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Existence of seven solutions for an asymptotically linear Dirichlet problem without symmetries

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Abstract In this paper, we establish sufficient conditions for an asymptotically linear elliptic boundary value problem to have at least seven solutions. We use the mountain pass theorem, Lyapunov–Schmidt reduction arguments, existence of solutions that change sign exactly once, and bifurcation properties. No symmetry is assumed on the domain or the non-linearity.

Keywords Semilinear elliptic equation · Morse index · Sign-changing solutions · Bifurcation

Mathematics Subject classification (2010) 35J20 · 35J25 · 35J61 · 35B38

1 Introduction

In the quest for multiple solutions to equations without symmetries, a central role is played by the equation

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{R}^N$, $N > 2$, is a bounded and smooth domain, and $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1. We denote by $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ the sequence of eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω, and by $\{\varphi_k\}_{k\in\mathbb{N}}$ a corresponding complete orthonormal sequence of eigenfunctions in the Sobolev space $H_0^1(\Omega)$. We assume that $f(0) = 0$ and that

(f1) $f'(0) \leq 0$
(f2) $\lim_{|t|\to\infty} f'(t) = 1$
(f3) $tf''(t) > 0$ for all $t \in \mathbb{R} - \{0\}$.

Our main result is:

Theorem 1.1 *If $k \geq 3$ and $\lambda_k < \lambda_{k+1}$ then there exists $\epsilon > 0$ such that if $\lambda \in (\lambda_{k+1}, \lambda_{k+1}+\epsilon)$ then (1) has at least seven solutions.*

Corollary 1.2 *If $\lambda = \lambda_{k+1}$ then (1) has at least five solutions.*

The existence of three solutions for problems such as (1) was established in [6]. The results of [6] were extended in [7] using Morse theory. Also the results in [6] were extended in [3] to prove the existence of five solutions. Using the minmax principle developed in [4], further description of such five solutions was established in [5]. The main minmax principle proved in [4] was in turn motivated by the ideas in [14]. For a recent result on the existence of four solutions to problem (1) when f is *superlinear* and λ is near an eigenvalue, the reader is referred to [13]. When f is an odd function, using Liusternik-Schnierelmann methods one can establish that (1) has $2k+1$ solutions (see [12]). For related results where the *Morse index* of solutions to (1) is estimated and used to find additional solutions, the reader is referred to [1,2,8,9].

Our proofs use extensively the Lyapunov–Schmidt reduction method, critical groups and Morse indices. We take advantage of the maxmin characterization of a solution obtained using the Lyapunov–Schmidt reduction method to establish estimates on the L_∞ norm of such a solution depending on the location of λ with respect to λ_k and λ_{k+1} (see Sect. 4). The proofs of Theorem 1.1 and Corollary 1.2 are found in Sect. 6.

2 Preliminaries

Let $H_0^1(\Omega)$ denote the Hilbert space of square integrable functions having generalized first-order partial derivatives in $L^2(\Omega)$. We denote by $\|\cdot\|$ the norm in this space and by $\|\cdot\|_p$ the norm in $L^p(\Omega)$, $1 \leq p \leq +\infty$. The Euclidean norm in $\mathbb{R}^N$ will be denoted by $|\cdot|$.

The solutions to (1) are the critical point of the functional $J_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \lambda F(u)\right) \mathrm{d}x,$$

where $F(\xi) = \int_0^\xi f(s)\,\mathrm{d}s$. Because of (f2), $J \in C^2$ (see [12]) and, moreover,

$$DJ(u)\,v = \langle \nabla J(u), v\rangle = \int_\Omega (\nabla u \cdot \nabla v - \lambda f(u)\,v)\,\mathrm{d}x, \quad \forall u, v \in H_0^1(\Omega), \tag{2}$$

$$\left\langle D^2 J(u)\,v, w\right\rangle = \int_\Omega \left(\nabla v \cdot \nabla w - \lambda f'(u)\,vw\right) \mathrm{d}x, \quad \forall u, v, w \in H_0^1(\Omega). \tag{3}$$

We assume the critical points of J to be isolated. Without this assumption, problem (1) has infinitely many solutions. In Sect. 7, we clarify the nature of such solutions.

We recall that, if u_0 is a critical point of J, the *Morse index of J at u_0* is the maximal non-negative integer $m(J, u_0)$, or $m(u_0)$, such that there exists an $m(J, u_0)$-dimensional subspace of $H_0^1(\Omega)$ on which $D^2 J(u_0)$ is negative-definite. The *augmented Morse index* $m_a(u_0)$ is defined in a similar fashion, changing "negative-definite" by non-positive definite in the previous definition (see Sect. 3).

3 One-sign solutions

Let us note that (1) has a positive and a negative solution when $\lambda > \lambda_1$. This is a well-known consequence of the *Mountain Pass Theorem.* For the sake of completeness, we outline a proof of this result (see [3,7,10,12]). Let $f^+ : \mathbb{R} \to \mathbb{R}$ be the function defined in the following way: $f^+(t) := f(t)$, for $t \geq 0$, and $f^+(t) := f'(0)t$, for $t < 0$. Let $J_\lambda^+ : H_0^1(\Omega) \to \mathbb{R}$ be the functional defined by

$$J_\lambda^+(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - \lambda F^+(u) \right) \mathrm{d}x,$$

where $F^+(\xi) = \int_0^\xi f^+(s)\, \mathrm{d}s$. Because of (f1)–(f2), J_λ^+ satisfies the hypotheses of the Mountain Pass Theorem. Thus, J_λ^+ has a critical point u_+ of mountain pass type. That is, there exists U neighborhood of u_+ such that if $V \subset U$ is neighborhood of u_+, then$\{u \in B;\ J(u) < J(u_+)$ is not connected. By the Strong Maximum Principle, $u_+ > 0$ in Ω and u_+ is a critical point of mountain pass type of J_λ. A negative solution u_- is obtained in a similar fashion and the outline is complete.

Also, by Theorem 2 of [11], if W_+ (respectively, W_-) is a region containing u_+ (respectivley, u_-) and no other critical point, then

$$d(\nabla J, W_+, 0) = -1 \tag{4}$$

(respectively, $d(\nabla J, W_-, 0) = -1$). See also (3.10) of [3].

