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Concentration-compactness principles for Moser–Trudinger inequalities: new results and proofs

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Abstract We are concerned with the best exponent in Concentration-Compactness principles for the borderline case of the Sobolev inequality. We present a new approach, which both yields a rigorous proof of the relevant principle in the standard case when functions vanishing on the boundary are considered, and enables us to deal with functions with unrestricted boundary values.

Keywords Sobolev spaces · Sharp constants · Moser–Trudinger inequality · Concentration-Compactness Principle · Rearrangements · Isoperimetric inequalities

Mathematics Subject Classification (2000) 46E35 · 46E30

1 Introduction

The Concentration-Compactness Principle, as developed by P.-L.Lions in [9–12], is a powerful tool in proving existence of extremals in functional inequalities, typically of Sobolev type, and existence of solutions to boundary value problems for elliptic PDE's, in limiting situations when standard compactness arguments do not apply.

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A version of the Concentration-Compactness Principle corresponding to the borderline case of the Sobolev embedding theorem is related to the Moser inequality. The latter is a sharp form of an embedding for the Sobolev space $W_0^{1,n}(\Omega)$ into an Orlicz space of exponential type due to Yudovich [21], Pohozaev [17], and Trudinger [20]. Here, and in what follows, $n \geq 2$, Ω denotes an open bounded subset of $\mathbb{R}^n$, and $W_0^{1,n}(\Omega)$ stands for the standard Sobolev space of those functions in Ω whose weak derivatives belong to $L^n(\Omega)$, and which vanish in a suitable sense on $\partial\Omega$. On calling $\mathcal{L}_n$ the Lebesgue measure in $\mathbb{R}^n$, and denoting by ω_{n-1} the $(n-1)$-dimensional measure of the unit sphere in $\mathbb{R}^n$, the Moser inequality [15] asserts that a constant $C = C(n)$, i.e. depending only on n, exists such that

$$\int_\Omega \exp\left(n\omega_{n-1}^{\frac{1}{n-1}}|u|^{\frac{n}{n-1}}\right)\mathrm{d}x \leq C\mathcal{L}_n(\Omega) \tag{1.1}$$

for every $u \in W_0^{1,n}(\Omega)$ such that $\|\nabla u\|_{L^n(\Omega)} \leq 1$. Moreover, (1.1) does not hold if $n\omega_{n-1}^{\frac{1}{n-1}}$ is replaced with any larger number, whatever C is. A crucial point in (1.1) is that C does not depend on the trial function u. In fact, for each single function $u \in W_0^{1,n}(\Omega)$, the left-hand side of (1.1) turns out to be finite even if $n\omega_{n-1}^{\frac{1}{n-1}}$ is replaced with any arbitrarily large constant.

Roughly speaking, the Concentration-Compactness Principle associated with (1.1), contained in [11, Theorem I.6], tells us that, if a sequence $\{u_k\} \subset W_0^{1,n}(\Omega)$ converges pointwise and weakly to some function $u \in W_0^{1,n}(\Omega)$, and does not concentrate at one point in $\overline{\Omega}$, then an inequality like (1.1) holds along the sequence $\{u_k\}$, with a constant larger than $n\omega_{n-1}^{\frac{1}{n-1}}$, depending on $\|\nabla u\|_{L^n(\Omega)}$. A precise statement of this principle can be detailed as follows. Hereafter, $\mathcal{M}(\overline{\Omega})$ denotes the space of Radon measures on $\overline{\Omega}$.

Theorem (P-L.Lions) *Let $n \geq 2$ and let Ω be a bounded open subset of $\mathbb{R}^n$. Let $\{u_k\}$ be a sequence in $W_0^{1,n}(\Omega)$ such that $\int_\Omega |\nabla u_k|^n\,\mathrm{d}x \leq 1$, let $u \in W_0^{1,n}(\Omega)$ and $\mu \in \mathcal{M}(\overline{\Omega})$. Assume that*

$$u_k \rightharpoonup u \quad \text{in } W_0^{1,n}(\Omega),\ \ u_k \to u \text{ a.e. in } \Omega \quad \text{and} \quad |\nabla u_k|^n \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\overline{\Omega}). \tag{1.2}$$

(i) *If $u = 0$, $\mu = \delta_{x_0}$ for some $x_0 \in \overline{\Omega}$, and*

$$\int_\Omega \exp\left(n\omega_{n-1}^{\frac{1}{n-1}}|u_k|^{\frac{n}{n-1}}\right)\mathrm{d}x \to c + \mathcal{L}_n(\Omega)$$

for some $c \in [0,\infty)$, then

$$\exp\left(n\omega_{n-1}^{\frac{1}{n-1}}|u_k|^{\frac{n}{n-1}}\right) \overset{*}{\rightharpoonup} c\delta_{x_0} + \mathcal{L}_n|_\Omega \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

(ii) *If $u = 0$ and μ is not a Dirac mass concentrated at one point, then there exists $p > 1$ such that*

$$\exp\left(n\omega_{n-1}^{\frac{1}{n-1}}p\,|u_k|^{\frac{n}{n-1}}\right) \text{ is bounded in } L^1(\Omega).$$

(iii) *If $u \neq 0$, then there exists $p > 1$ such that*

$$\exp\left(n\omega_{n-1}^{\frac{1}{n-1}}p\,|u_k|^{\frac{n}{n-1}}\right) \text{ is bounded in } L^1(\Omega). \tag{1.3}$$

Moreover, in both cases (ii) *and* (iii),

$$\exp\left(n\omega_{n-1}^{\frac{1}{n-1}}|u_k|^{\frac{n}{n-1}}\right) \to \exp\left(n\omega_{n-1}^{\frac{1}{n-1}}|u|^{\frac{n}{n-1}}\right) \qquad \text{in } L^1(\Omega). \tag{1.4}$$

Our first result is a refinement of Lions's Theorem, which yields a sharp upper bound for the value of p in (1.3).

