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Multiple solutions for superlinear Dirichlet problems with an indefinite potential

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Abstract We consider a semilinear elliptic equation with an indefinite unbounded potential and a Carathéodory reaction term that exhibits superlinear growth near $\pm\infty$ without satisfying the AR-condition. Also, at the origin, the primitive of the reaction satisfies a nonuniform nonresonance condition with respect to the first eigenvalue of $\left(-\triangle, H_0^1(\Omega)\right)$. Using critical point theory and Morse theory, we show that the problem has at least three nontrivial smooth solutions. Our result extends that of Wang (Anal Nonlineaire 8:43–58, 1991).

Keywords Superlinear reaction · Critical groups · Mountainpass type critical point · C-condition · Morse relation · Three solutions · Indefinite potential

Mathematics Subject Classification (2000) 35J20 · 35J61 · 58E05

1 Introduction

The starting point of this paper is the work of Wang [21], in which the author studies superlinear elliptic equations. More precisely, let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a regular boundary $\partial\Omega$. Wang [21] considers the following Dirichlet problem:

$$-\triangle u(z) = f\big(u(z)\big) \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{1}$$

Wang [21] assumes that $f \in C^1(\mathbb{R})$, $f(0) = f'(0) = 0$, $|f'(x)| \le c(1 + |x|^{r-2})$ for all $x \in \mathbb{R}$ with $c > 0$ and $1 < r < 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \ge 3 \\ +\infty & \text{if } N \le 2 \end{cases}$, there exist $\mu > 2$ and $M > 0$ such that

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$0 < \mu F(x) \leq f(x)x$ for all $|x| \geq M$ where $F(x) = \int_0^x f(s)\mathrm{d}s$ (the Ambrosetti–Rabinowitz condition). Using critical point theory and Morse theory, Wang [21] proved the existence of at least three nontrivial solutions.

The aim of our work here is to extend the aforementioned result of Wang [21] in many different ways. So, let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We consider the following Dirichlet problem:

$$-\triangle u(z) + \beta(z)u(z) = f\big(z, u(z)\big) \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{2}$$

Here, $\beta \in L^q(\Omega)$ with $q > N/2$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., for all, $x \in \mathbb{R}$, $z \longrightarrow f(z, x)$ is measurable and for almost all, $z \in \Omega$, $x \longrightarrow f(z, x)$ is continuous), which exhibits a superlinear growth near $\pm\infty$, but without necessarily satisfying the Ambrosetti–Rabinowitz condition (AR-condition for short). Note that the potential β is in general sign changing. So, in our formulation (see (2)), the linear part of the equation is indefinite and the reaction term f is z-dependent with only measurable dependence in the z-variable and $f(z, \cdot)$ need not be C^1. Hence, the energy functional of the problem is not C^2 and this makes the use of Morse theory problematic. Finally, as we already indicated, we do not use the AR-condition to express the superlinearity of $f(z, \cdot)$ and instead we use an alternative condition involving the function $\xi(z, x) = f(z, x)x - 2F(z, x)(F(z, x) = \int_0^x f(z, s)\mathrm{d}s)$, which incorporates in our framework superlinear reactions with slower growth near $\pm\infty$. Recall that the AR-condition says that there exist $\mu > 2$ and $M > 0$ such that

$$0 < \mu F(z, x) \leq f(z, x)x \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M \quad \text{(see [2])}. \tag{3}$$

Integrating (3), we obtain the following weaker condition

$$c_1|x|^\mu \leq F(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M, \text{ and with } c_1 > 0. \tag{4}$$

It is clear from (3) and (4) that the AR-condition excludes superlinear functions that exhibit "slow" growth near $\pm\infty$. For this reason, there have been efforts to replace (3). An overview of the relevant literature in this direction can be found in the recent works of Miyagaki and Souto [13] and Li and Yang [11]. Motivated by these works, in this paper, we replace the AR-condition (3) by a quasi-monotonicity condition on the function $x \longrightarrow \xi(z, x) = f(z, x)x - 2F(z, x)$ (see hypothesis H(iii)).

Additional multiplicity results were obtained by Chen and Shen [6], Geng [8], Wang and Tang [22], and Zou [23]. Zou [23] deals with semilinear problems, and Geng [8] and Wang and Tang [22] consider equations driven by the p-Laplacian, and Chen and Shen [6] examine a problem involving the p-mean curvature differential operator. In Geng [8] and Zou [23], the reaction $f(z, x)$ belongs in $C(\overline{\Omega} \times \mathbb{R})$, while in Chen and Shen [6] and Wang and Tang [22], the reaction is a Carathéodory function. With the exception of Wang and Tang [22], all the other works impose a symmetry condition of $f(z, \cdot)$, (namely that $f(z, -x) = -f(z, x)$ (oddness)), and using the symmetric mountain pass theorem or the fountain theorem, they establish the existence of a sequence of nontrivial solutions for the equation. In Wang and Tang [22], the authors using variational methods prove the existence of one or two solutions. In all the aforementioned works, the authors avoid the AR-conditions and introduce alternative weaker conditions to express the superlinearity of the reaction. So, Zou [23] assumes monotonicity of $x \longrightarrow f(z, x)\big/x$. Geng [8] and Wang and Tang [22] employ a condition first introduced in the literature by Jeanjean [10] (for semilinear problems). Wang and Tang [22] use the more restrictive global version of the Jeanjean condition, while Geng [8] uses the local version. Chen and Shen [6] use a different condition involving the function $\xi(z, x)$. All these conditions are more restrictive than H(iii) used in this paper.

In the next section, for the convenience of the reader, we briefly review the main mathematical tools that we will use in this work and introduce the hypotheses on $f(z, x)$.

2 Mathematical background-hypotheses

We start with critical point theory. So, let X be a Banach space, X^* its topological dual, and let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Cerami condition* (the C-condition for short), if the following holds:

> Every sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|x_n\|)\varphi'(x_n) \longrightarrow 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

This condition is in general weaker than the *Palais-Smale condition* (the PS-condition for short), which is usually used in critical point theory. It was shown by Bartolo et al. [3] that the *Deformation Theorem* and consequently the minimax theory of critical values remains valid if the PS-condition is replaced by the C-condition.

Using this compactness-type condition, we can state the following Theorem, which is known in the literature as the *mountain pass theorem*, and which is slightly more general than the original result of Ambrosetti and Rabinowitz [2].

Theorem 1 **If** *X is a Banach space, $\varphi \in C^1(X)$ and satisfies the C-condition, $x_0, x_1 \in X$, $\|x_1 - x_0\| > r > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} \leq \inf[\varphi(x) : \|x - x_0\| = r] = \eta_r$$
$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi\big(\gamma(t)\big) \quad \textit{where } \Gamma = \big\{\gamma \in C([0, 1], X) : \gamma(0) = x_0, \gamma(1) = x_1\big\},$$

then *$c \geq \eta_r$ and c is a critical value of φ; moreover, if $c = \eta_r$, then there exists a critical point $x \in X$ of φ such that $\|x - x_0\| = r$ and $\varphi(x) = c$.*

For $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{x \in X : \varphi(x) \leq c\}, \quad \dot{\varphi}^c = \{x \in X : \varphi(x) < c\}, \quad K_\varphi = \{x \in X : \varphi'(x) = 0\}$$
$$\text{and} \quad K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

Let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$, we denote the $k^{\underline{\text{th}}}$ relative singular homology group with integer coefficients for the pair (Y_1, Y_2). Recall that for $k < 0$, $H_k(Y_1, Y_2) = 0$. The critical groups of φ at an isolated critical point $x \in X$ with $\varphi(x) = c$ (i.e., $x \in K_\varphi^c$), are defined by

$$C_k(\varphi, x) = H_k\big(\varphi^c \cap U, \varphi^c \cap U \setminus \{x\}\big) \qquad \text{for all} \;\; k \geq 0,$$

with U a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$ (see, for example, Chang [5] and Mawhin and Willem [12]). The excision property of singular homology theory implies that the above definition of critical groups is independent of the particular choice of the neighborhood U of x.

Next, suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $-\infty < \inf \varphi(K_\varphi)$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \qquad \text{for all} \;\; k \geq 0.$$

(see Bartsch and Li [4]). The second deformation theorem, (see, for example, Papageorgiou and Kyritsi [15, p. 349]), implies that this definition of critical groups at infinity is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$. Suppose that K_φ is finite. We set

$$M(t,x) = \sum_{k\geq 0} \operatorname{rank} C_k(\varphi, x)t^k \quad \text{for all } t \in \mathbb{R}, \text{ and all } x \in K_\varphi$$

$$\text{and} \quad P(t,\infty) = \sum_{k\geq 0} \operatorname{rank} C_k(\varphi, \infty)t^k \quad \text{for all } t \in \mathbb{R}.$$

The *Morse relation* says that

$$\sum_{x\in K_\varphi} M(t,x) = P(t,\infty) + (1+t)Q(t), \tag{5}$$

where $Q(t) = \sum_{k\geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients (see Chang [5, p. 337] and Mawhin and Willem [12, p. 184]).

