Annali di Matematica (2014) 193:1139–1146
DOI 10.1007/s10231-012-0320-8

Harmonic morphisms and shear-free perfect fluids coupled with gravity

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Received: 8 February 2012 / Accepted: 25 December 2012 / Published online: 13 January 2013

Abstract We employ the technique used in the classification of harmonic morphisms with one-dimensional fibers on four-dimensional Einstein manifolds (Pantilie in Commun Anal Geom 10:779–814, 2002) to give a simpler proof of the fact that the shear-free perfect fluids coupled to gravity are either irrotational or expansion free in the case when the equation of state is $\rho = -3p$.

Keywords Harmonic morphism · Einstein's equations · Relativistic perfect fluid · Shear-free vector

Mathematics Subject Classification (2000) 53B30 · 53C43 · 53Z05 · 83C55

1 Introduction

Harmonic (or wave) maps [4] are critical points of the Dirichlet's energy

$$\mathcal{E}(\varphi) = \frac{1}{2} \int_M |\mathrm{d}\varphi|^2 \mathrm{vol}_g$$

where $\varphi : (M, g) \to (N, h)$ denotes a smooth map between two (semi-) Riemannian manifolds M and N. Harmonic morphisms form a subclass of harmonic maps characterized by an additional condition of transversal conformality. Both notions enjoy an extensive study and a deep mathematical understanding, cf. [1–3]. Among their various generalizations, we distinguish the r-harmonic maps and morphisms [8], defined in a similar way with respect to the energy $\mathcal{E}_r(\varphi) = \frac{1}{r} \int_M |\mathrm{d}\varphi|^r \mathrm{vol}_g$, $r > 0$. These developments make harmonic maps and morphisms a very appealing tool in applications, as shown by the successfully concept of sigma model in theoretical physics. In [15], we showed that they can also play a rôle in

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general relativity. This rôle is due to the local duality between r-harmonic morphisms with one-dimensional timelike fibers defined on a space-time and shear-free relativistic perfect fluids $(U, p = \frac{1}{3}(r-3)\lambda^r, \rho = \lambda^r)$ on that space-time, where U is the unitary vector field tangent to the fibers and λ is the dilation of the harmonic morphism [15, Prop. 3.2]. This duality is known to geometers through the notion of foliations that produce (r-)harmonic morphisms [11].

Harmonic morphisms (and, to a lesser extent, r-harmonic morphisms) with one dimensional fibres have been classified under various curvature restrictions on the domain metric, see e.g. [9,12] and for a review [1,13]. In this note we want to point out that the local part of Pantilie's classification result [12] for harmonic morphisms on Riemannian four-manifolds satisfying Einstein (vacuum) condition extends to Lorentzian four-manifolds solving the Einstein field equations (with fluid source). In the dual perspective of relativistic fluids, the counterpart of this result is the positive answer to the following

> **Conjecture** ([16]). In general relativity, if the velocity vector field of a barotropic perfect fluid ($p + \rho \neq 0$ and $\rho = \rho(p)$) is shear free, then either the expansion or the rotation of the fluid vanishes.

in the case when $\rho = -3p$. This is already known to be true [7,18], but in the present approach, the proof is simpler and is written in coordinate-free differential geometric language and can provide a new insight into this conjecture still not established despite a long history of confirmations in particular cases.

For a recent discussion about the conjecture and the progress that have been made, see [5,19].

In the following section, we review the basic definitions and results needed thereafter; in Sect. 3, we give the proof of the main result, and we end with the proofs of some simple technical Lemmas.

Throughout the paper, (M, g) will be a Lorentzian manifold and Ric, Scal will denote its Ricci and scalar curvature, respectively. All the considerations are local.

