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Existence of minimizers of the Mumford-Shah functional with singular operators and unbounded data

Massimo Fornasier · Riccardo March · Francesco Solombrino

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Abstract We consider the regularization of linear inverse problems by means of the minimization of a functional formed by a term of discrepancy to data and a Mumford-Shah functional term. The discrepancy term penalizes the L^2 distance between a datum and a version of the unknown function which is filtered by means of a non-invertible linear operator. Depending on the type of the involved operator, the resulting variational problem has had several applications: image *deblurring*, or *inverse source problems* in the case of compact operators, and image *inpainting* in the case of suitable local operators, as well as the modeling of *propagation of fracture*. We present counterexamples showing that, despite this regularization, the problem is actually in general ill-posed. We provide, however, existence results of minimizers in a reasonable class of smooth functions out of piecewise Lipschitz discontinuity sets in two dimensions. The compactness arguments we developed to derive the existence results stem from geometrical and regularity properties of domains, interpolation inequalities, and classical compactness arguments in Sobolev spaces.

Keywords Mumford-Shah functional · Inverse problems · Inpainting · Deblurring · Image restoration

Mathematics Subject Classification (2000) 49J99 · 35J70 · 49N60 · 65J22 · 68U10

M. Fornasier
Faculty of Mathematics, Technical University of Munich, Boltzmannstrasse 3,
85748, Garching, Germany
e-mail: massimo.fornasier@ma.tum.de

R. March
Istituto per le Applicazioni del Calcolo "MauroPicone" of C.N.R.,
Via dei Taurini 19, 00185, Rome, Italy
e-mail: r.march@iac.cnr.it

F. Solombrino (✉)
Johann Radon Institute for Computational and Applied Mathematics,
Austrian Academy of Sciences, Altenbergerstrasse 69, 4040, Linz, Austria
e-mail: francesco.solombrino@oeaw.ac.at

1 Introduction

Free-discontinuity problems describe situations where the solution of interest is defined by a function and a lower dimensional set consisting of the discontinuities of the function [20]. Hence, the derivative of the solution is assumed to be a 'small' function almost everywhere except on sets where it concentrates as a singular measure. This is the case, for instance, in crack detection from fracture mechanics or in certain digital image processing problems (e.g., segmentation, denoising, deblurring, inpainting).

On the one hand, one of the best-known examples of free-discontinuity models is provided by the so-called Mumford-Shah functional [32] whose minimizers approximate a given *bounded* datum and are smooth everywhere except on a discontinuity surface. The theory of existence of minimizers in SBV for the classical Mumford-Shah functional has been extensively explored [3]. The proof of existence of minimizers is based on a deep compactness result in SBV by Ambrosio [2], which requires L^∞ boundedness of minimizing sequences, typically obtained by truncation arguments when the datum u_0 is also assumed in L^∞, combined with additional regularity arguments obtained by De Giorgi et al. [21]. The results have been generalized by Leaci [29] also for unbounded datum, but with at least a suitable high integrability. As the introduction of this latter work suggests, the interest for the case of an unbounded datum may come from the general consideration that Mumford-Shah-type models can be regarded as a possible schematization of many problems in mathematical physics other than image processing, in which both *volume* and *surface* energies are present.

On the other hand, it is also of great practical interest to be able to recover functions that are piecewise smooth also from *partial information*, provided, for instance, by suitable linear measurements via a singular operator, i.e., an operator which is not necessarily boundedly invertible. Actually, a Mumford-Shah regularization term is also used in image processing for modeling *inpainting* problems [22]. We analyse specifically these variational problems in Sect. 5. Furthermore, it has been widely used in practice as a regularization method, see, e.g., [6,22,34,35], as an alternative to *total variation minimization* [17,40,42], often with some advantages (see, e.g., Fig. 1 and [6] for a problem of deblurring/deconvolution as also analyzed in Sect. 4 below).

Also the finite-dimensional setting (derived from the discretization of the functional, e.g., by finite differences or finite elements [25,33]) has been recently widely explored. It has been shown in [25] that discrete linear inverse free-discontinuity problems always admit minimal solutions. In the same paper, some interesting segmentation properties of the discrete functional have been enlightened, and it has been proved that global minimizers are always isolated, while local ones can form a continuum. It is worth noticing, however, that although minimizers always exist, their computation is an NP-hard problem [1]. Perhaps this reflects the difficulties one encounters in the infinite-dimensional setting to prove existence of minimizers, as we clarify below, essentially due to lack of coerciveness.

Despite all these contributions in applications and in numerical methods, however, not many analytic results are so far available concerning Mumford-Shah functionals with discrepancy terms involving singular operators and/or unbounded datum, modeling inverse problems, and the mentioned fine techniques [2,21,29] seem not to apply straightforwardly in general. As we clarify in the first part of this paper, without additional constraints, the sole regularization by Mumford-Shah functionals is in fact not sufficient for the existence of minimizers. Perhaps the earliest contribution to the analysis of linear inverse free-discontinuity problems appeared in [39]. In this general setting, it seemed difficult to obtain the L^∞ boundedness required by the compactness result by Ambrosio, and in [39], the authors resumed it simply by *assuming* that the interesting solutions are in balls of L^∞.

Fig. 1 The case of a known (9-pixel horizontal motion) blur kernel. *Top-left*: Corrupted blurred image. *Top-right*: Restoration using total variation regularization. *Bottom-left*: Restoration using Mumford-Shah regularization. *Bottom-right*: Segmentation produced by applying the Mumford-Shah regularization. In this case, Mumford-Shah regularization gives sharper and better visual quality result than total variation minimization. This figure is reproduced from [6] with kind permission of the authors

Moreover, while looking quite natural in the case of image processing, this restriction does not always fit to some other situations of interest where this regularization has been used. This is the case of some inverse problems, where one is simultaneously interested in both the reconstruction of a *density distribution* (which, as a mathematical object, can be in principle a measure) and the extraction of its segmentation properties. For instance, in the recent papers [34] and [35], the authors, who constrain the model to the search for competitors within the set of *piecewise constant* solutions, consider a Mumford-Shah regularization for the inverse problem of determining a mass *density* from X-ray measurements, so that a singular operator T, precisely the Radon transform of the unknown density, appears. A priori a density can be highly concentrated, having a large L^∞ norm, not necessarily estimable a priori. As an example, some issues coming from the planning of surgery are borrowed. In this specific application, it is perhaps still possible to recover an a priori bound on the competitors from empirical considerations. Nevertheless, for being cautious, the authors preferred to keep a more general point of view on the problem, not imposing such a bound, and to recover instead the existence of solutions by adding topological regularity constraint on the partitions of the reference set Ω induced by the discontinuities of the competitors

In the same spirit, also in some works concerning the so-called *inverse gravimetric* problem, stabilizing functionals have been considered, which do not penalize sharp features of the solution and permit reconstruction of non-smooth density functions. We refer, for instance, to the work [10], where the unknown is related to the observed datum by a partial differential equation constraint (in particular, the operator T appearing in our notation is the convolution with a derivative of the Newtonian potential), and, alternatively to our approach, the total variation functional plays the role of the regularizing term. In the opinion of the authors, a good motivation for this approach comes from the fact that geological structures often have sharp contrasts (discontinuities) in properties; therefore, a smooth reconstruction is somewhat incorrect. Actually, this problem falls in a very general and well-known class of inverse problems, the so-called *inverse source* problems (see, e.g. [5]). In these problems, a mass, or heat, or electromagnetic source has to be reconstructed from linear measurements that are usually encoded by a convolution operator. This represents a field where, as mentioned before, regularization terms involving free discontinuities are a natural choice. Moreover, at least from an abstract point of view, allowing for unbounded data and solutions, that may correspond for instance to (approximations of) point sources, is surely meaningful. As a concrete example, one should consider the detection of hidden highly compact astronomical masses (neutron stars, black holes etc.) from gravitational effects [41]. In this case, the mass can be highly concentrated in space and it is not possible to impose a priori a bound on its density magnitude.

This finally leads us to the purpose of the present paper. Its aim is actually two-fold. First of all, in Sect. 2, we show that without imposing further restrictions on the competitors, there is in general no hope of getting existence of a minimizer for a Mumford-Shah functional with singular operators. In particular, Example 2.1 deals with a situation where this kind of regularization has been widely employed, that is, when T is a convolution with an integral kernel.

Secondly, under additional restrictions on the class of admissible discontinuities, which are strongly suggested by our counterexamples as natural constraints, we eventually provide new existence results of minimizers for some linear inverse problems regularized with Mumford-Shah-like functionals. In particular, we consider the two-dimensional case, i.e., $d = 2$, and we seek for such minimizers in the class introduced by Rondi in the recent works [36–38] on inverse crack and conductivity problems. There he introduced a new reasonable class of competitors, formed by smooth functions out of piecewise Lipschitz discontinuity sets. As we want to keep a strong formulation of the problems, differently from previous approaches [39] we address linear problems on L^p and we do not try to resume L^∞ boundedness in order to apply Ambrosio's results. In particular, in our setting, we will not require any high integrability of the datum, hence excluding the direct applicability of Leaci's extensions [29]. Instead, we can resume compactness/coerciveness by exploiting the special structure of the linear measurements, geometrical and regularity properties of domains, interpolation inequalities, and classical compactness arguments in Sobolev spaces $W^{1,p}$. While a precise class of functions (with additional geometrical properties) for being the natural solution space for the Mumford-Shah functional with singular operators is still elusive, our results constitute a first step indicating some necessary restrictions and sufficient properties.

We address two specific classes of singular operators. First of all, we deal with the case where T is a nonlocal compact operator, with the additional requirement of injectivity (this assumption has proved to be crucial for this kind of problems, see, e.g., [35]). Then, we provide a proof of existence of minimizers also for the case of local operators. We stress that, in this second case, the proof is simpler, since a L^p_{loc}-uniform estimate of minimizing sequences (that is, due to the gradient bound, compactness in the Deny-Lions space $L^{1,p}$) is enough to provide lower semicontinuity of the discrepancy term. Nonetheless, the rest of the proof of Theorem 5.2 clarifies that a priori stability assumptions on the discontinuity sets cannot be completely dropped in order to obtain an existence result.

Although the issue is not addressed in the present paper, it could be of some interest to explore whether it is possible to obtain similar existence results for any bounded linear operator, as it holds in finite dimensions [25] where no coerciveness is in fact used. There suitable modifications of the Frank-Wolfe theorem [7], valid presently only in finite dimension, could be exploited. In this latter context, we also stress that our results give a further explanation, perhaps alternative to the one proposed in [25], of the successful use of this regularization in a numerical setting, since actually a finite element discretization reduces the range of competitors to piecewise smooth functions outside of singular sets that preserve the cone property of the complementary domain, as required by the existence results presented in this paper. However, this does not necessarily hold uniformly with respect to the mesh size, reflecting the possible lack of existence of minimizers in infinite-dimensional settings.

Actually, the counterexamples we present in the paper can be interpreted as a final conclusion of this direction of research. Nevertheless, we hope that the analysis, that we carry on throughout the work, could serve for further exploration of linear ill-posed problems with regularization terms involving free discontinuities.

The paper is organized as follows. In Sect. 2, we introduce our notations and we present our counterexamples to the existence of minimizers. Moreover, Example 2.3 clarifies that we cannot expect to recover the nondegeneracy properties, that we need in order to prove an existence result, by trying to regularize the functional with the simple addition of pure geometrical terms involving the mean curvature of the discontinuity set (in the case of the classical Mumford-Shah functional, such a generalization has been studied in [18] and [12]). In Sect. 3, we recall classical notions of regularity of domains with a few relative results, useful for our analysis. We introduce the Rondi's class of admissible discontinuities, which are essentially piecewise Lipschitz continuous sets, with some additional stability properties to avoid degeneracies. We further recall certain interpolation inequalities for Sobolev spaces, involving compact subdomains. We then introduce our variational model for linear inverse free-discontinuity problems, specifically for local singular operators and compact injective operators. Sections 4 and 5 are devoted to the proofs of existence of minimizers for such variational problems in Rondi's classes of solutions. Section 6 collects a few open problems which stem from this analysis and previous results appeared in [25]. We conclude the paper with an Appendix which collects the technical proofs of the interpolation inequalities of Sect. 3.

2 Statement of the model and counterexamples to existence of solutions

2.1 Notation

We denote by $|\cdot|$ and $\langle\cdot,\cdot\rangle$ the usual Euclidean norm and scalar product in $\mathbb{R}^2$. We denote by dist the Euclidean distance in $\mathbb{R}^2$ and by $B_\rho(x_0) \subset \mathbb{R}^2$ the open ball centered at x_0 with radius ρ.