4 An augmented Morse index k solution

Let us recall a global version of the Lyapunov–Schmidt reduction method. We refer the reader to [6] for details.

Lemma 1 *Let H be a real Hilbert space and let $J : H \to \mathbb{R}$ be a function of the class $C^2(H, \mathbb{R})$. Let X and Y be closed subspaces of H such that $H = X \oplus Y$. Suppose there exists $c > 0$ such that*

$$\langle D^2 J(u) y, y \rangle \geq c \|y\|_H^2; \qquad \forall u \in H \quad \forall y \in Y. \tag{5}$$

Then:

(i) *There exists a function $\phi : X \to Y$, of the class C^1, such that*

$$J(x + \phi(x)) = \min_{y \in Y} J(x + y).$$

Moreover, given $x \in X$, $\phi(x)$ is the unique element of Y such that

$$\langle \nabla J(x + \phi(x)), y \rangle = 0 \quad \forall y \in Y. \tag{6}$$

(ii) *Functional $\widetilde{J} : X \to \mathbb{R}$, defined by $\widetilde{J}(x) := J(x + \phi(x))$ for $x \in X$, is of class C^2. Moreover,*

$$D\widetilde{J}(x)h = \left\langle \nabla \widetilde{J}(x), h \right\rangle = \langle \nabla J(x + \phi(x)), h \rangle \quad \forall x, h \in X. \tag{7}$$

(iii) *Given $x \in X$, x is a critical point of $\widetilde{J}$ if and only if $u = x + \phi(x)$ is a critical point of J.*

(iv) *If $x_0 \in X$ is an isolated critical point of $\widetilde{J}$, then the local Leray-Schauder degree is preserved under reduction, i.e.,*

$$d_{loc}\left(\nabla \widetilde{J}, x_0\right) = d_{loc}(\nabla J, u_0).$$

Let $\lambda \in (\lambda_k, \lambda_{k+1})$, X is the subspace of $H_0^1(\Omega)$ generated by $\{\varphi_1, \ldots, \varphi_k\}$ and Y is the closed subspace of $H_0^1(\Omega)$ generated by $\{\varphi_{k+1}, \ldots, \}$. Due to (f2)–(f3), J_λ satisfies (5) with $c = 1 - (\lambda/\lambda_{k+1}) > 0$. Also $\lim_{x \in X, \|x\| \to +\infty} J_\lambda(x) = -\infty$. Since $\widetilde{J}_\lambda(x) \leq J(x)$, $\widetilde{J}_\lambda$ attains it maximum value at some x_λ. Therefore, the Eq. (1) has a non-zero solution $v_\lambda = x_\lambda + \phi_\lambda(x_\lambda) \equiv v$ that satisfies

$$J_\lambda(v) = \max_{x \in X} \left(\min_{y \in Y} J_\lambda(x + y) \right). \tag{8}$$

In addition, the augmented Morse index of J at v is k and its local degree is $(-1)^k$ (see [6] and [3] for further details).

The following lemma allows us to distinguish v_λ from the solutions u_-, u_+ discussed in Sect. 3. Similar ideas are used to distinguish solutions that change sign exactly once from higher Morse index solutions, see Lemma 7.

Lemma 2 *If $\dim(X) \geq 2$ then v_λ is not a critical point of mountain pass type.*

Proof Let $v_\lambda = x_\lambda + y_\lambda$ with $x_\lambda \in X$ and $y_\lambda \in Y$. Since any neighborhood of v_λ contains a neighborhood of the form $A_\varepsilon = \{x + y; \|x - x_\lambda\| < \varepsilon, \|y - \phi(x)\| < \varepsilon\}$, it is sufficient to prove that $B_\varepsilon \equiv \{u \in A_\varepsilon; J_\lambda(u) < J_\lambda(v_\lambda)\}$ is connected.

Let $x_1+y_1 \in B_\varepsilon$, $x_2+y_2 \in B_\varepsilon$. Since, for $i = 1, 2$, $J_\lambda(x_\lambda+y_i) \geq J_\lambda(x_\lambda+y_\lambda) = J_\lambda(v_\lambda)$ we have $x_i \neq x_\lambda$ for $i = 1, 2$. Since $\dim(X) \geq 2$, $B = \{x \in X; \|x - x_\lambda\| \in (0, \varepsilon)\}$ is connected. Hence, there exists a continuous function $\sigma : [0, 1] \to B$ such that $\sigma(0) = x_1$, $\sigma(1) = x_2$. Since $\widetilde{J}$ attains a strict local maximum at x_λ and $x_\lambda \notin B$, $J_\lambda(\sigma(t) + \phi(\sigma(t))) = \widetilde{J}_\lambda(\sigma(t)) < \widetilde{J}_\lambda(x_\lambda) = J_\lambda(x_\lambda + y_\lambda)$. From (5), $J_\lambda(x_i + (1-s)y_i + s\phi(x_i)) \leq J_\lambda(x_i + y_i) < J_\lambda(x_\lambda + y_\lambda)$ for all $s \in [0, 1]$. Hence,

$$\sigma_1(t) = \begin{cases} x_1 + y_1 + (s+1)(\phi(x_1) - y_1) & s \in [-1, 0] \\ \sigma(s) + \phi(\sigma(s)) & s \in [0, 1] \\ x_2 + \phi(x_2) + (s-1)(y_2 - \phi(x_2)) & s \in [1, 2] \end{cases} \tag{9}$$

defines a continuous path in B_ε connecting $x_1 + y_1$ with $x_2 + y_2$. This proves that B_ε is connected. Hence, v_λ is not a critical point of mountain pass, proving the lemma. □

Lemma 3 *For $v = v_\lambda$ as in (8) we have*

$$\lim_{\lambda \to \lambda_k +} \|v_\lambda\|_\infty = +\infty. \tag{10}$$

Proof Assuming to the contrary, there exists a sequence $\{\mu_j\}_j$ in $(\lambda_k, \lambda_{k+1})$ converging to λ_k and a real number m such that $\|v_{\mu_j}\|_\infty \leq m$ for all $j's$. From (f2) to (f3), we see that $m_1 \equiv \max\{f'(u); |u| \leq m\} < 1$. Hence, for j, sufficiently large $|\mu_j f'(v_j(x))| \leq m_1 \mu_j <$