In the analysis of problem (2) in addition to the Sobolev space $H_0^1(\Omega)$, we will also use the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\}$. This is an ordered Banach space with positive cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \quad \text{for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\right\}$$

where by $n(\cdot)$, we denote the outward unit normal on $\partial\Omega$.

We consider the following linear eigenvalue problem

$$-\triangle u(z) + \beta(z)u(z) = \widehat{\lambda} u(z) \ \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{6}$$

The following condition will be valid for the potential $\beta(\cdot)$ throughout this work.

$\mathbf{H}_0$: $\beta \in L^q(\Omega), \quad q > N/2.$

For any $r \in (1,\infty)$ by r', we denote the conjugate exponent defined by $1/r + 1/r' = 1$. We have

$$2q' = 2\frac{q}{q-1} < 2^* = \frac{2N}{N-2}. \tag{7}$$

Then, the Sobolev embedding theorem implies that $H_0^1(\Omega)$ is embedded (compactly) in $L^{2q'}(\Omega)$. From this fact and Hölder's inequality, we have

$$\left|\int_\Omega \beta u^2 \mathrm{d}z\right| \leq \|\beta\|_q \|u\|_{2q'}^2. \tag{8}$$

We know that $2 < 2q' < 2^*$ (see (7)) and we have $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^{2q'}(\Omega) \hookrightarrow L^2(\Omega)$, where $\overset{c}{\hookrightarrow}$ denotes compact embedding. Invoking Ehrling's inequality (see, for example, Papageorgiou and Kyritsi [15, p. 698]), given $\varepsilon > 0$, we can find $c(\varepsilon) > 0$ such that

$$\|u\|_{2q'}^2 \leq \varepsilon\|u\|^2 + c(\varepsilon)\|u\|_2^2 \qquad \text{for all } u \in H_0^1(\Omega), \tag{9}$$

where $\|\cdot\|$ stands for the norm of $H_0^1(\Omega)$ defined by $\|u\| = \|Du\|_2$ for all $u \in H_0^1(\Omega)$ (by Poincaré's inequality). From (8) and (9), we have

$$\|Du\|_2^2 - \int_\Omega \beta u^2 dz \le \|Du\|_2^2 + \varepsilon\|\beta\|_q \|u\|^2 + c(\varepsilon)\|\beta\|_q \|u\|_2^2,$$
$$\Rightarrow \ \left(1 - \varepsilon\|\beta\|_q\right)\|u\|^2 \le \sigma(u) + c(\varepsilon)\|\beta\|_q \|u\|_2^2,$$

where $\sigma(u) = \|Du\|_2^2 + \int_\Omega \beta u^2 dz$ for all $u \in H_0^1(\Omega)$. Let $\varepsilon \in \left(0, \frac{1}{\|\beta\|_q}\right)$. Then,

$$\|u\|^2 \le c_1\left(\sigma(u) + \widehat{c}\,\|u\|_2^2\right) \ \text{ for some } \ c_1, \widehat{c} > 0, \ \text{all } u \in H_0^1(\Omega). \tag{10}$$

Consider the continuous bilinear form $\alpha : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\alpha(u, y) = c_1 \left[\int_\Omega (Du, Dy)_{\mathbb{R}^N} dz + \int_\Omega \beta u y dz \right] \qquad \text{for all } \ u, y \in H_0^1(\Omega).$$

Then, from (10), it follows that

$$\alpha(u, u) + c_1\widehat{c}\,\|u\|_2^2 \ge \|u\|^2 \qquad \text{for all } \ u \in H_0^1(\Omega). \tag{11}$$

Then, (11) and Corollary 7.D, p. 78 of Showalter [18], imply that problem (6) admits a sequence $\{\widehat{\lambda}_k\}_{k\ge1}$ of distinct eigenvalues such that

$$-c_1\widehat{c} < \widehat{\lambda}_1 < \widehat{\lambda}_2 < \cdots < \widehat{\lambda}_n \longrightarrow +\infty \ \text{ as } \ n \to \infty$$

and a corresponding sequence $\{\widehat{u}_n\}_{n\ge1} \subseteq H_0^1(\Omega)$ of eigenfunctions that form an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H_0^1(\Omega)$. Moreover, hypothesis H_0 and the regularity theory for semilinear Dirichlet problems (see Struwe [19, pp. 218–219]) imply that $\{\widehat{u}_n\}_{n\ge1} \subseteq C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. The eigenvalue $\widehat{\lambda}_1$ is simple and admits the following variational characterization

$$\widehat{\lambda}_1 = \inf \left[\frac{\sigma(u)}{\|u\|_2^2} : u \in H_0^1(\Omega), \ u \ne 0 \right]. \tag{12}$$

In (12), the infimum is realized on the one-dimensional eigenspace $E(\widehat{\lambda}_1)$. It is clear from (12) that every eigenfunction corresponding to $\widehat{\lambda}_1$ has constant sign. Let $\widehat{u}_1 \in C_0^{1,\alpha}(\overline{\Omega})$ be the L^2-normalized (i.e., $\|\widehat{u}_1\|_2 = 1$) eigenfunction for $\widehat{\lambda}_1$ such that $\widehat{u}_1(z) \ge 0$ for all $z \in \overline{\Omega}$. In fact, from Harnack's inequality (see Serrin [17] and Pucci and Serrin [16, p. 163]), we have $\widehat{u}_1(z) > 0$ for all $z \in \Omega$.

Lemma 1 **If** $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \le \widehat{\lambda}_1$ *a.e. in* Ω, $\vartheta \ne \widehat{\lambda}_1$, **then** *there exists* $c_2 > 0$ *such that* $\psi(u) = \sigma(u) - \int_\Omega \vartheta u^2 dz \ge c_2\|u\|^2$ *for all* $u \in H_0^1(\Omega)$.

Proof From (12) and the hypothesis on ϑ, we see that $\psi \ge 0$. Proceeding by contradiction, suppose that the Lemma is not true. Exploiting the 2-homogeneity of ψ, we can find $\{u_n\}_{n\ge1} \subseteq H_0^1(\Omega)$ such that

$$\|u_n\| = 1 \quad \text{for all } \ n \ge 1 \quad \text{and} \quad \psi(u_n) \longrightarrow 0^+ \ \text{ as } \ n \to \infty. \tag{13}$$

So, we may assume that

$$u_n \xrightarrow{w} u \ \text{ in } \ H_0^1(\Omega) \ \text{ and } \ u_n \longrightarrow u \ \text{ in } \ L^{2q'}(\Omega). \tag{14}$$

From (13) and (14), it follows that

$$\sigma(u) \leq \int_\Omega \vartheta u^2 dz \leq \widehat{\lambda}_1 \|u\|_2^2, \tag{15}$$
$$\Rightarrow \sigma(u) = \widehat{\lambda}_1 \|u\|_2^2 \quad \text{(see (12))},$$
$$\Rightarrow u \in E(\widehat{\lambda}_1).$$

If $u = 0$, then $Du_n \longrightarrow 0$ in $L^2(\Omega, \mathbb{R}^N)$ and so $u_n \longrightarrow 0$ in $H_0^1(\Omega)$ (see (14)), which contradicts the fact that $\|u_n\| = 1$ for all $n \geq 1$. Therefore, $u \in E(\widehat{\lambda}_1) \setminus \{0\}$ and so $|u(z)| > 0$ for all $z \in \Omega$. Then, from the first inequality in (15) and the hypothesis on ϑ, we have $\sigma(u) \leq \widehat{\lambda}_1 \|u\|_2^2$, which contradicts (12). This proves the Lemma. □

As we already mentioned, throughout this work for every $u \in H_0^1(\Omega)$, we set $\|u\| = \|Du\|_2$ and $u^\pm = \max\{\pm u, 0\}$. We know that $|u| = u^+ + -u^-$, $u = u^+ - u^-$, and $u^\pm \in H_0^1(\Omega)$. Also, by $|\cdot|_{\mathbb{R}^N}$, we denote the Lebesgue measure on $\mathbb{R}^N$.