For any subset A of $\mathbb{R}^2$, we denote by $\overline{A}$ the closure of A and by ∂A the topological boundary of A. The diameter of A is defined by $\mathrm{diam}(A) = \sup\{\mathrm{dist}(x, y) : x, y \in A\}$. For any set A, χ_A will be the *characteristic function* of A, that is, $\chi_A(x) = 1$ if $x \in A$, $\chi_A(x) = 0$ if $x \notin A$. Given two sets $A, B \subseteq \mathbb{R}^2$ by $A \subset\subset B$, we mean that $\overline{A}$ is a compact set contained in B. The *Hausdorff distance* between two closed sets C and K is defined as $\mathrm{d}_{\mathcal{H}}(C, K) = \inf\{r > 0 : C \subset (K)_r, K \subset (C)_r\}$, where for any set $A \subseteq \mathbb{R}^2$ and any $r > 0$ we denote

$$(A)_r = \{x \in \mathbb{R}^2 : \mathrm{dist}(x, A) < r\},$$

$\mathrm{dist}(\cdot, A)$ denoting the distance function from the set A.

We denote by $\mathrm{meas}(B)$ the Lebesgue measure of the measurable set $B \subseteq \mathbb{R}^2$, by $\mathcal{H}^1$ the one-dimensional Hausdorff measure in $\mathbb{R}^2$. We will use the standard notation for the Lebesgue and Sobolev spaces L^p and $W^{k,p}$, as well as for the spaces $\mathcal{C}^k$ of k-continuously differentiable functions. For any bounded open set $A \subset \mathbb{R}^2$, we write $L^{1,p}(A)$ as the following Deny-Lions space:

$$L^{1,p}(A) = \left\{u \in L^p_{\mathrm{loc}}(A) : \ \nabla u \in L^p(A; \mathbb{R}^2)\right\}.$$

For the properties of Deny-Lions spaces, we refer the reader to [30]. Let A be a measurable subset of $\mathbb{R}^2$. Given $u \in L^1(A)$, we denote

$$\overline{u}(A) = \frac{1}{\mathrm{meas}(A)} \int_A u(x)\mathrm{d}x.$$

Moreover, if A is bounded, connected, open, and with locally Lipschitz boundary (see Sect. 3.1 below), $1 \le p < \infty$, and if $u \in W^{1,p}(A)$, then there exists a constant $C = C(A)$ such that the following Poincaré-Wirtinger inequality holds: 7.7, pag. 380 in [3]):

$$\|u - \overline{u}(A)\|_{L^p(A)} \le C\|\nabla u\|_{L^p(A)}. \tag{1}$$

Let $L > 0$ be a fixed constant; we say that Γ is an *L-Lipschitz* arc if, up to a rigid transformation,

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 : \ -a/2 \le x \le a/2, \ y = \varphi(x)\right\},$$

where $0 < a$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant bounded by L and such that $\varphi(0) = 0$. The points $(a/2, \varphi(a/2))$ and $(-a/2, \varphi(-a/2))$ will be called the *endpoints* of the arc Γ. For $x_1, x_2, x_3 \in \mathbb{R}^2$, we denote $\widehat{x_1 x_2 x_3}$ the angle formed by the segments $\mathrm{conv}\{x_1, x_2\}$ and $\mathrm{conv}\{x_2, x_3\}$, and for two segments Γ_1 and Γ_2 intersecting at an endpoint V, we denote $\widehat{\Gamma_1 V \Gamma_2}$ their angle formed at V. In the following, Ω denotes a bounded open subset of $\mathbb{R}^2$.

2.2 A Mumford-Shah functional with singular operators in two space dimensions

Assume $1 < p < \infty$. Let $\overline{\mathcal{D}}$ be the domain of pairs (u, K) defined by

$$\overline{\mathcal{D}} = \{(u, K) : u \in W^{1,p}(\Omega \setminus K); \ K \in \mathcal{C}(\Omega)\}, \tag{2}$$

where $\mathcal{C}(\Omega)$ denotes the class of closed subsets of Ω. Let $u_0 \in L^2(\Omega)$ and for K fixed let $T : L^p(\Omega) \to L^2(\Omega)$ be a linear and continuous operator. We define the functional $\mathscr{E} : \overline{\mathcal{D}} \to [0, +\infty]$ as follows:

$$\mathscr{E}(u, K) = \|Tu - u_0\|^2_{L^2(\Omega)} + \lambda \int_{\Omega\setminus K} |\nabla u(x)|^p \mathrm{d}x + \alpha\mathcal{H}^1(K), \tag{3}$$

where λ, α are positive weights. In this paper, we limit our analysis to operators T mapping onto $L^2(\Omega)$, corresponding to discrepancy terms $\|Tu - u_0\|^2_{L^2(\Omega)}$ associated with Gaussian noise in a Baysian description of the inverse problem [19].

Our first goal is to show that without restricting K to a smaller class of closed subsets of Ω with additional geometrical properties, the minimization of (3) over $\overline{\mathcal{D}}$ may be in general an ill-posed problem.

2.3 A counterexample to existence of minimizers

In the following counterexample, we assume that T is a convolution operator. This is a remarkable case from the point of view of the applications (see, for instance [6]). It is worth noticing that ill-posedness happens for a generic choice of the datum u_0 and that this example also applies to the weak formulation of the problem in spaces of functions of bounded variation for our problem, see [3].

Example 2.1 Let $p = 2, \lambda = \alpha = 1$ and assume that, for any $u \in L^2(\Omega)$, Tu is defined as the convolution with a uniformly continuous kernel $\varphi \in L^1(\mathbb{R}^2)$:

$$(Tu)(x) = (\varphi * u)(x) = \int_\Omega u(\xi)\varphi(x-\xi)\mathrm{d}\xi, \quad x \in \Omega.$$

We also require that the convolution kernel φ is symmetric, so that the operator T is a self-adjoint operator from $L^2(\Omega)$ to $L^2(\Omega)$, and that T is injective. By well-known results of functional analysis, these two conditions imply that the range of T is a dense subset of $L^2(\Omega)$. We claim that

$$\inf\{\mathscr{E}(u, K) : (u, K) \in \overline{\mathcal{D}}\} = 0, \tag{4}$$

where $\overline{\mathcal{D}}$ is defined by (2). It is then obvious that, if u_0 is not in the range of the operator T, there is no minimizing pair $(u, K) \in \overline{\mathcal{D}}$.

To prove (4), it suffices to show that for every $v \in L^2(\Omega)$, we have

$$\int_\Omega |(Tv)(x) - u_0(x)|^2 \,\mathrm{d}x \geq \inf \mathscr{E}. \tag{5}$$

We will show actually that (5) holds true for any convolution operator T. In fact, only if we also assume that the range of T is dense, then it suffices to take the infimum of the left-hand side of (5) over all $v \in L^2(\Omega)$ to obtain (4).

To prove this, we fix $v \in L^2(\Omega)$ and we construct a sequence of piecewise constant functions v_k, whose jump set K_k is for every k a finite union of spheres (thus, a closed set), satisfying the following assumptions

$$\begin{aligned} \nabla v_k = 0 \text{ in } \Omega \setminus K_k \\ \mathcal{H}^1(K_k) \to 0 \text{ as } k \to +\infty \\ v_k \rightharpoonup v \text{ weakly}-* \text{ in } M_b(\Omega) \end{aligned} \tag{6}$$

where $M_b(\Omega)$ denotes the space of bounded Radon measures. The construction of such v_k's can be done as follows. First, we can take a sequence of atomic measures μ_k, supported for every k on a finite set of points $P_k := \{x_k^1, ..., x_k^{n_k}\}$ with n_k going to $+\infty$ when $k \to +\infty$, such that

$$\mu_k \rightharpoonup v \text{ weakly}-* \text{ in } M_b(\Omega).$$

Then, we define

$$v_k(x) = \sum_{i=1}^{n_k} \frac{1}{|B_k^i|} \mu_k(\{x_k^i\}) \chi_{B_k^i}(x)\,,$$

where $\mu_k(\{x_k^i\})$ is the (nonzero) value of the measure μ_k at the singleton $\{x_k^i\}$ and B_k^i are disjoint spheres of center x_k^i and radius $r_k^i \leq 2^{-n_k}$.

It is easy to check that (6) holds. Obviously $Tv_k \to Tv$ in the sense of distributions. By uniform continuity of the convolution kernel φ, the sequence Tv_k is equicontinuous; therefore, the Ascoli-Arzelà theorem yields actually that $Tv_k \to Tv$ uniformly in Ω. Since (v_k, K_k) is an admissible pair for $\mathcal{E}$, taking into account (6), we finally get

$$\inf \mathscr{E} \leq \mathscr{E}(v_k) \to \int_\Omega |(Tv)(x) - u_0|^2 \,\mathrm{d}x,$$

as k goes to $+\infty$, so that claim (5) is proved.

Remark 2.2 In the general case where the convolution operator T may be not injective, or not self-adjoint, we observe that (5) implies at least that if a minimizing pair $(\tilde{u}, \tilde{K})$ exists, then $\tilde{K}$ is empty, $\nabla\tilde{u} = 0$ a.e., and $\tilde{u}$ is a constant function. Indeed, if we assume the existence of such a minimizing pair $(\tilde{u}, \tilde{K})$, and in particular $\tilde{u} \in L^2(\Omega)$, from (5) with a trivial majorization we get

$$\inf \mathscr{E} = \mathscr{E}(\tilde{u}, \tilde{K}) \leq \int_\Omega |(T\tilde{u})(x) - u_0(x)|^2 \,\mathrm{d}x \leq \mathcal{E}(\tilde{u}, \tilde{K}).$$

But then $\mathcal{H}^1(\tilde{K}) = 0$ and $\int_{\Omega\setminus\tilde{K}} |\nabla\tilde{u}|^2(x) = 0$, which means that $\tilde{u}$ is a constant.

2.4 Counterexample with additional geometrical terms

As our further analysis will clarify, existence of minimizers for the functional (3) can be achieved imposing some geometrical restrictions to the class of admissible discontinuity sets K. As a further justification to our approach, we briefly show that these restrictions cannot be recovered by minimization of a functional whose geometrical part contains additional energy terms of curvature type. This one is a natural generalization to the Mumford-Shah functional ([12,18]) which excludes the applicability of the previous counterexample.

We confine our analysis to the dimension 2, and for any family C of $W^{2,2}$ curves, we define

$$\mathscr{G}(C) = \alpha \int_C (1 + \kappa^2(\sigma))\,\mathrm{d}\mathcal{H}^1(\sigma) + \beta\,\#P(C), \tag{7}$$

where $\kappa(\sigma)$ is the curvature at any point σ of C, $P(C)$ is the set of the endpoints of the curves in the family C, and α, β are positive parameters. The choice of the endpoints is free, except for the fact that transversal intersections between curves can happen only at endpoints (so, if two curves meet transversally, we are obliged to split them). In [18], a family C is defined to be admissible if for any couple of curves γ_i and γ_j in C

$$\gamma_i(s) = \gamma_j(t) \Rightarrow \dot{\gamma}_i(s) = \dot{\gamma}_j(t) \tag{8}$$

whenever s and t are in the interior of the domain of definition of γ_i and γ_j, respectively. Tangential intersection must be allowed for semicontinuity reasons. Observe that the case

$i = j$ is not excluded in (8), so that also tangential self-intersections are allowed. The presence of the term $\beta \# P(C)$ implies that the value of the functional $\mathscr{G}$ on an admissible family C of curves can depend on the chosen parametrization. To avoid this, denoting by $[C]$ the image of the family C, one can consider

$$\mathscr{G}_0(C) = \inf\{\mathscr{G}(C') : [C'] = [C]\}, \tag{9}$$

and try to minimize the functional

$$\mathscr{F}(u, C) := \int_{\Omega} |(Tu)(x) - u_0(x)|^2 \, dx + \lambda \int_{\Omega \setminus C} |\nabla u|^2(x) \, dx + \mathscr{G}_0(C) \tag{10}$$

over all admissible families C and all $u \in W^{1,2}(\Omega \setminus C)$. In the case where T is the identity operator, existence of minimizers has been shown in [18] (see also [12], where the functional (10) has been studied from the point of view of Γ-convergence relaxation). With these additional terms, it is easy to check that the contribution of the jump sets considered in Example 2.1 explodes, also using for instance squares in place of circles. Still, the following counterexample, which is modeled after the previous one, easily shows that also in this case, we cannot hope for an existence result without imposing some a priori restrictions on the competitors.

Example 2.3 Let $p = 2, \lambda = \alpha = \beta = 1$ and assume that T is a convolution with a uniformly continuous kernel $\varphi \in L^1(\mathbb{R}^2)$. Ω is a smooth connected bounded open set containing 0 as an interior point and we choose a datum $u_0(x) = \mu\varphi(x)$ where the positive parameter μ is chosen in such a way that

$$\min \left\{ \int_{\Omega} |(Tu)(x) - \mu\varphi(x)|^2 \, dx + \int_{\Omega} |\nabla u(x)|^2 \, dx : u \in W^{1,2}(\Omega) \right\} > 2. \tag{11}$$

We remark that this choice is always possible. Indeed by the Poincaré-Wirtinger inequality, it is easy to show that the minimum problem considered in (11) has a solution with nonzero minimum value for every choice of μ. Actually, no injectivity of T is needed here, since, if T annihilates constants, we can take a mean-free minimizing sequence. By the homogeneity of the functional is then possible to choose μ in a way that (11) is satisfied. To stress this choice of the datum, we will use throughout this example the notation $\mathscr{F}_{\mu,\varphi}$ for the functional $\mathscr{F}$ defined by (10) with $u_0(x) = \mu\varphi(x)$.