2 Basic material

2.1 Perfect fluids coupled with gravity

Definition 1 [10,17] Let (M, g) be a four-dimensional space-time. A triple (U, p, ρ) is called (relativistic) *perfect fluid* if

(i) U is a timelike future-pointing unit vector field on M, called the *flow vector field*,
(ii) $\rho, p : M \to \mathbb{R}$ are the real functions called *mass (energy) density* and *pressure*, respectively,
(iii) the *stress-energy tensor* of the fluid is conserved:

$$\operatorname{div}\left(p\, g + (p+\rho) U^\flat \otimes U^\flat\right) = 0.$$

If instead of (iii), we impose the stronger condition
$(iii)'$ the *Einstein equations* are satisfied

$$\mathrm{Ric} - \frac{1}{2}\,\mathrm{Scal} \cdot g = p\, g + (p+\rho) U^\flat \otimes U^\flat, \tag{1}$$

then (U, p, ρ) is called (relativistic) *perfect fluid coupled with gravity*.

Condition (iii) of the definition decomposes into the fluid's equations:

$$\begin{aligned}(\rho + p)\,\mathrm{div}\, U + U(\rho) &= 0 \quad \text{(the conservation of energy along the flow)},\\ (\rho + p)\nabla_U U + \mathrm{grad}^{\mathcal{H}} p &= 0 \quad \text{(the Euler equations)},\end{aligned} \tag{2}$$

where $\mathrm{grad}^{\mathcal{H}} p$ is the *spatial pressure gradient*, that is, the component orthogonal to U. If moreover the perfect fluid is coupled with gravity, we have the following block-diagonal structure of the Ricci tensor of (M, g):

$$\begin{aligned}\mathrm{Ric}(U, U) &= \frac{1}{2}(\rho + 3p),\\ \mathrm{Ric}(X, U) &= 0, \quad \forall X \perp U,\\ \mathrm{Ric}(X, Y) &= \frac{1}{2}(\rho - p)g(X, Y), \quad \forall X, Y \perp U.\end{aligned} \tag{3}$$

Note that $\mathrm{Scal} = \rho - 3p$.

In this paper, we are concerned with perfect fluids coupled with gravity satisfying $\rho = -3p$. This *equation of state* is supposed to represent a hypothetical form of matter called *quintessence* characterized by negative pressure and $-1 < p/\rho \leq 0$. Moreover, as shown by Vilenkin [20], a randomly oriented distribution of (infinitely thin) straight strings averaged over all directions behaves like a perfect fluid with $\rho = -3p$.

2.2 Harmonic morphisms and curvature restrictions

Recall [1,6] that a harmonic morphism $\varphi : (M, g) \to (N, h)$ between (semi-) Riemannian manifolds can be characterized as a *harmonic map* (i.e., trace $\nabla \mathrm{d}\varphi = 0$) which is moreover *horizontally weakly conformal* of *dilation* $\lambda : M \to \mathbb{R}_+$ (i.e., $\varphi^* h = \lambda^2 g|_{(\mathrm{Ker}\,\mathrm{d}\varphi)^\perp \times (\mathrm{Ker}\,\mathrm{d}\varphi)^\perp}$).

Let φ be a harmonic morphism defined on a four-dimensional space-time taking values in a three-dimensional Riemannian manifold, denote by U the unit *vertical* timelike vector that spans $\mathcal{V} = \mathrm{Ker}\,\mathrm{d}\varphi$ and by $\mathcal{H} = \mathcal{V}^\perp$ the *horizontal* distribution associated with φ. Then it is known that

$$-\nabla_U U + \mathrm{grad}^{\mathcal{H}}(\ln \lambda) = 0, \tag{4}$$

$$\mathrm{div}\, U + 3U(\ln \lambda) = 0, \tag{5}$$

$$(\mathcal{L}_U g)(X, Y) - \frac{2}{3}\,\mathrm{div}\, U \cdot g(X, Y) = 0, \quad \forall X, Y \in \mathcal{H}, \tag{6}$$

where (4) is the fundamental equation of the harmonic morphism φ, (5) is a geometric identity and (6) is equivalent to the horizontal conformality of φ.

Recall [1, p.59] also that a section σ of $(\otimes^r \mathcal{H}) \otimes (\otimes^s \mathcal{H}^*)$ is called *basic* if $(\mathcal{L}_V \sigma)^{\mathcal{H}} = 0$ for all $V \in \Gamma(\mathcal{V})$. In particular, a function f on M is basic if $V(f) = 0$ and a horizontal vector field X on M is basic if $[V, X]^{\mathcal{H}} = 0$, for all $V \in \Gamma(\mathcal{V})$; the latter condition means that X is *projectable* on N.