The hypotheses on the reaction $f(z,x)$ are the following:

H: $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathédory function such that for a.a. $z \in \Omega$, $f(z,0) = 0$ and

- **(i)** $|f(z,x)| \leq \alpha(z) + c|x|^{r-1}$ for a.a $z \in \Omega$, all $x \in \mathbb{R}$, with $\alpha \in L^\infty(\Omega)_+$, $c > 0$, and $2 < r < 2^*$;
- **(ii)** $\lim\limits_{|x| \to \infty} \frac{F(z,x)}{x^2} = +\infty$ uniformly for a.a. $z \in \Omega$;
- **(iii)** if $\xi(z,x) = f(z,x)x - 2F(z,x)$, then there exists $\beta^* \in L^1(\Omega)_+$ such that
$$\xi(z,x) \leq \xi(z,y) + \beta^*(z) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq y \text{ or } y \leq x \leq 0;$$
- **(iv)** there exist $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leq \widehat{\lambda}_1$ a.e. in Ω, $\vartheta \neq \widehat{\lambda}_1$ and $\gamma > 0$ such that
$$\limsup_{x \to 0} \frac{2F(z,x)}{x^2} \leq \vartheta(z) \quad \text{and} \quad \liminf_{x \to 0} \frac{2F(z,x)}{x} \geq -\gamma.$$

Remark 1 Hypothesis H(iii) implies that for a.a. $z \in \Omega$, $F(z,\cdot)$ is superquadratic near $\pm\infty$. Moreover, hypotheses H(ii), (iii) imply that $\lim\limits_{|x| \to \infty} \frac{F(z,x)}{x^2} = +\infty$ uniformly for a.a. $z \in \Omega$ (see Li and Yang [11, Lemma 2.4]). Hence, for a.a. $z \in \Omega$, $f(z,\cdot)$ is superlinear near $\pm\infty$. Hypothesis H(iii) is a quasi-monotonicity condition on $\xi(z,\cdot)$. It is satisfied if there exists $M > 0$ such that for a.a. $z \in \Omega$, $x \longrightarrow \frac{f(z,x)}{x}$ is increasing on $x \geq M$ and decreasing on $x \leq -M$ (see, for example, Li and Yang [11]).

Example 1 The following function $f(x)$ satisfies hypotheses H (for the sake of simplicity, we drop the z-dependence):

$$f(x) = \begin{cases} \vartheta x - \frac{\vartheta}{2}|x|^{\tau-2}x & \text{if } |x| \leq 1 \\ x\left(\ln|x| + \frac{1}{2}\right) & \text{if } |x| > 1, \end{cases}$$

with $\vartheta < \widehat{\lambda}_1$ and $2 < \tau < \infty$. Note that this function does not satisfy the AR-condition.

Our method of proof employs also truncation techniques. For this purpose, we introduce the following functions:

$$\widehat{f}_+(z,x) = \begin{cases} 0 & \text{if } x \leq 0 \\ f(z,x) + \widehat{c}x & \text{if } x > 0 \end{cases} \quad \text{and} \quad \widehat{f}_-(z,x) = \begin{cases} f(z,x) + \widehat{c}x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases} \tag{16}$$

Both are Carathéodory functions. We consider the C^1-functionals $\widehat{\varphi}_\pm : H_0^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_\pm(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{c}}{2}\|u\|_2^2 - \int_\Omega \widehat{F}_\pm\big(z, u(z)\big)\mathrm{d}z \qquad \text{for all } \ u \in H_0^1(\Omega).$$

Also, let $\varphi : H_0^1(\Omega) \longrightarrow \mathbb{R}$ be the energy functional for problem (2) defined by

$$\widehat{\varphi}(u) = \frac{1}{2}\sigma(u) - \int_\Omega F\big(z, u(z)\big)\mathrm{d}z \qquad \text{for all } \ u \in H_0^1(\Omega).$$

Evidently $\varphi \in C^1\big(H_0^1(\Omega)\big)$.

3 Solutions of constant sign

In this section, we produce two constant sign smooth solutions (one positive and the other negative).

Proposition 1 **If** *hypotheses* H_0 *and H(i), (ii), (iii) hold,* **then** $\widehat{\varphi}_\pm$ *satisfy the C-condition.*

Proof We do the proof for $\widehat{\varphi}_+$, the proof for $\widehat{\varphi}_-$ being similar.

So, let $\{u_n\}_{n\geq 1} \subseteq H_0^1(\Omega)$ be a sequence such that

$$|\widehat{\varphi}_+(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \quad \text{all } n \geq 1 \tag{17}$$

$$\text{and} \quad (1 + \|u_n\|)\widehat{\varphi}_+'(u_n) \longrightarrow 0 \ \text{ in } H^{-1}(\Omega) = H_0^1(\Omega)^* \ \text{ as } n \to \infty. \tag{18}$$

From (18), we have

$$\Big| \langle A(u_n), h\rangle + \int_\Omega (\beta + \widehat{c})u_n h \mathrm{d}z - \int_\Omega \widehat{f}_+(z, u_n)h \mathrm{d}z \Big| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \tag{19}$$

for all $h \in H_0^1(\Omega)$ with $\varepsilon_n \to 0^+$, where $A \in \mathcal{L}\big(H_0^1(\Omega), H^{-1}(\Omega)\big)$ is defined by $\langle A(u), y\rangle = \int_\Omega (Du, Dy)_{\mathbb{R}^N} \mathrm{d}z$ for all $u, y \in H_0^1(\Omega)$.

In (19), we choose $h = -u_n^- \in H_0^1(\Omega)$ and obtain

$$\sigma(u_n^-) + \widehat{c}\, \|u_n^-\|_2^2 \leq \varepsilon_n \quad \text{for all } \ n \geq 1,$$

$$\Rightarrow \ u_n^- \longrightarrow 0 \ \text{ in } \ H_0^1(\Omega) \ \text{ as } n \to \infty \quad (\text{see } (10)). \tag{20}$$

Next, in (19), we choose $h = u_n^+ \in H_0^1(\Omega)$. Then,

$$-\sigma(u_n^+) + \int_\Omega f(z, u_n^+)u_n^+ \mathrm{d}z \leq \varepsilon_n \qquad \text{for all } \ n \geq 1 \quad (\text{see } (16)). \tag{21}$$

On the other hand from (17), (20), and (16), we have

$$\sigma(u_n^+) - 2\int_\Omega F(z, u_n^+)\mathrm{d}z \leq M_2 \quad \text{for some } M_2 > 0, \quad \text{all } n \geq 1. \tag{22}$$

Adding (21) and (22), we obtain

$$\int_\Omega \big[f(z, u_n^+)u_n^+ - 2F(z, u_n^+)\big]\mathrm{d}z \le M_3 \quad \text{for some } M_2 > 0, \quad \text{all } n \ge 1,$$
$$\Rightarrow \int_\Omega \xi(z, u_n^+)\mathrm{d}z \le M_3 \qquad \text{for all } \ n \ge 1. \tag{23}$$

Claim: $\{u_n\}_{n\ge 1} \subseteq H_0^1(\Omega)$ is bounded.

Suppose that the Claim is not true. We may assume that $\|u_n^+\| \to \infty$. We set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \ge 1$. Then, $\|y_n\| = 1$ for all $n \ge 1$, and so we may assume that

$$y_n \xrightarrow{w} y \ \text{ in } \ H_0^1(\Omega) \ \text{ and } \ y_n \longrightarrow y \ \text{ in } L^{2q'}(\Omega), \quad y \ge 0. \tag{24}$$

First, suppose $y \ne 0$ and let $Z(y) = \{z \in \Omega : y(z) = 0\}$. Then, $|\Omega \setminus Z(y)|_N > 0$ and $u_n^+(z) \longrightarrow +\infty$ for a.a. $z \in \Omega \setminus Z(y)$. Hypothesis H(ii) implies that

$$\frac{F\big(z, u_n^+(z)\big)}{\|u_n^+\|^2} = \frac{F\big(z, u_n^+(z)\big)}{u_n^+(z)^2}\, y_n(z)^2 \longrightarrow +\infty \ \text{ for a.a. } z \in \Omega \setminus Z(y),$$
$$\Rightarrow \int_\Omega \frac{F\big(z, u_n^+\big)}{\|u_n^+\|^2}\mathrm{d}z \longrightarrow +\infty \quad \text{(by Fatou's lemma)}. \tag{25}$$

From (17) and (20), we have

$$-\sigma(u_n^+) + \int_\Omega F(z, u_n^+)\mathrm{d}z \le M_4 \quad \text{for some } M_4 > 0, \quad \text{all } n \ge 1,$$
$$\Rightarrow \ -\sigma(y_n) + \int_\Omega \frac{F\big(z, u_n^+\big)}{\|u_n^+\|^2}\mathrm{d}z \le \frac{M_4}{\|u_n^+\|^2} \qquad \text{for all } \ n \ge 1,$$
$$\Rightarrow \int_\Omega \frac{F\big(z, u_n^+\big)}{\|u_n^+\|^2}\mathrm{d}z \le M_5 \quad \text{for some } M_5 > 0, \quad \text{all } n \ge 1 \quad \text{(see (24))}. \tag{26}$$

Comparing (25) and (26), we reach a contradiction.

