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Quasiconvex variational functionals in Orlicz–Sobolev spaces

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Abstract We prove a $C^{1,\alpha}$ partial regularity result for minimizers of variational integrals of the type

$$J[u] := \int_\Omega f(\nabla u)\mathrm{d}x, \quad u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N,$$

where the integrand f is strictly quasiconvex and satisfies suitable growth conditions in terms of Young functions.

Keywords Quasiconvex · Young functions · Partial regularity

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1 Introduction

Let $n, N \in \mathbb{N}$ with $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be an open bounded set. In this paper we will consider variational integrals of the type

$$J[u] := \int_\Omega f(\nabla u)\,\mathrm{d}x, \tag{1.1}$$

where $u : \Omega \to \mathbb{R}^N$ and $f : \mathbb{R}^{nN} \to [0, \infty)$ is a C^2-function. In order to describe the special growth conditions of the density function f, let us consider two Young functions (see [2]) φ, ψ of class $C^2([0, \infty), [0, \infty))$ s.t. for $h \in \{\varphi, \psi\}$ it holds

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$$\lambda \frac{h'(t)}{t} \leq h''(t) \leq \Lambda \frac{h'(t)}{t} \quad \text{for all } t \geq 0, \tag{1.2}$$

where $\lambda, \Lambda > 0$ as well as

$$h(t) \geq bt^2 - c \quad \text{for all} \quad t \geq 0, \tag{1.3}$$

with $b > 0$ and $c \geq 0$. We will assume

$$c_1\varphi(|Z|) \leq f(Z) \leq c_2(1 + \psi(|Z|)) \quad \text{for all} \quad Z \in \mathbb{R}^{nN}, \tag{1.4}$$

with $c_1, c_2 > 0$, combined with the strict $W^{1,\varphi}$-quasiconvexity, i.e., for each $M > 0$ there is a constant $\gamma_M > 0$ such that

$$\fint_{B_1} f(A + \nabla\phi)\,\mathrm{d}x \geq f(A) + \gamma_M \fint_{B_1} \left(1 + \frac{\varphi'(|\nabla\phi|)}{|\nabla\phi|}\right) |\nabla\phi|^2\,\mathrm{d}x \tag{1.5}$$

for all $A \in \mathbb{R}^{nN}$ with $|A| \leq M + 1$ and all $\phi \in W_0^{1,\varphi}(B_1, \mathbb{R}^N)$. Furthermore, we need a condition between the functions φ and ψ limiting the range of the anisotropy; hence if $\mathcal{N} := \varphi \circ (\psi')^{-1}$ we will assume

$$\mathcal{N}^*(t) \leq c\varphi^\alpha(t) \quad \text{for all} \quad t \gg 1, \quad \alpha < \frac{n}{n-1}. \tag{1.6}$$

Here h^* denotes the complementary function of h. Note that from (1.6) we can immediately deduce the following inequality

$$\psi(t) \leq c\varphi^\alpha(t) \quad \text{for all} \quad t \gg 1. \tag{1.7}$$

In the special case $\varphi(t) = t^p$ and $\psi(t) = t^q$ (1.6) is equivalent to $q < p + \frac{1}{n}$.
Let us recall the definition of a local minimizer of J.

Definition 1.1 A map $u \in W^{1,\varphi}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is called a $W^{1,\varphi}$ local minimizer of J in Ω if one has $J[u] < \infty$ and

$$J[u] \leq J[u + \phi]$$

for every $\phi \in W_0^{1,\varphi}(\Omega, \mathbb{R}^N)$.

In this paper we are interested in proving partial $C^{1,\alpha}$-regularity of a local minimizer of J, i.e., regularity outside a closed set of Lebesgue measure zero. More precisely:

Theorem 1.1 *Let $u \in W^{1,\varphi}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of* (1.1) *under the assumptions* (1.2)–(1.6). *Then there is an open subset $\Omega_0 \subset \Omega$ s.t. $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$ for all $\alpha < 1$ and* $\mathcal{L}^n(\Omega \setminus \Omega_0) = 0$.

Let us observe that the existence of minima can be proved by using the direct methods in the Calculus of Variations. In fact, compactness theorems in $W^{1,\varphi}(\Omega, \mathbb{R}^N)$ combined with lower semicontinuity results guarantee the existence of such minima. In particular in [19] the authors studied the lower semicontinuity of the functional (1.1) with respect to weak $W^{1,p}$-convergence of $W^{1,q}$-functions under the condition $q < \frac{np}{n-1}$. In [18] it is also considered the case of general growth like (1.4) (see also [24]). Thus, starting from this existence result, the aim of this paper is to show a regularity theory for minimizers of quasiconvex variational integrals with (1.4)-growth. In this direction, it is worth to mention that in the power case when $p = q$ partial regularity results are originally developed by Evans [12] and next extended in [1] when $p \geq 2$ and in [3] and [4] when $1 < p < 2$ (see [20] for a good survey on this subject). In case of different growth $p - q$, we can refer to [21] and [22], but no corresponding regularity results were known if (1.4)-growth is supposed.

Let us observe that if this strong notion of $W^{1,\varphi}$-quasiconvexity is not available, one can think to a relaxation procedure to show that a similar result still holds following the ideas suggested in the papers [13] and [23], where an analysis with respect to the Lavrentiev phenomenon (non-occurring when the functional is convex and autonomous) is performed.

In the same way, very recently, there has been interest in functionals with non-standard growth depending also on (x, u); for this, a similar Lavrentiev phenomenon occurs. It is likely that the regularity analysis made in this paper extends to those as well. In particular, it could be interesting if bounds as $\frac{q}{p} < 1 + \frac{\alpha}{n}$ found in [13] admit a reformulation in the setting of Orlicz spaces too.

Now let us explain the main ideas of the proof.

The crucial points are essentially two: a Caccioppoli inequality and $\mathcal{A}$-harmonic approximation lemma. The main idea in proving Caccioppoli estimate is to construct suitable test functions by using smoothing operators from [15] revised in the Orlicz setting. Thus, by using these modified test functions, we gain the estimate desired with some perturbation terms on the right hand side.

Proceeding with the proof, we will use the $\mathcal{A}$-harmonic approximation method developed more recently in [11] (see also [7–9] and [10]). This method gives us a closeness in the norm L^2 of the minimizer u with respect to an $\mathcal{A}$-harmonic function w for which we have at our disposal an excess decay estimate. Next, by using an interpolation argument, we will be able to prove that this closeness holds also in the norm L^φ. Finally, the combination of these two arguments leads us to show an excess decay estimate also for ∇u. Next, by using Campanato's integral characterization of Hölder continuous functions, we obtain the desired result.