λ_k. Hence, $D^2 J_{\mu_j}(v_{\mu_j})$ is positive definite on the closed subspace of $H_0^1(\Omega)$ spanned by $\{\varphi_k, \varphi_{k+1}, \ldots\}$ which contradicts that v_j has augmented Morse index k. This proves the lemma. □

Lemma 4 *There exists a real number M such that if $\lambda \in ((\lambda_k + \lambda_{k+1})/2, \lambda_{k+1})$ then*

$$\max\{\|v_\lambda\|, \|v_\lambda\|_\infty\} \leq M. \tag{11}$$

Proof Let $\lambda_k < \alpha \leq \beta < \lambda_{k+1}$. Because $F(t) \geq 0$ for all $t \in \mathbb{R}$,

$$\begin{aligned} J_\beta(v_\beta) &= \max_{x\in X}\left(\min_{y\in Y}\int_\Omega\left(\frac{|\nabla(x+y)|^2}{2} - \beta F(x+y)\right)\mathrm{d}\zeta\right) \\ &\leq \max_{x\in X}\left(\min_{y\in Y}\int_\Omega\left(\frac{|\nabla(x+y)|^2}{2} - \alpha F(x+y)\right)\mathrm{d}\zeta\right) \\ &= J_\alpha(v_\alpha). \end{aligned} \tag{12}$$

Also, for $\lambda \in (\lambda_k, \lambda_{k+1}]$,

$$\begin{aligned} J_\lambda(v_\lambda) &= \max_{x\in X}\left(\min_{y\in Y}\int_\Omega\left(\frac{|\nabla(x+y)|^2}{2} - \lambda F(x+y)\right)\mathrm{d}\zeta\right) \\ &\geq \min_{y\in Y}\int_\Omega\left(\frac{|\nabla(y)|^2}{2} - \lambda F(y)\right)\mathrm{d}\zeta \\ &\geq \min_{y\in Y}\int_\Omega\left(\frac{|\nabla(y)|^2}{2} - \lambda \frac{y^2}{2}\right)\mathrm{d}\zeta \\ &\geq 0. \end{aligned} \tag{13}$$

Let $v_\lambda = x_\lambda + y_\lambda$ with $x_\lambda \in X$, $y_\lambda \in Y$. From (f2), we see that $F(t) = t^2/2 - G(t)$ with $G(t) \geq 0$ for all $t \in \mathbb{R}$ and $\lim_{|t|\to\infty} G(t)/t^2 = 0$.

From the definition of G, we see that there exists a positive number M_1 such that

$$G(t) \leq \frac{\lambda_{k+1} - \lambda_k}{8\lambda_{k+1}} t^2 + M_1 \quad \text{for all } t \in \mathbb{R}. \tag{14}$$

Now for $\lambda \in [(\lambda_{k+1} + \lambda_k)/2, \lambda_{k+1})$, we have

$$\begin{aligned} 0 &\leq J_\lambda(v_\lambda) \leq J_\lambda(x_\lambda) \\ &= \int_\Omega\left(\frac{|\nabla(x_\lambda)|^2}{2} - \lambda\frac{x_\lambda^2}{2} + \lambda G(x_\lambda)\right)\mathrm{d}\zeta \\ &\leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|x_\lambda\|^2 + \int_\Omega \lambda G(x_\lambda)\mathrm{d}\zeta. \\ &\leq \left(\frac{1}{2}\left(1 - \frac{\lambda_k + \lambda_{k+1}}{2\lambda_k}\right) + \frac{\lambda_{k+1} - \lambda_k}{8\lambda_k}\right)\|x_\lambda\|^2 + \lambda_{k+1}|\Omega|M_1 \\ &\leq \frac{\lambda_k - \lambda_{k+1}}{8\lambda_k}\|x_\lambda\|^2 + \lambda_{k+1}|\Omega|M_1, \end{aligned} \tag{15}$$

which proves that x_λ is bounded.

Now we write $y_\lambda = y_{\lambda,1} + y_{\lambda,2}$ with $\Delta y_{\lambda,1} + \lambda_{k+1} y_{\lambda,1} = 0$ and $\int_\Omega y_{\lambda,1} y_{\lambda,2} \mathrm{d}\zeta = 0$. Letting $\bar{\lambda} = (\lambda_k + \lambda_{k+1})/2$, we have

$$\begin{aligned} J_{\bar{\lambda}}(v_{\bar{\lambda}}) &\geq J_\lambda(v_\lambda) \\ &= \frac{1}{2}\int_\Omega \left(|\nabla(x_\lambda)|^2 - \lambda x_\lambda^2\right) \mathrm{d}\zeta + (1 - \lambda/\lambda_{k+1})\|y_{\lambda,1}\|^2 \\ &\quad + \frac{1}{2}\int_\Omega (|\nabla(y_{\lambda,2})|^2 - \lambda y_{\lambda,2}^2)\mathrm{d}\zeta + \int_\Omega \lambda G(v_\lambda)\mathrm{d}\zeta \\ &\geq \frac{1}{2}\left(\int_\Omega \left(|\nabla(x_\lambda)|^2 - \lambda x_\lambda^2\right) \mathrm{d}\zeta + \left[1 - \frac{\lambda}{\lambda_{k+j+1}}\right]\|y_{\lambda,2}\|^2\right), \end{aligned} \tag{16}$$

where j is the multiplicity of λ_{k+1}. Since $\|x_\lambda\|$ is bounded, $\|y_{\lambda,2}\|$ is also bounded. Replacing this in (16) and using that $1 - \lambda/\lambda_{k+1} > 0$ yields that $\int_\Omega G(v_\lambda) d\zeta$ is also bounded. Since $\lim_{|t|\to\infty} G(t) = +\infty$ and the $y_{\lambda,1}$'s belong to a finite dimensional subspace, we see that $\|y_{\lambda,1}\|$ is bounded. Thus, $\|v_\lambda\|$ is bounded. By standard regularity theory for second-order elliptic operators we have $\|v_\lambda\|_\infty$ is bounded, which proves the lemma. □

