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Invertibility criteria for Wiener–Hopf plus Hankel operators with different almost periodic Fourier symbol matrices

L. P. Castro · A. S. Silva

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Abstract Based on the different kinds of auxiliary operators and corresponding operator relations, we will present conditions which characterize the invertibility of matrix Wiener–Hopf plus Hankel operators having different Fourier symbols in the class of almost periodic elements. To reach such invertibility characterization, we introduce a new kind of factorization for AP matrix functions. Additionally, under certain conditions, we will obtain the one-sided and two-sided inverses of the matrix Wiener–Hopf plus Hankel operators in study.

Keywords Wiener–Hopf operator · Hankel operator · Almost periodic function · Invertibility

Mathematics Subject Classification (2000) 47B35 · 47A05 · 47A20 · 42A75

1 Introduction

The main goal of the present work is to obtain invertibility criteria for matrix Wiener–Hopf plus Hankel operators with (possibly different) almost periodic Fourier symbols (and acting between L^2 Lebesgue spaces), as well as to derive formulas for the (lateral) inverses of these operators.

We would like to point out that partial results are already known in this line of research. In [13], a characterization for the left-, right-, and two-sided invertibility of certain matrix Wiener–Hopf plus Hankel operators has already been obtained. This was done for Fourier symbols in the Wiener subclass of the almost periodic algebra, and under the condition that a certain almost periodic matrix-valued function (constructed from the initial Fourier symbols

L. P. Castro (✉) · A. S. Silva
Department of Mathematics, Center for Research and Development in Mathematics and Applications, University of Aveiro, 3810–193 Aveiro, Portugal
e-mail: castro@ua.pt

A. S. Silva
e-mail: anabela.silva@ua.pt

of the Hankel and Wiener–Hopf operators) must admit a numerical range bounded away from zero. In a different way, for weaker situations like the Fredholm property, several results are also already known. However, these strongly depend on the type of Fourier symbols and on particular conditions imposed on the symbols. This is the case of matrix Wiener–Hopf plus Hankel operators with (possibly different) Fourier symbols in the C^*-algebra of semi-almost periodic elements (which naturally include the almost periodic functions), where in [14] conditions to ensure the Fredholm property were obtained (but under the assumption that certain auxiliary matrix functions admit *right AP factorizations* [6]). Other works which include almost periodic symbols but under stronger constrains (like the circumstance of having equal symbols in the Wiener–Hopf and the Hankel operators) can be found in [8–10,12,20].

It is also clear that much more additional works could be referred as important contributions to the understanding of the structure of that kind of operators (cf. the pioneering works [3–5,7,17,18,22] and the references therein)—even if in a more classical or, sometimes, indirect way. For example, the classical work of Power [21] includes the study of the spectra and essential spectra of Hankel operators by investigating the C^*-algebra generated by the Toeplitz and Hankel operators (in the two cases of piecewise continuous symbols and almost periodic symbols).

Let us now identify—in a mathematical way—the main objects of this work. We will consider matrix Wiener–Hopf plus Hankel operators denoted by

$$W_{\Phi_1} + H_{\Phi_2} : \left[L^2_+(\mathbb{R})\right]^N \to \left[L^2(\mathbb{R}_+)\right]^N , \tag{1}$$

with W_{Φ_1} and H_{Φ_2} being matrix Wiener–Hopf and Hankel operators defined by

$$W_{\Phi_1} = r_+\mathcal{F}^{-1}\Phi_1\mathcal{F} : \left[L^2_+(\mathbb{R})\right]^N \to \left[L^2(\mathbb{R}_+)\right]^N , \tag{2}$$

$$H_{\Phi_2} = r_+\mathcal{F}^{-1}\Phi_2\mathcal{F}J : \left[L^2_+(\mathbb{R})\right]^N \to \left[L^2(\mathbb{R}_+)\right]^N , \tag{3}$$

respectively (where for convenience Φ_j stands for the matrix function and for the multiplication operator, as well). We consider $L^2(\mathbb{R})$ to be the usual space of square integrable Lebesgue measurable functions on the real line $\mathbb{R}$, and $L^2(\mathbb{R}_+)$ the corresponding one in the positive half-line $\mathbb{R}_+ = (0, +\infty)$. The subspace of $L^2(\mathbb{R})$ formed by all functions supported in the closure of $\mathbb{R}_+$ is being denoted by $L^2_+(\mathbb{R})$. Similarly, the subspace of $L^2(\mathbb{R})$ formed by all functions supported in the closure of $\mathbb{R}_- = (-\infty, 0)$ will be denoted by $L^2_-(\mathbb{R})$. In addition, $\mathcal{F}$ denotes the Fourier transformation, r_+ represents the operator of restriction from $\left[L^2_+(\mathbb{R})\right]^N$ into $\left[L^2(\mathbb{R}_+)\right]^N$, J is the reflection operator given by the rule $J\varphi(x) = \widetilde{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}$, Φ_1 and $\Phi_2 \in [L^\infty(\mathbb{R})]^{N\times N}$ are the so-called Fourier matrix symbols.

In this work, the entries of the matrices Φ_1 and Φ_2 will belong to the algebra AP of *almost periodic functions*, i.e., the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ ($\lambda \in \mathbb{R}$), where $e_\lambda(x) = e^{i\lambda x}$, $x \in \mathbb{R}$:

$$\mathrm{AP} := alg_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \in \mathbb{R}\}.$$

Moreover, we will be also using the notation

$$\mathrm{AP}_+ := alg_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \geq 0\}, \quad \mathrm{AP}_- := alg_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \leq 0\}$$

for these two subclasses of AP (which are still closed subalgebras of $L^\infty(\mathbb{R})$).