Theorem 1.1 *Under the same assumptions as in case* (iii) *of Lions' Theorem, define*

$$P = \begin{cases} \left(1 - \int_\Omega |\nabla u|^n \mathrm{d}x\right)^{-\frac{1}{n-1}} & \text{if } \int_\Omega |\nabla u|^n \mathrm{d}x < 1, \\ \infty & \text{if } \int_\Omega |\nabla u|^n \mathrm{d}x = 1. \end{cases} \tag{1.5}$$

Then Eq. (1.3) *holds for every* $p < P$*. Moreover, such upper bound for* p *is sharp.*

Let us notice that Theorem 1.1 was claimed in [11, Theorem I.6 and Remark I.18], and subsequently applied to derive various results in the theory of elliptic PDE's—see e.g. [1,8, 16,19]. The proof given in [11] for $n \geq 3$ involves the radially decreasing symmetral $u^\bigstar$ of u, and $\Omega^\bigstar$, the ball centered at 0 such that $\mathcal{L}_n(\Omega^\bigstar) = \mathcal{L}_n(\Omega)$. However, that proof leads to (1.3) only for $p < \overline{P}$, where $\overline{P}$ is defined as

$$\overline{P} = \begin{cases} \left(1 - \int_{\Omega^\bigstar} |\nabla u^\bigstar|^n \mathrm{d}x\right)^{-\frac{1}{n-1}} & \text{if } \int_{\Omega^\bigstar} |\nabla u^\bigstar|^n \mathrm{d}x < 1, \\ \infty & \text{if } \int_{\Omega^\bigstar} |\nabla u^\bigstar|^n \mathrm{d}x = 1. \end{cases} \tag{1.6}$$

One has that $\overline{P} < P$, in general, since, by the Pólya–Szegö inequality,

$$\int_{\Omega^\bigstar} |\nabla u^\bigstar|^n \mathrm{d}x \leq \int_\Omega |\nabla u|^n \mathrm{d}x \tag{1.7}$$

for $u \in W_0^{1,n}(\Omega)$ ([2,18]), and the inequality is strict, unless u has a very special form (see [2] and [7]). Recall that the radially decreasing symmetral $u^\bigstar : \Omega^\bigstar \to [0, \infty)$ of u is given by

$$u^\bigstar(x) = u^*\left(\frac{\omega_{n-1}}{n}|x|^n\right) \qquad \text{for } x \in \Omega^\bigstar,$$

where $u^* : [0, \mathcal{L}_n(\Omega)] \to [0, \infty]$, the decreasing rearrangement of u, obeys

$$u^*(s) = \sup\{t \geq 0 : \mathcal{L}_n(\{x \in \Omega : |u(x)| > t\}) > s\} \quad \text{for } s \geq 0.$$

A flaw in the symmetrization argument of [11] is illustrated by a counterexample in Sect. 2 below. Instead, the proof of Theorem 1.1 given in [11] for $n = 2$ makes use of different ideas, resting upon the Hilbert space structure of $W_0^{1,2}(\Omega)$, and has no drawback.

Theorem 1.1 substantiates the various applications of the Concentration-Compactness Principle in $W_0^{1,n}(\Omega)$, where the validity of (1.3) for the full range of those values of p smaller than P is actually exploited. The proof of Theorem 1.1 to be presented here, even though based on (1.7), relies upon a different argument. Loosely speaking, we show that the symmetrized sequence $\{u_k^\bigstar\}$ of any critical sequence $\{u_k\}$ has a a very special behavior. We then turn this piece of information to the original sequence $\{u_k\}$ without making use of the symmetral of u, and we eventually obtain the desired conclusion.

Such an approach also applies to derive a Concentration-Compactness Principle for sequences of functions which need not vanish on $\partial\Omega$, namely sequences in the whole Sobolev space $W^{1,n}(\Omega)$. This version of the Concentration-Compactness Principle is the main result of the present paper, and is connected with a counterpart of the Moser inequality (1.1) in $W^{1,n}(\Omega)$. The relevant inequality tells us that if Ω is a bounded connected domain in $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in (0,1]$, then there exists a constant $C = C(\Omega)$ such that

$$\int_\Omega \exp\left(n(\tfrac{1}{2}\omega_{n-1})^{\frac{1}{n-1}}|u - m(u)|^{\frac{n}{n-1}}\right) \mathrm{d}x \le C \tag{1.8}$$

for every $u \in W^{1,n}(\Omega)$ satisfying $\|\nabla u\|_{L^n(\Omega)} \le 1$. Here $m(u)$ denotes either the mean value of u over Ω, or its median on Ω, or some other analog normalizing operator applied to u. The constant $n\left(\frac{1}{2}\omega_{n-1}\right)^{\frac{1}{n-1}}$ is sharp in (1.8), since the integral on the left-hand side is not uniformly bounded with respect to u if $n\left(\frac{1}{2}\omega_{n-1}\right)^{\frac{1}{n-1}}$ is replaced with any larger constant, although it is finite for each fixed u. Inequality (1.8), with $m(u)$ equal to the mean value of u, was proved for $n = 2$ in domains with piecewise C^2 boundary in [3], and for any $n \ge 2$ and domains with $C^{1,\alpha}$ boundary in [5]. The latter paper also contains a version of (1.8) for domains with conical singularities. The case when m is a more general operator follows from [6].