2 Auxiliary results

The following definitions and results are standard in the context of Young functions (see for instance [2]). A real function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a Young function if it satisfies the following conditions: $\varphi(0) = 0$ and there exists the derivative φ' of φ. This derivative is right continuous, non-decreasing and satisfies $\varphi'(0) = 0$, $\varphi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \varphi'(t) = \infty$. Especially, φ is convex.

We say that φ satisfies the Δ_2–condition, if there exists $c_1 > 0$ such that for all $t \geq 0$ holds $\varphi(2t) \leq c_1\, \varphi(t)$. By $\Delta_2(\varphi)$ we denote the smallest constant c_1. Since $\varphi(t) \leq \phi(2t)$ the Δ_2 condition is equivalent to $\varphi(2t) \sim \varphi(t)$. By L^φ and $W^{1,\varphi}$ we denote the classical Orlicz and Orlicz–Sobolev spaces, i. e. $f \in L^\varphi$ iff $\int \varphi(|f|)\, \mathrm{d}x < \infty$ and $f \in W^{1,\varphi}$ iff $f, \nabla f \in L^\varphi$. By $W_0^{1,\varphi}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,\varphi}(\Omega)$.

By $(\varphi')^{-1} : [0, \infty) \to [0, \infty)$ we denote the function

$$(\varphi')^{-1}(t) := \sup\{s \in [0, \infty) : \varphi'(s) \leq t\}.$$

If φ' is strictly increasing, then $(\varphi')^{-1}$ is the inverse function of φ'. Then $\varphi^* : [0, \infty) \to [0, \infty)$ defined by

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\, \mathrm{d}s$$

is again a Young function and $(\varphi^*)'(t) = (\varphi')^{-1}(t)$ for $t > 0$. It is the complementary function of φ. Note that $\varphi^*(t) = \sup_{s\geq 0}(st - \varphi(s))$ and $(\varphi^*)^* = \varphi$. For all $\delta > 0$ there exists c_δ (only depending on $\Delta_2(\varphi)$, $\Delta_2(\varphi^*)$) such that for all $t, s \geq 0$ holds

$$t\,s \leq \delta\,\varphi(t) + c_\delta\,\varphi^*(s). \tag{2.1}$$

This inequality is called Young's inequality.

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we define the non-centered maximal function of f by

$$Mf(x) := \sup_{B \ni x} \fint_B |f(y)|\,\mathrm{d}y, \tag{2.2}$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ which contain x. The following Lemma can be found in [17].

Lemma 2.1 *Let φ be a Young function with $\Delta_2(\varphi^*) < \infty$, then there exists $c > 0$ which only depends on $\Delta_2(\varphi^*)$ such that*

$$\int_{\mathbb{R}^n} \varphi(Mf)\,\mathrm{d}x \leq c \int_{\mathbb{R}^n} \varphi(|f|)\,\mathrm{d}x \quad \text{and} \quad \|Mf\|_\varphi \leq c\|f\|_\varphi \tag{2.3}$$

for all $f \in L^\varphi(\mathbb{R}^n)$.

Now let us collect some basic properties of Young functions and their corresponding tensor-functions appearing in the growth condition of f.

Lemma 2.2 *Let h be a Young function satisfying (1.2) and (1.3). Then we have*

(*a*) *h fulfills $\Delta_2(h) < \infty$ and $\Delta_2(h^*) < \infty$;*
(*b*) *for all $t > 0$ it holds*

$$h(1)(t^p - 1) \leq h(t) \leq h(1)(t^q + 1),$$

where $p = \lambda + 1$ and $q = \Lambda + 1$;
(*c*) *for all $t > 0$ it holds $h'(t)t \sim h(t)$.*

For the proof see Lemma 3.1 in [14].
We define for $h \in \{\varphi, \psi\}$

$$V^h(\xi) := \sqrt{\left(1 + \tfrac{h'(|\xi|)}{|\xi|}\right)}\,\xi, \quad \xi \in \mathbb{R}^{nN}.$$

Some basic properties of V^h are collected in the following lemma.

Lemma 2.3 *Let $h \in C^2([0,\infty),[0,\infty))$ be a Young function with property (1.2) then we have*

$$|V^h(A+B)|^2 \leq c\left(|V^h(A)|^2 + |V^h(B)|^2\right)$$
$$\frac{|V^h(A+B)|^2}{|A+B|} \leq c'\left(\frac{|V^h(A)|^2}{|A|} + \frac{|V^h(B)|^2}{|B|}\right)$$

for all A, B with constants c, c' only depending on the (Δ_2)-constant of h resp. h'.

Proof From the monotonicity of h and its (Δ_2)-condition, one can deduce

$$\begin{aligned}|V^h(A+B)|^2 &= |A+B|^2 + h'(|A+B|)|A+B| \\ &\le c\left(|A|^2+|B|^2+h(|A+B|)\right) \\ &\le c\left(|A|^2+|B|^2+h(2|A|)+h(2|B|)\right) \\ &\le c\left(|A|^2+|B|^2+h(|A|)+h(|B|)\right) \\ &\le c\left(|A|^2+|B|^2+\frac{h'(|A|)}{|A|}|A|^2+\frac{h'(|B|)}{|B|}|B|^2\right) \\ &= c\left(|V^h(A)|^2+|V^h(B)|^2\right).\end{aligned}$$

The proof of the second inequality follows in a similar fashion.
The following Lemma will be useful in the sequel.

Lemma 2.4 *For a function f, satisfying our assumptions, we have*

$$|f(A+B)-f(A)-Df(A)B| \le c|V^\psi(B)|^2,$$
$$|Df(A+B)-Df(A)| \le c\frac{|V^\psi(B)|^2}{|B|}$$

for all $A, B \in \mathbb{R}^{nN}$ with $|A| \le M+1$. Here c depends only on the constants in (1.4) and (1.5).

Proof we have

$$\begin{aligned}|f(A+B)-f(A)-Df(A)B| &= \left|\int_0^1\int_0^1 D^2f(A+\tau\sigma B)\,\mathrm{d}\tau\,\mathrm{d}\sigma(B,B)\right| \\ &\le c\left(1+\int_0^1\int_0^1\frac{\psi'(|A+\tau\sigma B|)}{|A+\tau\sigma B|}\,\mathrm{d}\tau\,\mathrm{d}\sigma\right)|B|^2.\end{aligned}$$

From [6] (Appendix, Lemma 20) we quote

$$\begin{aligned}\int_0^1\frac{\psi'(|A+\tau\sigma B|)}{|A+\tau\sigma B|}\,\mathrm{d}\tau &\le \frac{\psi'(|A|+|\sigma B|)}{|A|+|\sigma B|} \le \frac{\psi'(2|A|)}{|A|}+\frac{\psi'(2|\sigma B|)}{|\sigma B|} \\ &\le c(M)\left(1+\frac{\psi'(|\sigma B|)}{|\sigma B|}\right),\end{aligned}$$

where we used convexity of ψ and (Δ_2)-condition of ψ'. This shows

$$\begin{aligned}|f(A+B)-f(A)-Df(A)B| &\le c\left(1+\int_0^1\frac{\psi'(|\sigma B|)}{|\sigma B|}\,\mathrm{d}\sigma\right)|B|^2 \\ &\le c\left(1+\frac{\psi'(|B|)}{|B|}\right)|B|^2 = c|V^\psi(B)|^2,\end{aligned}$$

where we used again [6] (Appendix, Lemma 20). The arguments leading to the second inequality are quite similar. □

The main tool in our regularity approach is the generalization of the extension operator from Fonseca-Maly [15] to Orlicz spaces (see also [18]).