Now, suppose $y = 0$. We can find $\lambda > 0$ such that

$$\|Dy_n\|_2^2 \ge \lambda > 0 \qquad \text{for all } \ n \ge 1. \tag{27}$$

Otherwise, for a subsequence $\{n_k\}_{k\ge 1}$, we have $Dy_{n_k} \longrightarrow 0$ in $L^2(\Omega, \mathbb{R}^N)$ and so $y_{n_k} \longrightarrow 0$ in $H_0^1(\Omega)$ (see (24)), contradicting the fact that $\|y_n\| = 1$ for all $n \ge 1$.

We consider the function $\zeta_n : [0, 1] \longrightarrow \mathbb{R}$ defined by

$$\zeta_n(t) = \varphi(tu_n^+) \qquad \text{for all } \ t \in [0, 1].$$

This is a continuous function and so we can find $t_n \in [0, 1]$ such that

$$\zeta_n(t_n) = \max\big[\zeta_n(t) : t \in [0, 1]\big] \quad n \ge 1. \tag{28}$$

For $\mu > 0$, we set $v_n = \left(\frac{2\mu}{\lambda}\right)^{1/2} y_n \in H_0^1(\Omega)$. Evidently, $v_n \longrightarrow 0$ in $L^r(\Omega)$ (recall that we assume $y = 0$). So, using hypothesis H(i), we have

$$\int_{\Omega} F(z, v_n)\mathrm{d}z \longrightarrow 0 \quad \text{as } n \to \infty. \tag{29}$$

Since $\|u_n^+\| \to +\infty$, we have $\left(\frac{2\mu}{\lambda}\right)^{1/2} \frac{1}{\|u_n^+\|} \in (0, 1)$ for all $n \geq n_0$. Hence,

$$\begin{aligned}
&\zeta_n(t_n) \geq \zeta_n\left(\left(\frac{2\mu}{\lambda}\right)^{1/2} \frac{1}{\|u_n^+\|}\right) \qquad \text{for all } n \geq n_0,\\
\Rightarrow\ &\varphi(t_n u_n^+) \geq \varphi\left(\left(\frac{2\mu}{\lambda}\right)^{1/2} y_n\right) = \varphi(v_n)\\
&\qquad = \frac{\mu}{\lambda}\sigma(y_n) - \int_{\Omega} F(z, v_n)\mathrm{d}z\\
&\qquad \geq \mu + \frac{\mu}{\lambda}\int_{\Omega} \beta y_n^2 \mathrm{d}z - \int_{\Omega} F(z, v_n)\mathrm{d}z \qquad \text{for all } n \geq n_0 \quad \text{(see (27))}.
\end{aligned}$$

But note that $\int_{\Omega} \beta y_n^2 \mathrm{d}z \longrightarrow 0$. This fact together with (29) imply that

$$\varphi(t_n u_n^+) \geq \frac{1}{2}\mu \qquad \text{for all } n \geq n_1 \geq n_0. \tag{30}$$

Since $\mu > 0$ is arbitrary from (30), it follows that

$$\varphi(t_n u_n^+) \longrightarrow +\infty \quad \text{as } n \to \infty. \tag{31}$$

Note that $0 \leq t_n u_n^+ \leq u_n^+$ for all $n \geq 1$. So, by virtue of hypothesis H(iii), we have

$$\int_{\Omega} \xi(z, t_n u_n^+)\mathrm{d}z \leq \int_{\Omega} \xi(z, u_n^+)\mathrm{d}z + \|\beta^*\|_1 \qquad \text{for all } n \geq 1. \tag{32}$$

Note $\varphi(0) = 0$ and by virtue of (17) and (20) we have $\widehat{\varphi}_+(u_n^+) = \varphi(u_n^+) \leq M_6$ for some $M_6 > 0$, all $n \geq 1$. Therefore, from (31), it follows that $t_n \in (0, 1)$ for all $n \geq n_2 \geq 1$. Then, from (28), we have

$$\begin{aligned}
0 = t_n \frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu_n^+)\Big|_{t=t_n} &= \langle \varphi'(t_n u_n^+), t_n u_n^+\rangle\\
&= \sigma(t_n u_n^+) - \int_{\Omega} f(z, t_n u_n^+)(t_n u_n^+)\mathrm{d}z \qquad \text{for all } n \geq n_2,
\end{aligned}$$

$$\Rightarrow\ \sigma(t_n u_n^+) = \int_{\Omega} f(z, t_n u_n^+)(t_n u_n^+)\mathrm{d}z \qquad \text{for all } n \geq n_2. \tag{33}$$

Returning to (32) and using (33), we obtain

$$\sigma(t_n u_n^+) - \int_\Omega 2F(z, t u_n^+)\mathrm{d}z \le \int_\Omega \xi(z, u_n^+)\mathrm{d}z + \|\beta^*\|_1 \quad \text{for all } n \ge n_2,$$

$$\Rightarrow \; 2\varphi(t_n u_n^+) \le \int_\Omega \xi(z, u_n^+)\mathrm{d}z + \|\beta^*\|_1 \quad \text{for all } n \ge n_2,$$

$$\Rightarrow \; \int_\Omega \xi(z, u_n^+)\mathrm{d}z \longrightarrow +\infty \text{ as } n \to \infty \quad (\text{see } (31)). \tag{34}$$

Comparing (23) and (34), we reach a contradiction. This proves the Claim.

By virtue of the Claim and (20), we have that $\{u_n\}_{n\ge1} \subseteq H_0^1(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } H_0^1(\Omega) \text{ and } u_n \longrightarrow u \text{ in } L^{2q'}(\Omega) \text{ and in } L^r(\Omega) \text{ as } n \to \infty. \tag{35}$$

In (19), we choose $h = u_n - u \in H_0^1(\Omega)$, pass to the limit as $n \to \infty$ and use (35). We obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$

$$\Rightarrow \; \|Du_n\|_2^2 \longrightarrow \|Du\|_2^2. \tag{36}$$

From (35), (36), and the Kadec–Klee property of Hilbert spaces, we have

$$Du_n \longrightarrow Du \text{ in } L^2(\Omega, \mathbb{R}^N),$$

$$\Rightarrow \; u_n \longrightarrow u \text{ in } H_0^1(\Omega).$$

This proves that $\widehat{\varphi}_+$ satisfies the C-condition. Similarly for $\widehat{\varphi}_-$. □

With minor modifications in the above proof (we omit the details), we obtain

Proposition 2 **If** *hypotheses* H_0 *and H(i), (ii), (iii) hold,* **then** φ *satisfies the C-condition.*

Proposition 3 **If** *hypotheses* H_0 *and H(i), (iv) hold,* **then** $u = 0$ *is a local minimizer of the functionals* $\widehat{\varphi}_\pm$ *and* φ.

Proof We do the proof for $\widehat{\varphi}_+$, the proofs for $\widehat{\varphi}_-$ and φ being similar.

Hypotheses H(i), (iv) imply that given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$F(z, x) \le \frac{1}{2}\big(\vartheta(z) + \varepsilon\big)x^2 + c_\varepsilon |x|^r \text{ for a.a. } z \in \Omega, \quad \text{all } x \in \mathbb{R}. \tag{37}$$

Then, for every $u \in H_0^1(\Omega)$, we have

$$\begin{aligned}
\widehat{\varphi}_+(u) &= \frac{1}{2}\sigma(u) + \frac{\widehat{c}}{2}\|u\|_2^2 - \int_\Omega \widehat{F}_+(z,u)\mathrm{d}z \\
&= \frac{1}{2}\sigma(u^+) + \frac{\widehat{c}}{2}\sigma(-u^-) + \frac{\widehat{c}}{2}\|u^-\|_2^2 - \int_\Omega F(z,u^+)\mathrm{d}z \quad \text{(see (16))} \\
&\geq \frac{1}{2}\sigma(u^+) - \frac{1}{2}\int_\Omega \vartheta(u^+)^2\mathrm{d}z - \frac{\varepsilon}{2}\|u\|^2 - c_3\|u\|^r \\
&\quad + \frac{1}{2}\left[\sigma(-u^-) + \widehat{c}\,\|u^-\|_2^2\right] \quad \text{for some } c_3 > 0 \quad \text{(see (37))} \\
&\geq \frac{c_2}{2}\|u^+\|^2 + \frac{1}{2c_1}\|u^-\|^2 - \frac{\varepsilon}{2}\|u\|^2 - c_3\|u\|^r
\end{aligned}$$

(see Lemma 1 and (10))

$$\geq \frac{1}{2}(c_4 - \varepsilon)\|u\|^2 - c_3\|u\|^r \quad \text{for some } c_4 > 0.$$

Choosing $\varepsilon \in (0, c_4)$, we infer that

$$\widehat{\varphi}_+(u) \geq c_5\|u\|^2 - c_3\|u\|^r \quad \text{for some } c_5 > 0, \quad \text{all } u \in H_0^1(\Omega). \tag{38}$$

Since $r > 2$, we can find $\varrho \in (0,1)$ small such that

$$\begin{aligned}
&\widehat{\varphi}_+(u) \geq 0 \quad \text{for all } u \in H_0^1(\Omega), \|u\| \leq \varrho \quad \text{(see (38))}, \\
\Rightarrow\ & u = 0 \ \text{ is a local minimizer of } \widehat{\varphi}_+.
\end{aligned}$$

Similarly for $\widehat{\varphi}_-$ and φ. □

We are ready to produce two nontrivial constant sign smooth solutions for problem (2).