The reminder part of this paper is organized as follows. Section 2 contains some auxiliary information mostly on operator identities for Wiener–Hopf plus Hankel operators and inverses of particular operators. In Sect. 3, we obtain our main results by using operator relations and introducing a specific matrix factorization. In Sect. 4, we present an example of the applicability of the main results in Section 3.

2 Operator identities for Wiener–Hopf plus Hankel operators

Let ℓ_0 denotes the zero extension operator from the space $\left[L^2(\mathbb{R}_+)\right]^N$ into the space $\left[L^2_+(\mathbb{R})\right]^N$,

$$\ell_0 : \left[L^2(\mathbb{R}_+)\right]^N \to \left[L^2_+(\mathbb{R})\right]^N. \tag{4}$$

We will denote by $\chi_\pm$ the canonical projections (considered as multiplication operators) of $\left[L^2(\mathbb{R})\right]^N$ onto $\left[L^2_\pm(\mathbb{R})\right]^N$, respectively.

Considering $H^2_\pm(\mathbb{R}) := \mathcal{F}L^2_\pm(\mathbb{R})$, the orthogonal projections of $\left[L^2(\mathbb{R})\right]^N$ onto $\left[H^2_\pm(\mathbb{R})\right]^N$ are given by

$$P := \mathcal{F}\chi_+\mathcal{F}^{-1} : \left[L^2(\mathbb{R})\right]^N \to \left[H^2_+(\mathbb{R})\right]^N,$$
$$Q := \mathcal{F}\chi_-\mathcal{F}^{-1} : \left[L^2(\mathbb{R})\right]^N \to \left[H^2_-(\mathbb{R})\right]^N$$

(where the operator P is also usually referred to as the Riesz projection).

We will be also using the spaces $H^\infty_\pm(\mathbb{R})$ consisting in all the elements of $L^\infty(\mathbb{R})$ that are non-tangential limits of bounded and analytic functions in $\mathbb{C}_\pm := \{z \in \mathbb{C} : \pm\Im m(z) > 0\}$.

We will now give some basic formulas for Wiener–Hopf plus Hankel operators in view of taking profit of certain factorization properties for such operators. The next proposition is based on [17, pp. 51] and [18, pp. 71].

Proposition 1 *Let $\Phi, \Psi \in [L^\infty(\mathbb{R})]^{N\times N}$ and $w \in \mathbb{C}^{N\times N}$ such that $w^2 = I_{N\times N}$. Then,*

$$W_{\Phi\Psi} + H_{\Phi\Psi w} = (W_\Phi + H_{\Phi w})\ell_0(W_\Psi + H_{\Psi w}) + H_{\Phi w}\ell_0(W_{w\widetilde{\Psi}w-\Psi} + H_{w\widetilde{\Psi}-\Psi w}). \tag{5}$$

Proof First, recall two well-known formulas from the theory of Wiener–Hopf-Hankel operators (cf., e.g., [6]):

$$W_{\Phi\Psi} = W_\Phi\ell_0W_\Psi + H_\Phi\ell_0H_{\widetilde{\Psi}}, \tag{6}$$
$$H_{\Phi\Psi} = W_\Phi\ell_0H_\Psi + H_\Phi\ell_0W_{\widetilde{\Psi}}. \tag{7}$$

Having also in mind that $w \in \mathbb{C}^{N\times N}$ and $w^2 = I_{N\times N}$, it follows

$$\begin{aligned} W_{\Phi\Psi} + H_{\Phi\Psi w} &= W_\Phi\ell_0W_\Psi + H_\Phi\ell_0H_{\widetilde{\Psi}} + W_\Phi\ell_0H_{\Psi w} + H_\Phi\ell_0W_{\widetilde{\Psi}w} \\ &= W_\Phi\ell_0(W_\Psi + H_{\Psi w}) + H_\Phi\ell_0(W_{\widetilde{\Psi}w} + H_{\widetilde{\Psi}}) \\ &= W_\Phi\ell_0(W_\Psi + H_{\Psi w}) + H_{\Phi w}\ell_0(W_{w\widetilde{\Psi}w} + H_{w\widetilde{\Psi}}) \\ &= (W_\Phi + H_{\Phi w})\ell_0(W_\Psi + H_{\Psi w}) + H_{\Phi w}\ell_0(W_{w\widetilde{\Psi}w-\Psi} + H_{w\widetilde{\Psi}-\Psi w}), \end{aligned}$$

which is the desired identity.

Theorem 1 *Let $\Phi, \Psi, \Theta \in [L^\infty(\mathbb{R})]^{N\times N}$, and $w \in \mathbb{C}^{N\times N}$ such that $w^2 = I_{N\times N}$. If $\Phi \in [H^\infty_-(\mathbb{R})]^{N\times N}$ and $\Theta = w\widetilde{\Theta}w$, then we have the factorization*

$$W_{\Phi\Psi\Theta} + H_{\Phi\Psi\Theta w} = W_\Phi\ell_0(W_\Psi + H_{\Psi w})\ell_0(W_\Theta + H_{\Theta w}). \tag{8}$$

Proof Using formula (5), we have

$$\begin{aligned} W_{\Phi\Psi\Theta} + H_{\Phi\Psi\Theta w} &= (W_\Phi + H_{\Phi w})\ell_0(W_{\Psi\Theta} + H_{\Psi\Theta w}) \\ &\quad + H_{\Phi w}\ell_0\Big(W_{w\widetilde{\Psi\Theta}w-\Psi\Theta} + H_{w\widetilde{\Psi\Theta}-\Psi\Theta}\Big). \end{aligned}$$

Recalling that $H_\Phi = 0$ if $\Phi \in \left[H_-^\infty(\mathbb{R})\right]^{N\times N}$, we obtain

$$W_{\Phi\Psi\Theta} + H_{\Phi\Psi\Theta w} = W_\Phi\ell_0(W_{\Psi\Theta} + H_{\Psi\Theta w}).$$

Now, relaying on the property of Θ and using (5) once again, we will have that

$$\begin{aligned} W_{\Phi\Psi\Theta} + H_{\Phi\Psi\Theta} &= W_\Phi\ell_0\left[(W_\Psi + H_{\Psi w})\ell_0(W_\Theta + H_{\Theta w})\right. \\ &\quad \left. + H_{\Psi w}\ell_0(W_{w\tilde{\Theta}w-\Theta} + H_{w\tilde{\Theta}-\Theta w})\right] \\ &= W_\Phi\ell_0(W_\Psi + H_{\Psi w})\ell_0(W_\Theta + H_{\Theta w}). \end{aligned}$$

For a unital algebra X, we shall denote by $\mathcal{G}X$ the group of invertible elements.