For every $k \in \mathbb{N}$, we consider $C_k := \gamma_1^k \cup \gamma_2^k$, where γ_1^k is simply the segment $[0, \frac{1}{k}]$ oriented by the horizontal axis (Fig. 2) and $\gamma_2^k := (s, \vartheta_k(s))$ with $0 \le s \le \frac{1}{k}$ and

$$\vartheta_k(s) := \frac{\sin(k\pi s)}{k^2}. \tag{12}$$

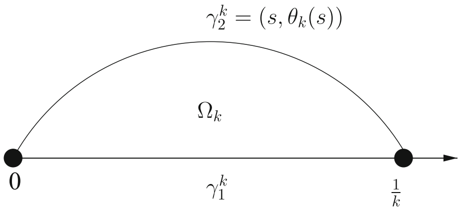

Fig. 2 The construction in Example 2.3

By easy computations,

$$|\dot{\vartheta}_k(s)| \le \frac{\pi}{k} \quad \text{and} \quad |\ddot{\vartheta}_k(s)| \le \pi^2$$

for every $0 \le s \le \frac{1}{k}$. We then immediately get that

$$\lim_{k\to+\infty} \mathscr{G}_0(C_k) = 2, \tag{13}$$

where $\mathscr{G}_0$ is defined by (7) and (9).

We now define Ω_k as the connected component of $\Omega \setminus C_k$ enclosed by C_k and the functions

$$u_k(x) := \begin{cases} \dfrac{\mu}{|\Omega_k|} & \text{if } x \in \Omega_k \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that $u_k \rightharpoonup \mu\delta_0$ weakly$-*$ in the space of bounded Radon measures so that, arguing as in Example 2.1, we have $(Tu_k)(x) \to \mu\varphi(x)$ uniformly with respect to $x \in \Omega$. As $\nabla u_k = 0$ in $\Omega \setminus C_k$ we conclude, using (13), that

$$\inf \mathscr{F}_{\mu,\varphi} \le \lim_{k\to+\infty} \mathscr{F}_{\mu,\varphi}(u_k, C_k) = 2. \tag{14}$$

Now, let us assume by contradiction that a minimizer $(\tilde{u}, \tilde{C})$ with $\tilde{u} \in W^{1,2}(\Omega \setminus \tilde{C})$ exists. If $\tilde{C}$ is the empty family of curves, then $\tilde{u} \in W^{1,2}(\Omega)$ so that (11) and (14) immediately give

$$\mathscr{F}_{\mu,\varphi}(\tilde{u}, \tilde{C}) > 2 \ge \inf \mathscr{F}_{\mu,\varphi}\,,$$

a contradiction. Thus, C must contain at least a curve γ. We define $L(\gamma)$ the *length* of γ and $K(\gamma)$ the *integral of the square of the curvature* along γ.

If γ is not closed, then it has 2 endpoints, so that $\mathscr{G}_0(\tilde{C}) \ge 2$. But then, since $\tilde{u} \in L^2(\Omega)$, using also (14) it must be

$$\mathscr{F}_{\mu,\varphi}(\tilde{u}, \tilde{C}) \ge \int_\Omega |(T\tilde{u})(x) - \mu\varphi(x)|^2 \, \mathrm{d}x + \mathscr{G}_0(\tilde{C}) > 2 \ge \inf \mathscr{F}_{\mu,\varphi}\,,$$

again a contradiction.

If γ is a regular closed curve (that is, its endpoints join smoothly, hence $\#P(\gamma) = 0$), by [8, Lemma 3.1], we have

$$L(\gamma)K(\gamma) \ge 4\pi^2\,,$$

which yields $L(\gamma) + K(\gamma) \ge 2\sqrt{2}\pi$ and finally

$$\mathscr{F}_{\mu,\varphi}(\tilde{u}, \tilde{C}) > \mathscr{G}_0(\tilde{C}) \ge 2\sqrt{2}\pi > 2 \ge \inf \mathscr{F}_{\mu,\varphi}\,,$$

so that the only possibility is that γ is closed but not regular. In this last case, it has en endpoint, then

$$\#P(\tilde{C}) \ge 1. \tag{15}$$

By [18, Section 3], we have

$$L(\gamma)K(\gamma) \ge 1.$$

Using this last inequality, (14), (15), and arguing as before, we find a contradiction, so that the nonexistence of a minimizer is proved.

3 A suitable class of discontinuities

This section, which is also a preliminary one, is devoted to introducing the Rondi's class of admissible discontinuities. This will turn out to be a suitable setting for our minimization problem, which, in the light of the previous fundamental examples, cannot be much relaxed. Eventually, we discuss some useful interpolation inequalities in Sobolev spaces that are needed in order to develop suitable compactness arguments.

3.1 Geometrical properties of domains

For the sake of clarity, we collect here useful notions of regularity of domains. For more details see, e.g., Chapter I, Section 2 in [43].

We say that Ω is *starshaped with respect to a ball* $B_\rho(x_0)$ contained in Ω if Ω is starshaped with respect to each point of this ball, i.e., every point in Ω can be reached by a segment fully included in Ω and originated by any point in $B_\rho(x_0)$.

We say that Ω has the *cone property with respect to a fixed finite cone* $\mathscr{C}$ if each point $x \in \Omega$ is the vertex of a finite cone $\mathscr{C}_x$ contained in Ω and congruent to $\mathscr{C}$.

We say that Ω has *the locally Lipschitz boundary property* if each point $x \in \partial\Omega$ has a neighborhood $\mathscr{U}_x$ such that $\partial\Omega \cap \mathscr{U}_x$ is the graph of a Lipschitz continuous function.

It is well known that domains with the locally Lipschitz boundary property have the cone property, while the converse is not true in general. However, we have the following result due to Gagliardo [27] which states that the cone property "almost" implies the locally Lipschitz boundary property.

Theorem 3.1 (Gagliardo) *Assume that Ω is a bounded open subset of $\mathbb{R}^2$ with the cone property. Then Ω is a finite union of domains with the locally Lipschitz boundary property.*

There are also similar relationships between domains with the cone property and starshaped domains.

Lemma 3.2 *If Ω is the union of an arbitrary family of domains G_α, each starshaped with respect to a ball $B_R(x_\alpha) \subset \Omega$ of a fixed radius $R > 0$, then for each $r < R$ there exists a finite number of domains Ω^ℓ $(1 \le \ell \le M)$ starshaped with respect to balls of radius r, contained in Ω^ℓ, and such that $\Omega = \bigcup_{\ell=1}^{M} \Omega^\ell$.*

Note in particular that a domain Ω with the cone property is necessarily arbitrary union of starshaped domains with respect to balls with a fixed radius. Hence, Ω with the cone property with respect to a cone $\mathscr{C}$ containing a ball $B_R(x)$ is necessarily a *finite* union of starshaped domains with respect to balls of radius r, for $r < R$. The proof of this lemma can be found, e.g., in [30, Section 1.1.9] and it is based on a greedy packing principle. In particular, it is important to notice that the number $M = M(r, R, \mathscr{D})$ of starshaped subdomains Ω^ℓ depends on the radii $r, R > 0$ and $\mathscr{D} = \operatorname{diam}(\Omega)$. In fact, the proof implies the construction of a sequence $B_r(x_1), \ldots, B_r(x_M)$ of balls in Ω with mutual distance of the centers larger than $R - r > 0$.

3.2 Rondi's class of admissible discontinuities

We consider a class of compact sets introduced by Rondi in [38]. For any positive constants $L > 0, \delta > 0$ and $0 < c < 1$, we define the following class $\mathcal{B}(\Omega) = \mathcal{B}(\Omega, L, \delta, c)$ of compact subsets of $\mathbb{R}^2$. We say that $B \in \mathcal{B}(\Omega, L, \delta, c)$ if and only if $B \subset \overline{\Omega}$, there exists a

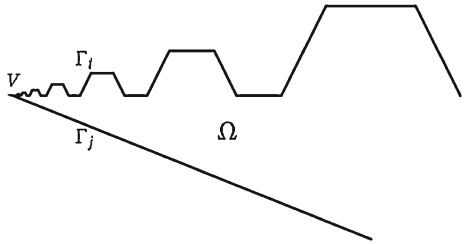

Fig. 3 Example of set a $K \in \mathcal{B}(\Omega)$ which does not allow for the cone property for $\Omega \setminus K$. We depict two arcs Γ_i and Γ_j with a common endpoint V. While Γ_j is simply a segment, Γ_i is the graph of the Lipschitz function

positive integer n, depending on B, such that

$$B = \bigcup_{i=1}^{n} \Gamma^i, \qquad \Gamma^i \text{ L-Lipschitz arc for all } i \in \{1, \dots, n\},$$

and the following conditions are satisfied

(i) for any $i, j \in \{1, \dots, n\}$ with $i \neq j$, we have that either $\Gamma^i \cap \Gamma^j$ is not empty or $\operatorname{dist}(\Gamma^i, \Gamma^j) \geq \delta$;
(ii) for any $i, j \in \{1, \dots, n\}$ with $i \neq j$, if $\Gamma^i \cap \Gamma^j$ is not empty, then $\Gamma^i \cap \Gamma^j$ is a common endpoint V. Moreover, for any $x \in \Gamma^i$ we have $\operatorname{dist}(x, \Gamma^j) \geq c|x - V|$;
(iii) for any $i \in \{1, \dots, n\}$ we have that either $\Gamma^i \cap \partial\Omega$ is not empty or $\operatorname{dist}(\Gamma^i, \partial\Omega) \geq \delta$;
(iv) for any $i \in \{1, \dots, n\}$, if $\Gamma^i \cap \partial\Omega$ is not empty, then $\Gamma^i \cap \partial\Omega$ is an endpoint V of Γ^i. Moreover, for any $x \in \Gamma^i$, we have $\operatorname{dist}(x, \partial\Omega) \geq c|x - V|$.

Remark 3.3 One would intuitively expect that, for $K \in \mathcal{B}(\Omega)$ and for Ω being a domain with locally Lipschitz boundary, the domain $\Omega \setminus K$ has the cone property. Unfortunately, in general this is not true as the example depicted in Fig. 3 shows: there we illustrate two arcs Γ_i and Γ_j with a common endpoint V. While Γ_j is simply a segment, Γ_i is the graph of the Lipschitz function given by

$$\Gamma_i(\xi) = \begin{cases} 2^{-k-2}, & |\xi - 2^{-k}| \leq 2^{-k-3} \\ 2\xi + (2^{-k-1} - 2^{-k+1}), & 2^{-k} - 2^{-k-2} \leq \xi \leq 2^{-k} - 2^{-k-3} \\ -2\xi + (2^{-k-1} + 2^{-k+1}), & 2^{-k} + 2^{-k-3} \leq \xi \leq 2^{-k} + 2^{-k-2} \\ 0, & \text{elsewhere} . \end{cases}$$

for $\xi \in [0, 1]$ and $k \in \mathbb{N}$. When the angle $\pi - 2\widehat{\Gamma_i V \Gamma_j}$, where $\widehat{\Gamma_i V \Gamma_j} \approx \arcsin(c)$ (note that $c < 1$) is the angle formed at any junction V of two Lipschitz arcs Γ_i and Γ_j, is too large, more precisely larger than the angle (roughly given by $\pi - 2\arctan(L)$) at the vertex of any finite cone $\mathscr{C}$ subtended by the arcs, then it is impossible to cover the area $(\Omega \setminus K) \cap B_\rho(V)$ with an arbitrary union of congruent cones with vertices at any $x \in (\Omega \setminus K) \cap B_\rho(V)$. Hence, in order to ensure that domains of the type $\Omega \setminus K$, for $K \in \mathcal{B}(\Omega)$, have the cone property, we need to assume that c is large enough, and elementary geometrical observations require roughly $\arctan(L) \lesssim \arcsin(c)$, where L is the Lipschitz constant of the Lipschitz arcs Γ_i. We clarify how these formulas are more precisely derived in Lemma 3.4 below. Moreover, according to Theorem 3.1 any such domain will be a finite union of subdomains with locally Lipschitz boundaries and, according to Lemma 3.2, it will be also a finite union of subdomains starshaped with respect to balls of a certain radius r. In particular, the radii $r = r(L, c, \delta) > 0$ of such balls and the number $M = M(L, c, \delta, \mathscr{D})$ of such starshaped domains will depend only on L, c, δ, and possibly on $\mathscr{D} = \operatorname{diam}(\Omega)$.