Lemma 5 *For $\lambda = \lambda_{k+1}$ the Eq.* (1) *has an augmented Morse index k solution.*

Proof Let $\{\mu_j\}_j$ be a sequence in $[(\lambda_k + \lambda_{k+1})/2, \lambda_{k+1})$ converging to λ_{k+1}. From Lemma 4, we may assume that the sequence $\{v_{\mu_j}\}$ converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Hence, $\{\lambda_j f \circ v_{\mu_j}\}$ converges in $L^2(\Omega)$. Thus, by the regularity properties of second-order elliptic operators, $\{v_{\mu_j}\}$ converges strongly in $H_0^1(\Omega)$ to some element v. Thus, for each $\psi \in H_0^1(\Omega)$,

$$\begin{aligned} 0 &= \lim_{j\to\infty} \int_\Omega (\langle \nabla v_{\mu_j}, \nabla\psi\rangle - \mu_j f(v_{\mu_j})\psi)\mathrm{d}\zeta \\ &= \int_\Omega (\langle \nabla v, \nabla\psi\rangle - \lambda_{k+1} f(v)\psi)\mathrm{d}\zeta \end{aligned} \tag{17}$$

Thus, v is a solution to (1) with $\lambda = \lambda_{k+1}$.

From Lemma 4, $\|v\|_\infty \leq M$. Since $m_2 \equiv \max\{|f'(t)|; |t| \leq 2M + 1\} < 1$, for $y \in Y$, we have

$$\begin{aligned} \langle D^2 J_{\lambda_{k+1}}(v)y, y\rangle &= \int_\Omega \left(|\nabla y|^2 - \lambda_{k+1} f'(v) y^2\right) \mathrm{d}\zeta \\ &\geq \int_\Omega \left(|\nabla y|^2 - \lambda_{k+1} m_2 y^2\right) \mathrm{d}\zeta \\ &> 0. \end{aligned} \tag{18}$$

Therefore, v is a critical point of $J_{\lambda_{k+1}}$ and $m_a(J_{\lambda_{k+1}}, v) \leq k$.

On the other hand, if $x \in X$,

$$\begin{aligned}\langle D^2 J_{\lambda_{k+1}}(v)x, x\rangle &= \int_\Omega \left(|\nabla x|^2 - \lambda_{k+1} f'(v)x^2\right) \mathrm{d}\zeta \\ &= \lim_{j\to\infty} \int_\Omega \left(|\nabla x|^2 - \lambda_{k+1} f'(v_{\lambda_j})x^2\right) \mathrm{d}\zeta \\ &\le 0. \end{aligned} \tag{19}$$

Hence, $m_a(J_{\lambda_{k+1}}, v) = k$, which proves Lemma 5. □

Let $\delta > 0$ be such that $\lambda_{k+1}(1-\delta) > (\lambda_k + \lambda_{k+1})/2$. Let $\alpha_+ > 0$ be the solution to $f(\alpha_+) = (1-\delta)\alpha_+$, and $\alpha_- < 0$ the solution to $f(\alpha_-) = (1-\delta)\alpha_-$ (see (f3)). We define

$$A = \max\left\{\frac{(1-\delta)\alpha_+^2}{2} - F(\alpha_+), \frac{(1-\delta)\alpha_-^2}{2} - F(\alpha_-)\right\}. \tag{20}$$

Let $n > 1$ be an integer such that

$$1 - \frac{1}{n} > \max\left\{f'(\alpha_+), f'(\alpha_-)\right\}, \quad \frac{n}{n-1}\lambda_k \le \lambda_{k+1} \le \frac{(n-1)\lambda_{k+j+1}}{n}. \tag{21}$$

Let $\beta_+ > \alpha_+$ be such that $f'(\beta_+) = (n-1)/n$, and $\beta_- < \alpha_-$ such that $f'(\beta_-) = (n-1)/n$.

We then define $\widehat{f_n}(t) \equiv \widehat{f}(t) = f(t)$ for $t \in (\beta_-, \beta_+)$, $\widehat{f}(t) = f(\beta_+) + (n-1)(t-\beta_+)/n$ for $t \ge \beta_+$, and $\widehat{f}(t) = f(\beta_-) + (n-1)(t-\beta_-)/n$ for $t \le \beta_-$. Let $\widehat{F}(t) = \int_0^t \widehat{f}(s)\mathrm{d}s$. Thus, for any n,

$$\widehat{F}(t) \ge \frac{(1-\delta)t^2}{2} - A \qquad \forall t \in \mathbb{R}. \tag{22}$$

Since $\widehat{f}'(t) \le (n-1)/n$ for all t, $\widehat{F}(t) \le (n-1)t^2/(2n) - (t-1)((n-1)/n - f(1))$ for $t \ge 1$. Similarly, $\widehat{F}(t) \le (n-1)t^2/(2n) + (t+1)((n-1)/n - f(-1))$ for $t \le -1$. Hence, there exists $a > 0$, $b \in \mathbb{R}$, independent of n, such that

$$\widehat{F}(t) \le \frac{(n-1)t^2}{2n} - a|t| + b \quad \text{for all } t \in \mathbb{R}. \tag{23}$$

Since $|\widehat{f}(u)| \le |f(u)| \le |u|$ for all $u \in \mathbb{R}$, by elliptic regularity theory, there exists $K > 0$ (independent of n) such that if $\lambda \in (0, \lambda_{k+j+1})$, where j is the multiplicity of the eigenvalue λ_{k+1}, and u is a solution to

$$\begin{cases} \Delta u + \lambda \widehat{f}(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{24}$$

then

$$\|u\|_\infty \le K\|u\|_2. \tag{25}$$

Since A, a, b, and K are independent of n, in addition to (21), we may assume that

$$K^4 \lambda_{k+j+1}^N C^{N+2} \left(\frac{\lambda_{k+j+1}\delta 4A|\Omega|\lambda_k}{2a\lambda_1(\lambda_{k+1} - \lambda_k)} + (A+b)|\Omega|\right)^{N+2} < \left(\frac{1}{6}\min\{-\beta_-, \beta_+\}\right)^4, \tag{26}$$

where C is the constant given by imbedding of $H_0^1(\Omega)$ into $L^{2N/(N-2)}(\Omega)$. That is

$$\|u\|_{L^{2N/(N-2)}} \le C\|u\| \quad \text{for all } u \in H_0^1(\Omega). \tag{27}$$