Theorem 2 *Let $\Phi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ such that $\Phi_e = w\widetilde{\Phi_e}w$, with $w \in \mathbb{C}^{N\times N}$ and $w^2 = I_{N\times N}$. Then, $W_{\Phi_e} + H_{\Phi_e w}$ is invertible with inverse being the following operator*

$$\ell_0\left(W_{\Phi_e^{-1}} + H_{\Phi_e^{-1}w}\right)\ell_0 : [L^2(\mathbb{R}_+)]^N \to [L_+^2(\mathbb{R})]^N.$$

Proof First, observe that

$$(W_{\Phi_e\Phi_e^{-1}} + H_{\Phi_e\Phi_e^{-1}w})\ell_0 = (W_{I_{N\times N}} + H_w)\ell_0 = r_+\ell_0 = I_{[L^2(\mathbb{R}_+)]^N}.$$

Recalling (5), we obtain that

$$\begin{aligned} \left(W_{\Phi_e\Phi_e^{-1}} + H_{\Phi_e\Phi_e^{-1}w}\right)\ell_0 &= (W_{\Phi_e} + H_{\Phi_e w})\ell_0\left(W_{\Phi_e^{-1}} + H_{\Phi_e^{-1}w}\right)\ell_0 \\ &\quad + H_{\Phi_e w}\ell_0\left(W_{w\widetilde{\Phi_e^{-1}}w-\Phi_e^{-1}} + H_{w\widetilde{\Phi_e^{-1}}-\Phi_e^{-1}w}\right)\ell_0. \end{aligned} \tag{9}$$

Because $\Phi_e = w\widetilde{\Phi_e}w$, we obtain that $w\widetilde{\Phi_e^{-1}}w - \Phi_e^{-1} = 0$ and $w\widetilde{\Phi_e^{-1}} - \Phi_e^{-1}w = 0$. It follows that (9) is equivalent to

$$W_{I_{N\times N}} + H_w = \left(W_{\Phi_e} + H_{\Phi_e w}\right)\ell_0\left(W_{\Phi_e^{-1}} + H_{\Phi_e^{-1}w}\right)$$

and thus

$$I_{[L^2(\mathbb{R}_+)]^N} = \left(W_{\Phi_e} + H_{\Phi_e w}\right)\ell_0\left(W_{\Phi_e^{-1}} + H_{\Phi_e^{-1}w}\right)\ell_0.$$

Similarly, we have that

$$\ell_0\left(W_{\Phi_e^{-1}} + H_{\Phi_e^{-1}w}\right)\ell_0\left(W_{\Phi_e} + H_{\Phi_e w}\right) = I_{[L^2(\mathbb{R}_+)]^N}.$$

Hence, we have explicitly shown that $W_{\Phi_e} + H_{\Phi_e w}$ is invertible and its inverse is given by the formula:

$$\ell_0\left(W_{\Phi_e^{-1}} + H_{\Phi_e^{-1}w}\right)\ell_0 : \left[L^2(\mathbb{R}_+)\right]^N \to \left[L_+^2(\mathbb{R})\right]^N.$$

3 Invertibility of Wiener–Hopf plus Hankel operators with AP symbols

3.1 Auxiliary operators and operator relations

In order to relate operators and to transfer certain operator properties between the related operators, we will be also using the known notion of equivalence after (one-sided) extension relation between bounded linear operators. In view of this, let us first recall the corresponding notion.

Two bounded linear operators acting between Banach spaces, e.g., $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$, are said to be *equivalent after (one-sided) extension* [1,15] if there are invertible bounded linear operators E and F such that

$$\begin{bmatrix} T & 0 \\ 0 & I_Z \end{bmatrix} = ESF, \tag{10}$$

for some additional Banach space Z and where I_Z represents the identity operator in Z. This is an important operator relation which—in particular—allows the transfer of invertibility properties between the related operators. Namely, it directly follows from (10) that if two operators are equivalent after one-sided extension, then they belong to the same *invertibility class*. More precisely, one of these operators is invertible, left-invertible or right-invertible, if and only if the other operator enjoys the same property. A bit more general is the (two-sided) equivalence after extension relation where S is extended in a similar way as T [1]. However, this is not needed here.

Moreover, for the readers familiar with the notion of *Schur coupling* [2], it is also interesting to observe that if (10) holds true, then T and S are Schur coupled.

Although the next proposition is basically known from [17, Theorem 3.2], we choose to present in here a direct and simple proof of it.