Inequality (1.8) entails that, in particular, for every $k < n(\frac{1}{2}\omega_{n-1})^{\frac{1}{n-1}}$, there exists a constant $C = C(\Omega, m(u), k)$ such that

$$\int_\Omega \exp\left(k|u|^{\frac{n}{n-1}}\right) \mathrm{d}x \le C \tag{1.9}$$

for every $u \in W^{1,n}(\Omega)$ satisfying $\|\nabla u\|_{L^n(\Omega)} \le 1$. This easily follows on making use of (1.8) to estimate the integral over the set where $k^{\frac{n-1}{n}}|u| \le n^{\frac{n-1}{n}}\left(\frac{1}{2}\omega_{n-1}\right)^{\frac{1}{n}}|u - m(u)|$, and on exploiting the fact that $|u|$ is bounded by a constant multiple of $m(u)$ on its complement.

The Concentration-Compactness Principle in $W^{1,n}(\Omega)$ is the content of the next theorem.

Theorem 1.2 *Let $n \ge 2$ and let Ω be a bounded connected open set in $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in (0,1]$. Let $\{u_k\}$ be a sequence in $W^{1,n}(\Omega)$ such that $\int_\Omega |\nabla u_k|^n \,\mathrm{d}x \le 1$, let $u \in W^{1,n}(\Omega)$ and let $\mu \in \mathcal{M}(\overline{\Omega})$. Assume that*

$$u_k \rightharpoonup u \ \text{ in } W^{1,n}(\Omega), \quad u_k \to u \text{ a.e. in } \Omega \quad \text{and} \quad |\nabla u_k|^n \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\overline{\Omega}). \tag{1.10}$$

(i) *If $u = a$ for some $a \in \mathbb{R}$, $\mu = \delta_{x_0}$ for some $x_0 \in \overline{\Omega}$, and*

$$\int_\Omega \exp\left(n\left(\tfrac{1}{2}\omega_{n-1}\right)^{\frac{1}{n-1}}|u_k - a|^{\frac{n}{n-1}}\right) \mathrm{d}x \to c + \mathcal{L}_n(\Omega) \tag{1.11}$$

for some $c \in [0,\infty)$, then

$$\exp\left(n\left(\tfrac{1}{2}\omega_{n-1}\right)^{\frac{1}{n-1}}|u_k - a|^{\frac{n}{n-1}}\right) \overset{*}{\rightharpoonup} c\delta_{x_0} + \mathcal{L}_n|_\Omega \ \text{ in } \mathcal{M}(\overline{\Omega}). \tag{1.12}$$

(ii) *If $u = a$ for some $a \in \mathbb{R}$, and μ is not a Dirac mass concentrated at one point, then there exists $p > 1$ such that*

$$\exp\left(n\left(\tfrac{1}{2}\omega_{n-1}\right)^{\frac{1}{n-1}} p\,|u_k|^{\frac{n}{n-1}}\right) \text{ is bounded in } L^1(\Omega). \tag{1.13}$$

(iii) *If u is not constant, then*

$$\exp\left(n\left(\tfrac{1}{2}\omega_{n-1}\right)^{\frac{1}{n-1}} p|u_k|^{\frac{n}{n-1}}\right) \text{ is bounded in } L^1(\Omega) \tag{1.14}$$

for every $p < P$, where P is defined as in (1.5). *Such upper bound for p is sharp. Moreover, in both cases* (ii) *and* (iii),

$$\exp\left(n\left(\tfrac{1}{2}\omega_{n-1}\right)^{\frac{1}{n-1}} |u_k|^{\frac{n}{n-1}}\right) \to \exp\left(n\left(\tfrac{1}{2}\omega_{n-1}\right)^{\frac{1}{n-1}} |u|^{\frac{n}{n-1}}\right) \quad \text{in } L^1(\Omega). \tag{1.15}$$

Our proof of Theorem 1.2 requires a substitute for the Pólya–Szegö inequality (1.7), which holds for any function $u \in W^{1,n}(\Omega)$ (that need not vanish on $\partial\Omega$) and involves both the signed decreasing rearrangement of u and the isoperimetric function $\lambda_\Omega : (0, \mathcal{L}_n(\Omega)) \to [0,\infty)$ of Ω. The signed decreasing rearrangement $u^\circ : [0, \mathcal{L}_n(\Omega)] \to [-\infty, \infty]$ of any $u \in W^{1,n}(\Omega)$ is given by

$$u^\circ(s) = \sup\{t \in \mathbb{R} : \mathcal{L}_n(\{x \in \Omega : u(x) > t\}) > s\} \quad \text{for } s \in [0, \mathcal{L}_n(\Omega)].$$

The isoperimetric function of an open set Ω of finite measure was introduced in [13], and is defined as

$$\lambda_\Omega(s) = \inf\{P(E;\Omega) : E \subset \Omega, \mathcal{L}_n(\Omega) = s\} \quad \text{for } s \in [0, \mathcal{L}_n(\Omega)], \tag{1.16}$$

where $P(E;\Omega)$ is the perimeter of $E \subset \mathbb{R}^n$ in Ω, which agrees with $\mathcal{H}^{n-1}(\partial^M E \cap \Omega)$, and $\partial^M E$ denotes the essential boundary of E. The relevant inequality tells us that u° is locally absolutely continuous, and

$$\int_0^{\mathcal{L}_n(\Omega)} \left(\lambda_\Omega(s)\left(-\frac{\mathrm{d}u^\circ}{\mathrm{d}s}\right)\right)^n \mathrm{d}s \le \int_\Omega |\nabla u|^n \,\mathrm{d}x \tag{1.17}$$

for every $u \in W^{1,n}(\Omega)$ (see e.g. [4, Lemma 1]).