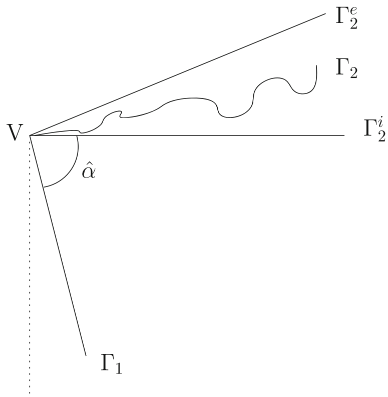

Fig. 4 Simplified situation where Γ_1 is a segment and Γ_2 is a Lipschitz curve

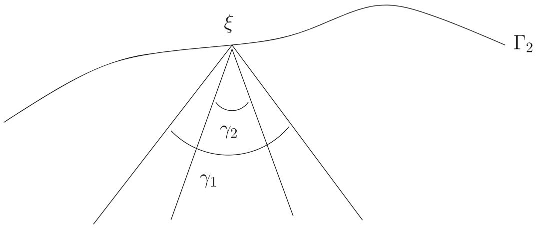

Fig. 5 Two cones $\mathscr{C}_L$ and $\mathscr{C}_{\hat{\alpha}}$ are subtended by Γ_2 at the same point ξ and their difference is the union of two oblique cones

In the following, we assume that Ω is a bounded open subset of $\mathbb{R}^2$ with locally Lipschitz boundary and that the parameters L, c of Rondi's class have been always chosen in such a way that $\Omega \setminus K$ is a domain with the cone property. The following result provides explicit sufficient conditions on the constants L, c, for such property to hold.

Lemma 3.4 *Assume that* $0 < c < 1$ *is sufficiently large and* $L > 0$ *sufficiently small, more precisely the following inequality is satisfied:*

$$c > \sin(5 \arctan(L)). \tag{16}$$

Then $\Omega \setminus K$ *has the cone property for all* $K \in \mathcal{B}(\Omega) = \mathcal{B}(\Omega, L, \delta, c)$.

Remark 3.5 Note that condition (16) imposes implicitly an upper bound $L < \tan\left(\frac{\pi}{10}\right) < 0.33$ for the Lipschitz constant.

Proof First of all note that it is sufficient to analyze the case $K = \Gamma_1 \cup \Gamma_2$ and $\{V\} = \Gamma_1 \cap \Gamma_2$ is a common endpoint of the two L-Lipschitz arcs Γ_1, Γ_2. We divide the proof into two steps.

Step 1. One of the two arcs is a segment.

Assume for the moment to be again in the situation of Fig. 3, i.e., Γ_1 is a segment and Γ_2 is a Lipschitz curve. Clearly, we can assume Γ_2 fully contained in a cone delimited by two segments Γ_2^i and Γ_2^e originating from V with angle $\alpha(L) = 2\arctan(L)$, to be conservative. Moreover, without loss of generality, we assume that Γ_2^i is the abscissa of the coordinate system for which Γ_2 is an L-Lipschitz function. In fact, the case when this is not verified is even more advantageous in terms of ensuring the cone property. We sketch the described situation in Fig. 4. We denote $\hat{\alpha} = \widehat{\Gamma_1 V \Gamma_2^i}$, the angle formed by the segments Γ_1 and Γ_2^i at the endpoint V. We note now that there exists a cone $\mathscr{C}_L$ subtended by Γ_2 and with angle $\gamma_1 = 2(\frac{\pi}{2} - \arctan(L)) = \pi - 2\arctan(L)$ at every point ξ of the curve Γ_2, see Fig. 5.

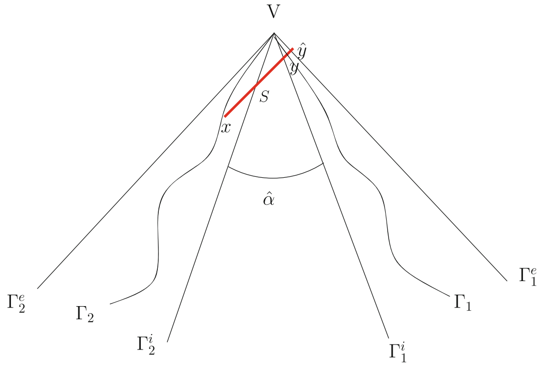

Fig. 6 The general situation of $K = \Gamma_1 \cup \Gamma_2$ and $\{V\} = \Gamma_1 \cap \Gamma_2$ where both Γ_1 and Γ_2 are L-Lipschitz arcs. We would like to establish distance relationships between Γ_1 and Γ_2 by arguing on their respective delimiting cone

Let us also consider another cone $\mathscr{C}_{\hat{\alpha}}$ subtended by Γ_2 at the same point ξ and with angle $\gamma_2 = 2(\frac{\pi}{2} - \hat{\alpha})$ and let us assume $\gamma_2 < \gamma_1$, see again Fig. 5. Note that in this case, $\mathscr{C}_L \setminus \mathscr{C}_{\hat{\alpha}}$ is a set formed by two oblique cones of nonzero angles at the vertex ξ. It is a simple observation to note that one of these two external and oblique cones can be transported along the curve Γ_2 in such a way to make ξ coinciding with V, the endpoint of Γ_2 and Γ_1, without ever intersecting the boundaries of the region limited by the curves Γ_1 and Γ_2. Hence in this case, $\Omega \setminus K$ has the cone property. The condition $\gamma_2 < \gamma_1$ is equivalent to

$$\hat{\alpha} > \arctan(L). \tag{17}$$

These observations also clarify the counterexample in Fig. 3 and Remark 3.3.

Step 2. The general situation.

We would like to show now that also for $K = \Gamma_1 \cup \Gamma_2$ and $\{V\} = \Gamma_1 \cap \Gamma_2$ where both Γ_1 and Γ_2 are L-Lipschitz arcs we can reduce the verification of the cone property of $\Omega \setminus K$ to the case of Step 1. For doing that we will show that Γ_1 and Γ_2 are both in a cone region delimited by two pairs of segments, respectively, (Γ_1^i, Γ_1^e) and (Γ_2^i, Γ_2^e) originating from V and with an angle $\alpha(L) = 2\arctan(L)$, to be conservative, where Γ_1^e, Γ_2^e are the external boundaries of the cone regions; moreover, we will show that for $c < 1$ large enough and $L > 0$ small enough, more precisely for c and L satisfying (16), Γ_1^i, Γ_2^i (the internal segments) form an angle $\hat{\alpha} = \hat{\alpha}(L, c)$ and $\hat{\alpha} > \arctan(L)$. We depict the described geometrical situation in Fig. 6. Without loss of generality, we assume $\hat{\alpha} \geq 0$, i.e., that the two cones limited by (Γ_1^i, Γ_1^e) and (Γ_2^i, Γ_2^e) do not intersect. In fact, with a similar argument as the one given below, we can always reduce the problem to such a case. We argue by contradiction, and we suppose that

$$\hat{\alpha} \leq \arctan(L). \tag{18}$$

Let us fix $x \in \Gamma_2$ and

$$\hat{y} = \arg\min_{\hat{\xi} \in \Gamma_1^e} |x - \hat{\xi}|.$$

We denote $S = \text{conv}\{x, \hat{y}\}$ the segment which connects x and $\hat{y}$. Now we define $y \in S \cap \Gamma_1$. We refer the reader to Fig. 6 for helping the understanding of the described situation. Since $K \in \mathcal{B}(\Omega)$, we have $\text{dist}(x, \Gamma_1) \geq c|x - V|$; hence, from (16), it follows

$$\text{dist}(x, \Gamma_1) > |x - V| \sin(5\arctan(L)). \tag{19}$$

By geometrical construction

$$d(x,\Gamma_1) \le |x-y| \le |x-\hat{y}| \text{ and } |x-\hat{y}| = |x-V|\sin(\theta), \tag{20}$$

where $\theta = \widehat{xV\hat{y}}$. Since $\theta \le \hat{\alpha} + 2\alpha(L) = \hat{\alpha} + 4\arctan(L)$, using (18), we get

$$\mathrm{dist}(x,\Gamma_1) \le |x-V|\sin(\hat{\alpha} + 4\arctan(L)) \le |x-V|\sin(5\arctan(L))$$

which contradicts (19). We conclude that $\hat{\alpha} > \arctan(L)$. From this geometrical situation, one can simply argue as in Step 1 by considering, for example, the segment Γ_1^i and the Lipschitz arc Γ_2 only. □

3.3 Interpolation inequalities for Sobolev spaces involving compact subdomains

In this section, we recall a classical interpolation inequality for Sobolev norms involving relatively compact subdomains, and we present a generalization which is tailored to Rondi's class and will be useful in our analysis.

Lemma 3.6 *Assume $1 \le p < \infty$. Let $A \subset \mathbb{R}^2$ be an open and bounded domain having the cone property with respect to a cone $\mathscr{C}$, and let D be an open set such that $D \subset\subset A$ and D has nonempty intersection with all the connected components of A. Then for $u \in L^{1,p}(A)$ we have*

$$\|u\|_{L^p(A)} \le \eta\left(\|\nabla u\|_{L^p(A)} + \|u\|_{L^p(D)}\right),$$

where η is a positive constant which depends essentially only on $\mathscr{C}$ and $\mathscr{D} = \mathrm{diam}(A)$, but not on u.

Lemma 3.6 in particular implies that the Sobolev space $W^{1,p}(A)$ and the Deny-Lions space $L^{1,p}(A)$ coincide for A having the cone property. The interested reader can refer to the proof of this lemma in [30, Section 1.1.11]; however, we include a detailed proof of it in the Appendix where we specifically make clearer the dependence of η on $\mathscr{C}$ and $\mathscr{D} = \mathrm{diam}(A)$ only. In particular, these arguments are useful in order to show the following stability result.

Proposition 3.7 *Assume $1 \le p < \infty$. Let $K \in \mathcal{B}(\Omega)$ and let $(K_h)_h \subset \mathcal{B}(\Omega)$ be a sequence of sets converging to K in the Hausdorff metric as $h \to +\infty$. Let $(u_h)_h \subset L^{1,p}(\Omega \setminus K_h)$ be a sequence of functions. If $D \subset\subset \Omega \setminus K$ and D has nonempty intersection with all the connected components of $\Omega \setminus K$, then, for h large enough, it holds*

$$\|u_h\|_{L^p(\Omega)} \le \eta\left(\|\nabla u_h\|_{L^p(\Omega\setminus K_h)} + \|u_h\|_{L^p(D)}\right),$$

where η is a constant independent of h.

The proof of this proposition follows directly from Lemma 3.6 and it is also postponed to the Appendix.

3.4 Existence of minimizers in the Rondi's class

In the following, except when explicitly required differently, we assume $1 < p < \infty$. Let $\mathcal{D}$ be the domain of pairs (u, K) defined by

$$\mathcal{D} = \{(u,K) : u \in W^{1,p}(\Omega \setminus K);\ K \in \mathcal{B}(\Omega)\}.$$

Let $u_0 \in L^2(\Omega)$ and for K fixed let $T : X \to L^2(\Omega)$ be a linear and continuous operator, where $X \in \{W^{1,p}(\Omega \setminus K), L^p(\Omega)\}$. We define the functional $\mathscr{E} : \mathcal{D} \to [0,+\infty]$ as in (3).

As anticipated, in this paper, we limit our analysis to operators T mapping onto $L^2(\Omega)$. Nevertheless, at the cost of additional technicalities, one may want to consider also discrepancy terms in $L^p(\Omega)$, i.e., $\|Tu - u_0\|^p_{L^p(\Omega)}$, for $1 < p < \infty$. In the following, we assume more specifically that the operator T belongs to one of the following two cases:

(i) (Nonlocal operator)for $1 < p < \infty$, we assume that $T : L^p(\Omega) \to L^2(\Omega)$ is a *compact* and *injective* operator.
(ii) (Local operator) for a fixed $K \in \mathcal{B}(\Omega)$, and for $2 \le p < \infty$, we define $T : W^{1,p}(\Omega \setminus K) \to L^2(\Omega)$ as the first-order differential operator

$$Tu = a_0 \cdot u + a_1 \cdot \frac{\partial u}{\partial x_1} + a_2 \cdot \frac{\partial u}{\partial x_2},$$

where $a_i \in L^\infty(\Omega)$ for $i = 0, 1, 2$.

Remark 3.8 Examples of operators belonging to the case (i) can be found, for instance, by considering convolution with suitable integral kernels (e.g., Green's functions of elliptic operators on the bounded domain Ω). This explains why we address this case as the one of nonlocal operators. Moreover, let $D \subset \Omega$ be a set of positive measure and $p \ge 2$. Then the operator T defined by

$$Tu = \chi_{\Omega \setminus D} \cdot u \qquad \text{for all } u \in W^{1,p}(\Omega \setminus K),$$

is an example of operator belonging to the class (ii) for $a_0 = \chi_{\Omega \setminus D}$ and $a_1 = a_2 \equiv 0$. Let us also observe that in this special case, if the datum $u_0 \in L^\infty$, as it is the case in image inpainting [22] or in variational models for fracture [26], then it is possible to apply classical truncation arguments (see formula (7.21) pag. 350 in [3]) to show that also a minimizer u has to be in L^∞, and one may resume well-known compactness results in SBV [2,21]. However, if u_0 is simply assumed in L^2 (but not with higher integrability which may allow to apply Leaci's extension [29]), also the case of such local operators is again nontrivial.

The main result of the paper is proving the existence of a minimizer (u, K) of the functional $\mathscr{E}$ in the domain $\mathcal{D}$, when T belongs to either the classes (i) and (ii).