Lemma 2.5 *Let* $0 \le r < s$, $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$, $B_s \Subset \Omega$ *and* $\alpha < \frac{n}{n-1}$. *Then there is* $T_{r,s}u \in W^{1,\varphi}(B_s, \mathbb{R}^N)$ *with the following properties:*

(a) $T_{r,s}u = u$ *on* $B_s \setminus \overline{B}_r$*;*
(b) $T_{r,s}u \in u + \mathring{W}^{1,\varphi}(B_s \setminus \overline{B}_r, \mathbb{R}^N)$*;*
(c) $|\nabla T_{r,s}u| \le c T_{r,s}|\nabla u|$*;*
(d) the following estimates hold:

$$\int_{B_s \setminus B_r} \varphi(|T_{r,s}u|)\,\mathrm{d}x \le c \int_{B_s \setminus B_r} \varphi(|u|)\,\mathrm{d}x,$$

$$\int_{B_s \setminus B_r} \varphi(|\nabla T_{r,s}u|)\,\mathrm{d}x \le c \int_{B_s \setminus B_r} \varphi(|\nabla u|)\,\mathrm{d}x,$$

$$\int_{B_s \setminus B_r} \varphi^{\alpha}(|T_{r,s}u|)\,\mathrm{d}x \le c(s-r)^{-n\alpha+n+\alpha}\left[\sup_{t\in(r,s)} \frac{\theta(t)-\theta(r)}{t-r} + \sup_{t\in(r,s)} \frac{\theta(s)-\theta(t)}{s-t}\right],$$

$$\int_{B_s \setminus B_r} \varphi^{\alpha}(|\nabla T_{r,s}u|)\,\mathrm{d}x \le c(s-r)^{-n\alpha+n+\alpha}\left[\sup_{t\in(r,s)} \frac{\Theta(t)-\Theta(r)}{t-r} + \sup_{t\in(r,s)} \frac{\Theta(s)-\Theta(t)}{s-t}\right].$$

where

$$\theta(t) := \int_{B_t} \varphi(|u|)\,\mathrm{d}x, \quad \Theta(t) := \int_{B_t} \varphi(|\nabla u|)\,\mathrm{d}x.$$

Proof We follow the main ideas of [15]. Let $\eta \in C_0^\infty(\Omega)$ and $[t_1, t_2] \subset (0, \|\eta\|_\infty)$ with $0 < |\nabla\eta| < A$ on $\{t_1 \le \eta \le t_2\}$. If $(a, b) \subset [t_1, t_2]$ write $Z_a^b = \{a < \eta < b\}$. We define

$$T_{r,s}u(x) := \fint_{B_1(0)} u(x + \xi(x)y)\,\mathrm{d}y$$

where

$$\xi(x) := \frac{1}{2A}\max\{0, \min\{\eta(x) - r, s - \eta(x)\}\}.$$

We can quote (a)–(c) from [15]. Note that $T_{r,s}u = u$ if $x \notin Z_r^s$ and

$$T_{r,s}u(x) = \fint_{B_{\xi(x)}(x)} u(z)\,\mathrm{d}z, \quad x \in Z_r^s.$$

We denote by $\tilde{u}$ the extension of $u|_{Z_r^s}$ by zero to $\mathbb{R}^n$ and taking into account Lemma 2.1 we get

$$\begin{aligned}\int_{B_s \setminus B_r} \varphi(|T_{r,s}u|)\,\mathrm{d}x &= \int_{B_s \setminus B_r} \varphi(|T_{r,s}\tilde{u}|)\,\mathrm{d}x \le \int_{B_s \setminus B_r} \varphi(M(|\tilde{u}|))\,\mathrm{d}x \\ &\le \int_{\mathbb{R}^n} \varphi(M(|\tilde{u}|))\,\mathrm{d}x \le c\int_{\mathbb{R}^n} \varphi(|\tilde{u}|)\,\mathrm{d}x = c\int_{B_s \setminus B_r} \varphi(|u|)\,\mathrm{d}x,\end{aligned}$$

which is the first inequality in d).

Let $c := \frac{a+b}{2}$ and abbreviate

$$M_0 := \sup_{t \in (r,s)} (t-r)^{-1} \int_{Z_r^t} \varphi(|u|)\, \mathrm{d}x.$$

W.l.o.g. we can assume that u is smooth, since the general case is a consequence of a standard approximation argument. If $\rho \in (0, \frac{1}{4}(b-a))$ and $z \in \{\eta = a + 2\rho\}$ then $\xi(z) = \frac{\rho}{A}$ and $B_{\xi(z)}(z) \subset Z_{a+\rho}^{a+3\rho}$. Thus, Jensen's inequality implies for $\alpha \geq 1$

$$\begin{aligned}\varphi^\alpha(|T_{r,s}u|) &\leq c\rho^{-n\alpha} \left[\int_{B_{\xi(z)}(z)} \varphi(|u|)\, \mathrm{d}y \right]^\alpha \\ &\leq \rho^{-n\alpha} \left[\int_{Z_{a+\rho}^{a+3\rho}} \varphi(|u|)\, \mathrm{d}y \right]^{\alpha-1} \left[\int_{B_{\xi(z)}(z)} \varphi(|u|)\, \mathrm{d}y \right].\end{aligned}$$

Lemma 2.1 in [15] applied to the last integral implies

$$\int_{\eta = a+2\rho} \varphi^\alpha(|T_{r,s}u|)\, \mathrm{d}\mathcal{H}^{n-1} \leq \rho^{-n\alpha+n-1} \left[\int_{Z_{a+\rho}^{a+3\rho}} \varphi(|u|)\, \mathrm{d}y \right]^\alpha.$$

This finally leads us to

$$\begin{aligned}\int_{Z_a^c} \varphi^\alpha(|T_{r,s}u|)\, \mathrm{d}x &\leq c \int_0^{\frac{1}{4}(s-r)} \rho^{-n\alpha+n-1} \left[\int_{Z_{a+\rho}^{a+3\rho}} \varphi(|u|)\, \mathrm{d}y \right]^\alpha \mathrm{d}\rho \\ &\leq c M_0^\alpha (s-r)^{-n\alpha+n+\alpha}.\end{aligned}$$

This shows the third inequality in d). The estimations for $\nabla T_{r,s}u$ are consequences of c). □

Another important tool in our proof will be the following generalization of the Sobolev–Poincaré inequality due to Diening and Ettwein [6] (see also [5]).

