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A maximum modulus theorem for the Oseen problem

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Abstract Classical solutions of the Oseen problem are studied on an exterior domain Ω with Ljapunov boundary in R^3. It is proved a maximum modulus estimate of the following form: If $\mathbf{u} \in C^2(\Omega)^3 \cap C^0(\overline{\Omega})^3$ and $p \in C^1(\Omega)$, $-\Delta \mathbf{u} + 2\lambda\partial_1 \mathbf{u} + \nabla p = 0$, $\nabla \cdot \mathbf{u} = 0$ in Ω, and if $|\mathbf{u}| \leq M$ on $\partial\Omega$, $\limsup |\mathbf{u}(\mathbf{x})| \leq M$ as $|\mathbf{x}| \to \infty$, then $|\mathbf{u}(\mathbf{x})| \leq cM$ in Ω. Here the constant c depends only on Ω and λ.

Keywords Oseen problem · Maximum modulus theorem · Oseen potentials · Uniqueness · Non-tangential limit · Theorem of Liouville type

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1 Introduction

In the theory of partial differential equations, the classical maximum principle is well-known. It states that each harmonic function u takes its maximum and minimum values always at the boundary $\partial\Omega$ of the corresponding bounded domain Ω. This result remains true also for solutions of more general elliptic equations of second order with regular coefficients. However, for solutions of higher-order equations or for solutions of elliptic systems, it is not true in general (see e.g., [38]). In these cases, a so-called maximum modulus estimate of the form

$$\max_{\mathbf{x}\in\overline{\Omega}} |\mathbf{u}(\mathbf{x})| \ \leq \ c_\Omega \ \max_{\mathbf{x}\in\partial\Omega} |\mathbf{u}(\mathbf{x})|$$

might be valid, with some constant $c = c_\Omega$ depending only on Ω.

Concerning the linearized steady Stokes system

$$-\Delta\mathbf{u} + \nabla p = 0 \quad \text{in } \Omega, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1}$$

a maximum modulus estimate has been proved, recently (see [30,31,33]): Let $\Omega \subset R^3$ be a bounded or an unbounded domain with a compact boundary $\partial\Omega \in C^{1,\alpha}$, $0 < \alpha < 1$. Let $\mathbf{u} \in C^2(\Omega)^3 \cap C^0(\overline{\Omega})^3$ and $p \in C^1(\Omega)$ satisfy the Stokes system (1), where in case of unbounded Ω, we require $|\mathbf{u}(\mathbf{x})| = O(|\mathbf{x}|^{-1})$, $|\nabla\mathbf{u}(\mathbf{x})| + |p(\mathbf{x})| = O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}| \to \infty$, in addition. Then

$$\sup_{\mathbf{x}\in\Omega} |\mathbf{u}(\mathbf{x})| \leq c_\Omega \max_{\mathbf{x}\in\partial\Omega} |\mathbf{u}(\mathbf{x})|$$

with a constant c_Ω depending only on Ω. Moreover, if Ω is a ball, special statements about the size of c_Ω are possible (see Kratz [26–28]).

It is the aim of the present paper to prove a maximum modulus estimate for the Oseen equations. These equations represent a mathematical model describing the motion of a viscous incompressible fluid flow around an obstacle. They are obtained by linearizing the steady Navier–Stokes equations at a nonzero constant vector $\mathbf{u} = \mathbf{u}_\infty$, where $\mathbf{u}_\infty$ represents the velocity at infinity, and have the form

$$-\nu\Delta\mathbf{u} + \mathbf{u}_\infty\cdot\nabla\mathbf{u} + \nabla p = 0 \quad \text{in } \Omega, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega. \tag{2}$$

Here $\Omega \subset R^3$ denotes an exterior domain, that is, a domain having a compact complement $R^3 \setminus \Omega$. The velocity field $\mathbf{u}$ and the pressure function p are unknown, while the kinematic viscosity $\nu > 0$ and the nonzero constant velocity $\mathbf{u}_\infty$ are given data.

The system (2) is well-known in hydrodynamics. It has been introduced in 1910 by Oseen [36] as a linearization at $t = \infty$ of the nonstationary Navier–Stokes equations describing the motion of a viscous incompressible fluid. In contrast to the simpler Stokes approximation (1), the Oseen system (2) avoids certain paradoxes related to the flow behavior at infinity and shows, in particular, a paraboloidal wake region behind the obstacle, extending with axis directed to $\mathbf{u}_\infty$. The Oseen equations have mostly been studied in exterior domains with Dirichlet boundary conditions. Early fundamental works are due to Finn [17–19] and Babenko [4] who considered these equations in two- and three-dimensional exterior domains using a weighted L^2-approach. Further important contributions are due to Farwig [14,15] introducing anisotropically weighted spaces in an L^2-framework, and Farwig and Sohr [16], Kračmar et al. [25] using weighted Sobolev spaces. Galdi considered the system in $W^{m,p}_{loc}$ spaces and, moreover, investigated a generalized Oseen system recently (see [20]). Enomoto and Shibata [12] and Kobayashi and Shibata [23] studied the corresponding Oseen semigroup. Concerning the scalar Oseen equation, important results in weighted Sobolev spaces are given by Amrouche and Bouzit [1,2] and Amrouche and Razafison [3]. The stationary Oseen system has been studied using a potential approach by Deuring, Kračmar [9,10], the corresponding nonstationary Oseen system has been considered recently by Deuring [6–8].

Choosing $\nu = 1$ and $\mathbf{u}_\infty = (2\lambda, 0, 0)$, without loss of generality from (2), we obtain the Oseen system in the form

$$-\Delta \mathbf{u} + 2\lambda \partial_1 \mathbf{u} + \nabla p = 0 \quad \text{in } \Omega, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{3}$$

in an exterior domain Ω. Here $0 \neq \lambda \in R$ is fixed (for $\lambda = 0$ the system (3) reduces to (1)). Without loss of generality, the Oseen equations are usually studied for $\lambda > 0$: If (3) holds true for $\mathbf{u}$ and p, then for $\tilde{\mathbf{u}}(\mathbf{x}) = \mathbf{u}(-\mathbf{x})$ and $\tilde{p}(\mathbf{x}) = -p(-\mathbf{x})$ we find $-\Delta\tilde{\mathbf{u}} - 2\lambda\partial_1\tilde{\mathbf{u}} + \nabla\tilde{p} = 0$, $\nabla \cdot \tilde{\mathbf{u}} = 0$ in $\tilde{\Omega} = \{\mathbf{x}; -\mathbf{x} \in \Omega\}$.

We study the Dirichlet problem for the Oseen equations (3) in an exterior domain $\Omega \subset R^3$ with a compact Ljapunov boundary $\partial\Omega$ (i.e., of class $C^{1,\alpha}$, $0 < \alpha < 1$) by the method of integral equations. We look for a solution in form of a linear combination of an Oseen single layer potential and an Oseen double layer potential both with the same density $\mathbf{\Psi}$. This leads to a system of boundary integral equations of the form $S\mathbf{\Psi} = \mathbf{g}$ in $\mathcal{C}^0(\partial\Omega)^3$, where $\mathbf{g} \in \mathcal{C}^0(\partial\Omega)^3$ is the prescribed Dirichlet boundary value. The operator $S - (1/2)I$ is a compact operator in $\mathcal{C}^0(\partial\Omega)^3$, where I means the identity. To study the properties of the operator S, we can use Fredholm's alternative theorem. For this reason, we investigate the Robin problem for the adjoint equations $-\Delta\mathbf{u} - 2\lambda\partial_1\mathbf{u} + \nabla p = 0$, $\nabla \cdot \mathbf{u} = 0$ in the complementary bounded open set $G = R^3 \setminus \overline{\Omega}$. We look for a solution of the Robin problem in form of an Oseen single layer potential with an unknown density $\mathbf{\Phi}$. This leads to the boundary integral equations' system $S'\mathbf{\Phi} = \mathbf{f}$, where $\mathbf{f}$ is the Robin boundary value. We prove the unique solvability of the Robin problem and the corresponding integral equations $S'\mathbf{\Phi} = \mathbf{f}$. Since S' is the operator adjoint to S, we conclude that the operator S is continuously invertible, too. Thus, we have proved that for each $\mathbf{g} \in \mathcal{C}^0(\partial\Omega)^3$, there exists a solution of the Dirichlet problem for the Oseen equations (3) with boundary value $\mathbf{g}$ such that $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$.

To prove a maximum modulus estimate for all classical solutions $\mathbf{u}$, p of the Oseen equations we start with a Liouville-type theorem as follows: If $\mathbf{u}$ and p are tempered distributions satisfying the Oseen equations (in a distributional sense) in the whole space R^3, then $\mathbf{u}$ and p are polynomials. In particular, if $\mathbf{u}$ is bounded, then $\mathbf{u}$ and p are constant. Similar results have been proved recently for the scalar Oseen equation (see [1,3]). Using this result, we prove that if $\mathbf{u}$, p are solving the Oseen equations in an exterior domain and if $\mathbf{u}$ is bounded, then there are constants $\mathbf{u}_\infty$, p_∞ with $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$, $p(\mathbf{x}) \to p_\infty$ as $|\mathbf{x}| \to \infty$. This implies that for $\mathbf{g} \in \mathcal{C}^0(\partial\Omega)^3$, $\mathbf{u}_\infty \in R^3$, $p_\infty \in R$, there exists a unique solution of the Dirichlet problem for the Oseen equations (3) with the boundary condition $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$ such that $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$, $p(\mathbf{x}) \to p_\infty$ as $|\mathbf{x}| \to \infty$. Moreover, we also know the integral representation of this solution.