Proposition 4 **If** *hypotheses* H_0 *and* H *hold,* **then** *problem* (2) *has two nontrivial constant sign smooth solutions* $u_0, v_0 \in C_0^{1,\alpha}(\overline{\Omega})(0 < \alpha < 1)$ *with* $v_0(z) < 0 < u_0(z)$ *for all* $z \in \Omega$.

Proof From Proposition 3, we know that $u = 0$ is a local minimizer of $\widehat{\varphi}_+$. We may assume that $u = 0$ is an isolated critical point of $\widehat{\varphi}_+$. Otherwise, we can find a sequence $\{u_n\}_{n\geq 1} \subseteq K_{\widehat{\varphi}_+}$ such that $u_n \to 0$ in $H_0^1(\Omega)$. Then, $A(u_n) + (\beta + \widehat{c})u_n = N_{\widehat{f}_+}(u_n)$ where $N_{\widehat{f}_+}(u)(\cdot) = \widehat{f}_+(\cdot, u(\cdot))$ for all $u \in H_0^1(\Omega)$. Acting with $-u_n^- \in H_0^1(\Omega)$, we obtain $u_n \geq 0$. Regularity theory (see [7]) and the Harnack's inequality (see [16,17]) imply that $u_n \in C_0^{1,\alpha}(\overline{\Omega})(0 < \alpha < 1)$ and $u_n(z) > 0$ for all $z \in \Omega$, all $n \geq 1$. Therefore, we are done. Assuming that $u = 0$ is an isolated critical point of $\widehat{\varphi}_+$ and reasoning as in Aizicovici et al. [1, see the proof of Proposition 29]), we can find $\varrho \in (0,1)$ small such that

$$0 = \widehat{\varphi}_+(0) < \inf\left[\widehat{\varphi}_+(u) : \|u\| = \varrho\right] = \widehat{\eta}_+. \tag{39}$$

Hypothesis H(ii) implies that

$$\widehat{\varphi}_+(t\widehat{u}_1) \longrightarrow -\infty \quad \text{as } t \to +\infty. \tag{40}$$

Proposition 1 together with (39) and (40) permits the use of Theorem 1 (the mountain pass theorem). So, we obtain $u_0 \in H_0^1(\Omega)$ such that

$$\widehat{\varphi}_+(0) = 0 < \widehat{\eta}_+ \leq \widehat{\varphi}_+(u_0) \tag{41}$$

$$\text{and } \widehat{\varphi}_+'(u_0) = 0. \tag{42}$$

From (41), we have $u_0 \neq 0$. From (42), it follows that

$$A(u_0) + (\beta + \widehat{c})u_0 = N_{\widehat{f}_+}(u_0),$$

from which as above we infer that $u_0 \geq 0$ and so

$$A(u_0) + \beta u_0 = N_f(u_0)$$

where $N_f(u)(\cdot) = f\big(\cdot, u(\cdot)\big)$ for all $u \in H_0^1(\Omega)$ (see (16)). Hence, u_0 solves problem (2)) and the regularity theory (see Struwe [19, pp. 218–219]) implies that $u_0 \in C_0^{1,\alpha}(\overline{\Omega})(0 < \alpha < 1)$. Moreover, Harnack's inequality (see Serrin [17]; Pucci and Serrin [16, p. 163]) implies that $u_0(z) > 0$ for all $z \in \Omega$.

Similarly, working this time with $\widehat{\varphi}_-$, we obtain one more constant sign smooth solution $v_0 \in C_0^{1,\alpha}(\overline{\Omega})(0 < \alpha < 1)$ with $v_0(z) < 0$ for all $z \in \Omega$. □

We can improve the conclusion of the above proposition, if we place an additional restriction on the potential $\beta(\cdot)$.

Proposition 5 **If** *hypotheses H_0 and H(i), (iv) hold, and $\beta^+ \in L^\infty(\Omega)$,* **then** *problem* (2) *has two nontrivial constant sign smooth solutions $u_0, v_0 \in C_0^{1,\alpha}(\overline{\Omega})$ with $u_0 \in intC_+$, $v_0 \in -intC_+$.*

Proof From Proposition 4, we know that problem (2) has two solutions

$$u_0, v_0 \in C_0^{1,\alpha}(\overline{\Omega}) \quad (0 < \alpha < 1) \quad \text{with} \quad v_0(z) < 0 < u_0(z) \quad \text{for all} \quad z \in \Omega.$$

Hypotheses H(i), (iv) imply that there exists $\gamma_0 > 0$ such that

$$f(z, x) \geq -\gamma_0 x \quad \text{for a.a. } z \in \Omega, \ \text{ all } x \in [0, \|u_0\|_\infty]. \tag{43}$$

We have

$$-\Delta u_0(z) + \big(\beta(z) + \gamma_0\big)u_0(z) = f\big(z, u_0(z)\big) + \gamma_0 u_0(z) \geq 0$$

a.e. in Ω (see (43)),

$$\Rightarrow \quad \Delta u_0(z) \leq \big(\beta^+(z) + \gamma_0\big)u_0(z) \leq \big(\|\beta^+\|_\infty + \gamma_0\big)u_0(z) \quad \text{a.e. in } \Omega,$$

$$\Rightarrow \quad u_0 \in \text{int } C_+ \quad \text{(see Vazquez [20] and Pucci and Serrin [16, p. 120])}.$$

Similarly, we show that $v_0 \in -\text{int } C_+$. □

4 Three-solutions theorem

In this section, using Morse theory, we produce a third nontrivial smooth solution and so we have the full multiplicity theorem for problem (2) (three-solutions theorem).

First, we compute the critical groups of φ at infinity.

Proposition 6 **If** *hypotheses H_0 and H hold,* **then** $C_k(\varphi, \infty) = 0$ *for all* $k \geq 0$.

Proof Hypothesis H(ii) implies that for every $u \in H_0^1(\Omega)$, $u \neq 0$, we have

$$\varphi(tu) \longrightarrow -\infty \quad \text{as } t \to +\infty. \tag{44}$$

Hypothesis H(iii) implies that for all $u \in H_0^1(\Omega)$, we have

$$0 = \xi(z,0) = \xi\big(z, u^+(z)\big) + \beta^*(z) \ \text{ and } \ 0 = \xi(z,0) \le \xi\big(z, -u^-(z)\big) + \beta^*(z)$$

a.e. in Ω,

$$\begin{aligned} &\Rightarrow \quad 0 \le \xi(z,0) \le \xi\big(z, u(z)\big) + \beta^*(z) \quad \text{a.e. in } \Omega, \\ &\Rightarrow \ -\xi\big(z, u(z)\big) = 2F\big(z, u(z)\big) - f\big(z, u(z)\big)u(z) \le \beta^*(z) \quad \text{a.e. in } \Omega. \end{aligned} \tag{45}$$

Then, for every $u \in H_0^1(\Omega)$ and every $t > 0$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,\varphi(tu) &= \langle \varphi'(tu), u\rangle \\ &= \frac{1}{t}\langle \varphi'(tu), tu\rangle \\ &= \frac{1}{t}\Big[\sigma(tu) - \int_\Omega f(z,tu)tu\mathrm{d}z\Big] \\ &\le \frac{1}{t}\Big[\sigma(tu) - \int_\Omega 2F(z,tu)\mathrm{d}z + \|\beta^*\|_1\Big] \quad \text{(see (45))} \\ &= \frac{1}{t}\big[2\varphi(tu) + \|\beta^*\|_1\big]. \end{aligned} \tag{46}$$

By virtue of (44) for $t > 0$ large, we have $\varphi(tu) < \widehat{\gamma} < -\frac{\|\beta^*\|_1}{2}$ and so from (46) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\varphi(tu) < 0. \tag{47}$$

Let $\partial B_1 = \big\{u \in H_0^1(\Omega) : \|u\| = 1\big\}$. For $u \in \partial B_1$, we can find a unique $\lambda(u) > 0$ such that $\varphi\big(\lambda(u)u\big) = \widehat{\gamma}$ and the implicit function theorem (see (47)) implies that $\lambda \in C(\partial B_1)$. We extend λ on $H_0^1(\Omega) \setminus \{0\}$ by setting

$$\lambda_0(u) = \frac{1}{\|u\|}\,\lambda\Big(\frac{u}{\|u\|}\Big) \quad \text{for all } \ u \in H_0^1(\Omega) \setminus \{0\}.$$