Note that, unlike (1.7), inequality (1.17) is not associated with any n-dimensional symmetrization of u. In fact, no special symmetry of both Ω and u entails equality in (1.17). Let us also point out that the explicit form of the isoperimetric function is known only for few very special domains Ω. However, the asymptotic behavior of λ_Ω can be precisely described under the assumptions of Theorem 1.2 [5, Theorem 1.3], and this piece of information suffices for (1.17) to be used in the proof of Theorem 1.2. In a sense, this is possible since the sharp exponential constants in inequality (1.8) and in the corresponding Concentration-Compactness Principle turn out to depend on the behavior of trial functions only for large values of their absolute value. Inasmuch as λ_Ω comes into play in estimating the measure of the level sets of u in terms of their perimeter in Ω, only the behavior of $\lambda_\Omega(s)$ as s tends to 0 is really relevant.

2 Concentration-compactness in $W_0^{1,n}(\Omega)$

The same argument as in the original proof of [11, Theorem I.6] will lead to Theorem 1.1, once the following proposition is established. We thus limit ourselves to the proof of this proposition. Further details can be provided along the same lines as in the proof of Theorem 1.2, Sect. 3.

Proposition 2.1 *Let $n \geq 2$ and let Ω be a bounded open set in $\mathbb{R}^n$. Let $u \in W_0^{1,n}(\Omega)$, $u \neq 0$, and let $\{u_k\} \subset W_0^{1,n}(\Omega)$ be a sequence such that*

$$\int_\Omega |\nabla u_k|^n \, dx \leq 1 , \qquad u_k \rightharpoonup u \quad \text{in } W_0^{1,n}(\Omega) \qquad \text{and} \qquad u_k \to u \quad \text{a.e. in } \Omega. \tag{2.1}$$

Let P be defined as in (1.5). Then, for every $p < P$, there exists a constant $C = C(n, p)$ such that

$$\int_\Omega \exp(n\omega_{n-1}^{\frac{1}{n-1}} p \, |u_k|^{\frac{n}{n-1}}) \, dx \leq C. \tag{2.2}$$

Moreover, this conclusion fails if $p \geq P$.

Proof We begin by assuming that $0 < \int_\Omega |\nabla u|^n < 1$, and proceed by contradiction. Suppose that there exists a sequence $\{u_k\} \subset W_0^{1,n}(\Omega)$ which satisfies (2.1), and causes (2.2) to fail for some $p_1 < P$. Thus,

$$\int_{\Omega^\bigstar} \exp\left(n\omega_{n-1}^{\frac{1}{n-1}} p_1 |u_k^\bigstar|^{\frac{n}{n-1}}\right) dx \to \infty, \tag{2.3}$$

since

$$\int_{\Omega^\bigstar} \exp\left(n\omega_{n-1}^{\frac{1}{n-1}} p_1 |u_k^\bigstar|^{\frac{n}{n-1}}\right) dx = \int_\Omega \exp\left(n\omega_{n-1}^{\frac{1}{n-1}} p_1 |u_k|^{\frac{n}{n-1}}\right) dx,$$

inasmuch as $u_k^\bigstar$ is equimeasurable with u_k for $k \in \mathbb{N}$. The Pólya–Szegö inequality (1.7), with u replaced with u_k, gives

$$\begin{aligned} &\left(\int_0^{\mathcal{L}_n(\Omega)} \left(n^{\frac{n-1}{n}} \omega_{n-1}^{\frac{1}{n}} \left(-\frac{du_k^*}{dr} \right) \right)^n r^{n-1} \, dr \right)^{\frac{1}{n}} \\ &= \|\nabla u_k^\bigstar\|_{L^n(\Omega^\bigstar)} \leq \|\nabla u_k\|_{L^n(\Omega)} \leq 1 \qquad \text{for } k \in \mathbb{N}. \end{aligned} \tag{2.4}$$

Since $u_k^*(\mathcal{L}_n(\Omega)) = 0$, and u_k^* is locally absolutely continuous,

$$u_k^*(s) = \int_s^{\mathcal{L}_n(\Omega)} -\frac{du_k^*}{dr} \, dr \qquad \text{for } s \in (0, \mathcal{L}_n(\Omega)). \tag{2.5}$$

Hölder's inequality and Eq. (2.4) yield

$$\begin{aligned} u_k^*(s) &\leq \left(\int_s^{\mathcal{L}_n(\Omega)} \left(n^{\frac{n-1}{n}} \omega_{n-1}^{\frac{1}{n}} \left(-\frac{du_k^*}{dr} \right) \right)^n r^{n-1} \, dr \right)^{\frac{1}{n}} \left(\int_s^{\mathcal{L}_n(\Omega)} \frac{1}{n\omega_{n-1}^{\frac{1}{n-1}}} \frac{dr}{r} \right)^{\frac{n-1}{n}} \\ &\leq \|\nabla u_k\|_{L^n(\Omega)} \left(\frac{1}{n\omega_{n-1}^{\frac{1}{n-1}}} \log\left(\frac{\mathcal{L}_n(\Omega)}{s} \right) \right)^{\frac{n-1}{n}} \\ &\leq \left(\frac{1}{n\omega_{n-1}^{\frac{1}{n-1}}} \log\left(\frac{\mathcal{L}_n(\Omega)}{s} \right) \right)^{\frac{n-1}{n}} \qquad \text{for } s \in (0, \mathcal{L}_n(\Omega)). \end{aligned} \tag{2.6}$$

Observe that (2.6) yields inequality (1.1) with $n\omega_{n-1}^{\frac{1}{n-1}}$ replaced with any smaller constant. We now make use of an analogous observation to show, by contradiction, that, given any $p_2 \in (p_1, P)$, for every $k_0 \in \mathbb{N}$ and every $s_0 \in (0, \mathcal{L}_n(\Omega))$ there exist $k \in \mathbb{N}$, $k > k_0$, and $s \in (0, s_0)$ such that

$$u_k^*(s) \geq \left(\frac{1}{p_2 n\omega_{n-1}^{\frac{1}{n-1}}}\right)^{\frac{n-1}{n}} \log^{\frac{n-1}{n}}\left(\frac{\mathcal{L}_n(\Omega)}{s}\right). \tag{2.7}$$