Lemma 2.6 (*Theorem* 7, [6]) *Let $B \subset \mathbb{R}^n$ be some ball with diameter R. Further, let φ be a Young function with $\Delta_2(\varphi), \Delta_2(\varphi^*) < \infty$. Then for all $1 < r < \frac{n}{n-1}$ there is $K > 0$, which only depend on $\Delta_2(\varphi), \Delta_2(\varphi^*)$, R and r, such that for all $v \in W^{1,\varphi}(B)$ holds*

$$\left(\fint_B \varphi^r \left(\frac{|v - v_B|}{R} \right) \mathrm{d}x \right)^{\frac{1}{r}} \leq K \fint_B \varphi(|\nabla v|) \mathrm{d}x. \tag{2.4}$$

Note that this Lemma is actually a slight modification of Theorem 7 in [6].

3 Caccioppoli inequality

In this section we prove a Caccioppoli inequality, which will be the main tool in the proof of the partial regularity in the next section.

Lemma 3.1 *Let the assumptions of Theorem 1.1 be true and* $v = u - \zeta - A(x - x_0)$ *where* $\zeta \in \mathbb{R}^N$ *and* $A \in \mathbb{R}^{nN}$ *with* $|A| \le M + 1$. *Then we have for all* $B_\rho \Subset \Omega$

$$\fint_{B_{\rho/2}} |V^\varphi(\nabla v)|^2 \,\mathrm{d}x \le c \fint_{B_\rho} \left| V^\varphi\left(\frac{v}{s-r}\right)\right|^2 \mathrm{d}x + c\left\{\fint_{B_\rho}\left[|V^\varphi(\nabla v)|^2 + \left|V^\varphi\left(\frac{v}{s-r}\right)\right|^2\right]\mathrm{d}x\right\}^\alpha .$$

Proof We follow the lines of Schmidt [21] and consider $\frac{\rho}{2} \le r < s \le \rho$ as well as

$$\Xi(t) := \int_{B_t}\left[\varphi(|\nabla u|) + \varphi\left(\frac{|u|}{s-r}\right)\right]\mathrm{d}x.$$

In accordance with [15], Lemma 2.3, there are $\tilde{r}$ and $\tilde{s}$ with $r \le \tilde{r} \le \tilde{s} \le s$ such that

$$\begin{aligned} \frac{\Xi(t) - \Xi(\tilde{r})}{t - \tilde{r}} &\le 3\frac{\Xi(s) - \Xi(r)}{s - r}, \\ \frac{\Xi(\tilde{s}) - \Xi(t)}{\tilde{s} - t} &\le 3\frac{\Xi(s) - \Xi(r)}{s - r} \end{aligned} \tag{3.1}$$

for every $t \in (\tilde{r}, \tilde{s})$ as well as

$$\frac{s-r}{3} \le \tilde{s} - \tilde{r} \le s - r. \tag{3.2}$$

Now we choose a cutoff function $\eta \in C_0^\infty(B_{\tilde{s}})$ with $\eta \equiv 1$ on $B_{\tilde{r}}$, $0 \le \eta \le 1$ and $|\nabla \eta| \le c/(\tilde{s} - \tilde{r})$. For the function $(1-\eta)v$, where $v(x) = u(x) - \zeta - A(x - x_0)$ with $\zeta \in \mathbb{R}^N$ and $A \in R^{nN}$, $|A| \le M$, we define, using the operator from Lemma 2.5,

$$\Psi := T_{\tilde{r},\tilde{s}}((1-\eta)v), \quad \phi := v - \Psi.$$

Hence we have $\phi \in \mathring{W}^{1,\varphi}(B_{\tilde{s}}, \mathbb{R}^N)$ and $\phi = v$ on $B_{\tilde{r}}$. As a consequence of (1.5), we get

$$\begin{aligned} \gamma_M \int_{B_{\tilde{r}}} |V^\varphi(\nabla v)|^2 \,\mathrm{d}x &\le \int_{B_{\tilde{s}}} [f(A + \nabla\phi) - f(A)]\,\mathrm{d}x \\ &= \int_{B_{\tilde{s}}} [f(\nabla u - \nabla\Psi) - f(\nabla u)]\,\mathrm{d}x + \int_{B_{\tilde{s}}} [f(\nabla u) - f(\nabla u - \nabla\phi)]\,\mathrm{d}x \\ &\quad + \int_{B_{\tilde{s}}} [f(A + \nabla\Psi) - f(A)]\,\mathrm{d}x, \end{aligned}$$

where we also used $\phi = v$ on $B_{\tilde{r}}$ and

$$\nabla u - A = \nabla v = \nabla\phi + \nabla\Psi \text{ on } B_{\tilde{s}}.$$

By the minimality of u and Lemma 2.4, we obtain

$$\gamma_M \int_{B_{\tilde{s}}} |V^{\varphi}(\nabla v)|^2 \, \mathrm{d}x \leq \int_{B_{\tilde{s}}} \int_0^1 [Df(A) - Df(A - \tau\nabla\Psi)] \, \mathrm{d}\tau : \nabla\Psi \, \mathrm{d}x$$
$$+ \int_{B_{\tilde{s}}} [f(A + \nabla\Psi) - f(A) - Df(A) : \nabla\Psi] \, \mathrm{d}x$$
$$\leq c \int_{B_{\tilde{s}}} \int_0^1 \frac{|V^{\psi}(\nabla v - \tau\nabla\Psi)|^2}{|\nabla v - \tau\nabla\psi|} \, \mathrm{d}\tau |\nabla\Psi| \, \mathrm{d}x$$
$$+ c \int_{B_{\tilde{s}}} |V^{\psi}(\nabla\Psi)|^2 \, \mathrm{d}x.$$

As a consequence of Lemma 2.3, we have

$$\int_{B_{\tilde{s}}} \int_0^1 \frac{|V^{\psi}(\nabla v - \tau\nabla\Psi)|^2}{|\nabla v - \tau\nabla\Psi|} \, \mathrm{d}\tau |\nabla\Psi| \, \mathrm{d}x \leq \int_{B_{\tilde{s}}} \frac{|V^{\psi}(\nabla v)|^2}{|\nabla v|} |\nabla\Psi| \, \mathrm{d}x$$
$$+ \int_{B_{\tilde{s}}} \int_0^1 \frac{|V^{\psi}(\tau\nabla\Psi)|^2}{|\tau\nabla\Psi|} \, \mathrm{d}\tau |\nabla\Psi| \, \mathrm{d}x.$$

