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Almost automorphy for abstract neutral differential equations via control theory

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Abstract In this paper we study the existence of almost automorphic solutions for a class of linear neutral functional differential equations with finite delay and values in a Banach space. We show that the existence of an almost automorphic mild solution is related to the approximate controllability of a distributed control system. We applied our results to establish the existence of an almost automorphic solution for a neutral wave equation with delay.

Keywords Neutral functional differential equations · Equations in abstract spaces · Almost automorphic functions · Semigroups of operators · Compact operators · Controllability of systems

Mathematics Subject Classification (2000) Primary 34K14 · 34K40; Secondary 34K30 · 35R10 · 47D06

1 Introduction

Motivated by the fact that abstract neutral functional differential equations (abbreviated ANFDE) arise in many areas of applied mathematics, this type of equations has received much attention in recent years [9,12,17,22–24,38]. In particular, the problem of the existence of almost periodic and almost automorphic solutions has been considered by several authors. We refer the reader to the papers [1,5,6,13–16,19,20,27–29,32] and references

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listed therein for recent information on this subject. Our objective in this paper is to establish the existence of almost automorphic mild solutions for a class of first order linear ANFDEs.

The notion of almost automorphic function was introduced by Bochner in [8] to avoid some assumptions of uniform convergence that arise when using almost periodic functions. From that time the theory of almost automorphic functions has been studied by numerous authors. In connection with differential equations, the great importance from both the applied and theoretical points of view of the existence of periodic solutions is well known. However, either because models are only an approximation of reality or due to numerical errors, in practice it is impossible to verify whether a solution is exactly periodic. The concept of almost automorphic function allows relaxing some assumptions to obtain solutions that have properties similar to those of a periodic function. The reader can see Definition 2.2 for the concept of almost automorphic function.

Throughout this work, we denote by X a complex Banach space endowed with a norm $\|\cdot\|$. Henceforth we represent by A the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X, and C stands for the space of continuous functions $C([-r,0];X)$, $r>0$, provided with the norm of uniform convergence. We will be concerned with the existence of almost automorphic solutions to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}D(x_t) = AD(x_t) + L(x_t) + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $x(t) \in X$, the function $x_t : [-r,0] \to X$, that denotes the segment of $x(\cdot)$ at t, is given by $x_t(\theta) = x(t+\theta)$, $D, L : C \to X$ are bounded linear maps, and $f : \mathbb{R} \to X$ is an appropriate function. This equation, and also the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}D(x_t) = AD(x_t) + L(x_t) + f(t), \quad t \geq \sigma, \tag{1.2}$$

$$x_\sigma = \varphi \tag{1.3}$$

for $\sigma \in \mathbb{R}$, have been studied by several authors. Equation (1.1) and problem (1.2)–(1.3) arise in references [25,26,39,40]. From these early works, both Eq. (1.1) and the initial value problem (1.2)–(1.3) have been studied by several authors. Some papers are devoted to establishing general properties such as well posed of the equation, the existence of solutions, properties of the solution operator, etc. [2,4,33], while other work is oriented at establishing specific properties like the existence of almost periodic and almost automorphic solutions [2,3,5,6,15,20,27]. In addition, some papers consider equations with finite delay [2–6,20] while others are concerned with equations with infinite delay, both in phase spaces defined axiomatically [15,27] and in concrete function spaces [33]. A usual condition to obtain these results is that the semigroup generated by A is uniformly exponentially stable and immediately compact.

Our aim in this paper is to use the theory developed in [2–4] and also the well establish mathematical control theory for distributed control systems to show that for a wide class of equations of type (1.1) there exist almost automorphic mild solutions though the semigroup $T(\cdot)$ is not uniformly exponentially stable and not compact.

As a model we consider the wave equation with delay

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}(w(\xi,t) - d_0 w(\xi,t-r)) + \beta\frac{\partial}{\partial t}(w(\xi,t) - d_0 w(\xi,t-r))\\ &\quad = \frac{\partial^2}{\partial \xi^2}(w(\xi,t) - d_0 w(\xi,t-r)) + \int_0^\pi\int_{-r}^0 p(\xi,\eta,\theta)\frac{\partial}{\partial t}w(\eta,t+\theta)\mathrm{d}\theta\mathrm{d}\eta + b(\xi,t),\end{aligned} \tag{1.4}$$

$$w(0, t) = w(\pi, t) = 0, \tag{1.5}$$

for $0 \le \xi \le \pi$ and $t \in \mathbb{R}$, where $\beta, d_0 \in \mathbb{R}$, and the scalar functions $p : [0, \pi] \times [0, \pi] \times [-r, 0] \to \mathbb{R}$ and $b : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ satisfy appropriate conditions.