Now, using the integral representation just mentioned and the closed graph theorem, we can prove a maximum modulus estimate of the following form: Let $\Omega \subset R^3$ be an exterior domain with $\partial\Omega$ of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, $\lambda \in R \setminus \{0\}$. Then there exists a constant $c = c_\Omega$ with the following property: If $\mathbf{u} \in C^2(\Omega)^3 \cap C^0(\overline{\Omega})^3$ and $p \in C^1(\Omega)$ solve the Oseen equations (3) in Ω, and if

$$|\mathbf{u}| \le M \quad \text{on } \partial\Omega, \quad \limsup_{|\mathbf{x}|\to\infty} |\mathbf{u}(\mathbf{x})| \le M,$$

then

$$|\mathbf{u}(\mathbf{x})| \le c_\Omega\, M \quad \text{in } \Omega.$$

2 Stokes potentials

Let $\mathbf{x} = [x_1, x_2, x_3] \in R^3$ and $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Then for $0 \neq \mathbf{x} \in R^3$ and $j, k \in \{1, 2, 3\}$, we define the Stokes fundamental solution by

$$E_{jk}(\mathbf{x}) = \frac{1}{8\pi} \left\{ \delta_{jk} \frac{1}{|\mathbf{x}|} + \frac{x_j x_k}{|\mathbf{x}|^3} \right\}, \tag{4}$$

$$Q_k(\mathbf{x}) = \frac{x_k}{4\pi |\mathbf{x}|^3}. \tag{5}$$

If $\mathbf{f} \in \mathcal{C}^0(R^3)^3$ has a compact support, then the convolution integrals (Stokes volume potentials)

$$E * \mathbf{f}(\mathbf{x}) = \int_{R^3} E(\mathbf{x} - \mathbf{y}) \mathbf{f}(\mathbf{y}) \, d\mathbf{y}, \quad \mathbf{Q} * \mathbf{f}(\mathbf{x}) = \int_{\mathbf{R^3}} \mathbf{Q}(\mathbf{x} - \mathbf{y}) \mathbf{f}(\mathbf{y}) \, d\mathbf{y}$$

are well defined, and it holds $E * \mathbf{f} \in \mathcal{C}^0(R^3)^3$, $Q * \mathbf{f} \in \mathcal{C}^0(R^3)$, $\partial_j(E * \mathbf{f}) = (\partial_j E) * \mathbf{f} \in \mathcal{C}^0(R^3)^3$ (see e.g., [40], II.2.3) and $-\Delta E * \mathbf{f} + \nabla Q * \mathbf{f} = \mathbf{f}$, $\nabla \cdot E * \mathbf{f} = 0$ in Ω in the sense of distributions. If $\mathbf{f} \in W^{m,q}(R^3)^3$ with $1 < q < \infty$, $m \geq 0$, then $E * \mathbf{f} \in W^{m+2,q}_{loc}(R^3)^3$, $Q * \mathbf{f} \in W^{m+1,q}_{loc}(R^3)$ (see e.g., [20], Chapter IV, Theorem 4.1).

Let $\Omega \subset R^3$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, and $\mathbf{\Psi} \in \mathcal{C}^0(\partial\Omega)^3$. Define the hydrodynamical single layer potential with density $\mathbf{\Psi}$ by

$$(E_\Omega \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} E(\mathbf{x} - \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \, d\sigma_{\mathbf{y}}$$

and the corresponding pressure by

$$(Q_\Omega \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} Q(\mathbf{x} - \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \, d\sigma_{\mathbf{y}}$$

whenever it makes sense. Then the pair $(E_\Omega \mathbf{\Psi}, Q_\Omega \mathbf{\Psi}) \in C^\infty(R^3 \setminus \partial\Omega)^4$ solves the Stokes system in $R^3 \setminus \partial\Omega$. Moreover, $E_\Omega \mathbf{\Psi} \in \mathcal{C}^0(R^3)^3$ and $E_\Omega \mathbf{\Psi} \in \mathcal{C}^\alpha(\partial\Omega)^3$ (see [35]).

For $\mathbf{u}$, p we define the stress tensor

$$T(\mathbf{u}, p) = 2\hat{\nabla}\mathbf{u} - pI, \tag{6}$$

where I denotes the identity matrix and

$$\hat{\nabla}\mathbf{u} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$$

is the deformation tensor, with $(\nabla\mathbf{u})^T$ as the matrix transposed to $\nabla\mathbf{u} = (\partial_j u_k)$, $j, k = 1, 2, 3$.

For $\mathbf{y} \in \partial\Omega$ we define $K^\Omega(\cdot, \mathbf{y}) = T(E(\cdot - \mathbf{y}), Q(\cdot - \mathbf{y}))\, \mathbf{n}^\Omega(\mathbf{y})$ on $R^3 \setminus \{\mathbf{y}\}$. Here and in the following, $\mathbf{n}^\Omega(\mathbf{y})$ is the outward unit normal of Ω at $\mathbf{y} \in \partial\Omega$. We set

$$K^\Omega_{k,j}(\mathbf{x}, \mathbf{y}) = \frac{3}{4\pi} \frac{(y_k - x_k)(y_j - x_j)(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}^\Omega(\mathbf{y})}{|\mathbf{y} - \mathbf{y}|^5}$$

for $j, k = 1, 2, 3$, and

$$\Pi_j(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\left\{-3\frac{(y_j-x_j)(\mathbf{y}-\mathbf{x})\cdot\mathbf{n}^{\Omega}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^5} + \frac{\mathbf{n}_j^{\Omega}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\right\}$$

for $j = 1, 2, 3$.

For $\mathbf{\Psi} \in \mathcal{C}^0(\partial\Omega)^3$ we define the hydrodynamical double layer potential with density $\mathbf{\Psi}$ by

$$(D_\Omega\mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} K^{\Omega}(\mathbf{x},\mathbf{y})\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}}, \quad \mathbf{x}\in R^3\setminus\partial\Omega$$

and the corresponding pressure by

$$(\Pi_\Omega\mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} \Pi^{\Omega}(\mathbf{x}-\mathbf{y})\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}}, \quad \mathbf{x}\in R^3\setminus\partial\Omega.$$

Then the pair $(D_\Omega\mathbf{\Psi}, \Pi_\Omega\mathbf{\Psi}) \in C^\infty(R^3\setminus\partial\Omega)^4$ solves the Stokes system in $R^3\setminus\partial\Omega$. For $\mathbf{x}\in\partial\Omega$ we denote the so-called directed values of the above potentials by

$$(K_\Omega\mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} K^{\Omega}(\mathbf{x},\mathbf{y})\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}},$$

$$(K'_\Omega\mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} K^{\Omega}(\mathbf{y},\mathbf{x})\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}}.$$

Then we find

$$\lim_{\substack{\mathbf{x}\to\mathbf{z}\\ \mathbf{x}\in\Omega}} D_\Omega\mathbf{\Psi}(\mathbf{x}) = \frac{1}{2}\mathbf{\Psi}(\mathbf{z}) + K_\Omega\mathbf{\Psi}(\mathbf{z}) \tag{7}$$

for $\mathbf{z}\in\partial\Omega$ (see [35,29], Chapter III, §2).

For $\mathbf{x}\in\partial\Omega$, $\beta > 0$ denote the non-tangential approach region of opening β at the point $\mathbf{x}$ by

$$\Gamma_\beta(\mathbf{x}) := \{\mathbf{y}\in\Omega;\, |\mathbf{x}-\mathbf{y}| < (1+\beta)\,\mathrm{dist}(\mathbf{y},\partial\Omega)\}.$$

Suppose that β is large enough. If

$$c = \lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in\Gamma_\beta(\mathbf{x})}} u(\mathbf{y}),$$

we call c the non-tangential limit of u at $\mathbf{x}\in\partial\Omega$. Note that $\mathbf{x}\in\overline{\Gamma_\beta(\mathbf{x})}$ for every $\mathbf{x}\in\partial\Omega$. If now u is a function defined in Ω, we denote the non-tangential maximal function of u on $\partial\Omega$ by

$$u^*(\mathbf{x}) = \sup\{|u(\mathbf{y})|;\, \mathbf{y}\in\Gamma_\beta(\mathbf{x})\}.$$

If $\mathbf{\Psi}\in\mathcal{C}^0(\partial\Omega)^3$, then we obtain

$$\||E_\Omega\mathbf{\Psi}|^* + |\nabla E_\Omega\mathbf{\Psi}|^* + |Q_\Omega\mathbf{\Psi}|^*\|_{L^2(\partial\Omega)} \le C\|\mathbf{\Psi}\|_{L^2(\partial\Omega)^3}$$

with some constant C depending only on Ω (see [5], Lemma 6.1). If $\mathbf{z}\in\partial\Omega$, then $\mathbf{\Psi}(\mathbf{z})/2 - K'_\Omega\Psi(\mathbf{z})$ is the non-tangential limit of $T(E_\Omega\mathbf{\Psi}, Q_\Omega\mathbf{\Psi})\mathbf{n}^{\Omega}(\mathbf{z})$ (see [13] or [22]).

3 Oseen fundamental solution and potentials

In this section, we recall some basic facts about the fundamental solution to the Oseen problem. Denote by $O(\cdot\,;2\lambda) = \big(O_{ij}(\cdot\,;2\lambda)\big)$, $Q = (Q_i)$ its fundamental solution; it satisfies the identities

$$-\Delta O_{ij} + 2\lambda\partial_1 O_{ij} + \partial_j Q_i = \delta_{ij}\delta, \qquad \partial_j O_{ij} = 0 \tag{8}$$

in the sense of distributions, where δ_{ij} denotes the Kronecker delta, while δ denotes the Dirac delta distribution.

We can easily verify (see e.g., [20], Chapter VII, §VII.3) that for $\lambda > 0$, the fundamental solution can be written as

$$Q_i(\mathbf{x}) = \frac{1}{4\pi}\frac{x_i}{|\mathbf{x}|^3} \tag{9}$$

$$O_{ij}(\mathbf{x};2\lambda) = (\delta_{ij}\Delta - \partial_i\partial_j)\phi_O(\mathbf{x};2\lambda), \tag{10}$$

where

$$\phi_O(\mathbf{x};2\lambda) = \frac{-1}{8\pi\lambda}\psi\Big(\lambda s(\mathbf{x})\Big) \tag{11}$$

with

$$\psi(z) = \int\limits_0^z \frac{1-e^{-t}}{t}\mathrm{d}t = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!\,k}z^k \tag{12}$$

and

$$s(\mathbf{x}) = |\mathbf{x}| - x_1. \tag{13}$$

The formulas (11)–(13) yield useful rescaling property

$$|2\lambda|\,O(2\lambda\mathbf{x};1) = O(\mathbf{x};2\lambda), \qquad \lambda \in R. \tag{14}$$

Since $O(\mathbf{x},-2\lambda) = O(-\mathbf{x},2\lambda)$, an easy calculation yields that $(O(\mathbf{x},2\lambda), Q(\mathbf{x}))$ is the fundamental solution of the Oseen equation (3) for arbitrary $\lambda \neq 0$.