Let

$$\widehat{J_\lambda}(u) = \int_\Omega \left(\frac{|\nabla u|^2}{2} - \lambda \widehat{F}(u)\right) \mathrm{d}\zeta, \tag{28}$$

Thus, for $\|u\|_\infty \leq \min\{\beta_+, -\beta_-\}$, u is a critical point of $\widehat{J_\lambda}$ if and only if u is a critical point of J_λ. Now, for each $\lambda \in (n\lambda_k/(n-1), n\lambda_{k+1}/(n-1))$, the functional $\widehat{J_\lambda}$ has a critical point v_λ satisfying

$$\widehat{J_\lambda}(v_\lambda) = \max_{x\in X} \left(\min_{y\in Y} \widehat{J_\lambda}(x+y)\right), \tag{29}$$

Hence, by (22) and (29),

$$\begin{aligned} 0 \leq \widehat{J_\lambda}(v_\lambda) &\equiv \widehat{J_\lambda}(x_\lambda + y_\lambda) \leq \widehat{J_\lambda}(x_\lambda) \\ &= \int_\Omega \left(\frac{|\nabla x_\lambda|^2}{2} - \lambda \widehat{F}(x_\lambda)\right) \mathrm{d}\zeta \\ &\leq \int_\Omega \left(\frac{|\nabla x_\lambda|^2}{2} - \frac{\lambda(1-\delta)x_\lambda^2}{2}\right) \mathrm{d}\zeta + A|\Omega|. \end{aligned} \tag{30}$$

Hence, for $\lambda \geq \lambda_{k+1}$,

$$\|x_\lambda\|^2 \leq \frac{2A|\Omega|\lambda_k}{\lambda(1-\delta) - \lambda_k} \leq \frac{4A|\Omega|\lambda_k}{\lambda_{k+1} - \lambda_k} \equiv M. \tag{31}$$

This and the definition of v_λ give for $\lambda \in [\lambda_{k+1}, n\lambda_{k+1}/(n-1))$,

$$\begin{aligned} \widehat{J_\lambda}(x_\lambda) &\geq \widehat{J_\lambda}(v_\lambda) \\ &= \int_\Omega \left(\frac{|\nabla (x_\lambda + y_\lambda)|^2}{2} - \frac{\lambda(n-1)}{2n}\left(x_\lambda^2 + y_\lambda^2\right) + a|v_\lambda|\right) \mathrm{d}\zeta - b|\Omega| \\ &\geq \int_\Omega \left(\frac{|\nabla x_\lambda|^2}{2} - \frac{\lambda(n-1)x_\lambda^2}{2n} + a|v_\lambda|\right) \mathrm{d}\zeta - b|\Omega|. \end{aligned} \tag{32}$$

This and (30) yield

$$\begin{aligned} \|v_\lambda\|_1 &\leq \frac{1}{2a}\lambda(\delta - (1/n))\|x_\lambda\|_2 + (A+b)|\Omega| \\ &\leq \frac{1}{2a\lambda_1}\lambda_{k+j+1}\delta M + (A+b)|\Omega| \\ &\equiv M_1. \end{aligned} \tag{33}$$

Therefore, by Holder's inequality and the continuous imbedding of $H^1(\Omega)$ into $L^{2N/(N-2)}(\Omega)$ (see 27)

$$\begin{aligned} \|v_\lambda\|_2^2 &= \int_\Omega |v_\lambda|^{4/(N+2)} |v_\lambda|^{2N/(N+2)} \mathrm{d}\zeta \\ &\leq \left(\int_\Omega |v_\lambda|\right)^{4/(N+2)} \left(\int_\Omega |v_\lambda|^{2N/(N-2)} \mathrm{d}\zeta\right)^{(N-2)/(N+2)} \\ &\leq M_1 C \|v_\lambda\|^{2-4/(N+2)}. \end{aligned} \tag{34}$$

Since v_λ satisfies (1),

$$\|v_\lambda\|^2 = \int_\Omega \lambda \widehat{f}(v_\lambda)v_\lambda \mathrm{d}\zeta \leq \lambda_{k+j+1}\|v_\lambda\|_2^2. \tag{35}$$

Combining (34) and (35), we conclude that

$$\|v_\lambda\|_2 \leq \lambda_{k+j+1}^{N/4}(M_1 C)^{(N+2)/4}. \tag{36}$$

Hence (see 25, 26, 33), we see that

$$\|v_\lambda\|_\infty \leq \frac{1}{6}\min\{-\beta_-, \beta_+\} \equiv M_2 \quad \text{for all } \lambda \in \lambda \in [\lambda_{k+1}, n\lambda_{k+1}/(n-1)). \tag{37}$$

This proves that, for each $\lambda \in [\lambda_{k+1}, n\lambda_{k+1}/(n-1))$, v_λ is a solution to (1) and its augmented Morse index is k.

Also for each $\lambda \in (n\lambda_k/(n-1), n\lambda_{k+1}/(n-1))$ let $\phi_\lambda : X \equiv \langle\varphi_1, \ldots, \varphi_k\rangle \longrightarrow Y \equiv X^\perp$ be as in Lemma 1 for the functional $\widehat{J_\lambda}$ defined as above. For later purposes, we state the following lemma that says that ϕ_λ is also L^∞-continuous.