Throughout this work we denote by $\mathcal{L}(X)$ the Banach algebra of bounded linear operators defined on X and by X^* the dual space of X. For a linear operator A with domain $D(A)$ and range $\mathcal{R}(A)$ in X, we represent by $\sigma(A)$ (resp. $\sigma_p(A)$, $\rho(A)$) the spectrum (resp. point spectrum, resolvent set) of A. For $\lambda \in \rho(A)$ we denote by $R(\lambda, A) = (\lambda I - A)^{-1}$ the resolvent operator of A [21].

For the necessary concepts related with the abstract Cauchy problem and the theory of strongly continuous semigroup of operators we refer to Engel and Nagel [18] and Pazy [35]. We only mention here a few concepts and results directly related to our development. Let $(G(t))_{t\ge 0}$ be a strongly continuous semigroup defined on a Banach space X with infinitesimal generator A_G. We say that $(G(t))_{t\ge 0}$ is strongly stable if $G(t)x \to 0$, $t \to \infty$, for all $x \in X$, and we say that $(G(t))_{t\ge 0}$ is uniformly exponentially stable if $\|G(t)\| \to 0$, $t \to \infty$. Moreover, we employ the terminology and notations for spectral bound $s(A_G)$, growth bound $\omega_0(G)$ and essential growth bound $\omega_{\text{ess}}(G)$ from [18]. Specifically, $s(A_G) = \sup\{Re(\lambda) : \lambda \in \sigma(A_G)\}$; $\omega_0(G) = \lim_{t\to\infty} \frac{\ln \|G(t)\|}{t}$ and $\omega_{\text{ess}}(G) = \lim_{t\to\infty} \frac{\ln \|G(t)\|_{\text{ess}}}{t}$, where the symbol $\|\cdot\|_{\text{ess}}$ denotes the essential norm of an operator. Consequently, in terms of these notations, semigroup $(G(t))_{t\ge 0}$ is uniformly exponentially stable if, and only if, $\omega_0(G) < 0$.

For completeness we also regard here that a strongly continuous semigroup $(G(t))_{t\ge 0}$ is said to be immediately compact if $G(t)$ is a compact operator for all $t > 0$ and that $(G(t))_{t\ge 0}$ is said to be quasi-compact if there is $t_0 > 0$ and a compact operator R such that $\|G(t_0) - R\| < 1$. We collect in the following lemma two fundamental results [18, Corollary IV.2.11, Proposition V.3.5] for our further development.

Lemma 1.1 *The following conditions are fulfilled.*

(i) *The semigroup $(G(t))_{t\ge 0}$ is quasi-compact if and only if $\omega_{\text{ess}}(G) < 0$.*
(ii) $\omega_0(G) = \max\{\omega_{\text{ess}}(G), s(A_G)\}$.

This paper is organized as follows. In Sect. 2 we have collected some technical results about spectral properties of ANFDE of type (1.2), most of which are included in [2–4]; in Sect. 3 we apply these properties to study the existence of almost automorphic solutions of ANFDE of type (1.1), and in the last Section 4 we have included an application of our results to the neutral wave equation.

2 Preliminaries

Throughout the rest of this paper $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a semigroup of bounded linear operators $(T(t))_{t\ge 0}$ on X, and $L, D : C \to X$ are bounded linear maps. We assume that D is defined by

$$D(\varphi) = \varphi(0) - \int_{-r}^{0} [d_\theta N(\theta)]\varphi(\theta),$$

where $N : [-r, 0] \to \mathcal{L}(X)$ is a map of bounded variation and non-atomic at zero (see [24] for the terminology). Moreover, $f : \mathbb{R} \to X$ is a continuous function. We refer to [2–4] for the basic properties of the Eq. (1.2). We only mention here that problem (1.2)–(1.3) has a unique mild solution $x = x(\cdot, \sigma, \varphi, f)$. This means that $x : [\sigma - r, \infty) \to X$ is a continuous function that verifies (1.3) and the restriction of $x(\cdot)$ on $[\sigma, \infty)$ satisfies the integral equation

$$Dx_t = T(t-\sigma)D\varphi + \int_{\sigma}^{t} T(t-s)(L(x_s) + f(s))\mathrm{d}s, \quad t \geq \sigma. \tag{2.1}$$

In particular, if $x(\cdot, \varphi)$ denotes the mild solution of the homogeneous problem

$$\frac{\mathrm{d}}{\mathrm{d}t} D(x_t) = AD(x_t) + L(x_t), \ t \geq 0, \tag{2.2}$$

$$x_0 = \varphi, \tag{2.3}$$

then the solution operator $(V(t))_{t\geq 0}$ defined by $V(t)\varphi = x_t(\cdot, \varphi)$ is a strongly continuous semigroup of bounded linear operators on C (see [2]). We will represent by A_V its infinitesimal generator. Moreover, it is well known that $(V(t))_{t\geq 0}$ satisfies the following translation property.