4 Existence of minimizers for nonlocal operators

In this section, we prove the existence of minimizers of the functional $\mathscr{E}$ in the domain $\mathcal{D}$ when the operator T belongs to the class (i). A remarkable difficulty in the proof is that the the operator T may fail to be lower semicontinuous with respect to the convergence in the Deny-Lions space $L^{1,p}$; therefore, we have to recover a uniform L^p estimate of minimizing sequences, which stems from Proposition 3.7.

Theorem 4.1 *Assume $1 < p < \infty$. Let the bounded operator $T : L^p(\Omega) \to L^2(\Omega)$ belong to the case* (i). *Then there exists a pair $(u, K) \in \mathcal{D}$ that minimizes the functional $\mathscr{E}$ over the domain $\mathcal{D}$.*

Proof Without loss of generality, we set $\lambda = \alpha = 1$.
Let $\mathscr{M} = \inf_{(u,K)\in\mathcal{D}} \mathscr{E}(u, K) \ge 0$ and let $((u_h, K_h))_h \subset \mathcal{D}$ denote a minimizing sequence for the functional $\mathscr{E}$, i.e.,

$$\lim_{h\to+\infty} \mathscr{E}(u_h, K_h) = \mathscr{M}.$$

Step 1. Limit of the sequence of sets K_h.

Using properties (i)-(iv) of sets $B \in \mathcal{B}(\Omega)$, it follows that there exists an integer N_0, depending on Ω, L, δ and c only, such that for any $B \in \mathcal{B}(\Omega, L, \delta, c)$, with $B = \bigcup_{i=1}^n \Gamma^i$, Γ^i an L-Lipschitz arc for any $i = 1, \dots, n$, we have that $n \leq N_0$ (see Section 3 of [38]). Hence, there exist a subsequence $\left(K_{h_k}\right)_k$ and an integer $N \leq N_0$, independent of both h and k, such that

$$K_{h_k} = \bigcup_{i=1}^{N} \Gamma^i_{h_k}, \qquad \Gamma^i_{h_k} \text{ L-Lipschitz arc for all } i \in \{1, \dots, N\}, \quad \text{for all } k \in \mathbb{N}. \tag{21}$$

Moreover, possibly extracting a finite number of further subsequences, we may assume:

(a) for any $i, j \in \{1, \dots, N\}$ with $i \neq j$, we have that either $\Gamma^i_{h_k} \cap \Gamma^j_{h_k}$ is empty for any $k \in \mathbb{N}$, or $\Gamma^i_{h_k} \cap \Gamma^j_{h_k}$ is a common endpoint V_{h_k} for any $k \in \mathbb{N}$;

(b) for any $i \in \{1, \dots, N\}$, we have that either $\Gamma^i_{h_k} \cap \partial\Omega$ is empty for any $k \in \mathbb{N}$ or $\Gamma^i_{h_k} \cap \partial\Omega$ is an endpoint V_{h_k} of $\Gamma^i_{h_k}$ for any $k \in \mathbb{N}$.

The class of sets $\mathcal{B}(\Omega)$ is compact with respect to the Hausdorff distance (see Section 3 of [38]). Hence, possibly extracting a further subsequence, there exists a set $K \in \mathcal{B}(\Omega)$ such that

$$\lim_{k \to +\infty} \mathrm{d}_{\mathcal{H}}(K_{h_k}, K) = 0. \tag{22}$$

Using again properties (i)-(iv) of sets $B \in \mathcal{B}(\Omega)$, we have

$$K = \bigcup_{i=1}^{N} \Gamma^i, \qquad \Gamma^i \text{ L-Lipschitz arc for all } i \in \{1, \dots, N\},$$

and

$$\Gamma^i_{h_k} \to \Gamma^i \quad \text{as } k \to +\infty, \qquad \text{for all } i \in \{1, \dots, N\}, \tag{23}$$

in the sense of the Hausdorff metric.

Step 2. Uniform L^p estimate of a minimizing sequence.

In this step, we prove that without loss of generality, we can assume the following claim:

$$\textit{the sequence } u_{h_k} \textit{ is weakly compact in } L^p(\Omega). \tag{24}$$

Indeed, if this is not the case, we can argue as follows.
As a first simple observation, the set $\Omega \setminus K$ has a finite number $\widehat{N}$ of connected components, which we denote by A^ℓ for $\ell = 1, \dots, \widehat{N}$. Then, using properties (a) and (b) of sets K_{h_k} and (23), we have

$$\Omega \setminus K_{h_k} = \bigcup_{\ell=1}^{\widehat{N}} A^\ell_k, \qquad \text{for all } k \in \mathbb{N},$$

where the sets $A^\ell_k \subset \Omega$ are open and connected for any $\ell = 1, \dots, \widehat{N}$ and any $k \in \mathbb{N}$. Then, we have

$$\overline{A^\ell_k} \to \overline{A^\ell} \quad \text{as } k \to +\infty, \qquad \text{for all } \ell \in \{1, \dots, \widehat{N}\}, \tag{25}$$

in the sense of the Hausdorff metric. Now, for every $\ell = 1, \ldots, \widehat{N}$, we consider a smooth connected set $D^\ell \subset\subset A^\ell$ so that for k sufficiently large we can also assume that $D^\ell \subset\subset A_k^\ell$. We put

$$D := \bigcup_{\ell=1}^{\widehat{N}} D^\ell$$

and we define

$$v_{h_k} := u_{h_k} - \sum_{\ell=1}^{\widehat{N}} \bar{u}_k^\ell \chi_{A_k^\ell}, \tag{26}$$

where $\bar{u}_k^\ell$ is the mean value of the functions u_{h_k} over the sets D^ℓ. Clearly, the functions v_{h_k} belong to the space $W^{1,p}(\Omega \setminus K_{h_k})$, and $\nabla v_{h_k} = \nabla u_{h_k}$. We now apply Proposition 3.7 and the Poincaré-Wirtinger inequality to get

$$\begin{aligned}
\|v_{h_k}\|_{L^p(\Omega)} &\le \eta\big(\|\nabla u_{h_k}\|_{L^p(\Omega\setminus K_{h_k})} + \|v_{h_k}\|_{L^p(D)}\big) \\
&\le \eta\Big(\|\nabla u_{h_k}\|_{L^p(\Omega\setminus K_{h_k})} + \sum_{\ell=1}^{\widehat{N}} \|u_{h_k} - \bar{u}_k^\ell\|_{L^p(D^\ell)}\Big) \\
&\le \eta\Big(\|\nabla u_{h_k}\|_{L^p(\Omega\setminus K_{h_k})} + \sum_{\ell=1}^{\widehat{N}} C^\ell \|\nabla u_{h_k}\|_{L^p(\Omega\setminus K_{h_k})}\Big),
\end{aligned}$$

where C^ℓ are the Poincaré constants of the sets D^ℓ. It immediately follows that there exists a constant C independent of k such that

$$\|v_{h_k}\|_{L^p(\Omega)} \le C(\mathscr{M} + 1). \tag{27}$$

The following section of the proof is inspired by [35, Section 3]. For fixed k, we define the $\widehat{N}$-dimensional subspaces Y_k of $L^2(\Omega)$ by

$$Y_k := \operatorname{span}\{\chi_{A_k^\ell}, \ell = 1, \ldots, \widehat{N}\}.$$

and we consider

$$\widehat{w}_k := \arg\min \left\{ \int_\Omega |(Tw)(x) + (Tv_{h_k})(x) - u_0(x)|^2 \, \mathrm{d}x : w \in Y_k \right\}.$$

Observe that, since T is injective, it is also coercive when it is restricted to the finite-dimensional space Y_k, even if the coercivity constant may depend on k. Therefore, existence and uniqueness of $\widehat{w}_k$ can be easily proved (see [35, Proposition 7]). Notice that being Y_k a linear space, $0 \in Y_k$ and

$$\int_\Omega |(T\hat{w}_k)(x) + (Tv_{h_k})(x) - u_0(x)|^2 \, \mathrm{d}x \le \int_\Omega |(Tv_{h_k})(x) - u_0(x)|^2 \, \mathrm{d}x. \tag{28}$$

Now, clearly $v_{h_k} + \widehat{w}_k \in W^{1,p}(\Omega \setminus K_{h_k})$, and $\nabla(v_{h_k} + \widehat{w}_k) = \nabla u_{h_k}$. With this, by minimality of $\widehat{w}_k$, and since $u_{h_k} - v_{h_k} \in Y_k$, we easily get

$$\mathscr{E}(v_{h_k} + \widehat{w}_k, K_{h_k}) \le \mathscr{E}(u_{h_k}, K_{h_k}) \tag{29}$$

for every k.

Since the operator T is compact, possibly extracting a subsequence, by (27) we can assume that there exists $g \in L^2(\Omega)$ such that

$$\|T v_{h_k} - g\|_{L^2(\Omega)} \to 0 \tag{30}$$

as k tends to infinity. We now consider

$$\widetilde{w}_k := \arg\min \left\{ \int_\Omega |(Tw)(x) + g(x) - u_0(x)|^2 \, \mathrm{d}x : w \in Y_k \right\}.$$

Again, we observe that, by minimality of $\widetilde{w}_k$ in Y_k and $0 \in Y_k$, we have

$$\int_\Omega |(T\widetilde{w}_k)(x) + g(x) - u_0(x)|^2 \, \mathrm{d}x \le \int_\Omega |g(x) - u_0(x)|^2 \, \mathrm{d}x. \tag{31}$$

As before, $v_{h_k} + \widetilde{w}_k \in W^{1,p}(\Omega \setminus K_{h_k})$, and $\nabla(v_{h_k} + \widetilde{w}_k) = \nabla u_{h_k}$. With this, by Cauchy-Schwarz inequality, we get

$$\begin{aligned} \mathscr{E}(v_{h_k} + \widetilde{w}_k, K_{h_k}) &= \|T\widetilde{w}_k + g - g + T v_{h_k} - u_0\|^2_{L^2(\Omega)} + \int_{\Omega\setminus K_{h_k}} |\nabla u_{h_k}|^p + \mathcal{H}^1(K_{h_k}) \\ &\le \Big(\|T\widetilde{w}_k + g - u_0\|^2_{L^2(\Omega)} + \|T v_{h_k} - g\|^2_{L^2(\Omega)} + 2\|T v_{h_k} - g\|_{L^2(\Omega)} \|T\widetilde{w}_k + g - u_0\|_{L^2(\Omega)} \Big) \\ &\quad + \int_{\Omega\setminus K_{h_k}} |\nabla u_{h_k}|^p + \mathcal{H}^1(K_{h_k}). \end{aligned}$$

By using (31) and the minimality of $\widetilde{w}_k$, we can estimate the latter expression by

$$\begin{aligned} &\le \Big(\|T\widetilde{w}_k + g - u_0\|^2_{L^2(\Omega)} + \|T v_{h_k} - g\|^2_{L^2(\Omega)} + 2\|T v_{h_k} - g\|_{L^2(\Omega)} \|g - u_0\|_{L^2(\Omega)} \Big) \\ &\quad + \int_{\Omega\setminus K_{h_k}} |\nabla u_{h_k}|^p + \mathcal{H}^1(K_{h_k}) \\ &\le \Big(\|T\widehat{w}_k + g - u_0\|^2_{L^2(\Omega)} + \|T v_{h_k} - g\|^2_{L^2(\Omega)} + 2\|T v_{h_k} - g\|_{L^2(\Omega)} \|g - u_0\|_{L^2(\Omega)} \Big) \\ &\quad + \int_{\Omega\setminus K_{h_k}} |\nabla u_{h_k}|^p + \mathcal{H}^1(K_{h_k}). \end{aligned}$$

By adding and subtracting $T v_{h_k}$, using again Cauchy-Schwarz inequality, and (28), we can further conclude the estimate

$$\begin{aligned} &\Big(\|T\widehat{w}_k + g - u_0\|^2_{L^2(\Omega)} + \|T v_{h_k} - g\|^2_{L^2(\Omega)} + 2\|T v_{h_k} - g\|_{L^2(\Omega)} \|g - u_0\|_{L^2(\Omega)} \Big) \\ &\quad + \int_{\Omega\setminus K_{h_k}} |\nabla u_{h_k}|^p + \mathcal{H}^1(K_{h_k}) \\ &\le \mathscr{E}(v_{h_k} + \widehat{w}_k, K_{h_k}) + 2\|T v_{h_k} - g\|^2_{L^2(\Omega)} \\ &\quad + 2\|T v_{h_k} - g\|_{L^2(\Omega)} (\|T v_{h_k} - u_0\|_{L^2(\Omega)} + \|g - u_0\|_{L^2(\Omega)}). \end{aligned}$$

Eventually, an application of (29) to the previous estimate yields

$$\begin{aligned}&\mathscr{E}(v_{h_k}+\widetilde{w}_k, K_{h_k}) \leq \mathscr{E}(u_{h_k}, K_{h_k})\\&+2\|Tv_{h_k}-g\|^2_{L^2(\Omega)}+2\|Tv_{h_k}-g\|_{L^2(\Omega)}(\|Tv_{h_k}-u_0\|_{L^2(\Omega)}+\|g-u_0\|_{L^2(\Omega)}).\end{aligned}$$

Thanks to (30) and the boundedness of $(\|Tv_{h_k}-u_0\|_{L^2(\Omega)}+\|g-u_0\|_{L^2(\Omega)})$ due to (27), we conclude that $(v_{h_k}+\widetilde{w}_k, K_{h_k})$ is a minimizing sequence for $\mathscr{E}$. Using (25), the strong L^p-compactness of $\widetilde{w}_k$ follows from the results in [35, Section 3]. Together with (27), this shows that we can assume that the claim (24) holds true, replacing if necessary the sequence u_{h_k} with $v_{h_k}+\widetilde{w}_k$.