Proposition 2 *Let $\Phi_1, \Phi_2 \in [L^\infty(\mathbb{R})]^{N\times N}$. The Wiener–Hopf plus Hankel operator $W_{\Phi_1} + H_{\Phi_2} : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N$ is equivalent after one-sided extension with $W_\Psi + H_{\Psi w} : [L^2_+(\mathbb{R})]^{2N} \to [L^2(\mathbb{R}_+)]^{2N}$, where*

$$\Psi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ 0 & I \end{bmatrix}, \qquad w = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{11}$$

Proof A direct computation yields

$$\begin{bmatrix} W_{\Phi_1} + H_{\Phi_2} & 0 \\ 0 & I_{[L^2_+(\mathbb{R})]^N} \end{bmatrix} = \begin{bmatrix} r_+ & -W_{\Phi_2} - H_{\Phi_1} \\ 0 & I_{[L^2_+(\mathbb{R})]^N} \end{bmatrix} \ell_0 (W_\Psi + H_{\Psi w}),$$

which shows the equivalence after one-sided extension between $W_{\Phi_1} + H_{\Phi_2}$ and $W_\Psi + H_{\Psi w}$ simply because $E = \begin{bmatrix} r_+ & -W_{\Phi_2} - H_{\Phi_1} \\ 0 & I_{[L^2_+(\mathbb{R})]^N} \end{bmatrix} \ell_0$ is an invertible bounded linear operator.

3.2 Matrix AP asymmetric factorization

We introduce now a new kind of AP factorization, the AP *asymmetric factorization with respect to* $w \in \mathbb{C}^{N\times N}$ (motivated by other types of asymmetric factorizations; cf. [9–11,16–19]).

Definition 1 Let $\Phi \in \mathcal{G}AP^{N\times N}$ and $w \in \mathbb{C}^{N\times N}$ such that $w^2 = I_{N\times N}$. We say that Φ admits an AP asymmetric factorization with respect to w if it can be represented in the form

$$\Phi = \Phi_- D \Phi_e ,$$

where $\Phi_- \in \mathcal{G}\mathrm{AP}_-^{N\times N}$, $\Phi_e = w\widetilde{\Phi_e}w$ and $D = \mathrm{diag}[e_{\lambda_1}, \ldots, e_{\lambda_N}]$, $\lambda_j \in \mathbb{R}$. The numbers λ_j are called the partial indices of the factorization. If $\lambda_1 = \ldots = \lambda_N = 0$, then the factorization is referred to as a canonical AP asymmetric factorization with respect to w.

3.3 Invertibility criteria

Considering a bounded linear operator $T : X \to Y$, acting between Banach spaces, we recall that T is normally solvable if ImT is closed (Theorem of Hausdorff). For a normally solvable operator T, the *deficiency numbers* of T are given by $n(T) := \dim \mathrm{Ker}T$ and $d(t) := \dim \mathrm{Coker}T$. If at least one of the deficiency numbers is finite, then the operator is said to be a *semi-Fredholm* operator. Additionally, we also recall that a normally solvable operator T is said to be: (i) *properly n-normal* if $n(T)$ is finite and $d(T)$ is infinite; (ii) *properly d-normal* if $d(T)$ is finite and $n(T)$ is infinite.

Theorem 3 *Let us assume that* $\Phi_1, \Phi_2 \in \mathcal{G}AP^{N\times N}$, $w = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ *and that* $\Psi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ 0 & I \end{bmatrix}$ *admits an AP asymmetric factorization with respect to* w, $\Psi = \Psi_- D\Psi_e$, *with* $\Psi_- \in \mathcal{G}AP_-^{2N\times 2N}$, $\Psi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{2N\times 2N}$ *such that* $\Psi_e = w\widetilde{\Psi_e}w$ *and* $D = \mathrm{diag}[e_{\lambda_1}, \ldots, e_{\lambda_{2N}}]$.

(a) *If there exist positive and negative partial indices* λ_i *(*$i = 1, \ldots, 2N$*), then* $W_{\Phi_1} + H_{\Phi_2}$ *is not semi-Fredholm;*
(b) *If* $\lambda_i \le 0$*, for all* $i = 1, \ldots, 2N$*, and if for at least one index* i *we have* $\lambda_i < 0$*, then* $W_{\Phi_1} + H_{\Phi_2}$ *is properly d-normal and right-invertible.*
(c) *If* $\lambda_i \ge 0$*, for all* $i = 1, \ldots, 2N$*, and if for at least one index* i *we have* $\lambda_i > 0$*, then* $W_{\Phi_1} + H_{\Phi_2}$ *is properly n-normal and left-invertible.*
(d) *If* $\lambda_i = 0$*, for all* $i = 1, \ldots, 2N$*, then* $W_{\Phi_1} + H_{\Phi_2}$ *is two-sided invertible.*

Proof To prove this theorem, we initially make use of the fact that $W_{\Phi_1} + H_{\Phi_2}$ is equivalent after one-sided extension with $W_\Psi + H_{\Psi w}$, which allows us to transfer the above-mentioned regularity properties from the operator $W_\Psi + H_{\Psi w}$ to the operator $W_{\Phi_1} + H_{\Phi_2}$ (within the present context, by regularity properties of a certain operator we mean that properties that depend on the kernel and cokernel of this operator).

Let $\Phi_1, \Phi_2 \in \mathcal{G}\mathrm{AP}^{N\times N}$, $\Psi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ 0 & I \end{bmatrix}$ such that Ψ admits an AP asymmetric factorization with respect to w:

$$\Psi = \Psi_- D\Psi_e,$$

with $\Psi_- \in \mathcal{G}\mathrm{AP}_-^{2N\times 2N}$, $D = \mathrm{diag}[e_{\lambda_1}, \ldots, e_{\lambda_{2N}}]$, and $\Psi_e = w\widetilde{\Psi_e}w$.

From (8), we have that

$$W_\Psi + H_{\Psi w} = W_{\Psi_-}\ell_0(W_D + H_{Dw})\ell_0(W_{\Psi_e} + H_{\Psi_e w}), \tag{12}$$

where W_{Ψ_-} is invertible because $\Psi_- \in \mathcal{G}\mathrm{AP}_-^{2N\times 2N}$ and $W_{\Psi_e} + H_{\Psi_e w}$ are also invertible due Theorem 2.

Thus, (12) shows us an equivalence relation between $W_\Psi + H_{\Psi w}$ and $W_D + H_{Dw}$.