Proposition 3.1 ([20, VII.3]). *If β is a multi-index, then we have*

$$\partial^\beta O(\mathbf{x},2\lambda)| = O(|\mathbf{x}|^{-1-|\beta|/2}) \quad \textit{as } |\mathbf{x}| \to \infty. \tag{15}$$

If $r > 0$ and $q > 4/3$, then we have

$$|\nabla O(\cdot,2\lambda)| \in L^q(R^3 \setminus B(0;r)). \tag{16}$$

Here $B(\mathbf{z};\mathbf{r}) = \{\mathbf{y} \in \mathbf{R}^3;\, |\mathbf{x}-\mathbf{y}| < \mathbf{r}\}$ denotes the open ball with center $\mathbf{z}$ and radius $r > 0$.

The integral representation (12) implies

$$\psi'(t) = \tfrac{1-e^{-t}}{t}, \quad \psi''(t) = \tfrac{-1+e^{-t}+te^{-t}}{t^2}, \quad \psi'''(t) = \tfrac{2-2e^{-t}-2te^{-t}-t^2e^{-t}}{t^3}.$$

The representation by the sum in (12) yields,

$$\psi^{(k)}(t) = \frac{(-1)^{k+1}}{k} + O(t) \quad \text{as } t \to 0\,, \quad k = 1, 2, \ldots. \tag{17}$$

When differentiating (13), we obtain

$$\frac{\partial s(\mathbf{x})}{\partial x_i} = \frac{x_i}{|\mathbf{x}|} - \delta_{1i}. \tag{18}$$

From here we get the estimates

$$\left|\frac{\partial s(\mathbf{x})}{\partial x_k}\right| \leq \begin{cases} \frac{s(\mathbf{x})}{|\mathbf{x}|} & (k=1) \\ \sqrt{2}\sqrt{\frac{s(\mathbf{x})}{|\mathbf{x}|}} & (k \neq 1) \end{cases} \qquad \left|D^\alpha s(\mathbf{x})\right| \leq \frac{c(\alpha)}{|\mathbf{x}|^{|\alpha|-1}}. \tag{19}$$

From (11)–(13) and (19) it is seen that $O(\cdot\,;\cdot) \in C^\infty\big((R^3 \setminus \{0\}) \times R\big)$ and for fixed $\mathbf{x} \neq \mathbf{0}$, $O(\mathbf{x};\cdot)$ is an analytic function.

Now we calculate the derivatives of $\phi_O(\cdot\,;\lambda)$ in order to establish the asymptotic behavior of the difference $R(\mathbf{x}, 2\lambda) = O(\cdot\,; 2\lambda) - E(\mathbf{x})$ and of its first derivatives near zero. The behavior of this difference gives us the possibility to prove (24) analogous to (7), that is, the jump relation property of the double layer potential of the Oseen problem, see proofs of Propositions 3.3 and 3.4. The asymptotic of this difference near zero implies also compactness of operator $L^{2\lambda}_{\Omega} - K_{\Omega}$ and its dual operator, see proofs of Lemma 5.2 and Theorem 5.3. We follow here the approach used in [24, §2], for another approach based on the explicit expressions of the Oseen fundamental solution see [37, § II.1.2]. The both approaches are applicable for the asymptotic of the second-order derivatives.

$$\begin{aligned}
-\partial_i \phi_O(\mathbf{x}; 2\lambda) &= \frac{1}{8\pi}\psi'(\lambda s(\mathbf{x}))\,\partial_i s(\mathbf{x}) \\
-\partial_r\partial_i \phi_O(\mathbf{x}; 2\lambda) &= \frac{\lambda}{8\pi}\psi''(\lambda s(\mathbf{x}))\,\partial_r s(\mathbf{x})\,\partial_i s(\mathbf{x}) + \frac{1}{8\pi}\psi'(\lambda s(\mathbf{x}))\,\partial_r\partial_i s(\mathbf{x}) \\
-\partial_k\partial_r\partial_i \phi_O(\mathbf{x}; 2\lambda) &= \frac{\lambda^2}{8\pi}\psi'''(\lambda s(\mathbf{x}))\,\partial_k s(\mathbf{x})\partial_r s(\mathbf{x})\partial_i s(\mathbf{x}) \\
&\quad + \frac{\lambda}{8\pi}\psi''(\lambda s(\mathbf{x}))\big[\partial_k\partial_r s(\mathbf{x})\,\partial_i s(\mathbf{x}) + \partial_k\partial_i s(\mathbf{x})\,\partial_r s(\mathbf{x}) \\
&\quad + \partial_r\partial_i s(\mathbf{x})\,\partial_k s(\mathbf{x})\big] + \frac{1}{8\pi}\psi'(\lambda s(\mathbf{x}))\,\partial_k\partial_r\partial_i s(\mathbf{x})
\end{aligned}$$

These formulas together with (17), (19), and (10) yield

$$\begin{aligned}
|R(\mathbf{x}; 2\lambda)| &= |O(\mathbf{x}; 2\lambda) - E(\mathbf{x})| = \lambda O(1) && \text{as} \quad \lambda|\mathbf{x}| \to 0, \\
|\nabla R(\mathbf{x}; 2\lambda)| &= |\nabla O(\mathbf{x}; 2\lambda) - \nabla E(\mathbf{x})| = \lambda^2 O\left(\frac{1}{\lambda|\mathbf{x}|}\right) && \text{as} \quad \lambda|\mathbf{x}| \to 0,
\end{aligned} \tag{20}$$

where (E, Q) is the Stokes fundamental solution. In particular, for $\lambda \in (0; \lambda_0)$, $R > 0$, $k = 0, 1$ and $|\lambda\mathbf{x}| \leq R$

$$\left|\nabla^k O(\mathbf{x}; 2\lambda)\right| \leq \frac{c(R; \lambda_0, k)}{|\mathbf{x}|^{k+1}}. \tag{21}$$

Since $E(\mathbf{x}) = |2\lambda| E(2\mathbf{x})$ we obtain this relation also for $\lambda < 0$. Formulas (20), (21) and Proposition 3.1 give us in particular that $O(\cdot\,; 2\lambda)$, $R(\cdot\,; 2\lambda)$, and $\nabla R(\cdot\,; 2\lambda)$ are weakly singular kernels of integral operators in R^3 and in R^2.

Remark that

$$O_{11}(\mathbf{x}, 1) = \frac{1}{4\pi|\mathbf{x}|}\left\{e^{-(|\mathbf{x}|-x_1)/2} + \frac{x_1(1 - e^{-(|\mathbf{x}|-x_1)/2})}{|\mathbf{x}|^2} - \frac{(|\mathbf{x}| - x_1)e^{-(|\mathbf{x}|-x_1)/2}}{2|\mathbf{x}|}\right\},$$

$$O_{22}(\mathbf{x},1)=\frac{e^{-(|\mathbf{x}|-x_1)/2}}{4\pi|\mathbf{x}|}-\frac{(x_1^2+x_3^2)(1-e^{(|\mathbf{x}|-x_1)/2})}{4\pi(|\mathbf{x}|-x_1)|\mathbf{x}|^3}-\frac{x_2^2e^{-(|\mathbf{x}|-x_1)/2}}{8\pi(|\mathbf{x}|-x_1)|\mathbf{x}|^2}+\frac{[1-e^{-(|\mathbf{x}|-x_1)/2}]x_2^2}{4\pi(|\mathbf{x}|-x_1)^2|\mathbf{x}|^2},$$

$$O_{33}(\mathbf{x},1)=\frac{e^{-(|\mathbf{x}|-x_1)/2}}{4\pi|\mathbf{x}|}-\frac{(x_1^2+x_2^2)(1-e^{-(|\mathbf{x}|-x_1)/2})}{4\pi(|\mathbf{x}|-x_1)|\mathbf{x}|^3}-\frac{x_3^2e^{-(|\mathbf{x}|-x_1)/2}}{8\pi(|\mathbf{x}|-x_1)|\mathbf{x}|^2}+\frac{[1-e^{-(|\mathbf{x}|-x_1)/2}]x_3^2}{4\pi(|\mathbf{x}|-x_1)^2|\mathbf{x}|^2},$$

$$O_{12}(\mathbf{x},1)=O_{21}(\mathbf{x},1)=\frac{x_2}{4\pi|\mathbf{x}|^2}\left[\frac{e^{-(|\mathbf{x}|-x_1)/2}}{2}-\frac{1-e^{-(|\mathbf{x}|-x_1)/2}}{|\mathbf{x}|}\right],$$

$$O_{13}(\mathbf{x},1)=O_{31}(\mathbf{x},1)=\frac{x_3}{4\pi|\mathbf{x}|^2}\left[\frac{e^{-(|\mathbf{x}|-x_1)/2}}{2}-\frac{1-e^{-(|\mathbf{x}|-x_1)/2}}{|\mathbf{x}|}\right],$$

$$O_{23}(\mathbf{x},1)=O_{32}(\mathbf{x},1)=\frac{x_2x_3}{4\pi|\mathbf{x}|^3}\left\{\frac{1-e^{-(|\mathbf{x}|-x_1)/2}}{(|\mathbf{x}|-x_1)}-\frac{|\mathbf{x}|e^{-(|\mathbf{x}|-x_1)/2}}{2(|\mathbf{x}|-x_1)}+\frac{|\mathbf{x}|[1-e^{-(|\mathbf{x}|-x_1)/2}]}{(|\mathbf{x}|-x_1)^2}\right\}.$$

(See [20]), Chapter VII, §VII.3).

Let $\Omega\subset R^3$ be an open set with compact boundary $\partial\Omega\in\mathcal{C}^{1,\alpha}, 0<\alpha<1$, and $\mathbf{\Psi}\in\mathcal{C}^0(\partial\Omega)^3$. Define the Oseen single layer potential with density $\mathbf{\Psi}$ by

$$(O_\Omega^{2\lambda}\mathbf{\Psi})(\mathbf{x})=\int_{\partial\Omega}O(\mathbf{x}-\mathbf{y},2\lambda)\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}},$$

whenever it makes sense. Then the pair $(O_\Omega^{2\lambda}\mathbf{\Psi},Q_\Omega\mathbf{\Psi})\in C^\infty(R^3\setminus\partial\Omega)^4$ solves the Oseen system (3) in $R^3\setminus\partial\Omega$. Let $R_\Omega^{2\lambda}\mathbf{\Psi}=O_\Omega^{2\lambda}\mathbf{\Psi}-E_\Omega\mathbf{\Psi}$ denote the difference of the Oseen and the Stokes single layer potentials. If β is a multi-index, then we have

$$|\partial^\beta O_\Omega^{2\lambda}\mathbf{\Psi}(\mathbf{x})|=O(|\mathbf{x}|^{-1-|\beta|/2})\quad\text{as}\quad|\mathbf{x}|\to\infty.$$

Moreover, if $r>0$, $\partial\Omega\subset B(0;r)$ and $q>4/3$, then $|\nabla O_\Omega^{2\lambda}\mathbf{\Psi}|\in L^q(R^3\setminus B(0;r))$.