Since the last integral is bounded by

$$\int_{B_{\tilde{s}}} \left(|\nabla\Psi|^2 + \psi'(|\nabla\Psi|)|\nabla\Psi| \right) \, \mathrm{d}x,$$

we arrive at

$$\int_{B_{\tilde{r}}} |V^{\varphi}(\nabla v)|^2 \, \mathrm{d}x \leq c \int_{B_{\tilde{s}}} |V^{\psi}(\nabla\Psi)|^2 \, \mathrm{d}x + c \int_{B_{\tilde{s}}} \frac{|V^{\psi}(\nabla v)|^2}{|\nabla v|} |\nabla\Psi| \, \mathrm{d}x =: J_1 + J_2.$$

In order to handle J_1, we decompose as follows: from (1.7) we deduce for $K \gg 1$

$$J_1 = c \int_{[|\nabla\Psi| \leq K]} \ldots \mathrm{d}x + c \int_{[|\nabla\Psi| > K]} \ldots \mathrm{d}x$$
$$\leq c \int_{B_{\tilde{s}}} |\nabla\Psi|^2 \, \mathrm{d}x + c \int_{B_{\tilde{s}}} \varphi^{\alpha}(|\nabla\Psi|) \, \mathrm{d}x$$
$$=: J_1^1 + J_1^2.$$

Using Lemma 2.5 we obtain by definition of Ψ, (3.1) and (3.2)

$$\begin{aligned} J_1^2 \le & c(\tilde{s}-\tilde{r})^{-n\alpha+n+\alpha}\left\{\sup_{t\in(\tilde{r},\tilde{s})}(t-\tilde{r})^{-1}\int_{B_t\setminus B_{\tilde{r}}}\varphi(|\nabla[(1-\eta)v]|)\,\mathrm{d}x\right\}^{\alpha} \\ & + c(\tilde{s}-\tilde{r})^{-n\alpha+n+\alpha}\left\{\sup_{t\in(\tilde{r},\tilde{s})}(\tilde{s}-t)^{-1}\int_{B_{\tilde{s}}\setminus B_t}\varphi(|\nabla[(1-\eta)v]|)\,\mathrm{d}x\right\}^{\alpha} \\ = & c(\tilde{s}-\tilde{r})^{-n\alpha+n+\alpha}\left\{\sup_{t\in(\tilde{r},\tilde{s})}\frac{\Xi(t)-\Xi(\tilde{r})}{t-\tilde{r}}\right\}^{\alpha} + c(\tilde{s}-\tilde{r})^{-n\alpha+n+\alpha}\left\{\sup_{t\in(\tilde{r},\tilde{s})}\frac{\Xi(\tilde{s})-\Xi(t)}{\tilde{s}-t}\right\}^{\alpha} \\ \le & c(s-r)^{-n\alpha+n}\left\{\Xi(s)-\Xi(r)\right\}^{\alpha}. \end{aligned} \tag{3.3}$$

Due to the superquadratic growth of φ (see (1.3)) and Lemma 2.5, we have

$$J_1^1 \le c\int_{B_s\setminus B_r}|V^{\varphi}(\nabla v)|^2\,\mathrm{d}x + c\int_{B_\rho}\left|V^{\varphi}\left(\frac{v}{s-r}\right)\right|^2\,\mathrm{d}x. \tag{3.4}$$

The term J_2 is bounded by

$$\begin{aligned} J_2 \le & c\int_{B_{\tilde{s}}\setminus B_{\tilde{r}}}|\nabla v||\nabla\Psi|\,\mathrm{d}x + c\int_{B_{\tilde{s}}\setminus B_{\tilde{r}}\cap[|\nabla v|\ge 1]}\psi'(|\nabla v|)|\nabla\Psi|\,\mathrm{d}x \\ =: & J_2^1 + J_2^2. \end{aligned}$$

The first term can be estimated as J_1^1 by Lemma 2.5. For the second one we get

$$\begin{aligned} J_2^2 \le & c\int_{B_{\tilde{s}}\setminus B_{\tilde{r}}}\varphi(|\nabla v|)\,\mathrm{d}x + c\int_{B_{\tilde{s}}\setminus B_{\tilde{r}}\cap[|\nabla v|\ge 1]}\mathcal{N}*(|\nabla\Psi|)\,\mathrm{d}x \\ \le & c\int_{B_{\tilde{s}}\setminus B_{\tilde{r}}}\varphi(|\nabla v|)\,\mathrm{d}x + c\int_{B_{\tilde{s}}\setminus B_{\tilde{r}}}\varphi^{\alpha}(|\nabla\Psi|)\,\mathrm{d}x \\ \le & c(s-r)^{-n\alpha+n}\left\{\Xi(s)-\Xi(r)\right\}^{\alpha}, \end{aligned}$$

where we took into account (1.6), (3.3), (3.4), Young's inequality and again Lemma 2.5. Plugging all together we get

$$\begin{aligned} \int_{B_r}|V^{\varphi}(\nabla v)|^2\,\mathrm{d}x \le & c\int_{B_s\setminus B_r}|V^{\varphi}(\nabla v)|^2\,\mathrm{d}x + c\int_{B_\rho}\left|V^{\varphi}\left(\frac{v}{s-r}\right)\right|^2\,\mathrm{d}x \\ & + c(s-r)^n\left\{(s-r)^{-n}\int_{B_\rho}\left[|V^{\varphi}(\nabla v)|^2 + \left|V^{\varphi}\left(\frac{v}{s-r}\right)\right|^2\right]\mathrm{d}x\right\}^{\alpha}. \end{aligned}$$

Hence the hole-filling method implies for some $\chi < 1$

$$\int_{B_r} |V^\varphi(\nabla v)|^2\,\mathrm{d}x \le \chi \int_{B_s} |V^\varphi(\nabla v)|^2\,\mathrm{d}x + c\int_{B_\rho} \left|V^\varphi\left(\frac{v}{s-r}\right)\right|^2\,\mathrm{d}x + c(s-r)^n\left\{(s-r)^{-n}\int_{B_\rho}\left[|V^\varphi(\nabla v)|^2 + \left|V^\varphi\left(\frac{v}{s-r}\right)\right|^2\right]\mathrm{d}x\right\}^\alpha.$$