We will now consider the corresponding appropriate cases separately:

(i) Suppose that at least some of the partial indices are greater that zero, some of them may be equal to zero and some of them are less than zero; for instance (without loss of generalization):

$$\lambda_1, \ldots, \lambda_i > 0, \; \lambda_{i+1} = \cdots = \lambda_j = 0 \quad \text{and} \quad \lambda_{j+1}, \ldots, \lambda_{2N} < 0$$

(for some $i \geq 1$, $i + 1 \leq j < 2N$). This means that

$$\begin{aligned} W_D + H_{Dw} &= W_D(I + wJ) \\ &= \operatorname{diag}\left[W_{e_{\lambda_1}}, \ldots, W_{e_{\lambda_i}}, W_{e_{\lambda_{i+1}}}, \ldots, W_{e_{\lambda_j}}, W_{e_{\lambda_{j+1}}}, \ldots, W_{e_{\lambda_{2N}}}\right](I + wJ) \\ &= \operatorname{diag}\left[W_{e_{\lambda_1}}, \ldots, W_{e_{\lambda_i}}, r_+, \ldots, r_+, W_{e_{\lambda_{j+1}}}, \ldots, W_{e_{\lambda_{2N}}}\right](I + wJ). \end{aligned}$$

Employing Gohberg-Feldman-Coburn-Douglas Theorem [6, Theorem 2.28], we have that $W_{e_{\lambda_1}}, \ldots, W_{e_{\lambda_i}}$ are properly n-normal and left-invertible, and $W_{e_{\lambda_{j+1}}}, \ldots, W_{e_{\lambda_{2N}}}$ are properly d-normal and right-invertible. Thus, $W_D + H_{Dw}$ is not semi-Fredholm and from the equivalence relations between $W_{\Phi_1} + H_{\Phi_2}$, $W_\Psi + H_{\Psi w}$ and $W_D + H_{Dw}$, it also follows that $W_{\Phi_1} + H_{\Phi_2}$ is not semi-Fredholm in this case.

(ii) Suppose now that $\lambda_i \leq 0$, for all $i = 1, \ldots, 2N$. This implies that $D \in \mathrm{AP}_-^{2N\times 2N}$. Since $\mathrm{AP}_-^{2N\times 2N} = \mathrm{AP}^{2N\times 2N} \cap [H_-^\infty(\mathbb{R})]^{2N\times 2N}$, it holds that $D \in [H_-^\infty(\mathbb{R})]^{2N\times 2N}$ and hence, $W_D + H_{Dw} = W_D$. Thus, in this case, $W_\Psi + H_{\Psi w}$ is equivalent to W_D. If we employ the Gohberg-Feldman-Coburn-Douglas Theorem to each the operators in the main diagonal of the operator W_D, it follows the assertion (b) of the theorem.

(iii) Part (c) can be deduced from the reasoning in (ii) by passing to adjoints.

(iv) If all partial indices are zero, we have that $W_D + H_{Dw}$ is equivalent to the identity operator which is two-sided invertible – obtaining therefore the assertion (d).

3.4 Formulas for the inverses

Theorem 4 *If $\Psi \in \mathcal{G}AP^{2N\times 2N}$ (defined in (11)) admits an AP asymmetric factorization with respect to $w = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$,*

$$\Psi = \Psi_- D \Psi_e\,,$$

with $D = \operatorname{diag}[e_{\lambda_1}, \ldots, e_{\lambda_{2N}}]$, then

$$(W_\Psi + H_{\Psi w})^- = \ell_0(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w})\ell_0 W_{D^{-1}}\ell_0 W_{\Psi_-^{-1}}\ell_0 : [L^2(\mathbb{R}_+)]^{2N} \to [L_+^2(\mathbb{R})]^{2N} \quad (13)$$

is the:

(i) *inverse of $W_\Psi + H_{\Psi w}$, if $\lambda_1 = \cdots = \lambda_{2N} = 0$;*
(ii) *the right-inverse of $W_\Psi + H_{\Psi w}$, if $\lambda_i \leq 0$ for all $i = 1, \ldots, 2N$;*
(iii) *the left-inverse of $W_\Psi + H_{\Psi w}$, if $\lambda_i \geq 0$ for all $i = 1, \ldots, 2N$.*

Proof From the hypothesis, we have $\Psi = \Psi_- D\Psi_e$, where $\Psi_- \in \mathcal{G}AP_-^{2N\times 2N}$, $\Psi_e = w\widetilde{\Psi_e}w$ and $D = \operatorname{diag}[e_{\lambda_1}, \ldots, e_{\lambda_{2N}}]$.

(i) If $\lambda_1 = \cdots = \lambda_{2N} = 0$, then

$$(W_\Psi + H_{\Psi w})^- = \ell_0(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w})\ell_0 W_{\Psi_-^{-1}}\ell_0 : [L^2(\mathbb{R}_+)]^{2N} \to [L_+^2(\mathbb{R})]^{2N} \quad (14)$$

and from (12) it follows

$$(W_\Psi + H_{\Psi w})(W_\Psi + H_{\Psi w})^- = W_{\Psi_-}\ell_0\left(W_{\Psi_e} + H_{\Psi_e w}\right)\ell_0\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\ell_0 W_{\Psi_-^{-1}}\ell_0 = W_{\Psi_-}\ell_0 W_{\Psi_-^{-1}}\ell_0 = I_{[L^2(\mathbb{R}_+)]^{2N}},$$

due to the use of Theorem 2 and due to the fact that $\Psi_- \in \mathcal{G}AP_-^{2N\times 2N}$.
In addition, by the same reason,

$$(W_\Psi + H_{\Psi w})^-(W_\Psi + H_{\Psi w}) = \ell_0\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\ell_0 W_{\Psi_-^{-1}}\ell_0 W_{\Psi_-}\ell_0\left(W_{\Psi_e} + H_{\Psi_e w}\right) = I_{[L_+^2(\mathbb{R})]^{2N}}.$$