For $\mathbf{y}\in\partial\Omega$ define $L^\Omega(\cdot,\mathbf{y};2\lambda)=T(O(\cdot-\mathbf{y};2\lambda),Q(\cdot-\mathbf{y}))\mathbf{n}^\Omega(\mathbf{y})$ in $R^3\setminus\{\mathbf{y}\}$. If $G=R^3\setminus\overline{\Omega}$, then $L^\Omega(\mathbf{x},\mathbf{y};2\lambda)=-L^G(\mathbf{x},\mathbf{y};2\lambda)$.

For $\mathbf{\Psi}\in\mathcal{C}^0(\partial\Omega)^3$ and $\mathbf{x}\in\partial\Omega$ denote

$$(L_\Omega^{2\lambda}\mathbf{\Psi})(\mathbf{x})=\int_{\partial\Omega}L^\Omega(\mathbf{x},\mathbf{y};2\lambda)\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}},\tag{22}$$

$$(\tilde{L}_\Omega^{2\lambda}\mathbf{\Psi})(\mathbf{x})=\int_{\partial\Omega}L^\Omega(\mathbf{y},\mathbf{x};2\lambda)\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}}.\tag{23}$$

Although the following statement is needed for the case $m=3$ only, we give the proof for general m.

Lemma 3.2 *Let $\Omega\subset R^m$ be an open set with bounded Lipschitz boundary. Let $k(\mathbf{x},\mathbf{y})$ be defined for $[\mathbf{x},\mathbf{y}]\in R^m\times\partial\Omega$; $\mathbf{x}\neq\mathbf{y}$ and $|k(\mathbf{x},\mathbf{y})|\le C|\mathbf{x}-\mathbf{y}|^{1-m+\beta}$ with positive constants C,β. Suppose that $k(\mathbf{x},\cdot)$ is measurable and $k(\cdot,\mathbf{y})$ is continuous. Let $f\in L^\infty(\partial\Omega)$. Then*

$$kf(\mathbf{x}) = \int_{\partial\Omega} k(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\sigma_{\mathbf{y}}$$

is a continuous function in R^m.

Proof The function kf is continuous in $R^m \setminus \partial\Omega$. Fix $\mathbf{z} \in \partial\Omega$, $\epsilon > 0$ and choose $M > 0$ such that $|f| \leq M$. Since $\partial\Omega$ is Lipschitz, there exists a constant c such that $\sigma(B(\mathbf{x}; r) \cap \partial\Omega) \leq cr^{m-1}$ for each $\mathbf{x} \in R^m$ and $r > 0$. Here σ denotes the surface measure.

Now fix $\mathbf{x} \in R^m$, $r > 0$, and set $B(j) = \partial\Omega \cap B(\mathbf{x}; 2^{-j+1}r) \setminus B(\mathbf{x}; 2^{-j}r)$ for $j \in N$. Then

$$\int_{\partial\Omega \cap B(\mathbf{x};r)} |k(\mathbf{x}, \mathbf{y}) f(\mathbf{y}|\, d\sigma_{\mathbf{y}} \leq CM \sum_{j=1}^{\infty} \int_{B(j)} |x - y|^{\beta+1-m}\, d\sigma_{\mathbf{y}}$$

$$\leq CM \sum_{j=1}^{\infty} (2^{-j} r)^{\beta+1-m} c(2^{-j+1} r)^{m-1} = \frac{CcM2^{m-1-\beta}}{1 - 2^{-\beta}}\, r^{\beta}.$$

Fix $r > 0$ such that $(2r)^{\beta} CcM2^{m-1-\beta}/(1 - 2^{-\beta}) < \epsilon/2$. If $|\mathbf{x} - \mathbf{z}| < r$ then

$$\int_{\partial\Omega \cap B(\mathbf{z};r)} |k(\mathbf{z}, \mathbf{y}) f(\mathbf{y})|\, d\sigma_{\mathbf{y}} + \int_{\partial\Omega \cap B(\mathbf{z};r)} |k(\mathbf{x}, \mathbf{y}) f(\mathbf{y})|\, d\sigma_{\mathbf{y}} \leq \epsilon.$$

Since

$$\int_{\partial\Omega \setminus B(\mathbf{z};r)} k(\mathbf{x}, \mathbf{y}) f(\mathbf{y}\, d\sigma_{\mathbf{y}} \to \int_{\partial\Omega \setminus B(\mathbf{z};r)} k(\mathbf{z}, \mathbf{y}) f(\mathbf{y}\, d\sigma_{\mathbf{y}}$$

as $\mathbf{x} \to \mathbf{z}$, we infer that kf is continuous. □

Proposition 3.3 *Let* $\Omega \subset R^3$ *be an open set with bounded boundary of class* $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$. *If* $\mathbf{\Psi} \in \mathcal{C}^0(\partial\Omega)^3$, *then* $O_{\Omega}^{2\lambda}\mathbf{\Psi} \in \mathcal{C}^0(R^3)^3$ *and* $|\nabla O_{\Omega}^{2\lambda}\mathbf{\Psi}|^* \in L^2(\partial\Omega)$. *If* $\mathbf{z} \in \partial\Omega$, *then* $\mathbf{\Psi}(\mathbf{z})/2 - \tilde{L}_{\Omega}^{-2\lambda}\mathbf{\Psi}(\mathbf{z})$ *is the non-tangential limit of* $T(O_{\Omega}\mathbf{\Psi}(\mathbf{x}), Q_{\Omega}\mathbf{\Psi}(\mathbf{x}))\mathbf{n}^{\Omega}(\mathbf{z})$ *at* $\mathbf{z}$.

Proof It holds $|R(\mathbf{x}, 2\lambda)| = O(1)$, $|\nabla R(\mathbf{x}, 2\lambda)| = O(|\mathbf{x}|^{-1})$ as $\mathbf{x} \to 0$. Thus $R_{\Omega}^{2\lambda}\mathbf{\Psi}$, $\nabla R_{\Omega}^{2\lambda}\mathbf{\Psi}$ are continuous in R^3 by Lemma 3.2. The properties of $E_{\Omega}\mathbf{\Psi}$ imply $O_{\Omega}^{2\lambda}\mathbf{\Psi} \in \mathcal{C}^0(R^3)^3$ and $|\nabla O_{\Omega}^{2\lambda}\mathbf{\Psi}|^* \in L^2(\partial\Omega)$. If $\mathbf{z} \in \partial\Omega$ then

$$\lim_{\substack{\mathbf{x} \to \mathbf{z} \\ \mathbf{x} \in \Gamma_{\beta}}} T(O_{\Omega}\mathbf{\Psi}(\mathbf{x}), Q_{\Omega}\mathbf{\Psi}(\mathbf{x}))\mathbf{n}^{\Omega}(\mathbf{z}) = \lim_{\substack{\mathbf{x} \to \mathbf{z} \\ \mathbf{x} \in \Gamma_{\beta}}} T(E_{\Omega}\mathbf{\Psi}(\mathbf{x}), Q_{\Omega}\mathbf{\Psi}(\mathbf{x})\mathbf{n}^{\Omega}(\mathbf{z})$$

$$+ \lim_{\substack{\mathbf{x} \to \mathbf{z} \\ \mathbf{x} \in \Gamma_{\beta}}} T(R_{\Omega}\mathbf{\Psi}(\mathbf{x}), 0)\mathbf{n}^{\Omega}(\mathbf{z}) = \frac{\mathbf{\Psi}(\mathbf{z})}{2} - K_{\Omega}'\mathbf{\Psi}(\mathbf{z})$$

$$+ \int_{\partial\Omega} 2\hat{\nabla}_{\mathbf{z}} R(\mathbf{z} - \mathbf{y}, 2\lambda)\mathbf{n}^{\Omega}(\mathbf{z})\mathbf{\Psi}(\mathbf{y})\, d\sigma_{\mathbf{y}} = \frac{\mathbf{\Psi}(\mathbf{z})}{2} - \tilde{L}_{\Omega}^{-2\lambda}\mathbf{\Psi}(\mathbf{z}).$$

□

Proposition 3.4 *Let* $\Omega \subset R^3$ *be an open set with compact boundary of class* $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$. *For* $\mathbf{\Psi} \in \mathcal{C}^0(\partial\Omega)^3$, $\mathbf{x} \in R^3 \setminus \partial\Omega$ *define*

$$W_{\Omega}^{2\lambda}\mathbf{\Psi}(\mathbf{x}) = \int_{\partial\Omega} L^{\Omega}(\mathbf{x},\mathbf{y};2\lambda)\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}},$$

$$w_{\Omega}^{2\lambda}\mathbf{\Psi}(\mathbf{x}) = \int_{\partial\Omega} [2\mathbf{n}^{\Omega}(\mathbf{y})\cdot\nabla_{\mathbf{y}}Q(\mathbf{x}-\mathbf{y}) + 2\lambda Q_1(\mathbf{x}-\mathbf{y})\mathbf{n}^{\Omega}(\mathbf{y})]\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}}.$$

Then $(W_{\Omega}^{2\lambda}\mathbf{\Psi}, w_{\Omega}^{2\lambda}\mathbf{\Psi}) \in C^{\infty}(R^3\setminus\partial\Omega)^4$ *solves the Oseen system (3) in* $R^3\setminus\partial\Omega$. *If* $\mathbf{z}\in\partial\Omega$, *then*

$$\lim_{\substack{\mathbf{x}\to\mathbf{z}\\ \mathbf{x}\in\Omega}} W_{\Omega}^{2\lambda}\mathbf{\Psi}(\mathbf{x}) = \frac{1}{2}\mathbf{\Psi}(\mathbf{z}) + L_{\Omega}^{2\lambda}\mathbf{\Psi}(\mathbf{z}). \tag{24}$$

If β *is a multi-index, then*

$$|\partial^{\beta}W_{\Omega}^{2\lambda}\mathbf{\Psi}(\mathbf{x})| = O(|\mathbf{x}|^{-3/2-|\beta|/2}) \quad as \quad |\mathbf{x}|\to\infty.$$