Thus we may deduce by a well-known lemma of Giaquinta (see [16], Chapter V, Lemma 3.1)

$$\fint_{B_{\rho/2}} |V^\varphi(\nabla v)|^2\,\mathrm{d}x \le c\fint_{B_\rho} \left|V^\varphi\left(\frac{v}{s-r}\right)\right|^2\,\mathrm{d}x + c\left\{\fint_{B_\rho}\left[|V^\varphi(\nabla v)|^2 + \left|V^\varphi\left(\frac{v}{s-r}\right)\right|^2\right]\mathrm{d}x\right\}^\alpha.$$

□

4 Partial regularity

Before coming to the proof of the decay estimate, we present the concept of the $\mathcal{A}$-harmonic approximation (see for instance [11]). We consider a bilinear form $\mathcal{A}$ defined on $\mathbb{R}^{nN}$, which is bounded i.e. there is $L > 0$ s.t.

$$|\mathcal{A}| \le L$$

and fulfills the Legendre-Hadamard condition i.e. for all $a \in \mathbb{R}^n$, $b \in \mathbb{R}^N$

$$\mathcal{A}_{ij}^{\alpha\beta} a^i b_\alpha a^j b_\beta \ge k_{\mathcal{A}} |a|^2 |b|^2$$

for some $k_{\mathcal{A}} > 0$. Let us observe that the biggest possible constant $k_{\mathcal{A}}$ is called the ellipticity constant of $\mathcal{A}$.

We call a function $w \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ $\mathcal{A}$-harmonic if

$$\int_\Omega \mathcal{A}(\nabla w, \nabla \phi)\,\mathrm{d}x = 0$$

for all $\phi \in C_0^\infty(\Omega, \mathbb{R}^d)$. Here we have the following standard result.

Lemma 4.1 *Every $\mathcal{A}$-harmonic function w belongs to the space $C^\infty(\Omega, \mathbb{R}^d)$ and for all $B_\rho \Subset \Omega$ we have*

$$\sup_{B_{\rho/2}} |\nabla w| + \rho \sup_{B_{\rho/2}} |\nabla^2 w| \le c \fint_{B_\rho} |\nabla w|\,\mathrm{d}x,$$

where c depends only on $n, N, k_{\mathcal{A}}, L$.

The crucial tool for our approach is the following lemma that corresponds to Lemma 6.8 in [21].

Lemma 4.2 *Fix $\epsilon > 0$. Then there is $\delta = \delta(\epsilon)$ such that the following assertion holds: $\forall s \in (0, 1]$ and $\forall u \in W^{1,\varphi}(B_\rho, \mathbb{R}^N)$ such that*

$$\fint_{B_\rho} |V^\varphi(\nabla u)|^2 \,\mathrm{d}x \le s^2$$

as well as

$$\left| \fint_{B_\rho} \mathcal{A}(\nabla u, \nabla\phi) \,\mathrm{d}x \right| \le s\delta \sup_{B_\rho} |\nabla\phi|$$

for all $\phi \in C_0^\infty(B_\rho, \mathbb{R}^d)$, there is an $\mathcal{A}$-harmonic function $w \in C^\infty(B_\rho, \mathbb{R}^N)$ with

$$\sup_{B_{\rho/2}} |\nabla w| + \rho \sup_{B_{\rho/2}} |\nabla^2 w| \le c$$

$$\fint_{B_{\rho/2}} \left| V^\varphi\left(\frac{u - sw}{\rho}\right)\right|^2 \mathrm{d}x \le s^2\epsilon.$$

Proof Without loss of generality we assume $B_\rho(x_0) = B_1(0) =: B_1$, otherwise we can argue by a scaling argument. If we define $v := \frac{u}{s}$ we have $\fint_{B_1} |\nabla v|^2 \,\mathrm{d}x \le 1$ and

$$\left| \fint_{B_\rho} \mathcal{A}(\nabla v, \nabla\phi) \,\mathrm{d}x \right| \le \delta \sup_{B_\rho} |\nabla\phi|$$

for all $\phi \in C_0^\infty(B_\rho, \mathbb{R}^d)$. From [11] it follows the existence of an $\mathcal{A}$-harmonic function $w \in W^{1,2}(B_1, \mathbb{R}^N)$ with

$$\fint_{B_1} |\nabla w|^2 \,\mathrm{d}x \le 1, \quad \fint_{B_1} |v - w|^2 \,\mathrm{d}x \le \epsilon. \tag{4.1}$$

Note that we may assume $\fint_{B_{1/2}} (v - w) \,\mathrm{d}x = 0$. Hence we get by Lemma 4.1 $w \in C^\infty(B_1, \mathbb{R}^N)$ and

$$\sup_{B_{1/2}} |\nabla w| + \rho \sup_{B_{1/2}} |\nabla^2 w| \le c.$$

Now we use an interpolation argument. Since from (4.1) we can deduce

$$\fint_{B_{1/2}\cap[|u-sw|\le 1]} \left|V^\varphi\,(u - sw)\right|^2 \,\mathrm{d}x \le cs^2\epsilon,$$

we only consider integrals over $\mathcal{B} := B_{1/2} \cap [|u - sw| > 1]$. For $t \in (0, 1)$ we get

$$\begin{aligned}\int_{\mathcal{B}} |V^\varphi(|u - sw|)|^2 \mathrm{d}x &\le c \int_{\mathcal{B}} \varphi(|u - sw|)\mathrm{d}x = c \int_{\mathcal{B}} \varphi(|u - sw|)^t \varphi(|u - sw|)^{1-t} \mathrm{d}x \\ &\le c \left(\int_{B_{1/2}} \varphi(|u - sw|)^r \mathrm{d}x \right)^{\frac{t}{r}} \cdot \left(\int_{\mathcal{B}} \varphi(|u - sw|)^{\frac{(1-t)r}{r-t}} \mathrm{d}x \right)^{\frac{r-t}{r}}\end{aligned}$$

where $r > 1$. Next, let us observe that Lemma 2.2 implies $\varphi^{\frac{(1-t)r}{r-t}}(z) \le cz^2$ on $(1,\infty)$ for t such that $\frac{(1-t)r}{r-t} = \frac{2}{q}$. Thus, applying Sobolev–Poincaré inequality (2.4) we get

$$\le c\left(\int_{B_1} \varphi(|\nabla(u-sw)|)\mathrm{d}x\right)^t \left(\int_{\mathcal{B}} |u-sw|^2\mathrm{d}x\right)^{\frac{r-t}{r}}.$$

Now we use (4.1) in order to get

$$\begin{aligned}
&\le c\left(\int_{B_1} \varphi(|\nabla(u-sw)|)\mathrm{d}x\right)^t (s^2\epsilon)^{\frac{r-t}{r}}\\
&\le c\left(\int_{B_1} \varphi(|\nabla u|)+\varphi(s|\nabla w|)\mathrm{d}x\right)^t s^{2(1-t)}\epsilon^{1-t}\\
&\le cs^{2t}s^{2(1-t)}\epsilon^{1-t}
\end{aligned}$$

where we used $s\cdot\epsilon^{\frac{1}{2}} \le 1$ and $\frac{r-t}{r} > 1-t$.