Indeed, suppose that there exist $k_0 \in \mathbb{N}$ and $s_0 \in (0, \mathcal{L}_n(\Omega))$ such that

$$u_k^*(s) < \left(\frac{1}{p_2 n\omega_{n-1}^{\frac{1}{n-1}}}\right)^{\frac{n-1}{n}} \log^{\frac{n-1}{n}}\left(\frac{\mathcal{L}_n(\Omega)}{s}\right) \quad \text{for every } s \in (0, s_0),\, k \geq k_0.$$

By the latter estimate and inequality (2.6), one has that, if $p_1 < p_2$ and $k \geq k_0$, then

$$\int_0^{\mathcal{L}_n(\Omega)} \exp\left(n\omega_{n-1}^{\frac{1}{n-1}} p_1 |u_k^*|^{\frac{n}{n-1}}\right) \mathrm{d}s \leq \int_0^{s_0} \left(\frac{\mathcal{L}_n(\Omega)}{s}\right)^{-\frac{p_1}{p_2}} \mathrm{d}s + \int_{s_0}^{\mathcal{L}_n(\Omega)} \left(\frac{\mathcal{L}_n(\Omega)}{s_0}\right)^{p_1} \mathrm{d}s < \infty,$$

contradicting (2.3). Our claim is proved.
Thus, possibly passing to a subsequence, there exists a sequence $\{s_k\}$, such that

$$u_k^*(s_k) \geq \left(\frac{1}{p_2 n\omega_{n-1}^{\frac{1}{n-1}}}\right)^{\frac{n-1}{n}} \log^{\frac{n-1}{n}}\left(\frac{\mathcal{L}_n(\Omega)}{s_k}\right) \quad \text{and} \quad s_k \leq \frac{1}{k} \quad \text{for every } k \in \mathbb{N}. \tag{2.8}$$

Now, given $L > 0$, define the truncation operators T^L and T_L acting on any function $v : \Omega \to \mathbb{R}$ as

$$T^L(v) = \min\{|v|, L\}\,\mathrm{sign}(v) \quad \text{and} \quad T_L(v) = v - T^L(v).$$

It is not difficult to verify that

$$\int_\Omega |\nabla u_k|^n \,\mathrm{d}x = \int_\Omega |\nabla(T^L(u_k))|^n \,\mathrm{d}x + \int_\Omega |\nabla(T_L(u_k))|^n \,\mathrm{d}x,$$
$$T^L(u_k) \to T^L(u) \text{ a.e. in } \Omega, \quad \text{and} \quad T_L(u_k) \to T_L(u) \text{ a.e. in } \Omega. \tag{2.9}$$

Moreover, $\{T^L(u_k)\}$ is a bounded sequence in $W^{1,n}(\Omega)$, and hence there exists a weakly convergent subsequence. Since it converges almost everywhere to $T^L(u)$, one also has that

$$T^L(u_k) \rightharpoonup T^L(u) \text{ in } W^{1,n}(\Omega) \quad \text{and} \quad T_L(u_k) \rightharpoonup T_L(u) \text{ in } W^{1,n}(\Omega).$$

Next, fix any $p_3 \in (p_2, P)$, and choose L so large that

$$\frac{1 - \int_\Omega |\nabla u|^n}{1 - \int_\Omega |\nabla(T^L(u))|^n} > \left(\frac{p_3}{P}\right)^{n-1}. \tag{2.10}$$

By (2.8), on passing to a subsequence if necessary, we have that $u_k^*(s_k) > L$ for every $k \in \mathbb{N}$. Consequently, there exists $r_k \in (s_k, \mathcal{L}_n(\Omega))$ such that $u_k^*(r_k) = L$ for every $k \in \mathbb{N}$. Owing to

(2.8) and to Hölder's inequality, via the same argument as in the proof of (2.6) we obtain

$$\left(\frac{1}{p_2 n\omega_{n-1}^{\frac{1}{n-1}}}\right)^{\frac{n-1}{n}} \log^{\frac{n-1}{n}}\left(\frac{\mathcal{L}_n(\Omega)}{s_k}\right) - L \le u_k^*(s_k) - u_k^*(r_k) = \int\limits_{s_k}^{r_k} -\frac{\mathrm{d}u_k^*}{\mathrm{d}s}\,\mathrm{d}s$$
$$\le \left\| -\frac{\mathrm{d}u_k^*}{\mathrm{d}s}\left(n^{\frac{n-1}{n}}\omega_{n-1}^{\frac{1}{n}}\right)s^{\frac{n-1}{n}}\right\|_{L^n(s_k,r_k)} \frac{1}{n^{\frac{n-1}{n}}\omega_{n-1}^{\frac{1}{n}}} \log^{\frac{n-1}{n}}\left(\frac{\mathcal{L}_n(\Omega)}{s_k}\right) \quad \text{for } k \in \mathbb{N}.$$

Hence, if k is sufficiently large,

$$\left(\frac{1}{p_3}\right)^{\frac{n-1}{n}} \le \left\| -\frac{\mathrm{d}u_k^*}{\mathrm{d}s}\left(n^{\frac{n-1}{n}}\omega_{n-1}^{\frac{1}{n}}\right)s^{\frac{n-1}{n}}\right\|_{L^n(s_k,r_k)} \le \left\| -\frac{\mathrm{d}u_k^*}{\mathrm{d}s}\left(n^{\frac{n-1}{n}}\omega_{n-1}^{\frac{1}{n}}\right)s^{\frac{n-1}{n}}\right\|_{L^n(0,r_k)}. \tag{2.11}$$