Lemma 6 *For all $x \in X$, $\phi_\lambda(x) \in L^\infty(\Omega)$. Moreover, given $x_0 \in X$ and $\eta > 0$, there exists $\epsilon > 0$ such that*

$$\|x - x_0\| < \epsilon \Rightarrow \|\phi_\lambda(x) - \phi_\lambda(x_0)\|_\infty < \eta.$$

Proof For a given $x \in X$, $\phi_\lambda(x) \in Y$ satisfies

$$\int_\Omega (\nabla\phi_\lambda(x)\cdot\nabla y - f(x+\phi_\lambda(x))y)\mathrm{d}\zeta = 0 \quad \forall y \in Y. \tag{38}$$

This means that, in the weak sense, $\phi_\lambda(x) \in H_0^1(\Omega)$ satisfies

$$-\Delta(\phi_\lambda(x)) = P_Y(f(x+\phi_\lambda(x)) \tag{39}$$

where $P_Y : L^2(\Omega) \to Y \subset L^2(\Omega)$ is the projection operator. Using that f is Lipschitzian (see (f2)) and standard regularity theory for elliptic operators, a boot-strap argument shows that $\phi_\lambda(x) \in L^\infty(\Omega)$. Similarly, given $x, x_0 \in X$, $\phi_\lambda(x) - \phi_\lambda(x_0) \in H_0^1(\Omega)$ satisfies

$$-\Delta(\phi_\lambda(x) - \phi_\lambda(x_0)) = P_Y(f(x+\phi_\lambda(x)) - f(x_0+\phi_\lambda(x_0))). \tag{40}$$

The same kind of arguments and the continuity of $\phi_\lambda : X \longrightarrow Y \subset H_0^1(\Omega)$ imply the second assertion of Lemma 6. □

Given $\lambda \in [\lambda_{k+1}, n\lambda_{k+1}/(n-1))$, we assume that v_λ is an isolated critical point of $\widehat{J_\lambda}$. Hence, there exists $\epsilon_0 = \epsilon_0(\lambda)$ such that

$$\widehat{J_\lambda}(v_\lambda) = \widehat{J_\lambda}(x_\lambda + \phi_\lambda(x_\lambda)) > \widehat{J_\lambda}(x+\phi_\lambda(x)) \qquad \forall x \in D_{\epsilon_0}(x_\lambda) \cap X \tag{41}$$

and $D_{\epsilon_0}(x_\lambda) \cap X = \left\{u \in H_0^1(\Omega) : \|u - x_\lambda\| < \epsilon_0\right\} \cap X$ contains no other critical point of $x \mapsto J_\lambda(x+\phi_\lambda(x))$. We observe that without our assumption of isolation of this critical point, there would exist infinitely many solutions of (1).

By applying Lemma 6, there exists $\epsilon_1 \in (0, \epsilon_0)$ such that

$$\|x - x_\lambda\| < \epsilon_1 \Rightarrow \|x + \phi_\lambda(x) - v_\lambda\|_\infty < \frac{1}{6}\min\{-\beta_-, \beta_+\}. \tag{42}$$

5 A Morse index 2 solution

First, we note that, for $\lambda > \lambda_2$, (f1)–(f3) imply the hypotheses Theorem 1.3 of [5]. Hence, we have:

Theorem 5.1 *If $\lambda > \lambda_2$ the Eq. (1) has a solution w_λ that changes sign exactly once and whose Morse index is two. If isolated its local degree is* $+1$.

Let v_λ as defined in the previous section. In order to distinguish w_λ from v_λ for $\lambda \in [\lambda_{k+1}, n\lambda_{k+1}/(n-1))$ (see 8), we recall how w_λ is obtained and characterized in [5]. Let us take $m \in \mathbb{N}$ such that

$$\max\{-\beta_-, \beta_+\} < m. \tag{43}$$

Let $\sigma \in (1, 1 + 2/N)$ and let us define function $f^* = f^*_m : \mathbb{R} \to \mathbb{R}$ by $f^*(t) = f(t)$ for $t \in [-m, m]$, $f^*(t) = f(m) + f'(m)(t - m) + (t - m)^\sigma$ for $t \geq m$, and $f^*(t) = f(-m) + f'(-m)(t + m) - |t + m|^\sigma$ for $t \leq -m$. Let $F^*(t) = \int_0^t f^*(s)\mathrm{d}s$. We consider functional

$$J^*_\lambda(u) = \int_\Omega \left(\frac{|\nabla u|^2}{2} - \lambda F^*(u)\right) \mathrm{d}\zeta. \tag{44}$$

Because of the results of [5] (see Lemmas 2.1, 2.2 and 2.3), there exist a solution w_λ of (1) and $\epsilon_2 \in (0, \epsilon_1)$ such that

(a) If $W := span\left\{w^+_\lambda, w^-_\lambda\right\}$,

$$J^*_\lambda(w_\lambda + w) < J^*_\lambda(w_\lambda) \qquad \forall w \in W \cap D_{\epsilon_2}(0)$$

and $D^2 J^*_\lambda(w_\lambda)$ is negative-defined on W.

(b) There exists $\delta = \delta(\epsilon_2) > 0$ such that, if $Z := W^\perp$,

$$J^*_\lambda(w_\lambda + w + z) < J^*_\lambda(w_\lambda) \qquad \forall w \in W \cap \partial D_{\epsilon_2}(0) \qquad \forall z \in Z \cap D_\delta(0).$$

(c) There exists a continuous function $\psi : Z \cap D_\delta(0) \longrightarrow W \cap D_{\epsilon_2}(0)$ such that, for each $z \in Z \cap D_\delta(0)$,

$$J^*_\lambda(w_\lambda + \psi(z) + z) = \max_{w \in W \cap D_{\epsilon_2}(0)} J^*_\lambda(w_\lambda + w + z) > J^*_\lambda(w_\lambda).$$

(d) If $c_W > 0$ is a constant such that $\|.\|_{L^\infty} \leq c_W \|.\|$ in W,

$$c_W \epsilon_2 < \frac{1}{6} \min\{-\beta_-, \beta_+\}.$$

Let $c := c_\lambda = J^*_\lambda(w_\lambda)$ and $\epsilon_3 > 0$ such that $\epsilon_3 < \frac{1}{2} \min\{\epsilon_2, \delta\}$ and

$$\|x - x_\lambda\| \leq \epsilon_3 \Rightarrow \|\phi_\lambda(x) - \phi_\lambda(x_\lambda)\| < \frac{1}{2} \min\{\epsilon_2, \delta\}. \tag{45}$$

Lemma 7 *If $v_\lambda = w_\lambda$ and*

$$\sigma : [0, 1] \longrightarrow \partial D_{\epsilon_3}(w_\lambda) \cap W = \{w_\lambda + w \in W : \ \|w\| = \epsilon_3\} \subset H^1_0(\Omega)$$

a parametrization of $\partial D_{\epsilon_3}(w_\lambda) \cap W$, then σ is homotopic to a point (or contractible) in the set

$$E := \left(J^*_\lambda\right)^{-1}(-\infty, c) \cap \{w_\lambda + w + z : \ w \in W \cap \overline{D}_{\epsilon_2}(0), \ z \in Z \cap D_\delta(0)\}.$$