The *fundamental vector field* of the map φ is $V := \lambda U$. Analogously to [1, Lemma 11.7.2], the importance of this vector field is given by the fact that it allows us to rewrite Eq. (4) as follows:

$$[V, X] = 0, \quad \forall X \text{ basic vector field on } M. \tag{7}$$

Consider the 1-form dual to V, defined by $\vartheta(X) := -\lambda^{-2} g(X, V)$, $\forall X$, and the 2-form $\Omega := \mathrm{d}\vartheta$. We notice that $\Omega(X, Y) = \lambda^{-2} g([X, Y], V)$ for all $X, Y \in \Gamma(\mathcal{H})$.

Lemma 1 *Let $\varphi : (M^4, g) \to (N^3, h)$ be a harmonic morphism defined on a four-dimensional space-time. Then,*

(i) $i_W \Omega = 0$*, for all* $W \in \Gamma(\mathcal{V})$ *(Ω is a* horizontal 2*-form)*
(ii) $\mathcal{H}$ *is integrable if and only if* $\Omega = 0$ *(Ω is the* integrability 2*-form of* $\mathcal{H}$*);*
(iii) $\mathcal{L}_W \Omega = 0$ *or, equivalently,* $W(\Omega(X, Y)) = 0$*, for all* X, Y *basic vectors and for all* $W \in \Gamma(\mathcal{V})$ *(Ω is a* basic 2*-form).*

Analogously to [12], [1, p. 343], we can prove the following

Proposition 1 *Let (M, g) be four-dimensional space-time, (N, h) a Riemannian 3-manifold and $\varphi : (M, g) \to (N, h)$ a harmonic morphism with one-dimensional timelike fibers tangent to the unit vector U. Let $\lambda : M \to \mathbb{R}_+$ denote the dilation of φ. Then, we have the identities:*

$$\mathrm{Ric}(U, U) = \Delta \ln \lambda + 4U(U(\ln \lambda)) - 6[U(\ln \lambda)]^2 + \frac{\lambda^2}{2} |\Omega|^2, \tag{8}$$

$$\mathrm{Ric}(X, U) = 2X(U(\ln \lambda)) - \frac{\lambda}{2} \{\delta\Omega(X) + 2\Omega(X, \mathrm{grad} \ln \lambda)\}, \tag{9}$$

$$\mathrm{Ric}(X, Y) = (\varphi^* \mathrm{Ric}^N)(X, Y) + g(X, Y)\Delta \ln \lambda - 2X(\ln \lambda)Y(\ln \lambda) + \frac{\lambda^2}{2} \langle i_X \Omega, i_Y \Omega \rangle, \tag{10}$$

where Ric^N *is the Ricci curvature of* (N, h) *and* X, Y *are horizontal vectors.*

Let us recall [1,12] also that a φ is *of Killing type* iff $\mathrm{grad}^{\mathcal{V}} \lambda = 0$, of *warped product type* iff it has totally geodesic fibers (i.e., $\nabla_U U = 0$) and $\mathcal{H}$ is integrable and *of type three (T)* iff $|\mathrm{grad}^{\mathcal{V}} \lambda|$ is a nonzero function of λ. The local part of Pantilie's classification result [12] states that if (M, g) is an orientable (Rie-mannian) Einstein four-manifold, then φ belongs to one of these types.

2.3 The duality

In the setting of the previous subsection, from a dual perspective, Eqs. (4), (5) can be identified respectively with the Euler equations and the energy conservation along the flow (2) for the relativistic fluid ($U, p = -\frac{1}{3}\lambda^2, \rho = \lambda^2$), while (6) tells that the fluid is *shear free*. In this case, the additional assumption (11) for the fluid to be coupled with gravity becomes

$$\begin{aligned} &\mathrm{Ric}(U, U) = \mathrm{Ric}(X, U) = 0 \\ &\mathrm{Ric}(X, Y) = \frac{2}{3}\lambda^2 g(X, Y), \quad \forall X, Y \perp U. \end{aligned} \tag{11}$$

Conversely, given a perfect fluid ($U, p, \rho = -3p$), we can always integrate it locally to obtain a (local) harmonic morphism from M into some Riemannian 3-manifold (N, h) such that $\mathrm{d}\varphi(U) = 0$ and ρ equal the squared dilation of φ. For more details see [15].