Replacing ϵ by a smaller quantity conveniently, we obtain the claim. □

For $u \in W^{1,\varphi}(B_\varrho(x_0),\mathbb{R}^N)$ and $A \in \mathbb{R}^{nN}$, let us define the excess

$$\Phi_\varphi(x_0,\varrho,A) := \fint_{B_\varrho(x_0)} |V^\varphi(\nabla u - A)|^2\mathrm{d}x. \tag{4.2}$$

Since $D^2 f$ is uniformly continuous on bounded sets we get for each $M > 0$ the existence of a modulus of continuity $\nu_M : [0,\infty) \to [0,\infty)$ with the following properties

(i) $|D^2 f(A) - D^2 f(B)| \le \nu_M(|A-B|^2)$ for all $A, B \in \mathbb{R}^{nN}$ with $|A| \le M$ and $|B| \le M+1$;
(ii) ν_M is non-decreasing;
(iii) ν_M^2 is concave;
(iv) $\nu_M^2(t) \ge t$ for all $t \ge 0$.

This observation leads to the following lemma.

Lemma 4.3 *Under the assumptions of Theorem 1.1, if $u \in W^{1,\varphi}(B_\varrho(x_0),\mathbb{R}^N)$ is a $W^{1,\varphi}$-minimizer of J on $B_\varrho(x_0)$, we have for all $A \in \mathbb{R}^{nN}$ with $|A| \le M$ and all $\phi \in C_0^\infty(B_\rho)$*

$$\left|\fint_{B_\rho} D^2 f(A)(\nabla w,\nabla\phi)\,\mathrm{d}x\right| \le c_M\sqrt{\Phi_\varphi(x_0,\rho)}\,\nu_M(\Phi_\varphi(x_0,\rho))\|\nabla\phi\|_\infty.$$

Here we have abbreviated $\Phi_\varphi(x_0,\varrho,A)$ by $\Phi_\varphi(x_0,\varrho)$ and setted $w := u - Ax$.

Proof The Euler equation of J gives

$$\left|\ \fint_{B_\rho} D^2 f(A)(\nabla w, \nabla\phi)\,\mathrm{d}x\right|$$
$$= \left|\ \fint_{B_\rho} \left[D^2 f(A)(\nabla w, \nabla\phi) + Df(A) : \nabla\phi - Df(\nabla u) : \nabla\phi\right]\mathrm{d}x\right|. \tag{4.3}$$

If $|\nabla w| \le 1$ we have

$$\begin{aligned}
&|D^2 f(A)(\nabla w, \nabla\phi) + Df(A) : \nabla\phi - Df(\nabla u) : \nabla\phi| \\
&\le \int_0^1 |D^2 f(A) - D^2 f(A + t\nabla w)|\,\mathrm{d}t\,|\nabla w|\|\nabla\phi\|_\infty \\
&\le \nu_M(|\nabla w|^2)|\nabla w|\|\nabla\phi\|_\infty \\
&\le \nu_M(|V^\varphi(\nabla w)|^2)|V^\varphi(\nabla w)|\|\nabla\phi\|_\infty.
\end{aligned} \tag{4.4}$$

Otherwise we get, as a consequence of Lemma 2.4 and (1.7),

$$\begin{aligned}
&|D^2 f(A)(\nabla w, \nabla\phi) + Df(A) : \nabla\phi - Df(\nabla u) : \nabla\phi| \\
&\le c_M\left(|\nabla w| + \frac{|V^\psi(\nabla w)|^2}{|\nabla w|}\right)\|\nabla\phi\|_\infty \le c_M\psi'(|\nabla w|)\|\nabla\phi\|_\infty \\
&\le c_M\varphi(|\nabla w|)\|\nabla\phi\|_\infty \le c_M|V^\varphi(\nabla w)|^2\|\nabla\phi\|_\infty.
\end{aligned} \tag{4.5}$$

Combining (4.3)–(4.5) with property (iv) of ν_M we arrive at

$$\left|\ \fint_{B_\rho} D^2 f(A)(\nabla w, \nabla\phi)\,\mathrm{d}x\right| \le c_M\|\nabla\phi\|_\infty \fint_{B_\rho} \nu_M(|V^\varphi(\nabla w)|^2)|V^\varphi(\nabla w)|\,\mathrm{d}x$$

From (ii) by using Cauchy-Schwarz and Jensen inequalities we get

$$\left|\ \fint_{B_\rho} D^2 f(A)(\nabla w, \nabla\phi)\,\mathrm{d}x\right| \le c_M\sqrt{\Phi_\varphi(x_0,\rho)}\,\nu_M(\Phi_\varphi(x_0,\rho))\|\nabla\phi\|_\infty$$

which is the claim. □

Let us observe that quasiconvexity of f implies the Legendre–Hadamard condition for $D^2 f$.

Now we are ready to formulate the main lemma of this section from which the claim of Theorem 1.1 follows by standard iteration arguments.

Proposition 4.4 *Let the assumptions of Theorem 1.1 hold. Let us consider $M > 0$ and $\beta \in (0, 1/2)$. Then there is $\epsilon = \epsilon(M, \beta)$ and $\theta \in (0, 1)$ such that*

$$\Phi_\varphi(x_0,\rho) := \Phi_\varphi(x_0,\rho,(\nabla u)_{x_0,\varrho}) \le \epsilon, \quad |(\nabla u)_{x_0,\rho}| \le M \tag{4.6}$$

for a local $W^{1,\varphi}$-minimizer $u \in W^{1,\varphi}(B_\rho(x_0), \mathbb{R}^N)$ imply

$$\Phi_\varphi(x_0,\theta\rho) \le \theta^{2\beta}\Phi_\varphi(x_0,\rho).$$

Proof We define $A := (\nabla u)_{x_0,\rho}$ so that $|A| \leq M$, $v = u - Ax$ and $s := \sqrt{\Phi_\varphi(x_0, \rho)}$. Since the case $\Phi_\varphi(x_0, \rho) = 0$ is trivial, we can assume $\Phi_\varphi(x_0, \rho) \neq 0$. We have

$$\fint_{B_\rho} |V^\varphi(\nabla v)|^2 \, \mathrm{d}x = s^2.$$

We will use the $\mathcal{A}$-harmonic approximation, where $\mathcal{A} := D^2 f(A)$. On account of Lemma 4.2 we get

$$\left| \fint_{B_\rho} \mathcal{A}(\nabla v, \nabla \phi) \, \mathrm{d}x \right| \leq c s \nu_M(\Phi_\varphi(x_0, \rho)) \sup_{B_\rho} |\nabla \phi|$$

for all $\phi \in C_0^\infty(B_\rho, \mathbb{R}^N)$. For a given ϵ, which we will fix later, we can find, by Lemma 4.2, a $\delta > 0$ s.t.