Lemma 2.1 [2] *Under the preceding conditions,*

$$[V(t)\varphi](\theta) = \begin{cases} [V(t+\theta)\varphi](0), & t+\theta \geq 0, \\ \varphi(t+\theta), & t+\theta \leq 0. \end{cases}$$

To establish the variation of constants formula for (1.2), we consider the space $X \times C$ provided with the product norm, and the operator $\widetilde{A}_V$ defined on

$$D(\widetilde{A}_V) = \{(z, \varphi) : z = \varphi(0), \varphi \in C^1([-r, 0], X), D(\varphi) \in D(A)\}$$

by

$$\widetilde{A}_V(z, \varphi) = (AD(\varphi) + L(\varphi) - D(\varphi'), \varphi').$$

We also define $J : X \to X \times C$ by $J(z) = (z, 0)$. We need to consider the following condition.

(H1) For every $z \in D(A)$ and $\lambda \in \mathbb{C}$, $D(e^{\lambda\theta} z) \in D(A)$.

The following result has been established in [2, Theorem 16].

Lemma 2.2 *Assume that (H1) holds. Then the mild solution of* (1.2)–(1.3) *is given by*

$$x_t = V(t-\sigma)\varphi + \lim_{\lambda\to\infty} \int_{\sigma}^{t} V(t-s)\lambda R(\lambda, \widetilde{A}_V)(Jf(s))\mathrm{d}s, \ t \geq \sigma.$$

We denote by $(W(t))_{t\geq 0}$ the solution operator corresponding to $L = 0$ in (2.2). It is clear that $(W(t))_{t\geq 0}$ is given by

$$[W(t)\varphi](\theta) = \begin{cases} v(t+\theta), & -t \leq \theta \leq 0, \\ \varphi(t+\theta), & -r \leq \theta < -t, \end{cases}$$

where $v(\cdot)$ satisfies the problem

$$Dv_t = T(t)D\varphi, \ t \geq 0,$$
$$v_0 = \varphi.$$

System (2.2) is said to be (asymptotically) stable if the semigroup $(V(t))_{t\geq 0}$ is uniformly exponentially stable. Consequently, it follows from Lemma 1.1 that the study of the asymptotic stability of system (2.2) is reduced to the study of the spectral properties of the solution semigroup $(V(t))_{t\geq 0}$.

Definition 2.1 Operator D is said to be stable if the solution of the problem

$$\begin{aligned} Dy_t &= 0, \\ y_0 &= \varphi \in \ker(D), \end{aligned}$$

is exponentially stable.

To simplify the text we introduce the following condition.
(H2) Let $D_0 : C \to X$ be the linear operator given by

$$D_0(\varphi) = \int_{-r}^{0} [d_\theta N(\theta)]\varphi(\theta).$$

Then the spectral radius $r_e(D_0) < 1$.

The following property is an immediate consequence of the definitions.

Lemma 2.3 *Assume that condition (H2) holds. Then D is stable. If further, the semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable, then the semigroup $(W(t))_{t\geq 0}$ is also uniformly exponentially stable.*

2.1 Asymptotic behavior of the solution semigroup

We are in a position to establish the first result about asymptotic behavior of the solution semigroup.

Theorem 2.1 *Assume that condition (H2) holds, the semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable and that the operator $T(t)L : C \to X$ is compact for all $t > 0$. Then the semigroup $(V(t))_{t\geq 0}$ is quasi-compact.*

Proof We define the operator $U(t) : C \to C$ for $t \geq 0$ by

$$[U(t)\varphi](\theta) = \begin{cases} \displaystyle\int_0^{t+\theta} T(t+\theta-s)L(V(s)\varphi)\mathrm{d}s, & -t \leq \theta \leq 0, \\ 0, & -r \leq \theta < -t. \end{cases}$$

It is clear from (2.1) that

$$V(t) = W(t) + U(t), \quad t \geq 0,$$

which implies that $U(t)$ is a bounded linear operator. It follows from Lemma 2.3 that the semigroup $(W(t))_{t\geq 0}$ is uniformly exponentially stable. Moreover, from [30, Theorem 1] we can assert that $U(t)$ is a compact operator. □

Remark 2.1 There are many interesting situations in which the semigroup $(T(t))_{t\geq 0}$ is not compact but the operator $T(t)L : C \to X$ is compact for $t > 0$. Next we mention two general cases:

(i) The operator $L : C \to X$ is compact. For instance, $L(\varphi) = \sum_{i=1}^{k} A_i \varphi(-r_i)$, where $A_i : X \to X$, $i = 1, \dots, k$, are compact linear operators, or

$$L(\varphi) = \int_{-r}^{0} \eta(\theta)\varphi(\theta)\mathrm{d}\theta,$$

where $\eta : [-r, 0] \to \mathcal{L}(X)$ is a map continuous for the norm of operators and $\eta(\theta)$ is a compact operator for each $-r \leq \theta \leq 0$. As a matter of fact, this property is verified under more general conditions in η.