Proof An easy calculation yields that $(W_{\Omega}^{2\lambda}\mathbf{\Psi}, w_{\Omega}^{2\lambda}\mathbf{\Psi}) \in C^{\infty}(R^3\setminus\partial\Omega)^4$ solves the Oseen system (3) in $R^3\setminus\partial\Omega$. Since $|\nabla R(\mathbf{x},2\lambda)| = O(|\mathbf{x}|^{-1})$ as $\mathbf{x}\to 0$, we infer that $|K_{\Omega}(\mathbf{x},\mathbf{y}) - L_{\Omega}(\mathbf{x},\mathbf{y};2\lambda)| \le M|\mathbf{x}-\mathbf{y}|^{-1}$. Hence, the relation (24) is a consequence of (7) and Lemma 3.2. □

Remark 3.5 If $\mathbf{u}\in\mathcal{C}^1(\overline{\Omega})^3$, $p\in\mathcal{C}^0(\overline{\Omega})$ solve the homogeneous Oseen system (3), then

$$\mathbf{u} = O_{\Omega}^{2\lambda}[T(\mathbf{u},p)\mathbf{n}^{\Omega}] + W_{\Omega}^{2\lambda}\mathbf{u} - 2\lambda O_{\Omega}^{2\lambda}(n_1\mathbf{u})$$
$$p = Q_{\Omega}[T(\mathbf{u},p)\mathbf{n}^{\Omega}] + w_{\Omega}^{2\lambda}\mathbf{u} - 2\lambda Q_{\Omega}(n_1\mathbf{u})$$

in Ω (compare [20], Chapter VII, Lemma 6.2 or [37], Chapter II, Lemma 2.5). The fact that $W_{\Omega}^{2\lambda}\mathbf{\Psi}$, $w_{\Omega}^{2\lambda}\mathbf{\Psi}$ solve the Oseen system (3) in $R^3\setminus\partial\Omega$ can be deduced from these relations.

4 Unique solvability of the Oseen problem

Concerning the Stokes system, we have the following result (see [33], Theorem 5.5):

Lemma 4.1 . *Let* $\Omega\subset R^3$ *be a bounded domain with boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0<\alpha<1$, $\mathbf{g}\in\mathcal{C}^0(\partial\Omega)^3$, $\mathbf{u}\in\mathcal{C}^2(\Omega)^3\cap\mathcal{C}^0(\overline{\Omega})^3$, $p\in\mathcal{C}^1(\Omega)$, $\mathbf{u}$, p *solve the Stokes system* (1), $\mathbf{u}=\mathbf{g}$ *on* $\partial\Omega$. *Then*

$$\sup_{\mathbf{x}\in\Omega}|\mathbf{u}(\mathbf{x})| \le K \sup_{\mathbf{x}\in\partial\Omega}|\mathbf{g}(\mathbf{x})|,$$

where the constant K *depends only on* Ω.

We say that $\mathbf{u}\in\mathcal{C}^2(\Omega)^3$, $p\in\mathcal{C}^1(\Omega)$ are an L^2-solution of the Dirichlet problem for the Stokes system in Ω with the boundary condition $\mathbf{g}$ if (1) holds true, $\mathbf{u}^*\in L^2(\partial\Omega)$ and $\mathbf{g}(\mathbf{x})$ is the non-tangential limit of $\mathbf{u}$ at almost all $\mathbf{x}\in\partial\Omega$.

Lemma 4.2 *Let* $\Omega\subset R^3$ *be a bounded domain with boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0<\alpha<1$, $\mathbf{g}\in L^2(\partial\Omega)^3$. *Then there exist an* L^2*-solution* $\mathbf{u}\in\mathcal{C}^2(\Omega)^3$, $p\in\mathcal{C}^1(\Omega)$ *of the Dirichlet problem of the Stokes system in* Ω *with the boundary condition* $\mathbf{g}$ *if and only if*

$$\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n}^{\Omega}\,\mathrm{d}\sigma = 0.$$

The function $\mathbf{u}$ *is unique, and* p *is unique up to an additive constant. If* $\mathbf{g} \in \mathcal{C}^0(\partial\Omega)^3$*, then* $\mathbf{u} \in \mathcal{C}^0(\overline{\Omega})^3$*. If* $\mathbf{g} \in W^{1,2}(\partial\Omega)^3$*, then* $(\nabla\mathbf{u})^*$*,* $p^* \in L^2(\partial\Omega)$*.*

For the proof of this Lemma, see [33], Proposition 3.3 and [33], and Theorem 5.3.

Lemma 4.3 *Let* $\Omega \subset R^3$ *be a bounded domain with boundary of class* $\mathcal{C}^{1,\alpha}$*,* $0 < \alpha < 1$*, and let* $v \in \mathcal{C}^0(\Omega)$*. If* $v^* \in L^s(\partial\Omega)$*,* $1 < s < \infty$*, then* $v \in L^s(\Omega)$*.*

For the proof of this Lemma see [32], Lemma 2 or [34], and Lemma 4.1.

Lemma 4.4 *Let* $\Omega \subset R^3$ *be a bounded open set with boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$*,* $\mathbf{g} \in \mathcal{C}^0(\partial\Omega)^3$*,* $\mathbf{u} \in \mathcal{C}^2(\Omega)^3 \cap \mathcal{C}^0(\overline{\Omega})^3$*,* $p \in \mathcal{C}^1(\Omega)$*,* $\mathbf{u}$*,* p *solve the Stokes system* (1)*,* $\mathbf{u} = \mathbf{g}$ *on* $\partial\Omega$*. If* $\mathbf{g} \in H^{1/2}(\partial\Omega)^3$*, then* $\mathbf{u} \in W^{1,2}(\Omega)^3$*,* $p \in L^2(\Omega)$*.*

Proof Lemma 4.2 gives that $\mathbf{g}$ is orthogonal to the unit normal $\mathbf{n}^\Omega$. Choose a sequence $\mathbf{g}_k \in [W^{1,2}(\partial\Omega)]^3 \cap \mathcal{C}^0(\partial\Omega)^3$ orthogonal to the normal $\mathbf{n}^\Omega$ such that $\mathbf{g}_k \to \mathbf{g}$ in $H^{1/2}(\partial\Omega)^3$ and in $\mathcal{C}^0(\partial\Omega)^3$. Then there exist $\mathbf{u}_k \in C^2(\Omega)^3 \cap \mathcal{C}^0(\overline{\Omega})^3 \cap W^{1,2}(\Omega)^3$, $p_k \in C^1(\Omega) \cap L^2(\Omega)$ such that $\mathbf{u}_k$, p_k solve (1) and $\mathbf{u}_k = \mathbf{g}_k$ on $\partial\Omega$ (see Lemma 4.2 and Lemma 4.3). According to Lemma 4.1, we have $\mathbf{u}_k \to \mathbf{u}$. By virtue of [20], Chapter IV, Theorem 1.1, there exist $\mathbf{w} \in W^{1,2}(\Omega)^3$ and $q \in L^2(\Omega)$ such that $\mathbf{w}$, q solve (1) and $\mathbf{g}$ is the trace of $\mathbf{w}$. Moreover, $\mathbf{u}_k \to \mathbf{w}$ in $W^{1,2}(\Omega)^3$. Thus $\mathbf{u} = \mathbf{w} \in W^{1,2}(\Omega)^3$. Since $\nabla p - \nabla q = \Delta\mathbf{u} - \Delta\mathbf{w} = 0$, the function $p - q$ is constant. □

Now we are ready to state the uniqueness result for the Oseen equations:

Theorem 4.5 *Let* $\Omega \subset R^3$ *be an exterior domain with boundary of class* $\mathcal{C}^{1,\alpha}$*,* $0 < \alpha < 1$*. Let* $\lambda \in R \setminus \{0\}$ *and* $\mathbf{u} \in C^2(\Omega)^3 \cap C^0(\overline{\Omega})^3$*,* $p \in C^1(\Omega)$ *solve the Oseen equations* (3) *in* Ω*. Fix* $r > 0$ *such that* $\partial\Omega \subset B(0; r)$*. If* $\mathbf{u} = 0$ *on* $\partial\Omega$*, then* $\mathbf{u} \in [W^{1,2}(\Omega \cap B(0; r))]^3$*,* $p \in L^2(B(0; r))$*. If, moreover,* $|\mathbf{u}(\mathbf{x})| \to 0$ *as* $|\mathbf{x}| \to \infty$*,* $|\nabla\mathbf{u}| \in L^2(\Omega)$*, then* $\mathbf{u} \equiv 0$*,* p *is constant.*

Proof Without loss of generality, we can suppose $\lambda > 0$. Set $\mathbf{u} = 0$ on $R^3 \setminus \overline{\Omega}$. Then $\mathbf{u} \in \mathcal{C}^0(R^3)^3$. Moreover, $\mathbf{u} \in C^\infty(\Omega)^3$, $p \in C^\infty(\Omega)$ due to [20], Chapter VII, Theorem 1.1. Choose a cutoff function $\varphi \in \mathcal{C}^\infty(R^3)$ such that $\varphi = 1$ in $B(0; 2r)$, $\varphi = 0$ in $R^3 \setminus B(0; 3r)$. Set $\mathbf{v} = E * (\mathbf{u}\varphi)$, $q = Q * (\mathbf{u}\varphi)$. Then $\mathbf{v} \in \mathcal{C}^1(R^3)^3 \cap [W^{2,2}(B(0; 3r))]^3$, $q \in \mathcal{C}^0(R^3) \cap W^{1,2}(B(0; 3r))$. Since $\varphi\mathbf{u} \in \mathcal{C}^\infty(R^3 \setminus \partial\Omega)^3$, we have $\mathbf{v} \in \mathcal{C}^\infty(R^3 \setminus \partial\Omega)^3$, $q \in \mathcal{C}^\infty(R^3 \setminus \partial\Omega)$. Moreover, $-\Delta\mathbf{v} + \nabla q = \varphi\mathbf{u}$ in $R^3 \setminus \partial\Omega$. Define $\mathbf{w} = \mathbf{u} + 2\lambda\partial_1\mathbf{v}$, $\rho = p + 2\lambda\partial_1 q$. Then $\mathbf{w} \in \mathcal{C}^0(R^3)^3 \cap \mathcal{C}^\infty(\Omega)^3$, $\rho \in \mathcal{C}^\infty(\Omega)$ and

$$-\Delta\mathbf{w} + \nabla\rho = -\Delta\mathbf{u} + \nabla p + 2\lambda\partial_1(\varphi\mathbf{u}) = -2\lambda\partial_1 u + 2\lambda\partial_1(\varphi\mathbf{u}).$$

Therefore, $-\Delta\mathbf{w} + \nabla\rho = 0$ in $\Omega \cap B(0; 2r)$. Similarly, $\nabla\cdot\mathbf{w} = 0$ in $\Omega \cap B(0; 2r)$. Moreover, $\mathbf{w} \in H^{1/2}(\partial(\Omega \cap B(0; 2r)))^3$. Thus, $\mathbf{w} \in [W^{1,2}(\Omega \cap B(0; 2r))]^3$, $\rho \in L^2(\Omega \cap B(0; 2r))$ by Lemma 4.4. Since $\partial_1\mathbf{v} \in [W^{1,2}_{loc}(R^3)]^3$, $\partial_1 q \in L^2_{loc}(R^3)$, we conclude that $\mathbf{u} \in W^{1,2}(\Omega \cap B(0; 2r))^3$, $p \in L^2(\Omega \cap B(0; 2r))$. If $|\mathbf{u}(\mathbf{x})| \to 0$ as $|\mathbf{x}| \to \infty$, $|\nabla\mathbf{u}| \in L^2(\Omega)$, then $\mathbf{u} \equiv 0$ (see [20], Chapter VII, Theorem 2.1). Since $\nabla p = \Delta\mathbf{u} - 2\lambda\partial_1\mathbf{u} = 0$, we infer that p is constant.