Remark 1 If the associated harmonic morphism φ is of Killing type, then the fluid is *expansion free* (i.e., $\mathrm{div}\, U = 0$), while if it is of warped product type, then the perfect fluid is *irrotational* (i.e., $\Omega = 0$).

3 The main result

Following the same lines in [1,12], we prove the following

Theorem 1 *Let (M, g) be four-dimensional space-time and $\varphi : (M, g) \to (N, h)$ be a harmonic morphism into a Riemannian manifold, with one-dimensional timelike fibers tangent to the unit vector U. Let λ denote its dilation. If $\mathrm{Ric}(X, U) = 0$ and $\mathrm{Ric}(X, Y) = \nu g(X, Y)$ for all X, Y orthogonal to U and for some function $\nu : M \to \mathbb{R}$, then locally φ is either of Killing type or of warped product type.*

Proof Let (M, g) be a four-dimensional space-time and $\varphi : (M, g) \to (N, h)$ be a harmonic morphism into a Riemannian manifold.

Choose a local orthogonal frame $\{X, Y, Z\}$ of basic horizontal vector fields around a regular point x in M; we may suppose that their lengths satisfy $|X| = |Y| = |Z| = 1/\lambda$ and that Z satisfies $i_Z\Omega = 0$ [1]. Then, the contractions $i_X\Omega$ and $i_Y\Omega$ are both basic and orthogonal and $|i_X\Omega|^2 = |i_Y\Omega|^2 = \lambda^2\Omega(X, Y)^2$.

Since according to the hypothesis, $\mathrm{Ric}\,|_{\mathcal{H}\times\mathcal{H}}$ must be proportional to $g^{\mathcal{H}}$, Eq. (10) applied to all pairs of vectors from $\{X, Y, Z\}$ gives us: $(\varphi^*\,\mathrm{Ric}^N)(X, Y) - 2X(\ln\lambda)Y(\ln\lambda) = 0$, $(\varphi^*\,\mathrm{Ric}^N)(Y, Z) - 2Y(\ln\lambda)Z(\ln\lambda) = 0$, and $(\varphi^*\,\mathrm{Ric}^N)(X, Z) - 2X(\ln\lambda)Z(\ln\lambda) = 0$. In particular, $X(\ln\lambda)Y(\ln\lambda)$, $Y(\ln\lambda)Z(\ln\lambda)$, and $Z(\ln\lambda)X(\ln\lambda)$ are all basic functions. Let W be the domain of the frame $\{X, Y, Z, V\}$. Consider the following closed subset in W:

$$S = \{x \in W : X_x(\ln\lambda) = Y_x(\ln\lambda) = 0\}.$$

Denote its complement in W by S^c—this will be an open subset of M.

Case I - on the open set S^c. Suppose that $X_x(\ln\lambda) \neq 0$. By applying Equation (10) to the pair $\{X + Y, X - Y\}$, we deduce that $X(\ln\lambda)^2 - Y(\ln\lambda)^2$ is basic; moreover, since $X(\ln\lambda)Y(\ln\lambda)$ is basic too, $X(\ln\lambda)$ and $Y(\ln\lambda)$ have to be both basic. Then also $Z(\ln\lambda)$ is basic (since $Z(\ln\lambda)X(\ln\lambda)$ is basic). Hence $\mathrm{grad}(V(\ln\lambda))$ is vertical.

Assume that at some point, $\mathrm{grad}(V(\ln\lambda)) \neq 0$ so by continuity this holds in a neighborhood.