Proof Given $t \in [0,1]$, let us write $\sigma(t) = \sigma_X(t) + \sigma_Y(t)$, where $\sigma_X(t)$ (respectively $\sigma_Y(t)$) is the projection of $\sigma(t)$ on X (respectively on Y). First, we observe that for every $s \in [0,1]$,

$$\begin{aligned}\|\sigma_X(t) + ((1-s)\sigma_Y(t) + s\phi_\lambda(\sigma_X(t)))\|_\infty &\le \|(1-s)(\sigma(t) - w_\lambda)\|_\infty \\ &+\|s[w_\lambda - (\sigma_X(t) + \phi_\lambda(\sigma_X(t)))]\|_{L^\infty} + \|w_\lambda\|_{L^\infty}.\end{aligned} \tag{46}$$

Since $\|\sigma_X(t) - x_\lambda\| \le \|\sigma(t) - w_\lambda\| = \epsilon_3 < \epsilon_1$, because of the (37), (42), (46) and (d),

$$\|\sigma_X(t) + ((1-s)\sigma_Y(t) + s\phi_\lambda(\sigma_X(t)))\|_{L^\infty} \le \frac{1}{2}\min\{-\beta_-, \beta_+\}. \tag{47}$$

Let $h : [0,1] \times [0,1] \longrightarrow H_0^1(\Omega)$ be defined by $h(s,t) = \sigma_X(t) + [(1-s)\sigma_Y(t) + s\phi_\lambda(\sigma_X(t))]$. By definition (see 28, 43 and 44) $J_\lambda(u) = \widehat{J_\lambda}(u) = J_\lambda^*(u)$ if $\|u\|_{L^\infty} < \min\{-\beta_-, \beta_+\}$. From this, (47) and (a),

$$c = J_\lambda(w_\lambda) = J_\lambda^*(w_\lambda) > J_\lambda^*(\sigma(t)) = \widehat{J_\lambda}(\sigma(t)). \tag{48}$$

The convexity of $\widehat{J_\lambda}$ on Y implies that

$$\widehat{J_\lambda}(\sigma(t)) \ge \widehat{J_\lambda}\left(\sigma_X(t) + ((1-s)\sigma_Y(t) + s\phi_\lambda(\sigma_X(t)))\right). \tag{49}$$

Then, because of (47),

$$\widehat{J_\lambda}\left(\sigma_X(t) + ((1-s)\sigma_Y(t) + s\phi_\lambda(\sigma_X(t)))\right) = J_\lambda^*\left(h(s,t)\right). \tag{50}$$

From (48–50), we conclude that $h(s,t) \in \left(J_\lambda^*\right)^{-1}(-\infty, c)$ for every (s,t). To prove that $h(s,t) \in E$, it suffices to show that

$$\|\sigma_X(t) + ((1-s)\sigma_Y(t) + s\phi_\lambda(\sigma_X(t))) - w_\lambda\| < \min\{\epsilon_2, \delta\}. \tag{51}$$

From our choice of ϵ_3 (see 45),

$$\begin{aligned}\|\sigma_X(t) + ((1-s)\sigma_Y(t) + s\phi_\lambda(\sigma_X(t))) - w_\lambda\| &\le \|\sigma_X(t) - x_\lambda\| \\ &+\|(1-s)(\sigma_Y(t) - \phi_\lambda(x_\lambda)))\| + \|s[\phi_\lambda(\sigma_X(t)) - \phi_\lambda(x_\lambda))]\| \\ &\le \|\sigma(t) - w_\lambda\| + (1-s)\|\sigma(t) - w_\lambda\| + s\|\phi_\lambda(\sigma_X(t)) - \phi_\lambda(x_\lambda))\| \\ &< \epsilon_3 + (1-s)\epsilon_3 + s\frac{1}{2}\min\{\epsilon_2, \delta\} \le \min\{\epsilon_2, \delta\}.\end{aligned} \tag{52}$$

We observe that $h(0,\cdot) = \sigma(\cdot)$ and $h(1,\cdot) = \sigma_X(\cdot) + \phi_\lambda(\sigma_X(\cdot))$. Hence, it suffices to prove that this curve is homotopic to a point in E. Since $\sigma_X : [0,1] \longrightarrow X \cap \overline{D_{\epsilon_3}(x_\lambda)} \setminus \{x_\lambda\}$ is a closed curve and $\dim X \ge 3$, σ_X is homotopic to a point in $X \cap \overline{D_{\epsilon_3}(x_\lambda)} \setminus \{x_\lambda\}$. Let $k : [0,1] \times [0,1] \longrightarrow X \cap \overline{D_{\epsilon_3}(x_\lambda)} \setminus \{x_\lambda\}$ be a homotopy connecting σ_X to a point. Define $H : [0,1] \times [0,1] \longrightarrow H_0^1(\Omega)$ by

$$H(s,t) = k(s,t) + \phi_\lambda(k(s,t)).$$

This is a homotopy between $h(1,\cdot) = \sigma_X(\cdot) + \phi_\lambda(\sigma_X(\cdot))$ and a point. To complete the proof of the lemma, it simply remains to verify that $H([0,1]\times[0,1]) \subset E$. If $(s,t) \in [0,1]\times[0,1]$, then because of (41),

$$c = \widehat{J_\lambda}(w_\lambda) > \widehat{J_\lambda}(H(s,t)). \tag{53}$$

Also, since $k(s,t) \subset X \cap \overline{D_{\epsilon_3}(x_\lambda)} \subset X \cap \overline{D_{\epsilon_1}(x_\lambda)}$, as a consequence of (42) we have

$$\|k(s,t) + \phi_\lambda(k(s,t))\|_{L^\infty} < \frac{1}{3}\min\{-\beta_-, \beta_+\}. \tag{54}$$

Hence, $c > \widehat{J}_\lambda(H(s,t)) = J_\lambda^*(H(s,t))$. The fact that $H(s,t) \in E$ is again a consequence of (45) since $k(s,t) \in X \cap \overline{D_{\epsilon_3}(x_\lambda)}$, which completes the proof. □

Theorem 5.2 *For all* $\lambda \in [\lambda_{k+1}, n\lambda_{k+1}/(n-1))$, $v_\lambda \neq w_\lambda$.