5 Solution of the Oseen problem

Lemma 5.1 *Let* $G \subset R^3$ *be a bounded open set with boundary of class* $\mathcal{C}^{1,\alpha}$*,* $0 < \alpha < 1$*. Let* $c, \lambda \in R$*,* $0 < c$*. If* $\mathbf{u} \in \mathcal{C}^2(G)^3 \cap \mathcal{C}^0(\overline{G})^3$*,* $p \in \mathcal{C}^1(G)$*,* $|\nabla\mathbf{u}|^* + p^* \in L^2(\partial G)$*,* $\mathbf{u}$*,* p *solve* *the homogeneous Oseen system (3), $T(\mathbf{u}, p)\mathbf{n}^G - \lambda n_1 \mathbf{u} + c\mathbf{u} = 0$ on ∂G in the sense of the non-tangential limit, then $\mathbf{u} \equiv 0$, $p \equiv 0$ in G.*

Proof Without loss of generality, we can suppose that G is connected. According to [42], Theorem 1.12, there exists a sequence of open sets $G(j)$ with C^∞-boundary with the following properties:

1. $\overline{G}(j) \subset G$.
2. There exist homeomorphisms $\Lambda_j : \partial G \to \partial G(j)$ and $\beta > 0$ such that $\Lambda_j(\mathbf{y}) \in \Gamma_\beta(\mathbf{y})$ for every j and every $\mathbf{y} \in \partial G$, and
$$\sup\{|\mathbf{y} - \Lambda_j(\mathbf{y})|; \mathbf{y} \in \partial G\} \to 0, \qquad \text{as } j \to \infty.$$
3. There are positive functions σ_j on ∂G bounded away from zero and infinity uniformly in j such that for any measurable set $E \subset \partial G$, we have
$$\int_E \sigma_j \, \mathrm{d}\sigma = \int_{\Lambda_j(E)} \mathrm{d}\sigma,$$
and such that $\sigma_j \to 1$ pointwise a.e..
4. The normal vectors $\mathbf{n}^j(\Lambda_j(\mathbf{y}))$ to $G(j)$ converge point-wise almost everywhere to $\mathbf{n}^G(\mathbf{y})$.

Using Green's formula and Lebesque's lemma, we obtain

$$0 = \int_{\partial G} \mathbf{u} \cdot [T(\mathbf{u}, p)\mathbf{n}^G - \lambda n_1 \mathbf{u} + c\mathbf{u}] \, \mathrm{d}\sigma = \lim_{j\to\infty} \int_{\partial G(j)} \mathbf{u} \cdot [T(\mathbf{u}, p)\mathbf{n}^G - \lambda n_1 \mathbf{u} + c\mathbf{u}] \, \mathrm{d}\sigma$$
$$= \lim_{j\to\infty} \left\{ \int_{G(j)} [\mathbf{u} \cdot \Delta\mathbf{u} + |\hat{\nabla}\mathbf{u}|^2 - \mathbf{u}\nabla p - 2\lambda \mathbf{u} \cdot \partial_1 \mathbf{u}] \, \mathrm{d}\mathbf{y} + \int_{\partial G(j)} |\mathbf{u}|^2 c \, \mathrm{d}\sigma \right\}$$
$$= \lim_{j\to\infty} \left[\int_{G(j)} |\hat{\nabla}\mathbf{u}|^2 \, \mathrm{d}\mathbf{y} + \int_{\partial G(j)} |\mathbf{u}|^2 c \, \mathrm{d}\sigma \right] = \int_G |\hat{\nabla}\mathbf{u}|^2 \, \mathrm{d}\mathbf{y} + \int_{\partial G} |\mathbf{u}|^2 c \, \mathrm{d}\sigma.$$

Therefore, $\hat{\nabla}\mathbf{u} \equiv 0$ in G, $\mathbf{u} = 0$ on $\partial\Omega$. Since $\hat{\nabla}\mathbf{u} \equiv 0$ there exists a skew symmetric matrix A and a vector $\mathbf{b}$ such that $\mathbf{u} = A\mathbf{x} + \mathbf{b}$ (see [32], Lemma 6). Hence, u_1, u_2, u_3 are harmonic functions vanishing on ∂G. The maximum principle gives $\mathbf{u} \equiv 0$. Therefore, $\nabla p = \Delta\mathbf{u} - 2\lambda\partial_1\mathbf{u} \equiv 0$. Hence, there is a constant a such that $p \equiv a$. But $0 = T(\mathbf{u}, p)\mathbf{n}^G - \lambda n_1 \mathbf{u} + c\mathbf{u} = -a\mathbf{n}^G$ on ∂G yields $p \equiv a = 0$. □

Lemma 5.2 *Let $G \subset R^3$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, $\lambda \in R \setminus \{0\}$, $c \in R$, $c > 0$. Suppose, moreover, that $R^3 \setminus \overline{G}$ is connected. Denote by I the identity operator. Then the operator $\frac{1}{2}I - \tilde{L}_G^{-2\lambda} + (c - \lambda n_1^G)O_G^{2\lambda}$ is continuously invertible on $\mathcal{C}^0(\partial G)^3$. If $\mathbf{f} \in \mathcal{C}^0(\partial G)^3$, then there exist unique $\mathbf{u} \in \mathcal{C}^2(G)^3 \cap \mathcal{C}^0(\overline{G})^3$, $p \in \mathcal{C}^1(G)$ such that $|\nabla\mathbf{u}|^* + p^* \in L^2(\partial G)$, $\mathbf{u}$, p solve the homogeneous Oseen system (3), and $T(\mathbf{u}, p)\mathbf{n}^G - \lambda n_1 \mathbf{u} + c\mathbf{u} = \mathbf{f}$ on ∂G in the sense of the non-tangential limit. This solution is given by $\mathbf{u} = O_G^{2\lambda}\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$, where $\mathbf{\Psi} = \left[\frac{1}{2}I - \tilde{L}_G^{-2\lambda} + (c - \lambda n_1^G)O_G^{2\lambda}\right]^{-1}\mathbf{f}$.*

Proof Proposition 3.3 gives that $\mathbf{u} = O_G^{2\lambda}\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$ with $\mathbf{\Psi} \in \mathcal{C}^0(\partial G)^3$ is a solution of the Robin problem for the Oseen system with the boundary condition $\mathbf{f}$ if and only if

$$\frac{1}{2}\mathbf{\Psi} - \tilde{L}_G^{-2\lambda}\mathbf{\Psi} + (c - \lambda n_1^G)O_G^{2\lambda}\mathbf{\Psi} = \mathbf{f}.$$

If $\mathbf{f} \equiv 0$, then the uniqueness of a solution of the Robin problem (see Lemma 5.1) implies $O_G^{2\lambda}\mathbf{\Psi} = 0$, $Q_G\mathbf{\Psi} = 0$ in G. Since $O_G^{2\lambda}\mathbf{\Psi}$ is continuous in R^3, the functions $O_G^{2\lambda}\mathbf{\Psi}$, $Q_G\mathbf{\Psi}$ solve the Oseen problem with zero boundary condition in $\Omega = R^3 \setminus \overline{G}$. From Theorem 4.5, we find that $O_G^{2\lambda}\mathbf{\Psi} \equiv 0$ and $Q_G\mathbf{\Psi}$ is constant in Ω. The behavior at infinity implies $Q_G\Psi = 0$ in Ω. The jump of the normal stresses of the single layer potential (Proposition 3.3) leads to

$$\mathbf{\Psi} = \left[\frac{\mathbf{\Psi}}{2} - \tilde{L}_G^{-2\lambda}\mathbf{\Psi}\right] + \left[\frac{\mathbf{\Psi}}{2} - \tilde{L}_\Omega^{-2\lambda}\mathbf{\Psi}\right] = 0.$$

Hence, the operator $\frac{1}{2}I - \tilde{L}_G^{-2\lambda} + (c - \lambda n_1^G)O_G^{2\lambda}$ is one to one. The integral operators K_G', $O_G^{2\lambda}$, $\tilde{L}_G^{-2\lambda} - K_G'$ have weakly singular kernels, hence compact on $\mathcal{C}^0(\partial G)^3$ (compare [41] or [43]). By the Riesz-Schauder theory, we obtain that the operator $\frac{1}{2}I - \tilde{L}_G^{-2\lambda} + (c - \lambda n_1^G)O_G^{2\lambda}$ is continuously invertible in $\mathcal{C}^0(\partial G)^3$. So, if $\mathbf{\Psi} = [\frac{1}{2}I - \tilde{L}_G^{-2\lambda} + (c - \lambda n_1^G)O_G^{2\lambda}]^{-1}\mathbf{f}$, then $\mathbf{u} = O_G^{2\lambda}\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$ solve the Robin problem for the Oseen system with the boundary value $\mathbf{f}$. □

