi\|_{\bar{p}^*} + \|u\|_r^{1/(p-1)}(1 + C|\Omega|^{1/\bar{p}^*}) \tag{38}$$

(note $w = \varphi + k$ and $k = \|u\|_r^{1/(p-1)}$). On the other hand, by taking $\pm v_\pm$ as test function in $T_\lambda(v) = u$,

$$\begin{aligned}
C_* \min\left\{\frac{C_0}{p-1}, \lambda\right\} \|\varphi\|_{\bar{p}^*}^p &\leq \min\left\{\frac{C_0}{p-1}, \lambda\right\} \|\varphi\|^p \leq \langle T_\lambda(v), \pm v_\pm \rangle \\
&\leq \|u\|_r \|\varphi\|_{r/(r-1)} \leq \|u\|_r \|\varphi\|_{\bar{p}^*} |\Omega|^\beta
\end{aligned} \tag{39}$$

(note that φ denotes v_+ or v_-), where we use Remark 6 (iii) in the second inequality and Hölder's inequality setting $\beta = ((r-1)\bar{p}^* - r)/(r-1)\bar{p}^*$. From (38) and (39), our conclusion follows. □

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