As $V(\ln\lambda)$ is non-constant and its level surfaces are horizontal, then $\mathcal{H}$ is integrable (so $\Omega = 0$). Applying Equation (9) combined with the hypothesis $\mathrm{Ric}(X, U) = 0$, we obtain that $X(U(\ln\lambda)) = 0$, or equivalently $X(\ln\lambda)V(\ln\lambda) = 0$. As we are on S^c, this would imply that $V(\ln\lambda) = 0$, a contradiction.
So $\mathrm{grad}(V(\ln\lambda)) = 0$ and then $V(\ln\lambda) = c$ a constant.
Assume that $c \neq 0$.
Replacing $V(\ln\lambda) = c$ in Equation (9), we obtain:

$$0 = -2cX(\ln\lambda) - \frac{\lambda^2}{2}\{\delta\Omega(X) + 2\Omega(X, \mathrm{grad}\ln\lambda)\}. \tag{12}$$

Notice that $\Omega(X, \mathrm{grad}\ln\lambda) = \lambda^2 Y(\ln\lambda)\Omega(X, Y)$ and this implies $V(\lambda^{-2}\Omega(X, \mathrm{grad}\ln\lambda)) = 0$. A more elaborate computation leads to:
So taking the derivative of (12) along V will give

$$0 = V(\lambda^4)\lambda^{-2}\{\delta\Omega(X) + 2\Omega(X, \mathrm{grad}\ln\lambda)\}.$$

Therefore, the term inside the brackets must vanish; substituting this back into (12) gives us $X(\ln\lambda) = 0$, contradiction.

Lemma 2 *On S^c, we have $V\left(\lambda^{-2}\delta\Omega(X)\right) = 0$.*

[1] This is an algebraic fact. As Ω basic, it locally descends to a 2-form on N; we may see it as a skew-symmetric linear mapping $TN \to TN$. Since $\dim N$ is odd, any such mapping is singular.

Therefore, the constant c must be zero; this implies $U(\ln\lambda) = 0$. We have proved that on S^c (so on $\overline{S^c}$), φ is of Killing type.

Case II: on the open set int$S$. Let

$$A = \{x \in \mathrm{int}S : Z_x(\ln\lambda) = 0\}.$$

(II_a) On the subset A, we have $\mathrm{grad}^{\mathcal{H}} \ln\lambda = 0$, so $\mathrm{grad}\ln\lambda$ is vertical. Thus, around a point where λ is non-constant, its level surfaces are horizontal so $\mathcal{H}$ is integrable and φ is of warped product type (if λ is constant, $\mathrm{div}\, U = 0$ and φ is of Killing type).
(II_b) On the open subset $A^c \cap \mathrm{int}S$, we have $X(\ln\lambda) = 0$, $Y(\ln\lambda) = 0$ and $Z(\ln\lambda) \neq 0$. The identity (10) and $\mathrm{Ric}(X, X) = \mathrm{Ric}(Z, Z)$ (consequence of our hypothesis) imply

$$4Z(\ln\lambda)^2 + \lambda^4\Omega(X, Y)^2 = 2\left((\varphi^* \mathrm{Ric}^N)(Z, Z) - (\varphi^* \mathrm{Ric}^N)(X, X)\right).$$

Since the right-hand term is a basic function, the derivative along V of the left-hand term must vanish. This gives us immediately, by using Lemma 1:

$$-\lambda^2\Omega(X, Y)^2 \frac{V(\ln\lambda)}{Z(\ln\lambda)} = 2\lambda^{-2} Z(V(\ln\lambda)). \tag{13}$$

Combine this relation with the following

Lemma 3 *On $A^c \cap \mathrm{int}S$, we have*

$$\delta\Omega(Z) = -\lambda^2\Omega(X, Y)^2 \frac{V(\ln\lambda)}{Z(\ln\lambda)}. \tag{14}$$

to obtain

$$\delta\Omega(Z) = 2\lambda^{-2} Z(V(\ln\lambda)). \tag{15}$$

Now Equation (9) with $\mathrm{Ric}(Z, U) = 0$ implies

$$4[Z(V(\ln\lambda)) - Z(\ln\lambda)V(\ln\lambda)] - \lambda^2\delta\Omega(Z) = 0.$$