Evidently, $\lambda_0 \in C\big(H_0^1(\Omega) \setminus \{0\}\big)$ and $\varphi\big(\lambda_0(u)u\big) = \widehat{\gamma}$. Moreover, $\varphi(u) = \widehat{\gamma}$ implies that $\lambda_0(u) = 1$. We set

$$\widehat{\lambda}_0(u) = \begin{cases} 1 & \text{if } \varphi(u) < \widehat{\gamma} \\ \lambda_0(u) & \text{if } \varphi(u) \ge \widehat{\gamma}. \end{cases} \tag{48}$$

Then, $\widehat{\lambda}_0 \in C^1\big(H_0^1(\Omega) \setminus \{0\}\big)$. We consider the homotopy $h : [0,1] \times \big(H_0^1(\Omega) \setminus \{0\}\big) \longrightarrow H_0^1(\Omega) \setminus \{0\}$ defined by

$$h(t,u) = (1-t)u + t\widehat{\lambda}_0(u)u.$$

Then,

$$\begin{aligned} &h(0,u), h(1,u) = \widehat{\lambda}_0(u)u \in \varphi^{\widehat{\gamma}} \quad \text{for all } \ u \in H_0^1(\Omega) \setminus \{0\} \\ &h(t,\cdot)\Big|_{\varphi^{\widehat{\gamma}}} = \mathrm{id}\Big|_{\varphi^{\widehat{\gamma}}} \quad \text{for all } \ t \in [0,1] \quad \text{(see (48))}. \end{aligned}$$

These properties imply that $\varphi^{\widehat{\gamma}}$ is a strong deformation retract of $H_0^1(\Omega) \setminus \{0\}$. Using the radial retraction, we see that ∂B_1 is a deformation retract of $H_0^1(\Omega) \setminus \{0\}$ (see Dugundji [7, Theorem 6.5, p. 325]). So, it follows that

$$\begin{aligned}&\varphi^{\widehat{\gamma}} \text{ and } \partial B_1 \text{ are homotopy equivalent,}\\ \Rightarrow\ & H_k\big(H_0^1(\Omega), \varphi^{\widehat{\gamma}}\big) = H_k\big(H_0^1(\Omega), \partial B_1\big) \quad \text{for all } k \geq 0.\end{aligned} \tag{49}$$

Since $H_0^1(\Omega)$ is infinite dimensional, ∂B_1 is contractible in itself. Hence,

$$\begin{aligned}&H_k\big(H_0^1(\Omega), \partial B_1\big) = 0 \quad \text{for all } k \geq 0 \quad \text{(see Granas and Dugundji [9, p. 389]),}\\ \Rightarrow\ & H_k\big(H_0^1(\Omega), \varphi^{\widehat{\gamma}}\big) = 0 \quad \text{for all } k \geq 0 \quad \text{(see (49)),}\\ \Rightarrow\ & C_k(\varphi, \infty) = 0 \quad \text{for all } k \geq 0 \quad (\text{choose } \widehat{\gamma} < 0 \text{ with } |\widehat{\gamma}| > 0 \text{ big}).\end{aligned}$$

□

Next, we compute the critical groups at infinity for the functionals $\widehat{\varphi}_+$ and $\widehat{\varphi}_-$.

Proposition 7 **If** *hypotheses* H_0 *and* H *hold,* **then** $C_k(\widehat{\varphi}_+, \infty) = C_k(\widehat{\varphi}_-, \infty) = 0$ *for all* $k \geq 0$.

Proof Let $\widehat{\psi}_+ = \widehat{\varphi}_+\Big|_{C_0^1(\overline{\Omega})}$. The regularity theory for Dirichlet problems (see Struwe [19, p. 219]) implies that $K_{\widehat{\varphi}_+} \subseteq C_0^1(\overline{\Omega})$ and in fact $K_{\widehat{\varphi}_+} \subseteq C_+$. So, $K_{\widehat{\psi}_+} = K_{\widehat{\varphi}_+} = K$.

Because $C_0^1(\overline{\Omega})$ is dense in $H_0^1(\Omega)$, we can apply Theorem 16 of Palais [14] and have

$$H_k\big(H_0^1(\Omega), \dot{\widehat{\varphi}}_+^{\alpha}\big) = H_k\big(C_0^1(\overline{\Omega}), \dot{\widehat{\psi}}_+^{\alpha}\big) \quad \text{for all } k \geq 0. \tag{50}$$

Choosing $\alpha < \inf\limits_K \widehat{\psi}_+ = \inf\limits_K \widehat{\varphi}_+$, from (50) we have

$$C_k(\widehat{\varphi}_+, \infty) = C_k(\widehat{\psi}_+, \infty) \quad \text{for all } k \geq 0. \tag{51}$$

So, from (51), we see that in order to prove the proposition, it suffices to show that

$$C_k(\widehat{\psi}_+, \infty) = H_k\big(C_0^1(\overline{\Omega}), \widehat{\psi}_+^{\alpha}\big) = 0 \quad \text{for all } k \geq 0.$$

To this end, we introduce the following sets

$$\partial B_1^C = \big\{u \in C_0^1(\overline{\Omega}) : \|u\|_{C_0^1(\overline{\Omega})} = 1\big\} \text{ and } \partial B_{1,+}^C = \big\{u \in \partial B_1^C : u^+ \neq 0\big\}.$$

Let $h_+ : [0,1] \times \partial B_{1,+}^C \longrightarrow \partial B_{1,+}^C$ be the homotopy defined by

$$h_+(t,u) = \frac{(1-t)u + t\widehat{u}_1}{\|(1-t)u + t\widehat{u}_1\|_{C_0^1(\overline{\Omega})}} \quad \text{for all } (t,u) \in [0,1] \times \partial B_{1,+}^C.$$

Note that $h_+(1,u) = \frac{u}{\|u\|_{C_0^1(\overline{\Omega})}} \in \partial B_{1,+}^C$. Hence $\partial B_{1,+}^C$ is contractible in itself.

Hypothesis H(ii) implies that for all $u \in \partial B_{1,+}^C$, we have

$$\widehat{\psi}_+(tu) \longrightarrow -\infty \text{ as } t \to +\infty. \tag{52}$$

From (45), we have

$$-f\big(z, u(z)\big)u(z) \leq \beta^*(z) - 2F\big(z, u(z)\big) \quad \text{a.e. in } \Omega. \tag{53}$$

Let $\langle\cdot,\cdot\rangle_C$ denote the duality brackets for the pair $\left(C_0^1(\overline{\Omega})^*, C_0^1(\overline{\Omega})\right)$. We fix $u \in \partial B_{1,+}^C$. For all $t>0$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\widehat{\psi}_+(tu) &= \left\langle \widehat{\psi}_+'(tu), u\right\rangle_C \\ &= \frac{1}{t}\left\langle \widehat{\varphi}_+'(tu), tu\right\rangle \\ &= \frac{1}{t}\left[\sigma(tu)+\widehat{c}\,\|tu\|_2^2 - \int_\Omega \widehat{f}_+(z,tu)tu\,\mathrm{d}z\right] \\ &\le \frac{1}{t}\left[\sigma(tu)+\widehat{c}\,\|tu\|_2^2 - \int_\Omega 2\widehat{F}_+(z,tu)\mathrm{d}z + \|\beta^*\|_1\right]\end{aligned}$$

(see (16) and (53))

$$= \frac{1}{t}\left[2\widehat{\psi}_+(tu)+\|\beta^*\|_1\right]. \tag{54}$$

By virtue of (52) for $t>0$ big, we have $\widehat{\psi}_+(tu) < -\frac{\|\beta^*\|_1}{2}$ and so from (54), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\widehat{\psi}_+(tu) < 0. \tag{55}$$

Let $\overline{B}_1^C = \left\{u \in C_0^1(\overline{\Omega}) : \|u\|_{C_0^1(\overline{\Omega})} \le 1\right\}$ and choose $\alpha \in \mathbb{R}$ such that

$$\alpha < \min\left\{-\frac{\|\beta^*\|_1}{2}, \inf_{\overline{B}_1^C} \widehat{\psi}_+\right\}. \tag{56}$$

From the above, it follows that for every $u \in \partial B_{1,+}^C$, we can find a unique $\tau(u) \ge 1$ such that

$$\widehat{\psi}_+(tu) = \begin{cases} >\alpha & \text{if } t \in [0,\tau(u)) \\ =\alpha & \text{if } t = \tau(u) \\ <\alpha & \text{if } t > \tau(u). \end{cases} \tag{57}$$