(ii) If $\lambda_i \geq 0,\ i = 1, \ldots, 2N$, then

$$\begin{aligned}
&(W_\Psi + H_{\Psi w})^-(W_\Psi + H_{\Psi w})\\
&= \ell_0\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\ell_0 W_{D^{-1}}\ell_0 W_{\Psi_-^{-1}}\ell_0 W_{\Psi_-}\ell_0(W_D + H_{Dw})\ell_0\left(W_{\Psi_e} + H_{\Psi_e w}\right)\\
&= \ell_0\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\ell_0 W_{D^{-1}}\ell_0(W_D + H_{Dw})\ell_0\left(W_{\Psi_e} + H_{\Psi_e w}\right)\\
&= \ell_0\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\ell_0\left(W_{\Psi_e} + H_{\Psi_e w}\right)\\
&= I_{[L_+^2(\mathbb{R})]^{2N}}.
\end{aligned}$$

To obtain this last result, we use the fact that $\Psi_- \in \mathcal{G}AP_-^{2N\times 2N}$, and $\lambda_i \geq 0$. This, in particular, yields

$$\ell_0 W_{D^{-1}}\ell_0(W_D + H_{Dw})\ell_0 = \ell_0\left(W_{D^{-1}D} + H_{D^{-1}Dw}\right)\ell_0 = \ell_0(W_I + H_w)\ell_0 = \ell_0 W_I\ell_0 = \ell_0$$

(cf. Proposition 1).

(iii) If $\lambda_i \leq 0,\ i = 1, \ldots, 2N$, then (having in mind that $\Psi_-, D \in AP_-^{2N\times 2N}$) it follows:

$$\begin{aligned}
(W_\Psi + H_{\Psi w})(W_\Psi + H_{\Psi w})^- &= W_{\Psi_-}\ell_0 W_D\ell_0\left(W_{\Psi_e} + H_{\Psi_e w}\right)\ell_0\\
&\quad\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\ell_0 W_{D^{-1}}\ell_0 W_{\Psi_-^{-1}}\ell_0\\
&= W_{\Psi_-}\ell_0 W_D\ell_0 W_{D^{-1}}\ell_0 W_{\Psi_-^{-1}}\ell_0\\
&= W_{\Psi_-}\ell_0 W_{\Psi_-^{-1}}\ell_0\\
&= I_{[L^2(\mathbb{R}_+)]^{2N}}.
\end{aligned}$$

Using the equivalence after one-sided extension relation between $W_{\Phi_1} + H_{\Phi_2}$ and $W_\Psi + H_{\Psi w}$, and Theorem 4, we are in position to derive the representation of the one-sided and two-sided inverses of $W_{\Phi_1} + H_{\Phi_2}$.

Within the framework of bounded linear operators

$$V = [V_{ij}]_{i,j=1}^2 : [L_+^2(\mathbb{R})]^{2N} \to [L_+^2(\mathbb{R})]^{2N},$$

with $V_{ij} : [L^2_+(\mathbb{R})]^N \to [L^2_+(\mathbb{R})]^N$, we will use the notation $R_{ij}(V) := V_{ij}$.

Corollary 1 *If $\Psi \in \mathcal{G}AP^{2N\times 2N}$ [defined in (11)] admits an AP asymmetric factorization with respect to w, $\Psi = \Psi_- D\Psi_e$, having $D = \operatorname{diag}[e_{\lambda_1}, \ldots, e_{\lambda_{2N}}]$, then*

$$\begin{aligned}(W_{\Phi_1} + H_{\Phi_2})^- = {} & R_{11}\Big(\ell_0\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\Big)\ R_{11}\big(\ell_0 W_{D^{-1}}\big)\ R_{11}\Big(\ell_0 W_{\Psi_-^{-1}}\Big)\,\ell_0 \\ & + R_{12}\Big(\ell_0\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\Big)\ R_{22}\big(\ell_0 W_{D^{-1}}\big)\ R_{21}\Big(\ell_0 W_{\Psi_-^{-1}}\Big)\,\ell_0 \\ & : [L^2(\mathbb{R}_+)]^N \to [L^2_+(\mathbb{R})]^N \end{aligned} \tag{15}$$

is the:

(i) *inverse of $W_{\Phi_1} + H_{\Phi_2}$ if $\lambda_1 = \cdots = \lambda_{2N} = 0$;*
(ii) *the right-inverse of $W_{\Phi_1} + H_{\Phi_2}$, if $\lambda_i \le 0$ for all $i = 1, \ldots, 2N$;*
(iii) *the left-inverse of $W_{\Phi_1} + H_{\Phi_2}$, if $\lambda_i \ge 0$ for all $i = 1, \ldots, 2N$.*

Proof From Proposition 2, we have that $W_{\Phi_1} + H_{\Phi_2}$ is equivalent after one-sided extension with $W_\Psi + H_{\Psi w}$ and the corresponding operator relation is given, e.g., in the following explicit form:

$$\begin{bmatrix} W_{\Phi_1} + H_{\Phi_2} & 0 \\ 0 & r_+ \end{bmatrix} = \begin{bmatrix} r_+ & -W_{\Phi_2} - H_{\Phi_1} \\ 0 & r_+ \end{bmatrix} \ell_0(W_\Psi + H_{\Psi w}).$$