Step 3. Compactness and lower semicontinuity results.

By (24), there exists a function $u \in L^p(\Omega)$ and a subsequence, still denoted by $\left(u_{h_k}\right)_k$, such that $u_{h_k} \rightharpoonup u$ weakly in $L^p(\Omega)$ as $k \to +\infty$. By compactness of the operator T, this immediately gives

$$\lim_{k\to+\infty}\|Tu_{h_k}-u_0\|^2_{L^2(\Omega)}=\|Tu-u_0\|^2_{L^2(\Omega)}. \tag{32}$$

We denote by Ω_ρ a net of open sets such that

$$\Omega_\rho \subset\subset \Omega\setminus K \text{ and } \Omega_\rho \nearrow \Omega\setminus K \tag{33}$$

as $\rho \to 0$. From (25), for fixed $\rho > 0$ and for k large enough, we have $\Omega_\rho \subset\subset \Omega\setminus K_{h_k}$; therefore,

$$\int_{\Omega_\rho}|\nabla u_{h_k}|^p\mathrm{d}x \leq \int_{\Omega\setminus K_{h_k}}|\nabla u_{h_k}|^p\mathrm{d}x \leq \mathscr{E}(u_{h_k}, K_{h_k}) \leq \mathscr{M}+1. \tag{34}$$

As $u_{h_k} \rightharpoonup u$ weakly in $L^p(\Omega)$, by (34) we have that $u_{h_k} \rightharpoonup u$ weakly in $W^{1,p}(\Omega_\rho)$, and by the weak lower semicontinuity of the L^p norm, we get

$$\mathscr{M}+1 \geq \liminf_{k\to+\infty}\int_{\Omega\setminus K_{h_k}}|\nabla u_{h_k}|^p\mathrm{d}x \geq \liminf_{k\to+\infty}\int_{\Omega_\rho}|\nabla u_{h_k}|^p\mathrm{d}x \geq \int_{\Omega_\rho}|\nabla u|^p\mathrm{d}x.$$

Since the net of sets $\left(\Omega_\rho\right)_\rho$ invades $\Omega\setminus K$, by letting $\rho \to 0^+$, we get

$$\mathscr{M}+1 \geq \liminf_{k\to+\infty}\int_{\Omega\setminus K_{h_k}}|\nabla u_{h_k}|^p\mathrm{d}x \geq \int_{\Omega\setminus K}|\nabla u|^p\mathrm{d}x. \tag{35}$$

Since $u \in L^p(\Omega)$, this gives in particular that $u \in W^{1,p}(\Omega\setminus K)$. As $K \in \mathcal{B}(\Omega)$, we then conclude that$(u, K) \in \mathcal{D}$.

From (21), we have that the number of connected components of the compact sets K_{h_k} is uniformly bounded by a constant N that is independent of k. It follows that the Hausdorff measure $\mathcal{H}^1(K_{h_k})$ is lower semicontinuous with respect to the Hausdorff convergence (22) of the sets K_{h_k} to the set K (see Theorem 3.18 of [23]):

$$\liminf_{k\to+\infty}\mathcal{H}^1(K_{h_k}) \geq \mathcal{H}^1(K). \tag{36}$$

Using the compactness result, together with (32), (35), and (36), the application of the direct method of the Calculus of Variations completes the proof of the theorem. □

5 Existence of minimizers for local operators

In this section, we prove the existence of minimizers of the functional $\mathscr{E}$ in the domain $\mathcal{D}$ when the operator T belongs to the class (i). We stress that the proof in this case is simpler than in the case of nonlocal operators, since now the operator T is lower semicontinuous with respect to the convergence in the Deny-Lions space. Existence can be achieved through an application of the direct method in this space and a localization technique, together with the observation that when K belongs to the Rondi's class, $\Omega \setminus K$ has the cone property and the Deny-Lions space $L^{1,p}(\Omega \setminus K)$ coincides with $W^{1,p}(\Omega \setminus K)$. In particular, since only a L^p_{loc} estimate is actually needed, the uniform estimate of Proposition 3.7 is not required, and we only rely on Lemma 3.6.

5.1 On differential operators and their locality properties

We now turn to first-order differential operators of the type:

$$Tu = a_0 \cdot u + a_1 \cdot \frac{\partial u}{\partial x_1} + a_2 \cdot \frac{\partial u}{\partial x_2},$$

where $a_i \in L^\infty(\Omega)$ for $i = 0, 1, 2$. Linear operators of this type are found, for instance, in the study of linear conservation laws with *rough* coefficients, i.e., in the analysis of equations of the type:

$$\frac{\partial u}{\partial t} + \operatorname{div}(b \cdot u) = u_0, \tag{37}$$

where $b \in [L^\infty(\Omega)]^2$ and $\operatorname{div}(b) \in L^\infty(\Omega)$, see, e.g., [9]. In fact, we may write $\operatorname{div}(b \cdot u) = a_0 \cdot u + a_1 \cdot \frac{\partial u}{\partial x_1} + a_2 \cdot \frac{\partial u}{\partial x_2}$, where $a_0 = \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2}$ and $a_i = b_i, i = 1, 2$. Hence, minimizers of (3) may be used as a regularization of rough solutions of (37). Here $\frac{\partial u}{\partial x_i}, i = 1, 2$ are the partial derivatives of the distribution u in weak sense. In the following, we would like to remind a few properties of these operators with respect to Sobolev spaces $W^{1,p}$, for $2 \leq p < \infty$.

If $A \subset \Omega \subset\subset \mathbb{R}^2$, then T can be defined on $W^{1,p}(\Omega)$ (resp. on $W^{1,p}(A)$), the weak derivatives are taken on Ω (resp. on A), i.e., with respect to smooth test functions compactly supported on Ω (resp. on A). Hence, in weak sense, the operator T depends on the domain of the functions on which it is applied. Nevertheless, if $u \in W^{1,p}(\Omega)$ then its restriction $u_{|A}$ on A belongs to $W^{1,p}(A)$ and

$$(Tu)_{|A} = Tu_{|A}, \tag{38}$$

where on the right-hand side, T is applied on a function defined on the domain A (with weak derivatives defined accordingly). This is called the *locality property* of T.

Observe now that the functional (for simplicity $\alpha = \lambda = 1$)

$$\mathscr{E}(u, K) = \|Tu - u_0\|^2_{L^2(\Omega)} + \int\limits_{\Omega \setminus K} |\nabla u|^p + \mathscr{H}^1(K),$$

is understood for T as operating on $W^{1,p}(\Omega \setminus K)$. Hence, changing K essentially means also changing T, and to be more precise, we should write $T \equiv T_K$, as depending on K. However, for each K fixed $T = T_K$ is a bounded linear operator on $W^{1,p}(\Omega \setminus K)$.

Let us further note that, given $K_1, K_2 \in \mathcal{B}(\Omega)$, and $\emptyset \neq A \subset (\Omega \setminus K_1) \cap (\Omega \setminus K_2)$, then, in force of the locality property of T, for every $u \in W^{1,p}(A)$ which is restriction on A of both $u_i \in W^{1,p}(\Omega \setminus K_i)$, $i = 1, 2$,

$$Tu \equiv (T_{K_1} u_1)_{|A} = (T_{K_2} u_2)_{|A},$$

is a well-defined L^2 function on A. In other words, the function $Tu \in L^2(A)$ essentially does not depend on the K_i's. Moreover, if A is an *extension domain* in Ω, i.e., there exists a bounded extension operator $E : W^{1,p}(A) \to W^{1,p}(\Omega)$, $u \mapsto Eu$, such that $(Eu)_{|A} = u$ and $\|Eu\|_{W^{1,p}(\Omega)} \leq C_A \|u\|_{W^{1,p}(A)}$, then it follows that $T_K = T := T \circ E$ can be understood as a bounded linear operator on $W^{1,p}(A)$, independent of K, as soon as $A \subset \Omega \setminus K$.

We would like to use the previous observations now to derive a certain *lower semicontinuity property* of $\|T_K u - u_0\|_{L^2}$ which will turn out to be useful in the following. Let (K_h) be a sequence in $\mathcal{B}(\Omega)$, $K \in \mathcal{B}(\Omega)$, and

$$d_{\mathcal{H}}(K_h, K) \to 0, \quad h \to \infty.$$

We define Ω_ρ as in (33), then $\Omega_\rho \subset\subset \Omega \setminus K_h$ for h large enough. Note further that, without loss of generality, we can assume Ω_ρ to be an extension domain. If $(u_h)_h$ converges weakly to u in $W^{1,p}(\Omega_\rho)$ for $u_h \in W^{1,p}(\Omega \setminus K_h)$ and $u \in W^{1,p}(\Omega \setminus K)$, then we have the following relevant inequalities

$$\liminf_{h\to\infty} \|T_{K_h} u_h - u_0\|_{L^2(\Omega)} \geq \liminf_{h\to\infty} \|T_{K_h} u_h - u_0\|_{L^2(\Omega_\rho)} \geq \|T_K u - u_0\|_{L^2(\Omega_\rho)}. \quad (39)$$

According to the observations above, here we have heavily used the fact that T does not depend on K_h or K as soon as its image is restricted on $L^2(\Omega_\rho)$; hence, by the extension property of Ω_ρ, T is a bounded operator on $W^{1,p}(\Omega_\rho)$, independent of K_h or K. Moreover, since u_h converges weakly to u in $W^{1,p}(\Omega_\rho)$ then by boundedness of T also Tu_h converges weakly to Tu in $L^2(\Omega_\rho)$ and the last inequality follows from the lower semicontinuity of the L^2 norm with respect to the weak convergence.

5.2 Existence result

We need the following lemma.

Lemma 5.1 *Assume $1 \leq p < \infty$. Let A be a bounded, connected, open subset of $\mathbb{R}^2$ with locally Lipschitz boundary, $u \in W^{1,p}(A)$, and let $T : W^{1,p}(A) \to L^2(A)$ be a linear and continuous operator. Let us assume that*

$$\|Tu - u_0\|^2_{L^2(A)} + \int_A |\nabla u(x)|^p dx \leq \mathscr{H},$$

where $\mathscr{H}$ is a positive constant. Then we have

$$\alpha := |\overline{u}(A)| \cdot \|T\chi_A\|_{L^2(A)} \leq \beta,$$

where

$$\beta := \gamma + \left(\gamma^2 + \mathscr{H}\right)^{1/2}, \qquad \gamma := \mathscr{H}^{1/p} C(A) \|T\|_{W^{1,p} \to L^2} + \|u_0\|_{L^2(A)}, \quad (40)$$

where $C(A)$ is the constant of inequality (1).

Proof For reader's convenience we sketch the proof which follows the lines of Step 1 of the proof of [42, Proposition 3.1]. Let $w = \overline{u}(A)\chi_A$ and $v = u - w$. Then $\overline{v}(A) = 0$ and

$\int_A |\nabla v(x)|^p \mathrm{d}x = \int_A |\nabla u(x)|^p \mathrm{d}x \leq \mathscr{H}$. By the Poincaré-Wirtinger inequality (1) we obtain $\|v\|_{L^p(A)} \leq C(A)\mathscr{H}^{1/p}$. We also have

$$\mathscr{H} \geq \|Tu - u_0\|^2_{L^2(A)} \geq \|Tw\|_{L^2(A)} \left[\|Tw\|_{L^2(A)} - 2(\|T\|_{W^{1,p}\to L^2}\|v\|_{L^p(A)} + \|u_0\|_{L^2(A)})\right]$$
$$\geq \alpha(\alpha - 2\gamma).$$

Solving the latter inequality for nonnegative values of α yields

$$0 \leq |\overline{u}(A)| \cdot \|T\chi_A\|_{L^2(A)} = \alpha \leq \gamma + \left(\gamma^2 + \mathscr{H}\right)^{1/2} = \beta.$$

□

Theorem 5.2 *Assume* $2 \leq p < \infty$ *and let the operator* T *belong to the case* (i). *Then, there exists a pair* $(u, K) \in \mathcal{D}$ *that minimizes the functional* $\mathscr{E}$ *over the domain* $\mathcal{D}$.

Proof Without loss of generality we set $\lambda = \alpha = 1$. Let $\mathscr{M} = \inf_{(u,K)\in\mathcal{D}} \mathscr{E}(u, K) \geq 0$ and let $((u_h, K_h))_h \subset \mathcal{D}$ denote a minimizing sequence for the functional $\mathscr{E}$, i.e.,

$$\lim_{h\to+\infty} \mathscr{E}(u_h, K_h) = \mathscr{M}. \tag{41}$$

We use the same notations as in the proof of Theorem 4.1. The proof of Step 1 is the same as in Theorem 4.1.