By (2.4) and the definition of T_L, one has that

$$\int\limits_{\Omega} |\nabla(T_L(u_k))|^n\,\mathrm{d}x \ge \int\limits_{\Omega^\bigstar} \left|\nabla\left((T_L(u_k))^\bigstar\right)\right|^n\,\mathrm{d}x = \int\limits_{\Omega^\bigstar} \left|\nabla\left(T_L(u_k^\bigstar)\right)\right|^n\,\mathrm{d}x$$
$$= \left\| -\frac{\mathrm{d}u_k^*}{\mathrm{d}s}\left(n^{\frac{n-1}{n}}\omega_{n-1}^{\frac{1}{n}}\right)s^{\frac{n-1}{n}}\right\|_{L^n(0,r_k)}^n. \tag{2.12}$$

Owing to (2.12), inequality (2.11) yields

$$\left(\frac{1}{p_3}\right)^{n-1} \le \int\limits_{\Omega} |\nabla(T_L(u_k))|^n\,\mathrm{d}x. \tag{2.13}$$

Therefore, (2.9) implies that

$$\int\limits_{\Omega} |\nabla(T^L(u_k))|^n\,\mathrm{d}x \le 1 - \left(\frac{1}{p_3}\right)^{n-1}.$$

The latter inequality, the weak lower semicontinuity of the L^n-norm of the gradient, and (2.10) yield

$$p_3 \ge \frac{1}{(1-\liminf_{k\to\infty}\int_\Omega |\nabla(T^L(u_k))|^n\,\mathrm{d}x)^{\frac{1}{n-1}}} \ge \frac{1}{(1-\int_\Omega |\nabla(T^L(u))|^n\,\mathrm{d}x)^{\frac{1}{n-1}}}$$
$$> \frac{p_3}{P}\frac{1}{(1-\int_\Omega |\nabla u|^n\,\mathrm{d}x)^{\frac{1}{n-1}}} = p_3\,,$$

a contradiction.

In the case when $\int_\Omega |\nabla u|^n\,\mathrm{d}x = 1$, the proof proceeds along the same lines, and we limit ourselves to sketching a few differences. Given any $p_1 > 0$, we fix any $p_2 > p_1$. In the final part of the argument, we fix an arbitrary $p_3 > p_2$, and choose $L > 0$ in such a way that

$$\int\limits_{\Omega} |\nabla(T^L(u))|^n\,\mathrm{d}x > 1 - \frac{1}{2}\left(\frac{1}{p_3}\right)^{n-1}.$$

Hence, if k is so large that (2.11) is satisfied we obtain similarly as above

$$\int_\Omega |\nabla(T^L(u_k))|^n \,\mathrm{d}x \le 1 - \left(\frac{1}{p_3}\right)^{n-1}.$$

Therefore, since $T^L(u_k) \rightharpoonup T^L(u)$ in $W_0^{1,n}(\Omega)$, the lower semicontinuity produces a contradiction.

We conclude by showing that the assumption $p < P$ cannot be relaxed. For every $\alpha \in (0,1)$, we exhibit a sequence $\{u_k\} \subset W_0^{1,n}(\Omega)$ and a function $u \in W_0^{1,n}(\Omega)$ such that

$$\|\nabla u_k\|_{L^n(\Omega)} = 1\,, \quad u_k \rightharpoonup u \text{ in } W_0^{1,n}(\Omega)\,, \quad u_k \to u \text{ a.e. in } \Omega,$$
$$\|\nabla u\|_{L^n(\Omega)} = \alpha \quad \text{and} \quad \int_\Omega \exp\left(n\omega_{n-1}^{\frac{1}{n-1}} \frac{1}{(1-\alpha^n)^{\frac{1}{n-1}}} |u_k|^{\frac{n}{n-1}}\right) \mathrm{d}x \to \infty.$$

Assume, without loss of generality, that $0 \in \Omega$. Let $R > 0$ be such that $\overline{B(0,R)} \subset \Omega$. Here, and in what follows, $B(x_0, R)$ denotes the ball, centered at $x_0 \in \mathbb{R}^n$, and having radius R. Set $\varrho = \frac{R}{3}$, and consider the sequence $v_k \in W_0^{1,n}(\Omega)$ introduced in [15], and defined, for $k \in \mathbb{N}$, as

$$v_k(x) = \begin{cases} 0 & \text{if } |x| \in [\varrho, \infty), \\ n^{\frac{1}{n}} \omega_{n-1}^{-\frac{1}{n}} \log\left(\frac{\varrho}{|x|}\right) k^{-\frac{1}{n}} & \text{if } |x| \in [\varrho \mathrm{e}^{-\frac{k}{n}}, \varrho) \\ n^{\frac{1-n}{n}} \omega_{n-1}^{-\frac{1}{n}} k^{\frac{n-1}{n}} & \text{if } |x| \in [0, \varrho \mathrm{e}^{-\frac{k}{n}}]. \end{cases}$$

We have that

$$\int_{B(0,\varrho)} |\nabla v_k|^n \,\mathrm{d}x = \int_{\varrho \mathrm{e}^{-\frac{k}{n}}}^{\varrho} \left(n^{\frac{1}{n}} \omega_{n-1}^{-\frac{1}{n}} \frac{1}{r} k^{-\frac{1}{n}}\right)^n \omega_{n-1} r^{n-1} \,\mathrm{d}r = \frac{n}{k} \left[\log(r)\right]_{\varrho \mathrm{e}^{-\frac{k}{n}}}^{\varrho} = 1.$$