(ii) A more general situation is the following. Assume that there exists a topological decomposition of $X = X_0 \oplus X_1$, where X_i are invariant spaces under $T(t)$, and X_1 has finite dimension. Let P_0 be the projection on X_0 with kernel X_1. If $T(t)P_0L$ is compact, then the product $T(t)L$ is also compact.

Combining Theorem 2.1 with Theorem V.3.7 in [18] we can establish the following property of asymptotic behavior for the solution semigroup associated to the homogeneous problem (2.2)–(2.3).

Corollary 2.1 *Assume that condition (H2) holds, the semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable and that the operator $T(t)L : C \to X$ is compact for all $t > 0$. Then the semigroup $(V(t))_{t\geq 0}$ is uniformly exponentially stable if and only if* $\sup Re\sigma_p(A_V) < 0$.

Remark 2.2 Assume that semigroup $(V(t))_{t\geq 0}$ is quasi-compact. In this case the set $\Lambda = \{\lambda \in \sigma(A_V) : Re(\lambda) \geq 0\}$ is finite and consists of poles of $R(\cdot, A_V)$ with finite algebraic multiplicity [18, Theorem V.3.7]. Therefore, the space C is decomposed as

$$C = P_\Lambda \oplus Q_\Lambda, \tag{2.4}$$

where P_Λ and Q_Λ are spaces invariant under $V(t)$ and the space P_Λ is the range of the spectral projection Π^P corresponding to Λ. Consequently, P_Λ consists of the generalized eigenvectors corresponding to the eigenvalues $\lambda_i \in \Lambda$. Specifically, if $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$, then

$$P_\Lambda = \bigoplus_{i=1}^{m} \ker(\lambda_i I - A_V)^{k_i} \tag{2.5}$$

for certain $k_i \in \mathbb{N}$. We denote by $V^P(t)$, (respectively, $V^Q(t)$) the restriction of $V(t)$ on P_Λ (respectively, on Q_Λ). Similarly, A_V^P and A_V^Q represent the restrictions of A_V on P_Λ and Q_Λ, respectively. Since P_Λ is a space of finite dimension d, the semigroup $(V^P(t))_{t\geq 0}$ is uniformly continuous and A_V^P is a bounded linear operator defined on P_Λ. Let $\varphi^1, \varphi^2, \dots, \varphi^d$ be a basis of P_Λ. We set $\Phi = (\varphi^1, \varphi^2, \dots, \varphi^d)$. It has been proved in [3, Theorem 11] that there is a $d \times d$ matrix G such that $V^P(t)\Phi = \Phi e^{Gt}$, for $t \geq 0$, and $\sigma_p(G) = \Lambda$. Let Ψ be the dual basis of Φ associated with decomposition (2.4), so that $\langle \Psi, \Phi \rangle = I$. Under these conditions, it has been established in [3, Theorem 16] that there exists $x^* = col(x_1^*, x_2^*, \dots, x_d^*) \in X^{*d}$ such that the projection $x_t^P = \Pi^P x_t$ on P_Λ of the solution of (1.1) is given by $x_t^P = \Phi z(t)$, where the d-vector $z(t) = \langle \Psi, x_t \rangle$ satisfies the ordinary differential equation

$$z'(t) = Gz(t) + \langle x^*, f(t) \rangle, \ t \in \mathbb{R}. \tag{2.6}$$

2.2 Almost automorphic functions

On the other hand, in this work we employ the concept of almost automorphic functions in the sense of Bochner [8].

Definition 2.2 A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for every sequence of real numbers $(s_n')_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s_n')_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\to\infty} f(t+s_n)$ is well defined for each $t \in \mathbb{R}$, and $f(t) = \lim_{n\to\infty} g(t-s_n)$ for each $t \in \mathbb{R}$.

For the properties of almost automorphic functions we refer the reader to [34]. We only mention here that the range of an almost automorphic function is a relatively compact set. Hence, the function g involved in the Definition 2.2 is bounded and measurable. We denote by $AA(X)$ the space consisting of all almost automorphic functions with values in X. The space $AA(X)$ endowed with the supremum norm is a Banach space.

To complete this section, we establish formally the following concept.

Definition 2.3 A continuous function $x : \mathbb{R} \to X$ is said to be a mild solution of the Eq. (1.1) if for each $\sigma \in \mathbb{R}$ the restriction $x : [\sigma, \infty) \to X$ is a mild solution of the Eq. (1.2) with initial condition x_σ at $t = \sigma$.

3 Existence of almost automorphic solutions

In this section we turn our attention to the existence of almost automorphic solutions of Eq. (1.1). Throughout this section we assume that A, D and L satisfy the general conditions considered in Sect. 2 and that $f : \mathbb{R} \to X$ is a continuous function.