Therefore,

$$\begin{bmatrix} W_{\Phi_1} + H_{\Phi_2} & 0 \\ 0 & r_+ \end{bmatrix}^- = (W_\Psi + H_{\Psi w})^- \begin{bmatrix} r_+ & -W_{\Phi_2} - H_{\Phi_1} \\ 0 & r_+ \end{bmatrix}^{-1},$$

and so

$$\begin{aligned}(W_{\Phi_1} + H_{\Phi_2})^- &= R_{11}\left((W_\Psi + H_{\Psi w})^- \begin{bmatrix} r_+ & W_{\Phi_2} + H_{\Phi_1} \\ 0 & r_+ \end{bmatrix}\right)\ell_0 \\ &= R_{11}\left((W_\Psi + H_{\Psi w})^- r_+\right)\ell_0. \end{aligned}$$

Using now Theorem 4, a straightforward computation yields

$$\begin{aligned}(W_{\Phi_1} + H_{\Phi_2})^- &= R_{11}\left(\ell_0\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\ell_0 W_{D^{-1}}\ell_0 W_{\Psi_-^{-1}}\right)\ell_0 \\ &= R_{11}\Big(\ell_0\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\Big)\ R_{11}\big(\ell_0 W_{D^{-1}}\big)\ R_{11}\Big(\ell_0 W_{\Psi_-^{-1}}\Big)\,\ell_0 \\ &\quad + R_{12}\Big(\ell_0\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\Big)\ R_{22}\big(\ell_0 W_{D^{-1}}\big)\ R_{21}\Big(\ell_0 W_{\Psi_-^{-1}}\Big)\,\ell_0. \end{aligned}$$

4 Example

In this last section, we will present an example to illustrate the results described above.

For $x \in \mathbb{R}$, let us consider

$$\Phi_1(x) = \begin{bmatrix} 2e^{-e^{-ix}+e^{2ix}} & 0 \\ e^{e^{-ix}+e^{ix}} & e^{e^{-ix}+e^{ix}+1} \end{bmatrix}, \quad \Phi_2(x) = \begin{bmatrix} ie^{e^{-2ix}} & 0 \\ -ie^{e^{-ix}} & -ie^{e^{-ix}+1} \end{bmatrix}. \tag{16}$$

Being clear that $\Phi_1,\ \Phi_2 \in \mathcal{G}\mathrm{AP}^{2\times 2}$, let us construct the corresponding matrix Ψ:

$$\Psi(x) = \begin{bmatrix} \Phi_1(x) & \Phi_2(x) \\ 0 & I_{2\times 2} \end{bmatrix} = \begin{bmatrix} 2e^{-e^{-ix}+e^{2ix}} & 0 & ie^{e^{-2ix}} & 0 \\ e^{e^{-ix}+e^{ix}} & e^{e^{-ix}+e^{ix}+1} & -ie^{e^{-ix}} & -ie^{e^{-ix}+1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (17)$$

Considering Ψ_- and Ψ_e given by

$$\Psi_-(x) = \begin{bmatrix} 2e^{-e^{-ix}} & 0 & i & 0 \\ 0 & e^{e^{-ix}} & 0 & -i \\ 0 & 0 & e^{-e^{-2ix}} & 0 \\ 0 & 0 & -e^{-e^{-2ix}-1} & e^{-e^{-ix}-1} \end{bmatrix}$$

and

$$\Psi_e(x) = \begin{bmatrix} e^{e^{2ix}} & 0 & 0 & 0 \\ e^{e^{ix}} & e^{e^{ix}+1} & 0 & 0 \\ 0 & 0 & e^{e^{-2ix}} & 0 \\ 0 & 0 & e^{e^{-ix}} & e^{e^{-ix}+1} \end{bmatrix},$$

for $x \in \mathbb{R}$, we have that

$$\Psi = \Psi_- \Psi_e$$

with $\Psi_- \in \mathcal{G}\mathrm{AP}_-^{2\times 2}$, $\Psi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{2\times 2}$ such that $\Psi_e = w\widetilde{\Psi_e}w$, and

$$w = \begin{bmatrix} 0 & I_{2\times 2} \\ I_{2\times 2} & 0 \end{bmatrix}.$$

Therefore, it follows that Ψ admits a canonical AP asymmetric factorization with respect to $w \in \mathbb{C}^{2\times 2}$.

As a consequence, using Corollary 1, we conclude that the Wiener–Hopf plus Hankel operator $W_{\Phi_1} + H_{\Phi_2}$ (with Fourier symbols given in (16)) is two-sided invertible.

Moreover, using the same corollary, we are able to determine the inverse of $W_{\Phi_1} + H_{\Phi_2}$. From (15), it follows that

$$\begin{aligned}\left(W_{\Phi_1} + H_{\Phi_2}\right)^{-1} = {} & R_{11}\Big(\ell_0\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\Big)\ R_{11}\Big(\ell_0 W_{\Psi_-^{-1}}\Big)\,\ell_0 \\ & + R_{12}\Big(\ell_0\left(W_{\Psi_e^{-1}} + H_{\Psi_e^{-1}w}\right)\Big)\ R_{21}\Big(\ell_0 W_{\Psi_-^{-1}}\Big)\,\ell_0. \end{aligned} \quad (18)$$