Next, define $u \in W_0^{1,n}(\Omega)$ by

$$u(x) = \begin{cases} 0 & \text{if } |x| \in [R, \infty) \\ 3A - \frac{3A}{R}|x| & \text{if } |x| \in [\frac{2R}{3}, R) \\ A & \text{if } |x| \in [0, \frac{2R}{3}]\,, \end{cases}$$

where $A > 0$ is chosen in such a way that $\|\nabla u\|_{L^n(\Omega)} = \alpha$. Finally, set

$$u_k = u + (1-\alpha^n)^{\frac{1}{n}} v_k \quad \text{for } k \in \mathbb{N}.$$

Consequently,

$$\begin{aligned} \int_\Omega \exp\left(n\omega_{n-1}^{\frac{1}{n-1}} \frac{|u_k|^{\frac{n}{n-1}}}{(1-\alpha^n)^{\frac{1}{n-1}}}\right) \mathrm{d}x &\ge \int_{B(0,\varrho \mathrm{e}^{-\frac{k}{n}})} \exp\left(n\omega_{n-1}^{\frac{1}{n-1}} \frac{|A + (1-\alpha^n)^{\frac{1}{n}} v_k|^{\frac{n}{n-1}}}{(1-\alpha^n)^{\frac{1}{n-1}}}\right) \mathrm{d}x \\ &= \int_{B(0,\varrho \mathrm{e}^{-\frac{k}{n}})} \exp\left(n\omega_{n-1}^{\frac{1}{n-1}} |C + v_k|^{\frac{n}{n-1}}\right) \mathrm{d}x \\ &= C' \mathrm{e}^{-k} \exp((C'' + k^{\frac{n-1}{n}})^{\frac{n}{n-1}}) \to \infty \end{aligned}$$

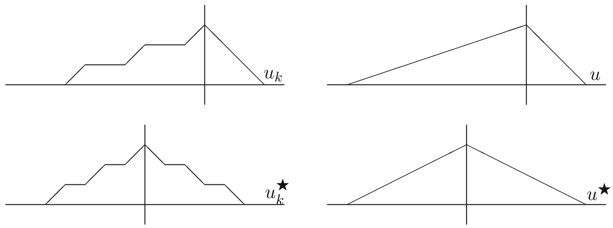

Fig. 1 Functions u_k, u and their symmetrals

for some positive constants C, C' and C''. Since ∇u and ∇v_k have disjoint supports,

$$\|\nabla u_k\|^n_{L^n(\Omega)} = \int\limits_{B(0,R)} |\nabla u|^n \,\mathrm{d}x + (1-\alpha^n) \int\limits_{B(0,R)} |\nabla v_k|^n \,\mathrm{d}x = 1.$$

The remaining properties of the sequence $\{u_k\}$ are easily verified. □

We conclude this section with a few comments on a proof of Proposition 2.1 given in [11]. The proof in the case when $n \geq 3$ rests upon the claim that, owing to (1.7), it suffices to establish the result for radially decreasing functions. However, as already observed in Sect. 1, such a proof only yields (1.3) for $p < \overline{P}$, where $\overline{P}$ is given by (1.6). The technical reason which prevents that proof from giving (1.3) for every $p < P$ is that, for a sequence $\{u_k\}$ fulfilling (1.2), one may have

$$\int\limits_{\Omega} |\nabla u_k|^n \,\mathrm{d}x = \int\limits_{\Omega^\star} |\nabla u_k^\star|^n \,\mathrm{d}x \quad \text{for every } k \in \mathbb{N}, \tag{2.14}$$

but

$$\int\limits_{\Omega} |\nabla u|^n \,\mathrm{d}x > \int\limits_{\Omega^\star} |\nabla u^\star|^n \,\mathrm{d}x. \tag{2.15}$$

Sequences $\{u_k\}$ fulfilling (1.2), (2.14) and (2.15), with $u \not\equiv 0$, are constructed in Example 2.2 ($n = 1$), and Example 2.3 ($n \geq 2$) below.

Example 2.2 Let $p > 1$. Define $u \in W_0^{1,\infty}(-3, 1)$ as

$$u(x) = \begin{cases} 1 + \frac{x}{3} & \text{if } x \in (-3, 0) \\ 1 - x & \text{if } x \in [0, 1). \end{cases}$$

Furthermore, for each $k \in \mathbb{N}$ let $u_k \in W_0^{1,\infty}(-3, 1)$ is given by

$$u_k(x) = \begin{cases} 1 - \frac{m}{k} & \text{for } x \in (-\frac{3m}{k}, -\frac{3(m-1)}{k} - \frac{1}{k}], \quad m \in \{1, \ldots, k\} \\ 1 - \frac{m-1}{k} + \left(x + \frac{3(m-1)}{k}\right) & \text{for } x \in (-\frac{3(m-1)}{k} - \frac{1}{k}, -\frac{3(m-1)}{k}], \quad m \in \{1, \ldots, k\} \\ 1 - x & \text{for } x \in (0, 1) \end{cases}$$

(Fig. 1). It is not difficult to see that

$$u_k \to u \text{ a.e. in } (-3, 1) \quad \text{and} \quad u_k \rightharpoonup u \text{ in } W_0^{1,p}(-3, 1).$$

Since $|(u_k)'|$ and $|(u_k^\bigstar)'|$ equal 1 on a set of measure 2, and vanish elsewhere,

$$\int_{-2}^{2} |(u_k^\bigstar)'|^p \, dx = \int_{-3}^{1} |u_k'|^p \, dx = 2.$$

On the other hand, since $u \neq u^\bigstar$, and $\mathcal{L}^1(\{(u^\bigstar)' = 0\}) = 0$, the characterization of the cases of equality in the Pólya–Szegö inequality given in [2] tells us that

$$\int_{-2}^{2} |(u^\bigstar)'|^p \, dx < \int_{-3}^{1} |u'|^p \, dx.$$

Example 2.3 We outline here how to adapt Example 2.2 for $n \geq 2$. We proceed in steps. Let u and $u_k : (-3, 1) \to [0, \infty)$ be the functions constructed in Example 2.2. Recall, in particular, that $|u_k'|$ equals either 1 or 0 for $k \in \mathbb{N}$. Define the functions $\widetilde{v}$ and $\widetilde{v}_k : B(0, 2) \to [0, \infty)$ as

$$\widetilde{v}(x) = u^\bigstar(|x|) \quad \text{and} \quad \widetilde{v}_k(x) = u_k^\bigstar(|x|)\,,\, k \in \mathbb{N}\,, \qquad \text{for } x \in B(0, 2).$$

The functions $\widetilde{v}$ and $\widetilde{v}_k$ are radially symmetric.