Theorem 5.3 *Let $\Omega \subset R^3$ be an exterior domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, $\lambda \in R \setminus \{0\}$, $c \in R$, $c > 0$. For $\mathbf{\Psi} \in \mathcal{C}^0(\partial\Omega)^3$ set $S\mathbf{\Psi} = \frac{1}{2}\mathbf{\Psi} + L_\Omega^{2\lambda}\mathbf{\Psi} + O_\Omega^{2\lambda}(c + \lambda n_1^\Omega)\mathbf{\Psi}$. Then S is a continuously invertible operator on $\mathcal{C}^0(\partial\Omega)^3$. For a fixed $\mathbf{f} \in \mathcal{C}^0(\partial\Omega)^3$ put $\mathbf{\Psi} = S^{-1}\mathbf{f}$. Then*

$$\mathbf{u} = W_\Omega^{2\lambda}\mathbf{\Psi} + O_\Omega^{2\lambda}(c + \lambda n_1^\Omega)\mathbf{\Psi}, \tag{25}$$
$$p = w_\Omega^{2\lambda}\mathbf{\Psi} + Q_\Omega(c + \lambda n_1^\Omega)\mathbf{\Psi} \tag{26}$$

are the unique solution of the problem $\mathbf{u} \in C^2(\Omega)^3 \cap C^0(\overline{\Omega})^3$, $p \in C^1(\Omega)$,

$$-\Delta\mathbf{u} + 2\lambda\partial_1\mathbf{u} + \nabla p = 0, \quad \nabla\cdot\mathbf{u} = 0 \quad in\ \Omega, \quad \mathbf{u} = \mathbf{f} \quad on\ \partial\Omega,$$
$$|\mathbf{u}(\mathbf{x})| + |p(\mathbf{x})|] = o(1) \quad as\ |\mathbf{x}| \to \infty,$$

$|\nabla\mathbf{u}| \in L^2(R^3 \setminus B(0; r))$ for some $r > 0$.

Proof Let $G = R^3 \setminus \overline{\Omega}$. The operator $\tilde{S} = \frac{1}{2}I - \tilde{L}_G^{2\lambda} + (c - \lambda n_1^G)O_G^{-2\lambda} = \frac{1}{2}I + \tilde{L}_\Omega^{2\lambda} + (c + \lambda n_1^\Omega)O_\Omega^{-2\lambda}$ is continuously invertible on $\mathcal{C}^0(\partial\Omega)^3$ (see Lemma 5.2). So, $\tilde{S}'$, the adjoint operator of $\tilde{S}$, is also continuously invertible (on the space of vector measures on $\partial\Omega$). If $\mathbf{\Psi}, \mathbf{\Phi} \in \mathcal{C}^0(\partial\Omega)^3$, then Fubini's theorem gives

$$\int_{\partial\Omega} \mathbf{\Psi}(\tilde{S}\mathbf{\Phi})\,\mathrm{d}\sigma = \int_{\partial\Omega} (S\mathbf{\Psi})\mathbf{\Phi}\,\mathrm{d}\sigma.$$

If we denote by σ the surface measure on $\partial\Omega$, then $\tilde{S}'(\mathbf{\Psi}\sigma) = (S\mathbf{\Psi})\sigma$. Since $\tilde{S}'$ is injective, the operator S is also one to one. The integral operators K_Ω, $O_\Omega^{2\lambda}$, $L_\Omega^{2\lambda} - K_\Omega$ have weakly singular kernels. Hence, they are compact on $\mathcal{C}^0(\partial\Omega)^3$ (see [41] or [43]). The Riesz-Schauder theory implies that the operator S is continuously invertible in $\mathcal{C}^0(\partial\Omega)^3$.

If $\mathbf{\Psi} = S^{-1}\mathbf{f}$, then $\mathbf{u}$, p given by (25), (26) are a solution of the Oseen problem with boundary value $\mathbf{f}$ (see Propositions 3.3 and 3.4). The uniqueness follows from Theorem 4.5. □

6 Theorems of Liouville type

Proposition 6.1 *Denote by $\mathcal{S}'(R^3)$ the space of complex tempered distributions on R^3. Suppose that $u_1, u_2, u_3, \in \mathcal{S}'(R^3)$ and p satisfy (3) in R^3 in the sense of distributions. Then u_1, u_2, u_3, p are polynomials.*

Proof Suppose first that $p \in \mathcal{S}'(R^3)$. Denote by $\mathcal{F}h$ the Fourier transformation of h. Then

$$0 = \mathcal{F}(\nabla \cdot \mathbf{u})(\mathbf{x}) = i\mathbf{x} \cdot \mathcal{F}\mathbf{u}(\mathbf{x}),$$
$$0 = \mathcal{F}[-\Delta \mathbf{u} + 2\lambda\partial_1 \mathbf{u} - \nabla p](\mathbf{x}) = [|\mathbf{x}|^2 + 2\lambda i x_1]\mathcal{F}u(x) - i\mathbf{x}\mathcal{F}p(\mathbf{x}).$$

Thus

$$i|\mathbf{x}|^2\mathcal{F}p(\mathbf{x}) = [|\mathbf{x}|^2 + 2\lambda i x_1]\mathbf{x} \cdot \mathcal{F}\mathbf{u}(\mathbf{x}) = 0.$$

Therefore, $\mathcal{F}p(\mathbf{x}) = 0$ in $R^3 \setminus \{0\}$. Hence, $[|\mathbf{x}|^2 + 2\lambda i x_1]\mathcal{F}u(x) = i\mathbf{x}\mathcal{F}p(\mathbf{x}) = 0$ in $R^3 \setminus \{0\}$.

Fix $j \in \{1, 2, 3\}$. Denote by v the real part of $\mathcal{F}u_j$ and by w the imaginary part of $\mathcal{F}u_j$. Then $|x|^2 v(x) - 2\lambda x_1 w(x) = 0$, $|x|^2 w + 2\lambda x_1 v = 0$. Hence $v(x) = 2\lambda x_1 w(x)/|x|^2 = -(2\lambda x_1)^2 v(x)/|x|^4$ and $w(x) = -2\lambda x_1 v(x)/|x|^2 = -(2\lambda x_1)^2 w(x)/|x|^4$ in $\{x \in R^3; |x| \neq 0\}$. Since $[|x|^4 + (2\lambda x_1)^2]v(x) = 0$, $[|x|^4 + (2\lambda x_1)^2]w(x) = 0$ in $\{x \in R^3; |x| \neq 0\}$, we infer that $\mathcal{F}u$ is supported in $\{0\}$. According to [39], Chapter II, §10, there exist $k \in N_0$ and constants a_α such that

$$\mathcal{F}u_j = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha \delta_0.$$

Set

$$P_j(x) = \sum_{|\alpha| \le k} a_\alpha (-ix)^\alpha.$$

Then

$$\mathcal{F}P_j = \sum_{|\alpha| \le k} a_\alpha \mathcal{F}[(-ix)^\alpha 1] = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha \delta_0 = \mathcal{F}u_j.$$

Since the Fourier transform is an isomorphism on $\mathcal{S}'(R^3)$, we infer that $u_j = P_j$.

Let p be general. Then $\partial_k u_j \in \mathcal{S}'(R^3)$, $\partial_k p = \Delta u_k - 2\lambda\partial_1 u_k \in \mathcal{S}'(R^3)$ by [11], Theorem 14.21. Moreover, $\partial_k \mathbf{u}$, $\partial_k p$ satisfy (3) in R^3. Thus, we have proved that $\partial_k u_j$ are polynomials. Hence, u_j are polynomials. Since $\partial_k p = \Delta u_j - 2\lambda\partial_1 u_j$ are polynomials, we infer that p is a polynomial, too. □

Corollary 6.2 *Let u_1, u_2, u_3, p be distributions in R^3. Suppose, moreover, that there exists a compact set $F \subset R^3$ such that $\mathbf{u} = (u_1, u_2, u_3) \in L^\infty(R^3 \setminus F)^3$. If $\mathbf{u}$, p satisfy in R^3 the homogeneous Oseen equations (3) in the sense of distributions, then $\mathbf{u}$, p are constant.*

Proof Consider $\varphi \in \mathcal{C}^\infty(R^3)$ with compact support such that $\varphi \equiv 1$ in a neighborhood of F. The distribution φu_j has a compact support, hence it is a tempered distribution. The function $(1 - \varphi)u_j \in L^\infty(R^3)$ is also a tempered distribution. Proposition 6.1 implies that u_1, u_2, u_3, p are polynomials. The behavior at infinity yields that u_j is constant ($j = 1, 2, 3$). Thus, $\nabla p = \Delta \mathbf{u} - 2\lambda\partial_1 \mathbf{u} = 0$, and it follows immediately that p is constant.

7 Maximum modulus estimate

Proposition 7.1 *Let $F \subset R^3$ be a compact set. Let $\mathbf{u}$, p solve the Oseen equations (3) in $R^3 \setminus F$, and let $\mathbf{u}$ be bounded. Then there exist constants $\mathbf{u}_\infty$, p_∞ such that $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$, $p(\mathbf{x}) \to p_\infty$ as $|\mathbf{x}| \to \infty$. If β is a multi-index, then $|\partial^\beta[\mathbf{u}(\mathbf{x}) - \mathbf{u}_\infty]| = O(|\mathbf{x}|^{-1-|\beta|/2})$, $|\partial^\beta[p(\mathbf{x}) - p_\infty]| = O(|\mathbf{x}|^{-2-|\beta|})$ as $|\mathbf{x}| \to \infty$. If $F \subset B(0; r)$ then $|\nabla \mathbf{u}| \in L^2(R^3 \setminus B(0; r))$.* □

Proof Fix $r > 0$ such that $F \subset B(0; r)$ and let $\Omega = R^3 \setminus \overline{B}(0; r)$. According to Theorem 5.3, there exists $\boldsymbol{\Psi} \in \mathcal{C}^0(\partial\Omega)^3$ such that $\mathbf{v} = W_\Omega^{2\lambda}\boldsymbol{\Psi} + O_\Omega^{2\lambda}(c + \lambda n_1^\Omega)\boldsymbol{\Psi}$, $q = w_\Omega^{2\lambda}\boldsymbol{\Psi} + Q_\Omega(c + \lambda n_1^\Omega)\boldsymbol{\Psi}$ are a classical solution of the Oseen problem in Ω with the boundary value $\mathbf{u}$. We have $\mathbf{v}(\mathbf{x}) \to 0$, $q(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to 0$, and $|\partial^\beta \mathbf{v}(\mathbf{x})| = O(|\mathbf{x}|^{-1-|\beta|/2})$, $|\partial^\beta q(\mathbf{x})| = O(|\mathbf{x}|^{-2-|\beta|})$ as $|\mathbf{x}| \to \infty$. Moreover, $|\nabla \mathbf{v}| \in L^2(R^3 \setminus B(0; r))$.