Therefore, introducing in (18), the elements

$$\Psi_e^{-1} = \begin{bmatrix} e^{-e^{2ix}} & 0 & 0 & 0 \\ -e^{-e^{2ix}-1} & e^{-e^{ix}-1} & 0 & 0 \\ 0 & 0 & e^{-e^{-2ix}} & 0 \\ 0 & 0 & -e^{-e^{-2ix}-1} & e^{-e^{-ix}-1} \end{bmatrix}$$

$$\Psi_-^{-1} = \begin{bmatrix} \frac{1}{2}e^{e^{-ix}} & 0 & -\frac{i}{2}e^{e^{-2ix}+e^{-ix}} & 0 \\ 0 & e^{-e^{-ix}} & i & ie \\ 0 & 0 & e^{e^{-2ix}} & 0 \\ 0 & 0 & e^{e^{-ix}} & e^{e^{-ix}+1} \end{bmatrix}$$

$$\Psi_e^{-1} w = \begin{bmatrix} 0 & 0 & e^{-e^{2ix}} & 0 \\ 0 & 0 & -e^{-e^{2ix}-1} & e^{-e^{ix}-1} \\ e^{-e^{-2ix}} & 0 & 0 & 0 \\ -e^{-e^{-2ix}-1} & e^{-e^{-ix}-1} & 0 & 0 \end{bmatrix},$$

we obtain the inverse of $W_{\Phi_1} + H_{\Phi_2}$ in the following explicit form

$$(W_{\Phi_1} + H_{\Phi_2})^{-1} = \ell_0 \begin{bmatrix} W_{e^{-e^{2ix}}} & 0 \\ W_{-e^{-e^{2ix}-1}} & W_{e^{-e^{ix}-1}} \end{bmatrix} \ell_0 \begin{bmatrix} \frac{1}{2} W_{e^{e^{-ix}}} & 0 \\ 0 & W_{e^{-e^{-ix}}} \end{bmatrix} \ell_0$$
$$= \frac{1}{2} \begin{bmatrix} \ell_0 W_{e^{-e^{2ix}}} \ell_0 W_{e^{e^{-ix}}} \ell_0 & 0 \\ -\ell_0 W_{e^{-e^{2ix}-1}} \ell_0 W_{e^{e^{-ix}}} \ell_0 & 2 \ell_0 W_{e^{-e^{ix}-1}} \ell_0 W_{e^{-e^{-ix}}} \ell_0 \end{bmatrix}.$$

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References

1. Bart, H., Tsekanovskii, V.E.: Matricial coupling and equivalence after extension. Oper. Theory Adv. Appl. **59**, 143–160 (1992)
2. Bart, H., Tsekanovskii, V.E.: Complementary Schur complements. Linear Algebra Appl. **197**, 651–658 (1994)
3. Basor, E.L., Ehrhardt, T.: On a class of Toeplitz + Hankel operators. N. Y. J. Math. **5**, 1–16 (1999)
4. Basor, E.L., Ehrhardt, T.: Factorization theory for a class of Toeplitz + Hankel operators. J. Oper. Theory. **51**, 411–433 (2004)
5. Basor, E.L., Ehrhardt, T.: Factorization of a class of Toeplitz + Hankel operators and the A_p–condition. J. Oper. Theory. **55**(2), 269–283 (2006)
6. Böttcher, A., Karlovich, Yu.I., Spitkovsky, I.M.: Convolution Operators and Factorization of Almost Periodic Matrix Functions. Birkhäuser, Basel (2002)
7. Böttcher, A., Silbermann, B., Karlovich, A.: Analysis of Toeplitz Operators, 2nd edn. In: Springer Monographs in Mathematics, Springer, Berlin (2006)
8. Bogveradze, G., Castro, L.P.: On the Fredholm index of matrix Wiener–Hopf plus/minus Hankel operators with semi-almost periodic symbols. Oper. Theory Adv. Appl. **181**, 143–158 (2008)
9. Bogveradze, G., Castro, L.P.: Invertibility properties of matrix Wiener–Hopf plus Hankel integral operators. Math. Model. Anal. **13**, 7–16 (2008)
10. Bogveradze, G., Castro, L.P.: Invertibility characterization of Wiener–Hopf plus Hankel operators via odd asymmetric factorizations. Banach J. Math. Anal. **3**, 1–18 (2009)
11. Castro, L.P., Duduchava, R., Speck, F.-O.: Asymmetric factorizations of matrix functions on the real line. Oper. Theory Adv. Appl. **170**, 53–74 (2007)
12. Castro, L.P., Nolasco, A.P.: A semi-Fredholm theory for Wiener–Hopf-Hankel operators with piecewise almost periodic Fourier symbols. J. Oper. Theory. **62**, 3–31 (2009)
13. Castro, L.P., Silva, A.S.: Invertibility of matrix Wiener–Hopf plus Hankel operators with symbols producing a positive numerical range. Z. Anal. Anwend. **28**, 119–127 (2009)
14. Castro, L.P., Silva, A.S.: Fredholm property of matrix Wiener–Hopf plus and minus Hankel operators with semi-almost periodic symbols. Cubo **12**, 217–234 (2010)
15. Castro, L.P., Speck, F.-O.: Regularity properties and generelized inverses of delta-related operators. Z. Anal. Anwend. **17**, 577–598 (1998)
16. Castro, L.P., Speck, F.-O., Teixeira, F.S.: A direct approach to convolution type operators with symmetry. Math. Nachr. **269/270**, 73–85 (2004)
17. Ehrhardt, T.: Factorization theory for Toeplitz plus Hankel operators and singular integral operators with flip. Habilitation Thesis, Technischen Universitität Chemnitz, Chemnitz (2004)

18. Ehrhardt, T.: Invertibility theory for Toeplitz plus Hankel operators and singular integral operators with flip. J. Funct. Anal. **208**, 64–106 (2004)
19. Nolasco, A.P., Castro, L.P.: Factorization theory for Wiener–Hopf plus Hankel operators with almost periodic symbols. Contemp. Math. **414**, 111–128 (2006)
20. Nolasco, A.P., Castro, L.P.: A Duduchava–Saginashvili's type theory for Wiener–Hopf plus Hankel operators. J. Math. Anal. Appl. **331**, 329–341 (2007)
21. Power, S.C.: C^*-algebras generated by Hankel operators and Toeplitz operators. J. Funct. Anal. **31**, 52–68 (1979)
22. Roch, S., Silbermann, B.: Algebras of convolution operators and their image in the Calkin algebra. Report MATH 90-05. Akademie der Wissenschaften der DDR, Karl-Weierstrass-Institut für Mathematik, Berlin (1990)