If the semigroup $(V(t))_{t\geq 0}$ is quasi-compact, we can apply the properties and notations introduced in Remark 2.2 in relation to the homogeneous Eq. (2.2). In particular, decomposition (2.5) does not depend on the compactness of the semigroup $(T(t))_{t\geq 0}$. In what follows we set $\Lambda = \{\lambda \in \sigma_p(A_V) : Re(\lambda) \geq 0\}$. Moreover, since in our case A generates a strongly continuous semigroup on X, the concept of mild solution as defined in (2.1) (see also [4]), and the concept of integral solution as defined in [2,3] coincide. Therefore, we can use the results in [3] for the concept of mild solution. The following result has been established in [3, Theorem 16]. In this statement we use the notations introduced in Remark 2.2.

Lemma 3.1 *Assume that conditions (H1) and (H2) hold. Assume further that the semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable and the operator $T(t)L$ is compact for $t > 0$. If $x : \mathbb{R} \to X$ is a mild solution of* (1.1) *on $\mathbb{R}$, then $z(t) = \langle \Psi, x_t\rangle$ is a solution of the differential equation*

$$z'(t) = Gz(t) + \langle x^*, f(t)\rangle, \ t \in \mathbb{R}. \tag{3.1}$$

Conversely, if f is a bounded function on $\mathbb{R}$ and $z(\cdot)$ is a solution of (3.1) *on $\mathbb{R}$, then the function $x : \mathbb{R} \to X$ given by*

$$x(t) = \left[\Phi z(t) + \lim_{n\to\infty} \int_{-\infty}^{t} V^Q(t-s)\Pi^Q n R(n, \widetilde{A}_V)(Jf(s))\, ds \right](0), \ t \in \mathbb{R}, \tag{3.2}$$

is a mild solution of (1.1) *on $\mathbb{R}$.*

Now we are in a position to establish the main result of this work.

Theorem 3.1 *Assume that conditions (H1) and (H2) hold. Assume further that the semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable and the operator $T(t)L$ is compact for $t > 0$. Let $f : \mathbb{R} \to X$ be an almost automorphic function. If Eq.* (1.1) *has a bounded mild solution on $\mathbb{R}^+$, then it has an almost automorphic mild solution.*

Proof Let $x(\cdot)$ be the mild solution of (1.1) given by (3.2). Since $z(t)$ satisfies Eq. (3.1) and $\sigma_p(G) \subseteq \{\lambda \in \mathbb{C} : Re(\lambda) \geq 0\}$, then $z(\cdot)$ is bounded on $\mathbb{R}$. It follows from [34, Theorem 2.4] that $z(\cdot)$ is an almost automorphic function. On the other hand, we denote

$$Y(t) = \lim_{n\to\infty} \int_{-\infty}^{t} V^Q(t-s)\Pi^Q nR(n, \widetilde{A}_V)(Jf(s))\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{3.3}$$

then we can write

$$Y(t) = \lim_{n\to\infty} \int_{0}^{\infty} V^Q(s)\Pi^Q nR(n, \widetilde{A}_V)(Jf(t-s))\,\mathrm{d}s, \quad t \in \mathbb{R}.$$

We denote

$$Y^n(t) = \int_{-\infty}^{t} V^Q(t-s)\Pi^Q nR(n, \widetilde{A}_V)(Jf(s))\,\mathrm{d}s, \quad t \in \mathbb{R} \quad n \in \mathbb{N}.$$

Let (s_k') be an arbitrary sequence of real numbers. Since f is almost automorphic, there exists a subsequence $(s_k) \subset (s_k')$ such that $\lim_{k\to\infty} f(t+s_k) = g(t)$ and $\lim_{k\to\infty} g(t-s_k) = f(t)$ pointwise on $\mathbb{R}$. Consequently, if we fix $t \in \mathbb{R}$, we can affirm that $\lim_{k\to\infty} f(t-s+s_k) = g(t-s)$ for each $s \in \mathbb{R}$. Since X_0 is a bounded linear map, we also have that $\lim_{k\to\infty} X_0 f(t-s+s_k) = X_0 g(t-s)$, and

$$Y^n(t+s_k) = \int_{0}^{\infty} V^Q(s)\Pi^Q nR(n, \widetilde{A}_V)(Jf(t-s+s_k))\,\mathrm{d}s.$$