Step 2. Construction of a partition of the set Ω.

Arguing as in Theorem 4.1, we write

$$\Omega \setminus K = \bigcup_{\ell=1}^{\widehat{N}} A^\ell,$$

where $A^\ell \subset \Omega$ is an open and connected set for any $\ell = 1, \dots, \widehat{N}$ and

$$\Omega \setminus K_{h_k} = \bigcup_{\ell=1}^{\widehat{N}} A^\ell_k, \qquad \text{for all } k \in \mathbb{N}, \tag{42}$$

where the sets $A^\ell_k \subset \Omega$ are open and connected for any $\ell = 1, \dots, \widehat{N}$ and any $k \in \mathbb{N}$. We recall that we have

$$\overline{A^\ell_k} \to \overline{A^\ell} \quad \text{as } k \to +\infty, \qquad \text{for all } \ell \in \{1, \dots, \widehat{N}\}, \tag{43}$$

in the sense of the Hausdorff metric.

We define the open sets Ω_ρ as in (33). We can assume that they have the locally Lipschitz boundary property and choose $\rho > 0$ small enough in such a way that we have

$$\Omega_\rho = \bigcup_{\ell=1}^{\widehat{N}} A^\ell_\rho,$$

where the sets A^ℓ_ρ are open, connected, and satisfy $\overline{A^\ell_\rho} \subset A^\ell$ for any $\ell = 1, \dots, \widehat{N}$.

Then there exists $k_0 \in \mathbb{N}$ such that $k_0 = k_0(\rho)$ and, using (23), we have

$$K_{h_k} \cap \Omega_\rho = \emptyset \qquad \text{for all } k \geq k_0,$$

and

$$u_{h_k} \in W^{1,p}(A^\ell_\rho) \qquad \text{for all } \ell \in \{1, \dots, \widehat{N}\} \quad \text{and} \quad \text{for all } k \geq k_0. \tag{44}$$

Step 3. Uniform L^p_{loc} estimate of u_{h_k} when $a_0 \neq 0$ in $L^\infty(A^\ell)$ for any ℓ.

If $a_0 \neq 0$ in $L^\infty(A^\ell)$ for any $\ell = 1, \dots, \widehat{N}$, then for any ℓ there exists an open set J^ℓ with the following properties: (i) $\overline{J^\ell} \subset A^\ell$; (ii) if $v \in W^{1,p}(A^\ell)$ is such that $v(x) = 1$ if $x \in J^\ell$, then $(Tv)_{|J^\ell} \neq 0$ in $L^2(J^\ell)$.

Let us fix a connected component A^ℓ_ρ of Ω_ρ. We may assume $\rho > 0$ small enough in such a way that we have $\overline{J^\ell} \subset A^\ell_\rho$. Let $v \in W^{1,p}(A^\ell)$ be such that $v(x) = 1$ if $x \in A^\ell_\rho$. Then, by using the locality property of the operator T, we have

$$(Tv)_{|J^\ell} = Tv_{|J^\ell},$$

from which it follows, being $v(x) = 1$ if $x \in J^\ell$,

$$\alpha^\ell_\rho := \|Tv_{|A^\ell_\rho}\|_{L^2(A^\ell_\rho)} \geq \|Tv_{|A^\ell_\rho}\|_{L^2(J^\ell)} = \|Tv_{|J^\ell}\|_{L^2(J^\ell)} = \|(Tv)_{|J^\ell}\|_{L^2(J^\ell)} \neq 0. \tag{45}$$

Using again the locality property of T for any $k \in \mathbb{N}$, we have

$$\|Tu_{h_k} - u_0\|^2_{L^2(\Omega)} \geq \|Tu_{h_k} - u_0\|^2_{L^2(A^\ell_\rho)} = \|Tu_{h_k|A^\ell_\rho} - u_0\|^2_{L^2(A^\ell_\rho)},$$

from which, for k large enough, we find

$$\begin{aligned}\|Tu_{h_k} - u_0\|^2_{L^2(A^\ell_\rho)} + \int_{A^\ell_\rho} |\nabla u_{h_k}|^p \mathrm{d}x &\leq \|Tu_{h_k} - u_0\|^2_{L^2(\Omega)} + \int_{\Omega\setminus K_{h_k}} |\nabla u_{h_k}|^p \mathrm{d}x \\ &\leq \mathscr{M} + 1.\end{aligned}$$

Hence, we can apply Lemma 5.1 to the set A^ℓ_ρ and, using (45), for k large enough, we find

$$|\overline{u}_{h_k}(A^\ell_\rho)| \leq \frac{\beta^\ell_\rho}{\alpha^\ell_\rho}, \tag{46}$$

where, using (40), the positive constant β^ℓ_ρ is given by

$$\beta^\ell_\rho = \gamma^\ell_\rho + \left(\gamma^{\ell\,2}_\rho + \mathscr{M} + 1\right)^{1/2}, \qquad \gamma^\ell_\rho = (\mathscr{M} + 1)^{1/p} C(A^\ell_\rho)\|T\| + \|u_0\|_{L^p(A^\ell_\rho)},$$

$C(A^\ell_\rho)$ being the constant of the Poincaré-Wirtinger inequality (1) evaluated for the set A^ℓ_ρ.

Since for k large enough we have

$$\|\nabla u_{h_k}\|_{L^p(A^\ell_\rho)} = \left(\int_{A^\ell_\rho} |\nabla u_{h_k}|^p \mathrm{d}x\right)^{1/p} \leq \left(\int_{\Omega\setminus K_{h_k}} |\nabla u_{h_k}|^p \mathrm{d}x\right)^{1/p} \leq (\mathscr{M} + 1)^{1/p}, \tag{47}$$

by using the Poincaré-Wirtinger inequality and (46), we find

$$\begin{aligned}\|u_{h_k}\|_{L^p(A^\ell_\rho)} &\leq \|u_{h_k} - \overline{u}_{h_k}(A^\ell_\rho)\|_{L^p(A^\ell_\rho)} + \left(\mathrm{meas}(A^\ell_\rho)\right)^{1/p} |\overline{u}_{h_k}(A^\ell_\rho)| \\ &\leq C(A^\ell_\rho)\|\nabla u_{h_k}\|_{L^p(A^\ell_\rho)} + \left(\mathrm{meas}(A^\ell_\rho)\right)^{1/p} |\overline{u}_{h_k}(A^\ell_\rho)| \\ &\leq C(A^\ell_\rho)(\mathscr{M} + 1)^{1/p} + \left(\mathrm{meas}(A^\ell_\rho)\right)^{1/p} \frac{\beta^\ell_\rho}{\alpha^\ell_\rho}.\end{aligned}$$

Eventually we get

$$\|u_{h_k}\|_{L^p(A^\ell_\rho)} \leq C^\ell_\rho, \tag{48}$$

for any $\ell \in \{1, \ldots, \widehat{N}\}$, where C_ρ^ℓ is a positive constant dependent on both ℓ and ρ, but independent of k.

Step 4. Compactness and lower semicontinuity results.

Arguing as in Step 3 of the proof of Theorem 4.1, we find that (34) holds again. By this and (48), the sequence $\left(u_{h_k}\right)_k$ is uniformly bounded in $W^{1,p}(\Omega_\rho)$ with respect to k for any ρ small enough. Since the sequence of sets $\left(\Omega_\rho\right)_\rho$ invades $\Omega \setminus K$, by letting $\rho \to 0^+$ and a diagonal argument we get that there exists a function $u \in W_{loc}^{1,p}(\Omega \setminus K)$ and a subsequence, still denoted by $\left(u_{h_k}\right)_k$ for simplicity, such that $u_{h_k} \rightharpoonup u$ weakly in $W^{1,p}(\Omega_\rho)$ for every ρ small enough and $u_{h_k} \to u$ pointwise a.e. on Ω as $k \to +\infty$.

Arguing as in the proof of (35), we then get

$$\mathscr{M} + 1 \geq \liminf_{k\to+\infty} \int\limits_{\Omega\setminus K_{h_k}} |\nabla u_{h_k}|^p \mathrm{d}x \geq \int\limits_{\Omega\setminus K} |\nabla u|^p \mathrm{d}x. \tag{49}$$

In particular, $\nabla u \in L^p(\Omega \setminus K)$. We already know that $u \in W_{\mathrm{loc}}^{1,p}(\Omega \setminus K)$; therefore, it follows $u \in L_{\mathrm{loc}}^p(\Omega \setminus K)$ and $u \in L^{1,p}(\Omega \setminus K)$. Since $\Omega \setminus K$ has the cone property, Lemma 3.6 implies that $L^{1,p}(\Omega \setminus K)$ coincides with $W^{1,p}(\Omega \setminus K)$, so that $u \in W^{1,p}(\Omega \setminus K)$. Moreover, being $K \in \mathcal{B}(\Omega)$ implies $(u, K) \in \mathcal{D}$.

Then, by the lower semicontinuity of the term $\|Tu - u_0\|^2_{L^2(\Omega_\rho)}$ with respect to the weak convergence in $W^{1,p}(\Omega_\rho)$ for any $\rho > 0$ small enough and, according to (39), we have

$$\liminf_{k\to+\infty} \|Tu_{h_k} - u_0\|^2_{L^2(\Omega)} \geq \liminf_{k\to+\infty} \|Tu_{h_k} - u_0\|^2_{L^2(\Omega_\rho)} \geq \|Tu - u_0\|^2_{L^2(\Omega_\rho)},$$

from which, by letting $\rho \to 0^+$ we get

$$\liminf_{k\to+\infty} \|Tu_{h_k} - u_0\|^2_{L^2(\Omega)} \geq \|Tu - u_0\|^2_{L^2(\Omega)}. \tag{50}$$

Arguing as in the proof of (36), we get

$$\liminf_{k\to+\infty} \mathcal{H}^1(K_{h_k}) \geq \mathcal{H}^1(K). \tag{51}$$

From (49), (50), and (51), a straightforward application of the direct method of the Calculus of Variations gives that the pair $(u, K) \in \mathcal{D}$ minimizes $\mathscr{E}$ over the domain $\mathcal{D}$.

Step 5. We consider the case $a_0 = 0$ in $L^\infty(A^\ell)$ for some ℓ.

In the following, $\left(u_{h_k}\right)_k$ denotes again the subsequence obtained at the end of Step 2. Let us fix a connected component A^ℓ of $\Omega \setminus K$ such that $a_0(x) = 0$ for $x \in A^\ell$ a.e.

Let E^ℓ denote an open set having the locally Lipschitz boundary property such that $E^\ell \subset\subset A^\ell$. Let $\rho > 0$ be such that $E^\ell \subset\subset A_\rho^\ell$ and let k be large enough such that $A_\rho^\ell \subset\subset A_k^\ell$.

The function $u_{h_k} \in L^2(\Omega)$ for all $k \in \mathbb{N}$ and the average value $\overline{u}_{h_k}(E^\ell)$ is finite for any k, though not necessarily uniformly bounded with respect to k. Now we define the function $\widetilde{u}_{h_k}$ by means of

$$\widetilde{u}_{h_k}(x) = \begin{cases} u_{h_k}(x) - \overline{u}_{h_k}(E^\ell) & \text{if } x \in A_k^\ell \\ u_{h_k}(x) & \text{if } x \notin A_k^\ell. \end{cases}$$

By construction, we have $(\widetilde{u}_{h_k}, K_{h_k}) \in \mathcal{D}$. Since $\overline{\widetilde{u}}_{h_k}(E^\ell) = 0$ for any k, by using the Poincaré-Wirtinger inequality (1) and (47), we have:

$$\|\widetilde{u}_{h_k}\|_{L^p(E^\ell)} \leq C(E^\ell)\|\nabla \widetilde{u}_{h_k}\|_{L^p(E^\ell)} \leq C(E^\ell)(\mathscr{M} + 1)^{1/p},$$

$C(E^\ell)$ being the constant of the inequality evaluated for the set E^ℓ.

As A_ρ^ℓ is a connected set having the locally Lipschitz boundary property, then it has also the cone property, so that using Lemma 3.6 and (47), we have

$$\|\widetilde{u}_{h_k}\|_{L^p(A_\rho^\ell)} \leq \eta_\rho^\ell \left(\|\nabla\widetilde{u}_{h_k}\|_{L^p(A_\rho^\ell)} + \|\widetilde{u}_{h_k}\|_{L^p(E^\ell)}\right) \leq \eta_\rho^\ell (1 + C(E^\ell))(\mathscr{M} + 1)^{1/p},$$

from which it follows,

$$\|\widetilde{u}_{h_k}\|_{L^p(A_\rho^\ell)} \leq C_\rho^\ell,$$

where η_ρ^ℓ and C_ρ^ℓ are positive constants dependent on both ℓ and ρ, but independent of k.

We repeat the above argument for each connected component A^ℓ of $\Omega \setminus K$, $\ell = 1, \ldots, \widehat{N}$, such that $a_0 = 0$ in $L^\infty(A^\ell)$. Eventually, using also the results of Step 3, we obtain a sequence of functions $\left(\widetilde{u}_{h_k}\right)_k$ such that

$$\|\widetilde{u}_{h_k}\|_{L^p(\Omega_\rho)} \leq C_\rho,$$

where C_ρ is a positive constant dependent on ρ, but independent of k.