Proof Suppose $v_\lambda = w_\lambda$ for some $\lambda \in [\lambda_{k+1}, n\lambda_{k+1}/(n-1))$. Because of Lemma 7, the function $\sigma : [0,1] \longrightarrow \partial D_{\epsilon_3}(w_\lambda) \cap W = \{w_\lambda + w \in W : \ \|w\| = \epsilon_3\}$ can be extended to a continuous function

$$\widetilde{\sigma} : \overline{D}_1(w_\lambda) \cap W \longrightarrow E.$$

Let us write $\widetilde{\sigma}(T) = w_\lambda + \widetilde{\sigma}_W(T) + \widetilde{\sigma}_Z(T)$ where $T \in \overline{D}_1(w_\lambda) \cap W$. Hence, $\widetilde{\sigma}_W(T) \in W \cap D_{\epsilon_2}(0)$ and $\widetilde{\sigma}_Z(T) \in Z \cap D_\delta(0)$. Let $G : [0,1] \times (\overline{D}_1(w_\lambda) \cap W) \longrightarrow E$ be defined by

$$G(s,T) = w_\lambda + \left[(1-s)\widetilde{\sigma}_W(T) + s\epsilon_2 \left(\frac{\psi\left(\widetilde{\sigma}_Z(T)\right) - \widetilde{\sigma}_W(T)}{\|\psi\left(\widetilde{\sigma}_Z(T)\right) - \widetilde{\sigma}_W(T)\|}\right)\right] + \widetilde{\sigma}_Z(T). \quad (55)$$

From (c) and (d) above, we see that $\|\psi(\widetilde{\sigma}_Z(T)) - \widetilde{\sigma}_W(T)\| \neq 0$. Hence, G is well-defined and continuous. Also, for each $T \in \overline{D}_1(w_\lambda) \cap W$, $G(0,T) = \widetilde{\sigma}(T)$ and $G(1,T) \in \{w_\lambda + w + z : \ w \in W \cap \partial D_{\epsilon_2}(0) \text{ and } z \in Z \cap D_\delta(0)\}$. Hence, $G(0, \overline{D}_1(w_\lambda) \cap W)$ is homotopic to S_W^1, which is a contradiction, which proves that $v_\lambda \neq w_\lambda$.

6 Proof of Theorem 1.1 and Corollary 1.2

Let $\lambda \in [\lambda_{k+1}, n\lambda_{k+1}/(n-1))$. From Sect. 3 and Theorem 5.1, we see that (1) has four solutions; $u_0 = 0$, a positive solution u_1, a negative solution u_2, and a solution $u_3 = w_\lambda$ that changes sign exactly once. As established in [5], Theorem 1.2, the local degree of ∇J_λ at u_3 is +1. From Sect. 4, we see that (1) has a solution $u_4 = v_\lambda$ whose augmented Morse index is k. Since $k \geq 3$, from Lemma 2 and Theorem 5.2, $u_4 \notin \{0, u_1, u_2, u_3\}$.

Let j be the multiplicity of the eigenvalue λ_{k+1}. For $\lambda \in (\lambda_{k+1}, \lambda_{k+j+1})$, arguing as in (8), we see that (1) has a solution z_λ with augmented Morse index $k+j$ and local degree $(-1)^{k+j}$. Moreover, $\lim_{\lambda\to\lambda_{k+1}} \|z_\lambda\| = +\infty$ (Lemma 3). Hence, there exists $\varepsilon \in (0, \lambda_{k+1}/(n-1))$ such that $\|z_\lambda\| > M_2$ for $\lambda \in (\lambda_{k+1}, \lambda_{k+1} + \varepsilon)$ (see 37). Also, by Lemma 2 and Theorem 5.2, z_λ is not a critical point of mountain pass type or a solution that changes sign exactly once. Thus, $z_\lambda \equiv u_5 \notin \{0, u_1, \ldots, u_4\}$.

From (f1), if B is a sufficiently large ball, $d(\nabla J_\lambda, B, 0) = (-1)^{k+j}$. Letting $\eta > 0$ be such that the only critical point of ∇J_λ in the ball centered at u_i with radius η is u_i, by the excision property of the Leray Schauder degree we have

$$\begin{aligned} d\left(\nabla J_\lambda, B - \cup_{i=0}^5 B(u_i,\eta)\right) &= d(\nabla J_\lambda, B, 0) - \sum_{i=0}^5 d\left(\nabla J_\lambda, \cup_{k=0}^5 B(u_i,\eta), 0\right) \\ &= (-1)^{k+j} - \sum_{i=0}^5 d(\nabla J_\lambda, B(u_i,\eta), 0) \\ &= (-1)^{k+j} - \left(1 + (-1) + (-1) + 1 + (-1)^k + (-1)^{k+j}\right) \\ &= (-1)^k. \end{aligned} \quad (56)$$

Here, we have also used that J_λ has a strict local minimum at $u_0 = 0$ and (4). Hence, by the existence property of the Leray-Schauder degree, J has a seventh critical point $u_6 \in B - \cup_{k=0}^5 B(u_i,\eta)$. This proves Theorem 1.1.

For $\lambda = \lambda_{k+1}$ the solutions $0, u_+, u_-, v_\lambda, w_\lambda$ persist, which proves Corollary 1.2. □

7 Non-isolated solutions

Double checking the above developments, one notes that the existence of a positive solution, a negative solution, and a solution that changes sign exactly once is independent of the assumption that the solutions are non-isolated. Also the Morse index $k + j$ solution may be chosen larger in norm that those that change sign exactly once as they are uniformly bounded while those of Morse index $k + j$ tend to ∞ as $\lambda > \lambda_{k+1}$ approaches λ_{k+1}. Thus, we conclude that when we assume the solutions to (1) to be non-isolated not only it has infinitely many solutions but also at least five of them are geometrically different: 0, a positive solution, a negative solution, a solution that changes sign exactly once, and a Morse index $k + j$ solution.

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