Moreover, since $(V^Q(t))_{t\geq 0}$ is a uniformly exponentially stable semigroup, we infer that there exist constants $\widetilde{M}, \alpha > 0$ such that

$$\|V^Q(s)\Pi^Q(Jf(t-s+s_k))\| \leq \widetilde{M}e^{-\alpha s} \sup_{t\in\mathbb{R}} \|f(t)\|$$

and the function defined by the right hand side of the above inequality is integrable on $[0, \infty)$. Now, applying the Lebesgue Dominate Convergence theorem ([34, Theorem 1.9]), we get

$$\lim_{k\to\infty} Y^n(t+s_k) = \int_{0}^{\infty} V^Q(s)\Pi^Q nR(n, \widetilde{A}_V)(Jg(t-s))\,\mathrm{d}s := X^n(t)$$

for each $t \in \mathbb{R}$. We can apply the same argument to obtain that

$$\lim_{k\to\infty} X^n(t-s_k) = Y^n(t),$$

for each $t \in \mathbb{R}$, which shows that the function $Y^n(\cdot)$ is almost automorphic.

On the other hand, using again that the semigroup $(V^Q(t))_{t\geq 0}$ is uniformly exponentially stable and the range of f is relatively compact [34, Theorem 1.31], we can assert that the convergence in (3.3) is uniform for $t \in \mathbb{R}$, which implies that $Y(\cdot)$ is also almost automorphic. □

We next state several immediate consequences of Theorem 3.1. In these statements we consider properties of semigroups which arise frequently in applications.

Corollary 3.1 *Assume that conditions (H1) and (H2) hold. Assume that $(T(t))_{t\geq 0}$ is a group uniformly exponentially stable and that the operator L is compact. Let $f:\mathbb{R}\to X$ be an almost automorphic function. If Eq. (1.1) has a bounded mild solution on $\mathbb{R}^+$, then it has an almost automorphic mild solution.*

Corollary 3.2 *Assume that conditions (H1) and (H2) hold. Assume that the semigroup $(T(t))_{t\geq 0}$ is eventually norm continuous, that $\sup Re(\sigma(A)) < 0$, and that the operator $T(t)L$ is compact for $t > 0$. Let $f:\mathbb{R}\to X$ be an almost automorphic function. If Eq. (1.1) has a bounded mild solution on $\mathbb{R}^+$, then it has an almost automorphic mild solution.*

Corollary 3.3 *Assume that conditions (H1) and (H2) hold. Assume that A is a bounded linear operator such that $\sup Re(\sigma(A)) < 0$, and that the operator L is compact. Let $f:\mathbb{R}\to X$ be an almost automorphic function. If Eq. (1.1) has a bounded mild solution on $\mathbb{R}^+$, then it has an almost automorphic mild solution.*

The condition that the semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable is somewhat demanding. However, we can apply the well established mathematical control theory to avoid this condition. Specifically, it is well known that there are many important non-delayed distributed control systems modeled by the equation

$$x'(t) = Ax(t) + Bu(t), \tag{3.4}$$

where $B:\mathbb{C}^m\to X$ is a linear map, which are stabilizable. We refer to [10,11,37] for a discussion of this subject. Briefly, this means that there is a feedback control $u = Fx$, where $F: X\to\mathbb{C}^m$ is a bounded linear map, such that the system

$$x'(t) = (A + BF)x(t)$$

is uniformly asymptotically stable or, equivalently, the semigroup generated by $A + BF$ is uniformly exponentially stable. Since B is a compact map, then $K = BF : X \to X$ is also a compact linear operator. We summarize this property in the following concept.

Definition 3.1 The semigroup $(T(t))_{t\geq 0}$ is said to be compact-stabilizable if there exists a compact linear operator $K : X \to X$ such that the semigroup generated by $A + K$ is uniformly exponentially stable.

Corollary 3.4 *Assume that conditions (H1) and (H2) hold. Assume further that the semigroup $T(\cdot)$ is compact-stabilizable and that the operator $T(t)L$ is compact for $t > 0$. Let $f:\mathbb{R}\to X$ be an almost automorphic function. If Eq. (1.1) has a bounded mild solution on $\mathbb{R}^+$, then it has an almost automorphic mild solution.*

Proof It follows from our hypotheses that there exists a compact linear operator $K : X \to X$ such that the semigroup $(\widetilde{T}(t))_{t\geq 0}$ generated by $A_1 = A + K$ is uniformly exponentially stable. Equation (1.1) can be written

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Dx_t &= (A+K)Dx_t + L_1(x_t) + f(t)\\ &= A_1 Dx_t + L_1(x_t) + f(t),\end{aligned} \tag{3.5}$$

where the operator $L_1 : C \to X$ is given by

$$L_1(\varphi) = L(\varphi) - KD(\varphi).$$

Moreover,

$$\widetilde{T}(t)L_1(\varphi) = \widetilde{T}(t)L(\varphi) - \widetilde{T}(t)KD(\varphi). \tag{3.6}$$