Now, arguing as in Step 4, there exist a function $\widetilde{u} \in W^{1,p}(\Omega \setminus K)$ and a subsequence, still denoted by $\left(\widetilde{u}_{h_k}\right)_k$ for simplicity, such that $\widetilde{u}_{h_k} \to \widetilde{u}$ pointwise a.e. on Ω and $\widetilde{u}_{h_k} \rightharpoonup \widetilde{u}$ weakly in $W^{1,p}(\Omega_\rho)$ for any $\rho > 0$ small enough, as $k \to +\infty$.

Since $\nabla u_{h_k} = \nabla\widetilde{u}_{h_k}$ in $\Omega \setminus K_{h_k}$, with the same argument used in Step 4 to prove (49), we get that

$$\liminf_{k\to+\infty} \int_{\Omega\setminus K_{h_k}} |\nabla u_{h_k}|^p \mathrm{d}x \geq \int_{\Omega\setminus K} |\nabla\widetilde{u}|^p \mathrm{d}x. \tag{52}$$

Moreover, we observe that, for every ℓ such that $a_0(x) = 0$ for $x \in A^\ell$ a.e., by the construction of $\widetilde{u}_{h_k}$, we have

$$T\widetilde{u}_{h_k} = a_1(x)\frac{\partial\widetilde{u}_{h_k}}{\partial x_1} + a_2(x)\frac{\partial\widetilde{u}_{h_k}}{\partial x_2} = a_1(x)\frac{\partial u_{h_k}}{\partial x_1} + a_2(x)\frac{\partial u_{h_k}}{\partial x_2} = Tu_{h_k}$$

for a.e. $x \in A_k^\ell \cap A^\ell$. Using (43), we get that $A_\rho^\ell \subset (A_k^\ell \cap A^\ell)$ for k large enough, therefore

$$T\widetilde{u}_{h_k} = Tu_{h_k}$$

for a.e. $x \in \Omega_\rho$. Together with the lower semicontinuity of the term $\|Tu - u_0\|^2_{L^2(\Omega_\rho)}$ with respect to the weak convergence in $W^{1,p}(\Omega_\rho)$ for any $\rho > 0$ small enough, and according to (39), this yields

$$\begin{aligned}\liminf_{k\to+\infty} \|Tu_{h_k} - u_0\|^2_{L^2(\Omega)} &\geq \liminf_{k\to+\infty} \|Tu_{h_k} - u_0\|^2_{L^2(\Omega_\rho)} \\ &= \liminf_{k\to+\infty} \|T\widetilde{u}_{h_k} - u_0\|^2_{L^2(\Omega_\rho)} \geq \|T\widetilde{u} - u_0\|^2_{L^2(\Omega_\rho)},\end{aligned}$$

from which, by letting $\rho \to 0^+$, we get

$$\liminf_{k\to+\infty} \|Tu_{h_k} - u_0\|^2_{L^2(\Omega)} \geq \|T\widetilde{u} - u_0\|^2_{L^2(\Omega)}. \tag{53}$$

Using (41), (51), (52), and (53), by the chain of inequalities

$$\begin{aligned}\mathscr{M} &= \liminf_{k\to+\infty} \mathscr{E}(u_{h_k}, K_{h_k}) \\ &\geq \liminf_{k\to+\infty} \int_{\Omega\setminus K_{h_k}} |\nabla u_{h_k}|^p \mathrm{d}x + \liminf_{k\to+\infty} \|Tu_{h_k} - u_0\|^2_{L^2(\Omega)} + \liminf_{k\to+\infty} \mathcal{H}^1(K_{h_k}) \\ &\geq \int_{\Omega\setminus K} |\nabla\widetilde{u}|^p \mathrm{d}x + \|T\widetilde{u} - u_0\|^2_{L^2(\Omega)} + \mathcal{H}^1(K) = \mathscr{E}(\widetilde{u}, K),\end{aligned}$$

the proof is concluded. □

6 Conclusions and open problems

In this section, we would like to collect a few interesting open problems that stem from the analysis provided by this paper and recent numerical results obtained in [1,25].

(i) We addressed two specific classes of linear operators. However, when the problem is discretized by means of finite differences or finite elements, solutions exist for every linear problem [25], despite the lack of coercivity also in the finite-dimensional setting. We wonder whether the NP-hardness of the finite-dimensional problem [1] is related to the difficulties one encounters in the analysis in the continuous setting.

(ii) In relationship to (ii), it is completely open also the Γ-convergence of the finite discrete problem to the continuous problem, as a generalization of results provided for the classical Mumford-Shah functional by Chambolle et al. [14–16]. In particular, it would be already very interesting to understand whether minimizers of the discrete problems can converge to minimizers of the continuous problem, if they do belong to a Rondi's class.

(iii) Numerical methods for minimizing the Mumford-Shah functional have been source of a large amount of results, see [4,11,13,16,31] for prominent approaches in this setting. For linear inverse free-discontinuity problems, the proved NP-hardness [1] makes the issue even more challenging. We address the interested reader to a new approach recently introduced in [25], which relates discrete linear inverse free-discontinuity problems and sparse recovery problems [24].

7 Appendix

This section is devoted to the proof of Lemma 3.6 and Proposition 3.7. We have first to recall the following integral representation formula, which is proved in [30, Theorem 1.1.10.1].

Theorem 7.1 *Assume* $1 \leq p < \infty$. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain starshaped with respect to a ball* $B_\rho(x_0) \subset \Omega$, *and let* $u \in L^{1,p}(\Omega)$. *Then for almost all* $x \in \Omega$

$$u(x) = \rho^{-2} \int_{B_\rho(x_0)} \varphi(y/\rho)u(y)dy + \sum_{|\alpha|=1} \int_{\Omega} \frac{f_\alpha(x; r, \theta)}{r} \frac{\partial^\alpha}{\partial y^\alpha} u(y)dy, \qquad (54)$$

where $r = |y - x|$, $\theta = \frac{y-x}{|y-x|}$, $\varphi \in C_c^\infty(B_1(x_0))$, *and* f_α *are bounded functions*

$$|f_\alpha| \leq K\frac{\mathscr{D}}{\rho},$$

where K *is a constant independent of* u, Ω, *and* $\mathscr{D} = \mathrm{diam}(\Omega)$.

A few observations are in order. The *Riesz potential* operator which appears in the right-hand side

$$\Lambda[v](x) = \int\limits_\Omega \frac{v(y)}{|y-x|}\mathrm{d}y, \quad x \in \Omega,$$

is known to be bounded on $L^p(\Omega)$ (see pag. 152 and following pages in [28], and the proof of Lemma in [30, Section 1.1.11]); let us denote its norm $\mathscr{N}(\Omega)$. Note that if $\Omega \subset \tilde{\Omega}$ then $\mathscr{N}(\Omega) \leq \mathscr{N}(\tilde{\Omega})$. In fact any function $v \in L^p(\Omega)$ can be trivially identified by zero extension with a function in $L^p(\tilde{\Omega})$. Moreover, we stress that also the function φ depends neither on u nor on Ω.

7.1 Proof of Lemma 3.6

Assume without loss of generality that A is connected and $D \subset\subset A$ is a ball. Let A^ℓ be any subdomain of A starshaped with respect to a ball as constructed in Lemma 3.2 and let $B_r(x_\ell)$ be the corresponding ball. We now construct a finite family of intersecting balls such that $B^{(0)} = B_r(x_\ell)$, $\emptyset \neq B^{(i)} \cap B^{(i+1)} \supset B_\rho(x_i^{i+1})$, for a fixed $\rho < r$, and $B^{(M)} = D$. Note that A^ℓ is starshaped with respect to the ball $B_\rho(x_0^1) \subset B^{(0)} \cap B^{(1)}$. Hence, we can apply the integral formula (54) and for $u \in L^{1,p}(A)$ we write

$$\begin{aligned}
\int\limits_{A^\ell} |u(x)|^p\mathrm{d}x \leq & 2^{p-1}\int\limits_{A^\ell} \rho^{-2p}\left|\int_{B_\rho(x_0^1)} \varphi(y/\rho)u(y)dy\right|^p \mathrm{d}x \\
& + 2^{2(p-1)}\sum_{|\alpha|=1}\int\limits_{A^\ell}\left|\int_{A^\ell}\frac{f_\alpha(x;r,\theta)}{r}\frac{\partial^\alpha}{\partial y^\alpha}u(y)\mathrm{d}y\right|^p dx \\
\leq & 2^{p-1}\rho^{-(2p-\frac{p}{q})}\,\mathrm{meas}(A^\ell)\left(\int\limits_{B_1(x_0^1)}|\varphi(y)|^q\mathrm{d}y\right)^{\frac{p}{q}}\left(\int\limits_{B_\rho(x_0^1)}|u(y)|^p\mathrm{d}y\right) \\
& + 2^{2(p-1)}\left(K\frac{\mathscr{D}}{\rho}\right)^p\mathscr{N}(A^\ell)^p\sum_{|\alpha|=1}\int\limits_{A^\ell}\left|\frac{\partial^\alpha}{\partial y^\alpha}u(y)\right|^p\mathrm{d}y,
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q} = 1$, $\mathscr{D} = \mathrm{diam}(A)$, and $\mathscr{N}(A^\ell)$ is the norm of the integral operator as mentioned above. Actually, we can estimate $\mathscr{N}(A^\ell) \leq \mathscr{N}(A)$. In short, we obtain

$$\|u\|_{L^p(A^\ell)} \leq \eta^\ell\left(\|\nabla u\|_{L^p(A^\ell)} + \|u\|_{L^p(B^{(0)}\cap B^{(1)})}\right),$$

where $\eta^\ell = \eta^\ell(\mathrm{meas}(A), \rho^{-1}, \mathscr{D}, \mathscr{N}(A))$ is an increasing function of its explicit parameters. (As it will be clear in the following, this explicit expression is particularly useful to show the stability result of Proposition 3.7.) Similarly, we can show that

$$\|u\|_{L^p(B^{(i)}\cap B^{(i-1)})} \leq \eta^\ell\left(\|\nabla u\|_{L^p(B^{(i)})} + \|u\|_{L^p(B^{(i)}\cap B^{(i+1)})}\right),$$

for all $i = 1, \dots, M-1$. Therefore, we can conclude by this chain of M extensions the following interpolation inequality

$$\|u\|_{L^p(A^\ell)} \le \eta \left(\|\nabla u\|_{L^p(A)} + \|u\|_{L^p(D)}\right), \tag{55}$$

where $\eta = \eta(\text{meas}(A), \rho^{-1}, \mathscr{D}, \mathscr{N}(A), M)$. However, let us note that the maximal number M of intersecting balls we need to reach D from $B^{(0)}$ is exclusively depending on $\mathscr{D}$, r, and $\rho < r$. Hence, we have actually $\eta = \eta(\text{meas}(A), \rho^{-1}, \mathscr{D}, \mathscr{N}(A))$. Now, summing over ℓ the inequality (55), we obtain the wanted result.

7.2 Proof of Proposition 3.7

We consider the net of sets $\Omega_\rho \subset\subset \Omega \setminus K$ as in (33). Note that for $\rho > 0$ small enough $D \subset\subset \Omega_\rho$ and $\Omega_\rho \subset\subset \Omega \setminus K_h$ for all $h \ge h_0$ sufficiently large. Hence, we can assume that $D \subset\subset \Omega \setminus K_h$ for all $h \ge h_0$ sufficiently large and that D has nontrivial intersection with all the connected components of $\Omega \setminus K_h$. Moreover, we recall from Lemma 3.4 that under our assumptions on L, c, all $\Omega \setminus K_h$ have the cone property as well as each of their connected components, with respect to a congruent cone $\mathscr{C}$, which depends only on L, c, δ, but not on h. In particular, the radius of a ball contained in $\mathscr{C}$ depends only on L, c, δ, but not on h. Applying Lemma 3.6, we obtain that for $u_h \in L^{1,p}(\Omega \setminus K_h)$

$$\|u_h\|_{L^p(\Omega)} \le \eta_h \left(\|\nabla u_h\|_{L^p(\Omega \setminus K_h)} + \|u_h\|_{L^p(D)}\right), \tag{56}$$

where η_h depends on $\mathscr{C} = \mathscr{C}(L, c, \delta)$, $\mathscr{D}_h = \text{diam}(\Omega \setminus K_h)$ and $\mathscr{N}(\Omega \setminus K_h)$. By convergence of K_h to K in the Hausdorff metric, we have $\mathscr{D}_h = \text{diam}(\Omega \setminus K_h) \to \mathscr{D} = \text{diam}(\Omega \setminus K)$. Moreover, $\mathscr{N}(\Omega \setminus K_h) \le \mathscr{N}(B_R(0))$ for any $R \gg 0$. Since also $\mathscr{C} = \mathscr{C}(L, c, \delta)$ does not depend on h, we eventually can conclude that $\eta_h \le \eta$ with η constant independent of h.

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