Substituting (17) in the above relation, we get

$$Z(V(\ln\lambda)) - 2Z(\ln\lambda)V(\ln\lambda) = 0. \tag{16}$$

Reinserting (16) in the derived constraint (13), we obtain

$$V(\ln\lambda)\left[4Z(\ln\lambda)^2 + \lambda^4\Omega(X, Y)^2\right] = 0.$$

Since $Z(\ln\lambda)$ cannot vanish, we must have $V(\ln\lambda) = 0$ so that φ is of Killing type on the considered subset. □

Taking Remark 1 into account and that coupling with gravity (11) is a special case of Theorem 1 hypothesis, we can state the following

Corollary 1 *If the velocity vector field of a perfect fluid coupled with gravity satisfying $\rho = -3p$ is shear free, then either the expansion or the rotation of the fluid vanishes.*

Notice that the condition $\mathrm{Ric}(U, U) = 0$ from (11) was not employed.

Corollary 2 *A harmonic morphism with one-dimensional timelike fibers from an Einstein space-time or from a vacuum space-time is either of Killing type or of warped product type.*

The above Corollary is the Lorentzian analog of the result in [12] (see also [1, p. 380]). Note that the change of sign in the identity (10) simplifies radically the Case (II_b) with respect to the Riemannian case so that type (T) morphisms are now excluded. Recall that type (T) harmonic morphisms allow both expansion and rotation (for instance on $\mathbb{R}^4$ with the Eguchi–Hanson-type metric [14]).

Remark 2 The relativistic fluids with $\rho = -3p$ (and harmonic morphisms) represent a *genuine particular case* of the shear-free conjecture. For an equation of state $\rho = w\,p$ with $w \neq -3$ (for r-harmonic morphisms with $r \neq 2$) in the Ricci identity (10), it will appear a second-order term in $\mathrm{Hess}(\ln\lambda)$, as we can check by performing a conformal change in metric to render the r-harmonic morphism a harmonic morphism. This will make inapplicable the entire argument used in the present case.

4 The Proofs of Lemmas

Proof of the Lemma 2 The basic ingredients are Lemma 1 and the following identity [1, p. 119]

$$\left(\nabla^{\varphi}_X \mathrm{d}\varphi(Y)\right)^{\widehat{}} - \nabla^{\mathcal{H}}_X Y = X(\ln\lambda)Y + Y(\ln\lambda)X - g(X,Y)\,\mathrm{grad}^{\mathcal{H}}\ln\lambda,$$

where $\widehat{}$ stands for the horizontal lift operator. Notice that the first term in the above identity is a basic vector field. With this in hand, we can check by straightforward but lengthy calculation that

$$\delta\Omega(X) = \lambda^2\Big[2Y(\ln\lambda)\Omega(X,Y) + Y(\Omega(X,Y)) + \Omega\left(\left(\nabla^{\varphi}_Y \mathrm{d}\varphi(Y)\right)^{\widehat{}} + \left(\nabla^{\varphi}_Z \mathrm{d}\varphi(Z)\right)^{\widehat{}}, X\right)$$
$$+\Omega\left(Y, \left(\nabla^{\varphi}_Y \mathrm{d}\varphi(X)\right)^{\widehat{}}\right)\Big]$$

and, since each term inside the brackets is basic, the conclusion follows. □

Proof of the Lemma 3 An easy computation shows us that

$$\delta\Omega(Z) = -\lambda^4\Omega(X,Y)g([X,Y],Z). \tag{17}$$

But $[X,Y](\ln\lambda) = 0$ on $\mathrm{int}S$ implies

$$[X,Y]^{\mathcal{H}}(\ln\lambda) - \Omega(X,Y)V(\ln\lambda) = 0,$$

where we have used $\Omega(X,Y) = \lambda^{-2}g([X,Y],V)$.

As on $A^c \cap \mathrm{int}S$, we have $[X,Y]^{\mathcal{H}}(\ln\lambda) = \lambda^2 g([X,Y],Z)Z(\ln\lambda)$, it follows that

$$g([X,Y],Z) = \Omega(X,Y)\frac{V(\ln\lambda)}{\lambda^2 Z(\ln\lambda)}.$$

Reinserting in (17) gives us the result. □

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