Since K is a compact operator, we can assert that the operator defined by the second term on the right hand side of (3.6) defines a compact operator. Similarly, it is well known [35] that

$$\widetilde{T}(t)x = T(t)x + \int_0^t \widetilde{T}(t-s)KT(s)x\mathrm{d}s$$

and arguing as in the proof of Theorem 2.1, we get that $\widetilde{T}(t)L$ is a compact operator. Consequently, $\widetilde{T}(t)L_1$ is a compact operator, and we can affirm that Eq. (3.5) satisfies the hypotheses of Theorem 3.1. □

Combining this result with the stabilizability criteria established in [10,37,36], we can present some results for the existence of almost automorphic solutions of Eq. (1.1) in terms of the controllability of the system (3.4). The system (3.4) is said to be approximately controllable in finite time if for every $x_1 \in X$ and $\varepsilon > 0$ there exist $t_1 > 0$ and a control function $u \in L^1([0,t_1],\mathbb{C}^m)$ such that $\|x(t_1) - x_1\| \leq \varepsilon$, where $x(\cdot)$ is the mild solution of (3.4) with initial condition $x(0) = 0$. In [10,31] the reader can find criteria for the approximate controllability of special classes of systems of type (3.4).

In the next result we assume that there is a topological decomposition $X = X_0 \oplus X_1$, where X_i are invariant subspaces under A, and X_1 is a finite dimensional space. Let $(T_i(t))_{t\geq 0}$ be the restriction of the semigroup $(T(t))_{t\geq 0}$ on X_i for $i = 0, 1$. Combining the Corollary 3.4 with [11, Corollary 3.33] we obtain the following result.

Corollary 3.5 *Assume that conditions (H1) and (H2) hold. Let $f : \mathbb{R} \to X$ be an almost automorphic function. Assume further that the following conditions hold:*

(a) *The semigroup $(T_0(t))_{t\geq 0}$ is uniformly exponentially stable.*
(b) *System (3.4) is approximately controllable in finite time.*
(c) *The operator $T(t)L$ is compact for $t > 0$.*
(d) *The Eq. (1.1) has a bounded mild solution on $\mathbb{R}^+$.*

Then the Eq. (1.1) has an almost automorphic mild solution.

In [10,37,36] the reader can find many systems that satisfy the conditions considered in the statement of Corollary 3.5. Similarly, combining Corollary 3.4 with the results in [7], we get the following consequence of the controllability.

Corollary 3.6 *Assume that X is a Hilbert space, $(T(t))_{t\geq 0}$ is a contraction semigroup such that $T(t_0)$ is compact for some $t_0 > 0$ and conditions (H1) and (H2) hold. Let $f : \mathbb{R} \to X$ be an almost automorphic function. Assume further that the following conditions are fulfilled:*

(a) *System (3.4) is approximately controllable in finite time.*
(c) *The operator $T(t)L$ is compact for $t > 0$.*
(d) *The Eq. (1.1) has a bounded mild solution on $\mathbb{R}^+$.*

Then the Eq. (1.2) has an almost automorphic mild solution.

Proof It follows from [7, Theorem 3.4.1] that the semigroup $(S(t))_{t\geq 0}$ generated by $A - BB^*$ is strongly stable. Since

$$S(t)x = T(t)x - \int_0^t S(t-s)BB^*T(s)x\mathrm{d}s,$$

we get that $S(t_0)$ is compact. Now using [7, Theorem 1.4.6] we get that $S(t)$ is uniformly stable. Therefore, the semigroup $(T(t))_{t\geq 0}$ is stabilizable, and the assertion is a consequence of Corollary 3.4. □

4 Applications

In this section we apply our results to study the existence of almost automorphic solutions of the abstract neutral wave equation.

Let H be a real Hilbert space and let $r > 0$. We consider the abstract neutral wave equation

$$\frac{d^2}{dt^2}Dx_t + \beta\frac{d}{dt}Dx_t + ADx_t = L\left(\frac{d}{dt}x_t\right) + f(t),\ t \in \mathbb{R}, \tag{4.1}$$

where $x(t) \in H$, $\beta > 0$, A is a positive self adjoint operator with domain $D(A)$ such that

$$\langle Ax, x\rangle \geq k\|x\|^2,\quad \forall x \in D(A),$$

for some constant $k > 0$, $L : C([-r, 0], H) \to H$ is a bounded linear map, and $f : \mathbb{R} \to H$ is a continuous function. We consider $D(A^{1/2})$ endowed with the graph norm, and we will assume that $D : C([-r, 0], H) \to H$ is a bounded linear operator which satisfies the following conditions:

(i) $D(\varphi) = \varphi(0) - D_0(\varphi)$, where $D_0 : C([-r, 0], H) \to H$ is a bounded linear map with $\|D_0\| < 1$ and $D_0 : C([-r, 0], D(A^{1/2})) \to D(A^{1/2})$ is also a bounded linear map with $\|D_0\| < 1$.
(ii) If $x \in D(A)$, then $D_0(e^{\lambda\theta}x) \in D(A)$, and if $x \in D(A^{1/2})$, then $D_0(e^{\lambda\theta}x) \in D(A^{1/2})$.