$$c\nu_M(\Phi_\varphi(x_0, \rho)) \leq \delta, \tag{4.7}$$

$$s = \sqrt{\Phi_\varphi(x_0, \rho)} \leq 1. \tag{4.8}$$

As a consequence of Lemma 4.2, we obtain an $\mathcal{A}$-harmonic function $w \in C^\infty(B_\rho, \mathbb{R}^N)$ with

$$\begin{aligned} \sup_{B_{\rho/2}} |\nabla w| + \rho \sup_{B_{\rho/2}} |\nabla^2 w| &\leq C^* \\ \fint_{B_{\rho/2}} \left| V^\varphi\left(\frac{v - sw}{\rho}\right)\right|^2 \mathrm{d}x &\leq s^2 \epsilon. \end{aligned} \tag{4.9}$$

For $\theta \in (0, \frac{1}{4}]$ we deduce by Taylor expansion

$$\sup_{B_{2\theta\rho}(x_0)} |w(x) - w(x_0) - \nabla w(x_0)(x - x_0)| \leq c\theta^2 \rho. \tag{4.10}$$

Hence by Lemma 2.3, the last inequality, (4.9) and the Δ_2-property of φ

$$\begin{aligned} &\fint_{B_{2\theta\rho}} \left| V^\varphi\left(\frac{v - sw(x_0) - s\nabla w(x_0)(x - x_0)}{2\theta\rho}\right)\right|^2 \mathrm{d}x \\ &\leq c \fint_{B_{2\theta\rho}} \left| V^\varphi\left(\frac{v - sw}{2\theta\rho}\right)\right|^2 \mathrm{d}x \\ &\quad + c \fint_{B_{2\theta\rho}} \left| V^\varphi\left(s\frac{w(x) - w(x_0) - \nabla w(x_0)(x - x_0)}{2\theta\rho}\right)\right|^2 \mathrm{d}x \\ &\leq cK(\theta) \fint_{B_{\rho/2}} \left| V^\varphi\left(\frac{v - sw}{\rho}\right)\right|^2 \mathrm{d}x \\ &\quad + c \fint_{B_{2\theta\rho}} \left| V^\varphi\left(s\frac{w(x) - w(x_0) - \nabla w(x_0)(x - x_0)}{2\theta\rho}\right)\right|^2 \mathrm{d}x \\ &\leq cK(\theta)s^2\epsilon + c\theta^2 s^2 \end{aligned}$$

We also used (4.10) in the last inequality. If we define $\epsilon := \frac{\theta^2}{K(\theta)}$ we get

$$\fint_{B_{2\theta\rho}} \left| V^{\varphi}\left(\frac{u - Ax - sw(x_0) - s\nabla w(x_0)(x - x_0)}{2\theta\rho}\right)\right|^2 \mathrm{d}x \le c\theta^2 \Phi_{\varphi}(x_0, \rho). \tag{4.11}$$

From (4.9) we deduce $s^2|\nabla w(x_0)|^2 \le C\Phi_{\varphi}(x_0, \rho)$ thus

$$\begin{aligned} &\Phi_{\varphi}(x_0, 2\theta\rho, A + s\nabla w(x_0)) \qquad (4.12)\\ &\quad \le c\left((2\theta)^{-n} \fint_{B_{\rho/2}} |V^{\varphi}(\nabla u - A)|^2 \mathrm{d}x + |V^{\varphi}(s\nabla w(x_0))|^2\right) \le c\theta^{-n}\Phi_{\varphi}(x_0, \rho, A). \end{aligned}$$

Taking into account (4.11), (4.12) and the Caccioppoli inequality (with $\zeta := sw(x_0)$ and $A + s\nabla w(x_0)$ instead of A), we get

$$\begin{aligned} &\Phi_{\varphi}(x_0, \theta\rho, A + s\nabla w(x_0))\\ &\quad \le c\left(\theta^2\Phi_{\varphi}(x_0, \rho) + \theta^{2\alpha}\Phi_{\varphi}(x_0, \rho)^{\alpha} + \theta^{-n\frac{n}{n-1}}\Phi_{\varphi}(x_0, \rho)^{\alpha}\right). \end{aligned} \tag{4.13}$$

Let us observe that

$$C^*\Phi_{\varphi}(x_0, \rho) \le 1. \tag{4.14}$$

implies $|A + s\nabla w(x_0)| \le M + 1$. In addition, if $\Phi_{\varphi}(x_0, \rho)$ is small enough (depending on θ), we have

$$\theta^{-n\alpha}\Phi_{\varphi}(x_0, \rho)^{\alpha} \le \theta^2\Phi_{\varphi}(x_0, \rho). \tag{4.15}$$

Since $\theta \le 1$ we finally arrive at

$$\Phi_{\varphi}(x_0, \theta\rho, A + s\nabla w(x_0)) \le \theta^2 c\Phi_{\varphi}(x_0, \rho).$$

The mean value is minimizing (up to a constant depending on φ), hence by the choice of A,

$$\Phi_{\varphi}(x_0, \theta\rho) = \Phi_{\varphi}(x_0, \theta\rho, A) \le c\theta^2\Phi_{\varphi}(x_0, \rho)$$

as well as

$$\Phi_{\varphi}(x_0, \theta\rho) \le \theta^{2\beta}\Phi_{\varphi}(x_0, \rho)$$

if θ is small enough. □

Proposition 4.5 *Let the assumptions of Theorem 1.1 hold. Let us consider $M > 0$ and $\beta \in (0, 1/2)$. Then there is $\epsilon = \epsilon(M, \beta)$ such that*

$$\Phi_{\varphi}(x_0, \rho) := \Phi_{\varphi}(x_0, \rho, (\nabla u)_{x_0,\varrho}) \le \epsilon, \quad |(\nabla u)_{x_0,\rho}| \le \frac{M}{2} \tag{4.16}$$

for a local $W^{1,\varphi}$-minimizer $u \in W^{1,\varphi}(B_{\rho}(x_0), \mathbb{R}^N)$ imply

$$\Phi_{\varphi}(x_0, r) \le c\left(\frac{r}{\rho}\right)^{2\beta}\Phi_{\varphi}(x_0, \rho) \quad \forall r \in (0, \rho].$$

Here c depends on $n, N, \varphi, M, \beta, c_2, \lambda, \Lambda, \gamma_M$.

The proof can be obtained by using a standard iteration argument applied to the previous Proposition.

Proof of Theorem 1.1 From Campanato's characterization of Hölder continuous functions, see chapter III of [16], we immediately get the result. □

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