Moreover, the implicit function theorem (see (55)) implies that the function $\tau : \partial B_{1,+}^C \longrightarrow [1,\infty)$ is continuous. Note that

$$\widehat{\psi}_+^\alpha = \left\{tu : u \in \partial B_{1,+}^C, t \ge \tau(u)\right\} \quad \text{(see (56) and (57))}. \tag{58}$$

Let $E_+ = \left\{tu : u \in \partial B_{1,+}^C, t \ge 1\right\}$. Then, $\widehat{\psi}_+^\alpha \subseteq E_+$ (see (58)). We consider the deformation $\widehat{h}_+ : [0,1]\times E_+ \longrightarrow E_+$ defined by

$$\widehat{h}_+(s,tu) = \begin{cases} (1-s)tu + s\tau(u)u & \text{if } t \in [0,\tau(u)] \\ tu & \text{if } t > \tau(u), \end{cases}$$

for all $s \in [0,1]$, all $t \ge 1$, all $u \in \partial B_{1,+}^C$. We have

$$\widehat{h}_+(0,tu) = tu, \quad \widehat{h}_+(1,tu) \in \widehat{\psi}_+^\alpha \quad \text{and} \quad \widehat{h}_+(s,\cdot)\Big|_{\widehat{\psi}_+^\alpha} = \mathrm{id}\Big|_{\widehat{\psi}_+^\alpha}$$

for all $s \in [0,1]$ (see (57))

$$\Rightarrow \widehat{\psi}_+^\alpha \text{ is a strong deformation retract of } E_+ \text{ and so}$$

$$H_k\left(C_0^1(\overline{\Omega}), E_+\right) = H_k\left(C_0^1(\overline{\Omega}), \widehat{\psi}_+^\alpha\right) \quad \text{for all } k \ge 0. \tag{59}$$

On the other hand, the deformation $\widetilde{h}_+ : [0,1] \times E_+ \longrightarrow E_+$ defined by

$$\widetilde{h}_+(s, tu) = (1-s)tu + s\,\frac{tu}{\|tu\|_{C_0^1(\overline{\Omega})}} \quad \text{for all } \ s \in [0,1], \ \text{all } t \geq 1, \ \text{all } u \in \partial B^C_{1,+},$$

shows that $\partial B^C_{1,+}$ is a deformation retract of E_+ (see Dugundji [7, p. 325]). Hence,

$$H_k\big(C_0^1(\overline{\Omega}), E_+\big) = H_k\big(C_0^1(\overline{\Omega}), \partial B^C_{1,+}\big) \quad \text{for all } \ k \geq 0. \tag{60}$$

From (59) and (60), it follows that

$$H_k\big(C_0^1(\overline{\Omega}), \widehat{\psi}_+^{\alpha}\big) = H_k\big(C_0^1(\overline{\Omega}), \partial B^C_{1,+}\big) \quad \text{for all } \ k \geq 0. \tag{61}$$

But recall that $\partial B^C_{1,+}$ is contractible in itself. Hence,

$$\begin{aligned} & H_k\big(C_0^1(\overline{\Omega}), \partial B^C_{1,+}\big) = 0 \quad \text{for all } \ k \geq 0 \quad \text{(see Granas and Dugundji [9, p.389])}, \\ \Rightarrow \ & H_k\big(C_0^1(\overline{\Omega}), \widehat{\psi}_+^{\alpha}\big) = 0 \quad \text{for all } \ k \geq 0 \quad \text{(see (61))}, \\ \Rightarrow \ & C_k(\widehat{\varphi}_+, \infty) = 0 \quad \text{for all } \ k \geq 0. \end{aligned}$$

Similarly, we show that $C_k(\widehat{\varphi}_-, \infty) = 0$ for all $k \geq 0$. □

Using this proposition and the stronger condition on the potential β (see Proposition 5), we can compute the critical groups of φ at the two constant sign smooth solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C+$ (see Proposition 5).

Proposition 8 **If** *hypotheses H_0 and H hold and $\beta^+ \in L^\infty(\Omega)$,* **then** $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ *for all $k \geq 0$.*

Proof First, we compute the critical groups of $\widehat{\varphi}_+$ at $u_0 \in \operatorname{int} C_+$. So, suppose that $K_{\widehat{\varphi}_+} = \{0, u_0\}$ (otherwise, recalling that $K_{\widehat{\varphi}_+} \subseteq C_+$ and since $\widehat{\varphi}_+\big|_{C_+} = \varphi\big|_{C_+}$, we see that we already have a third nontrivial (in fact positive) smooth solution of (2) and so we are done). Let $\mu < 0 < \nu < \widehat{\eta}_+$ (see (39)) and consider the following triple of sets

$$\widehat{\varphi}_+^{\mu} \subseteq \widehat{\varphi}_+^{\nu} \subseteq H_0^1(\Omega).$$

For this triple of sets, we consider the corresponding long exact sequence of singular homology groups

$$\ldots H_k\big(H_0^1(\Omega), \widehat{\varphi}_+^{\mu}\big) \xrightarrow{i_*} H_k\big(H_0^1(\Omega), \widehat{\varphi}_+^{\nu}\big) \xrightarrow{\partial_*} H_{k-1}\big(\widehat{\varphi}_+^{\nu}, \widehat{\varphi}_+^{\mu}\big) \longrightarrow \ldots \tag{62}$$

for all $k \geq 1$. Here, i_* is the group homomorphism induced by the inclusion $i : \big(H_0^1(\Omega), \widehat{\varphi}_+^{\mu}\big) \longrightarrow \big(H_0^1(\Omega), \widehat{\varphi}_+^{\nu}\big)$ and ∂_* is the boundary homomorphism. From (62) and the rank theorem, we have

$$\begin{aligned} \operatorname{rank} H_k\big(H_0^1(\Omega), \widehat{\varphi}_+^{\nu}\big) &= \operatorname{rank} \ker \partial_* + \operatorname{rank} \operatorname{im} \partial_* \\ &= \operatorname{rank} \operatorname{im} i_* + \operatorname{rank} \operatorname{im} \partial_* \quad \text{(by the exactness of (62))}. \end{aligned} \tag{63}$$

Since $\mu < 0$ and $K_{\widehat{\varphi}_+} = \{0, u_0\}$ (recall $\widehat{\varphi}_+(0) = 0 < \widehat{\eta}_+ \leq \widehat{\varphi}_+(u_0)$, see (41)), we have

$$\begin{aligned} & H_k\big(H_0^1(\Omega), \widehat{\varphi}_+^{\mu}\big) = C_k(\widehat{\varphi}_+, \infty) = 0 \quad \text{for all } \ k \geq 0 \quad \text{(see Proposition 7)}, \\ \Rightarrow \ & \operatorname{im} i_* = \{0\} \quad \text{(see (62))}. \end{aligned} \tag{64}$$

Also, since $\nu < \widehat{\eta}_+ \leq \widehat{\varphi}_+(u_0)$, we have

$$H_k\big(H_0^1(\Omega), \widehat{\varphi}_+^{\nu}\big) = C_k(\widehat{\varphi}_+, u_0) \quad \text{(see, for example, Chang [5]).} \tag{65}$$

Finally, since $\mu < 0 = \widehat{\varphi}_+(0) < \nu < \widehat{\eta}_+ \leq \widehat{\varphi}_+(u_0)$ and $u = 0$ is a local minimizer of $\widehat{\varphi}_+$ (see Proposition 3), it follows that

$$H_{k-1}\big(\widehat{\varphi}_+^{\nu}, \widehat{\varphi}_+^{\mu}\big) = C_{k-1}(\widehat{\varphi}_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \qquad \text{for all } k \geq 0. \tag{66}$$

Returning to (63) and using (64) through (66), we obtain

$$\operatorname{rank} C_1(\widehat{\varphi}_+, u_0) \leq 1. \tag{67}$$

Recall that u_0 is a critical point of $\widehat{\varphi}_+$ of mountain pass type (see the proof of Proposition 4). Hence,

$$\operatorname{rank} C_1(\widehat{\varphi}_+, u_0) \geq 1. \tag{68}$$

From (67) and (68) and since $C_k(\widehat{\varphi}_+, u_0) = 0$ for all $k \in \{0, 2, 3, \ldots\}$ (see (63) and (66)), we conclude that

$$C_k(\widehat{\varphi}_+, u_0) = \delta_{k,1}\mathbb{Z} \qquad \text{for all } k \geq 0. \tag{69}$$

We consider the homotopy $h_1 : [0, 1] \times H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$ defined by

$$h_1(t, u) = (1 - t)\varphi(u) + t\,\widehat{\varphi}_+(u) \qquad \text{for all } (t, u) \in [0, 1] \times H_0^1(\Omega).$$

Claim: We may assume that there exists $\varrho > 0$ small such that u_0 is the only critical point of $\big\{h_1(t, \cdot)\big\}_{t\in[0,1]}$ in $\overline{B}_{\varrho}(u_0) = \{u \in H_0^1(\Omega) : \|u - u_0\| \leq \varrho\|\}$.