Introducing the Hilbert space $\mathcal{H} = D(A^{1/2}) \times H$ with inner product

$$\left\langle \begin{bmatrix} x^1 \\ y^1 \end{bmatrix}, \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} \right\rangle = \left\langle A^{1/2}x^1, A^{1/2}x^2\right\rangle + \left\langle y^1, y^2\right\rangle,$$

we can write (4.1) as the first order system

$$\frac{d}{dt}\widetilde{D}w_t = \mathcal{A}\widetilde{D}w_t + \widetilde{L}(w_t) + \widetilde{f}(t),$$

where $w(t) = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} \in \mathcal{H}$, the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ -A & -\beta \end{bmatrix}$ is defined on $D(\mathcal{A}) = D(A) \times D(A^{1/2})$, operators $\widetilde{L}, \widetilde{D} : C([-r, 0], \mathcal{H}) \to \mathcal{H}$ are given in block form by

$$\widetilde{L} = \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix},\quad \widetilde{D} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix},$$

and $\widetilde{f}(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$.

It is known that $\mathcal{A}$ generates a strongly continuous group $(G(t))_{t\geq 0}$ on $\mathcal{H}$. Consequently, the group $(G(t))_{t\geq 0}$ is not compact. Moreover, if $Re(\mu) > 0$, then $\mu \in \rho(-A)$ and $\|R(\mu, -A)\| \leq \frac{C}{|\mu|}$. Hence, for every $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$ we have that $\lambda \in \rho(\mathcal{A})$,

$$(\lambda I - \mathcal{A})^{-1} = \begin{bmatrix} (\lambda + \beta)R(\lambda(\lambda + \beta), -A) & R(\lambda(\lambda + \beta), -A) \\ -AR(\lambda(\lambda + \beta), -A) & \lambda R(\lambda(\lambda + \beta), -A) \end{bmatrix}$$

and $\|(\lambda I - \mathcal{A})^{-1}\| \leq C$, where $C > 0$ is a generic constant. Thus, under the above conditions, it follows from [18, Theorem V.1.11] that $(G(t))_{t\geq 0}$ is uniformly exponentially stable.

The following property is an immediate consequence of Corollary 3.1.

Corollary 4.1 *Under the above conditions, let $f : \mathbb{R} \to H$ be an almost automorphic function. Assume that L is a compact operator, and that Eq.* (4.1) *has a bounded mild solution on $\mathbb{R}^+$. Then Eq.* (4.1) *has an almost automorphic mild solution.*

In particular, if we choose β, d_0, $b(\cdot)$ and $p(\cdot)$ appropriate, we can establish the following result.

Corollary 4.2 *Assume that $\beta > 0$, $|d_0| < 1$, the function $p : [0, \pi] \times [0, \pi] \times [-r, 0] \to \mathbb{R}$ is continuous and $b : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is a function that satisfies the conditions:*

(a) *For each $t \in \mathbb{R}$, the function $b(\cdot, t) \in L^2([0, \pi])$.*
(a) *There exists a positive function $\gamma \in L^2([0, \pi])$ such that $|b(\xi, t)| \leq \gamma(\xi)$ for all $t \in \mathbb{R}$.*
(a) *For each $\xi \in [0, \pi]$, the function $b(\xi, \cdot) : \mathbb{R} \to \mathbb{R}$ is almost automorphic uniformly for $\xi \in [0, \pi]$.*

If problem (1.4)–(1.5) *has a bounded mild solution on $\mathbb{R}^+$, then problem* (1.4)–(1.5) *has an almost automorphic mild solution.*

Proof We set $X = L^2([0, \pi])$. It is immediate from our hypotheses that the function $f : \mathbb{R} \to X$ given by $f(t) = b(\cdot, t)$ is almost automorphic. Besides, it is also clear that $L : C([-r, 0], X) \to X$ defined by $L(\varphi)(\xi) = \int_0^\pi \int_{-r}^0 p(\xi, \eta, \theta)\varphi(\eta, \theta)d\theta d\eta$ is a compact linear operator. On the other hand, the operator A given by $Az = \dfrac{\mathrm{d}^2}{d\xi^2}z(\xi)$ on the domain $D(A) = \{z \in H^2([0, \pi]) : z(0) = z(\pi) = 0\}$ satisfies the conditions considered in Corollary 4.1 and the group generated by $\mathcal{A}$ is uniformly exponentially stable. Finally, the operator $D(\varphi) = \varphi(0) - d_0\varphi(-r)$ satisfies conditions (i) and (ii) introduced at the beginning of this section. Therefore, the assertion is a consequence of Corollary 4.1. □

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