Set $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{v}$, $\tilde{p} = p - q$ in $R^3 \setminus B(0; r)$ and $\tilde{\mathbf{u}} = 0$, $\tilde{p} = 0$ in $B(0; r)$. Then $\tilde{\mathbf{u}} \in \mathcal{C}^0(R^3)^3 \cap L^\infty(R^3)^3$. Moreover, $\tilde{\mathbf{u}}$, $\tilde{p}$ solve the Oseen equations (3) in $R^3 \setminus \partial B(0; r)$. We have $\tilde{\mathbf{u}} \in W^{1,2}(B(0; 2r) \setminus \overline{B(0; r)})^3$, $\tilde{p} \in L^2(B(0; 2r))$ by Theorem 4.5, which implies $\tilde{\mathbf{u}} \in W^{1,2}(B(0; 2r))^3$. Therefore, $\nabla \cdot \tilde{\mathbf{u}} \in L^2(B(0; 2r))$. Since $\nabla \cdot \tilde{\mathbf{u}} = 0$ in $R^3 \setminus \partial B(0; r)$, we infer $\nabla \cdot \tilde{\mathbf{u}} = 0$ in R^3.

Define $\mathbf{f} = -\Delta \tilde{\mathbf{u}} + 2\lambda \partial_1 \tilde{\mathbf{u}} + \nabla \tilde{p}$. Since $\tilde{\mathbf{u}}$, $\tilde{p}$ satisfy (3) in $R^3 \setminus \partial B(0; r)$, the functions f_1, f_2, f_3 are distributions supported on $\partial B(0; r)$. Fix $\varphi \in \mathcal{C}^\infty(R^3)$ supported in $B(0; 2r)$ such that $\varphi = 1$ in a neighborhood of $\partial B(0; r)$. If $\mathbf{x} \in R^3 \setminus B(0; 2r)$, then for each multi-index β we have

$$
\begin{aligned}
|\partial^\beta O^{2\lambda} * \mathbf{f}(\mathbf{x})| &= |\langle \mathbf{f}, \varphi \partial_x^\beta O^{2\lambda}(\mathbf{x} - \cdot)\rangle| = \int_{R^3} \{\Delta_{\mathbf{y}}[\varphi(\mathbf{y})\partial_x^\beta O^{2\lambda}(\mathbf{x} - \mathbf{y})]\}\tilde{\mathbf{u}}(\mathbf{y})\, d\mathbf{y} \\
&\quad + \int_{R^3} \left\{2\lambda \frac{\partial}{\partial y_1}[\varphi(\mathbf{y})\partial_x^\beta O^{2\lambda}(\mathbf{x} - \mathbf{y})]\right\}\tilde{\mathbf{u}}(\mathbf{y})\, d\mathbf{y} \\
&\quad + \int_{R^3} \{\nabla_{\mathbf{y}} \cdot [\varphi(\mathbf{y})\partial_{\mathbf{x}}^\beta O^{2\lambda}(\mathbf{x} - \mathbf{y})]\}\tilde{p}(\mathbf{y})\, d\mathbf{y} = O(|\mathbf{x}|^{-1-|\beta|/2}), \quad |\mathbf{x}| \to \infty, \\
|\partial^\beta Q * \mathbf{f}(\mathbf{x})| &= |\langle \mathbf{f}(\mathbf{y}), \varphi(\mathbf{y})\partial_{\mathbf{x}}^\beta Q(\mathbf{x} - \mathbf{y})\rangle| = \int_{R^3} \{\Delta_{\mathbf{y}}[\varphi(\mathbf{y})\partial_{\mathbf{x}}^\beta Q(\mathbf{x} - \mathbf{y})]\}\tilde{\mathbf{u}}(\mathbf{y})\, d\mathbf{y} \\
&\quad + \int_{R^3} \left\{2\lambda \frac{\partial}{\partial y_1}[\varphi(\mathbf{y})\partial_x^\beta Q(\mathbf{x} - \mathbf{y})]\right\}\tilde{\mathbf{u}}(\mathbf{y})\, d\mathbf{y} \\
&\quad + \int_{R^3} \{\nabla_{\mathbf{y}} \cdot [\varphi(\mathbf{y})\partial_x^\beta Q(\mathbf{x} - \mathbf{y})]\}\tilde{p}(\mathbf{y})\, d\mathbf{y} = O(|\mathbf{x}|^{-2-|\beta|}), \qquad |\mathbf{x}| \to \infty.
\end{aligned}
$$

Moreover, $|\nabla O^{2\lambda} * \mathbf{f}| \in L^2(R^3 \setminus B(0; r))$.

Set $\tilde{\mathbf{v}} = \tilde{\mathbf{u}} + O^{2\lambda} * \mathbf{f}$, $\tilde{q} = \tilde{p} + Q * \mathbf{f}$. Since $(O^{2\lambda}, Q)$ is the fundamental tensor of the Oseen equations (3), $\tilde{\mathbf{v}}$, $\tilde{q}$ solve the Oseen system (3) in R^3. Thus, we have proved $O^{2\lambda} * \mathbf{f}(\mathbf{x}) = O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \to \infty$. Since $\tilde{\mathbf{v}}$ is bounded, $\tilde{\mathbf{v}}$, $\tilde{q}$ are constant by Corollary 6.2.
□

Corollary 7.2 *Let $\Omega \subset R^3$ be an exterior domain with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, $\lambda \in R \setminus \{0\}$. If $\mathbf{f} \in \mathcal{C}^0(\partial\Omega)^3$, $\mathbf{u}_\infty \in R^3$, $p_\infty \in R$, then there exists a unique solution of the problem $\mathbf{u} \in C^2(\Omega)^3 \cap C^0(\overline{\Omega})^3$, $p \in C^1(\Omega)$ satisfying*

$$-\Delta \mathbf{u} + 2\lambda \partial_1 \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \textit{in } \Omega, \quad \mathbf{u} = \mathbf{f} \quad \textit{on } \partial\Omega,$$
$$p(\mathbf{x}) \to p_\infty, \quad \mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty \quad \textit{as } |\mathbf{x}| \to \infty.$$

Proof The Corollary is an easy consequence of Theorem 5.3 and Proposition 7.1. □

Theorem 7.3 *Let $\Omega \subset R^3$ be an exterior domain with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, and $\lambda \in R \setminus \{0\}$. Then there exists a constant C such that the following statement holds true: If $\mathbf{u} \in C^2(\Omega)^3 \cap C^0(\overline{\Omega})^3$, $p \in C^1(\Omega)$ solve the Oseen equations (3) in Ω, and if $|\mathbf{u}| \le M$ on $\partial\Omega$,*

$$\limsup_{|\mathbf{x}| \to \infty} |\mathbf{u}(\mathbf{x})| \le M, \tag{27}$$

then $|\mathbf{u}| \le CM$ in Ω.

Proof For $\boldsymbol{\Psi} \in \mathcal{C}^0(\partial\Omega)^3$ set $S\boldsymbol{\Psi} = \frac{1}{2}\boldsymbol{\Psi} + L_\Omega^{2\lambda}\boldsymbol{\Psi} + O_\Omega^{2\lambda}(c + \lambda n_1^\Omega)\boldsymbol{\Psi}$ on $\partial\Omega$, $\tau\boldsymbol{\Psi} = W_\Omega^{2\lambda}\boldsymbol{\Psi} + O_\Omega^{2\lambda}(c + \lambda n_1^\Omega)\boldsymbol{\Psi}$ in Ω, $\tau\boldsymbol{\Psi} = S\boldsymbol{\Psi}$ on $\partial\Omega$. Then τ is a linear mapping from $\mathcal{C}^0(\partial\Omega)^3$ to $\mathcal{C}^0(\overline{\Omega})^3 \cap L^\infty(\Omega)^3$ equipped with the supremum norm (see Proposition 3.3 and Proposition 3.4). If $\boldsymbol{\Psi}_k \to \boldsymbol{\Psi}$ in $\mathcal{C}^0(\partial\Omega)^3$, $\tau\boldsymbol{\Psi}_k \to \mathbf{g}$ in $\mathcal{C}^0(\overline{\Omega})^3 \cap L^\infty(\Omega)^3$, then $\mathbf{g}(\mathbf{x}) = \lim \tau\boldsymbol{\Psi}_k(\mathbf{x}) = \tau\boldsymbol{\Psi}(\mathbf{x})$ for each $\mathbf{x} \in \Omega$. Thus, $\mathbf{g} = \tau\boldsymbol{\Psi}$ and τ is a closed operator. By the Closed Graph Theorem ([21], Theorem II.1.9), there is a constant C_1 such that

$$\sup_{\mathbf{x} \in \overline{\Omega}} |\tau\boldsymbol{\Psi}(\mathbf{x})| \le C_1 \sup_{\mathbf{y} \in \partial\Omega} |\boldsymbol{\Psi}(\mathbf{y})|.$$

Now let $\mathbf{u} \in C^2(\Omega)^3 \cap C^0(\overline{\Omega})^3$, $p \in C^1(\Omega)$ solve the Oseen equations (3) in Ω satisfying $|\mathbf{u}| \le M$ on $\partial\Omega$ and (27). According to Proposition 7.1, there exist $\mathbf{u}_\infty \in R^3$, $p_\infty \in R$ such that $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$, $p(\mathbf{x}) \to p_\infty$ as $|\mathbf{x}| \to \infty$ and $|\nabla(\mathbf{u} - \mathbf{u}_\infty)(\mathbf{x})| = O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}| \to \infty$. Clearly, $|\mathbf{u}_\infty| \le M$. According to Theorem 5.3 and Corollary 7.2, the operator S is continuously invertible and $\mathbf{u} - \mathbf{u}_\infty = \tau S^{-1}(\mathbf{u} - \mathbf{u}_\infty)$ in Ω. If $\mathbf{x} \in \Omega$, then

$$|\mathbf{u}(\mathbf{x})| \le |\mathbf{u}_\infty| + |\tau S^{-1}(\mathbf{u} - \mathbf{u}_\infty)(\mathbf{x})| \le M + C_1 \|S^{-1}\| 2M.$$

This proves the theorem. □

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