Suppose that we can find $\{t_n\}_{n\geq 1} \subseteq [0, 1]$ and $\{u_n\}_{n\geq 1} \subseteq H_0^1(\Omega)$ such that

$$t_n \longrightarrow t \in [0, 1], u_n \longrightarrow u_0 \text{ in } H_0^1(\Omega) \text{ and } (h_1)'_u(t_n, u_n) = 0 \qquad \text{for all } n \geq 1. \tag{70}$$

From the equality in (70), we have

$$\begin{aligned} &A(u_n) + \beta u_n + t_n u_n = (1 - t_n)N_f(u_n) + t_n N_{\widehat{f}_+}(u_n) \\ \Rightarrow\ & -\triangle u_n(z) + \beta(z)u_n(z) = f\big(z, u_n^+(z)\big) + (1 - t_n) f\big(z, -u_n^-(z)\big) + t_n u_n^-(z) \end{aligned}$$

a.e. in Ω, (see (16)). From regularity theory, we can find $\alpha \in (0, 1)$ and $M_6 > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}) \text{ and } \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq M_6 \qquad \text{for all } n \geq 1 \tag{71}$$

(see Struwe [19, pp. 217, 219]). From the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and (71), we infer that by passing to a suitable subsequence if necessary, we may assume that

$$\begin{aligned} & u_n \longrightarrow u_0 \text{ in } C_0^1(\overline{\Omega}) \quad \text{(see (70))}, \\ \Rightarrow\ & u_n \in \operatorname{int} C_+ \qquad \text{for all } n \geq n_0 \quad \text{(recall that } u_0 \in \operatorname{int} C_+, \text{ see Proposition 5).} \end{aligned}$$

Since $\widehat{\varphi}_+\big|_{C_+} = \varphi\big|_{C_+}$, it follows that $\{u_n\}_{n\geq 1} \subseteq C_0^{1,\alpha}(\overline{\Omega})$ are all distinct positive smooth solutions of (2) and so we are done.

Propositions 1 and 2 imply that $h_1(0, \cdot) = \varphi$ and $h_1(1, \cdot) = \widehat{\varphi}_+$ satisfy the C-condition. So, by virtue of the homotopy invariance property of critical groups (see Chang [5, p. 334]), we have

$$\begin{aligned} & C_k\big(h_1(0, \cdot), u_0\big) = C_k\big(h_1(1, \cdot), u_0\big) \qquad \text{for all } k \geq 0, \\ \Rightarrow\ & C_k(\varphi, u_0) = C_k(\widehat{\varphi}_+, u_0) = \delta_{k,1}\mathbb{Z} \qquad \text{for all } k \geq 0 \quad \text{(see (69)).} \end{aligned}$$

Similarly, using this time $\widehat{\varphi}_-$ and $v_0 \in -\text{int}\, C_+$, we show that $C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$. □

Now, we are ready for the multiplicity theorem (three-solutions theorem) that extends the result of Wang [21].

Theorem 2 **If** *hypotheses* H_0 *and* H *hold and* $\beta^+ \in L^\infty(\Omega)$, **then** *problem* (2) *has at least three nontrivial smooth solutions* $u_0 \in intC_+$, $v_0 \in -intC_+$ *and* $y_0 \in C_0^{1,\alpha}(\overline{\Omega}) \setminus \{0\}(0 < \alpha < 1)$.

Proof From Proposition 5, we already have two constant sign smooth solutions $u_0 \in \text{int}\, C_+$ and $v_0 \in -\text{int}\, C_+$. Moreover, from Proposition 8, we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \qquad \text{for all} \quad k \geq 0. \tag{72}$$

From Proposition 3, we know that $u = 0$ is a local minimizer of φ. Hence

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \qquad \text{for all} \quad k \geq 0. \tag{73}$$

Finally, from Proposition 6, we have

$$C_k(\varphi, \infty) = 0 \qquad \text{for all} \quad k \geq 0. \tag{74}$$

Suppose that $\{0, u_0, v_0\}$ are the only critical points of φ. Then, from (72) through (74) and the Morse relation (5) with $t = -1$, we have $2(-1)^1 + (-1)^0 = 0$, a contradiction. This means that we can find $y_0 \in K_\varphi$ such that $y_0 \notin \{0, u_0, v_0\}$. Then, y_0 is a solution of problem (2) and the regularity theory (see Struwe [19, p. 219]) implies that $y_0 \in C_0^{1,\alpha}(\overline{\Omega}) \setminus \{0\}(0 < \alpha < 1)$. □

Remark 2 It is an interesting open question, whether we can drop the extra condition $\beta^+ \in L^\infty(\Omega)$. This condition was used to apply the maximum principle (see [16,20]), which in turn implied that $u_0 \in \text{int}\, C_+$, $v_0 \in -\text{int}\, C_+$, and these properties were crucial in the calculation of the critical groups of φ at u_0 and v_0. If $f(z, \cdot) \in C^1$ for a.a. $z \in \Omega$, then $\varphi \in C^2\big(H_0^1(\Omega)\big)$ and since u_0, v_0 are critical points of mountain pass type, we have $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$ (see Mawhin and Willem [12, p.195]). So, in that case, the extra restriction $\beta^+ \in L^\infty(\Omega)$ can be dropped. In our general setting, we do not know whether this is possible.

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References

1. Aizicovici, S., Papageorgiou, N.S., Staicu, V.: Degree theory for operators of Monotone type and nonlinear elliptic equations with inequality constraints. Mem. AMS **196**(915), iv+70 pp (2008)
2. Ambrosetti, A., Rabinowitz, P.: Dual variational methods in the critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)
3. Bartolo, P., Benci, V., Fortunato, D.: Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity. Nonlinear Anal. **7**, 981–1012 (1983)
4. Bartsch, T., Li, S.: Critical point theory as asymptotically quadratic functionals and applications to problems with resonance. Nonlinear Anal. **28**, 419–441 (1997)
5. Chang, K.C.: Infinite Dimensional Morse Theory and Multiple Solution Problems. Birkhäuser, Boston (1993)
6. Chen, Z., Shen, Y.: Infinitely many solutions of Dirichlet problem for p-mean curvature operator. Appl. Math. J. Chin. Univ. Ser. B **18**, 161–172 (2003)
7. Dugundji, J.: Topology. Allyn and Bacon, Boston (1966)

8. Geng, D.: Infinitely many solutions of p-Laplacian equations with limit subcritical growth. Appl. Math. Mech. (English Ed.) **28**, 1373–1382 (2007)
9. Granas, A., Dugundji, J.: Fixed Point Theory. Springer, New York (2003)
10. Jeanjean, L.: On the existence of bounded palais—smale sequences and applications to the Ladesman–Lazer-type problem in $\mathbb{R}^N$. Proc. Royal Soc. Edinb. **129**, 787–809 (1999)
11. Li, G., Yang, C.: The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti–Rabinowitz condition. Nonlinear Anal. **72**, 4602–4613 (2010)
12. Mawhin, J., Willem, M.: Critical Point Theory and Hamiltonian Systems. Springer, New York (1989)
13. Miyagaki, O., Souto, M.A.S.: Superlinear problems without Ambrosetti–Rabinowitz growth condition. J. Differ. Equ. **245**, 3628–3638 (2008)
14. Palais, R.: Homotopy theory of infinite dimensional manifolds. Topology **5**, 1–16 (1966)
15. Papageorgiou, N.S., Kyritsi-Yiallourou, S.Th.: Handbook of Applied Analysis. Springer, New York (2009)
16. Pucci, P., Serrin, J.: The Maximum Principle. Birkhäuser, Basel (2007)
17. Serrin, J.: Local behavior of solutions of quasilinear elliptic equations. Acta Math. **111**, 247–302 (1964)
18. Showalter, R.: Hilbert Space Methods for Partial Differential Equations. Pitman, London (1977)
19. Struwe, M.: Variational Methods. Springer, Berlin (1990)
20. Vazquez, J.L.: A strong maximum principle for some quasilinear elliptic equations. Appl. Math. Optim. **12**, 191–202 (1984)
21. Wang, Z.Q.: On a superlinear elliptic equation. Anal Nonlineaire **8**, 43–58 (1991)
22. Wang, J., Tang, C.L.: Rxistence and multiplicity of solutions for a class of superlinear p-Laplacian equations. Boundary Value Problems (Article ID 47275), 1–12 (2006)
23. Zou, W.: Variant fountain theorems and their applications. Manuscr. Math. **104**, 343–358 (2001)