Finally, define v and $v_k : B(0, 3) \to [0, \infty)$ by shifting the level-sets of $\widetilde{v}$ and $\widetilde{v}_k$ along the direction of the x_1-axis in such a way that the restrictions of the functions v and v_k to the x_1-axis agree with u and u_k, respectively (up to translations).

Now, let $p > 1$. By construction, one has that

$$\int_{B(0,2)} |\nabla v_k^\bigstar|^p \, dx = \int_{B(0,3)} |\nabla v_k|^p \, dx \leq \mathcal{L}_n(B(0, 3)) \quad \text{for every } k \in \mathbb{N}.$$

Thus, there exists a subsequence of $\{v_k\}$ satisfying (1.2). Moreover, [2] gives

$$\int_{B(0,2)} |\nabla v^\bigstar|^p \, dx < \int_{B(0,3)} |\nabla v|^p \, dx,$$

since $v \neq v^\bigstar$, and $\mathcal{L}^n(\{\nabla v^\bigstar = 0\}) = 0$.

3 Concentration-compactness in $W^{1,n}(\Omega)$

Information on the isoperimetric function λ_Ω of a bounded open set Ω in $\mathbb{R}^n$, defined as in (1.16), is crucial in our proof of Theorem 1.2. The function λ_Ω satisfies

$$\lambda_\Omega(s) = \lambda_\Omega(\mathcal{L}_n(\Omega) - s) \quad \text{for } s \in [0, \mathcal{L}_n(\Omega)]. \tag{3.1}$$

λ_Ω is strictly positive in $(0, \mathcal{L}_n(\Omega))$ whenever Ω is connected [14, Lemma 5.2.4]. If $\partial\Omega$ is smooth enough, then half-balls centered on $\partial\Omega$ are approximate minimizers for the right-hand side of (1.16) when s is close to 0; this is heuristically the sense of the following estimate from [5, Theorem 1.3 and Corollary 2.4].

Proposition [5] *Let $n \geq 2$, and let Ω be a bounded connected open set in $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in (0, 1]$. Then there exist constants $C_0 > 0$, $\beta > 0$ and $s_1 \in (0, \frac{1}{2}\mathcal{L}_n(\Omega)]$ such* *that the function* $\lambda : (0, \mathcal{L}_n(\Omega)) \to \mathbb{R}$ *defined as*

$$\lambda(s) = \begin{cases} n^{\frac{n-1}{n}} (\frac{\omega_{n-1}}{2})^{\frac{1}{n}} s^{\frac{n-1}{n}} (1 - C_0 s^{\beta}) & \text{for } s \in (0, s_1], \\ \lambda(s_1) & \text{for } s \in [s_1, \frac{1}{2}\mathcal{L}_n(\Omega)], \\ \lambda(\mathcal{L}_n(\Omega) - s) & \text{for } s \in \left(\frac{1}{2}\mathcal{L}_n(\Omega), \mathcal{L}_n(\Omega)\right), \end{cases} \tag{3.2}$$

obeys

$$\lambda_\Omega(s) \geq \lambda(s) \quad \text{for } s \in (0, \mathcal{L}_n(\Omega)), \tag{3.3}$$

and $\lambda(s)$ *and* $\frac{s}{\lambda(s)}$ *are nonnegative and non-decreasing in* $(0, \frac{1}{2}\mathcal{L}_n(\Omega)]$.

The main new ingredient in the proof of Theorem 1.2 is Proposition 3.1 below. In what follows, we define the median of a measurable function $u : \Omega \to \mathbb{R}$ as

$$\text{med}(u) = \sup\left\{t \in \mathbb{R} : \mathcal{L}_n(\{x \in \Omega : u(x) > t\}) > \tfrac{1}{2}\mathcal{L}_n(\Omega)\right\}. \tag{3.4}$$

Proposition 3.1 *Let* $n \geq 2$ *and let* Ω *be a bounded connected open set in* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$ *for some* $\alpha \in (0, 1]$. *Let* $u \in W^{1,n}(\Omega)$, $u \neq 0$, *and let* $\{u_k\} \in W^{1,n}(\Omega)$ *be a sequence such that*

$$\int_\Omega |\nabla u_k|^n \mathrm{d}x \leq 1, \quad \text{med}(u_k) = 0, \quad u_k \rightharpoonup u \ \text{ in } \ W^{1,n}(\Omega) \ \text{ and } \ u_k \to u \text{ a.e. in } \ \Omega.$$

Let P *be defined as in* (1.5). *Then, for every* $p < P$, *there exists a constant* $C = C(\Omega, p)$ *such that*

$$\int_\Omega \exp\left(n(\tfrac{1}{2}\omega_{n-1})^{\frac{1}{n-1}} p\,|u_k|^{\frac{n}{n-1}}\right) \mathrm{d}x \leq C \quad \text{for } k \in \mathbb{N}. \tag{3.5}$$

Moreover, this conclusion fails if $p \geq P$.

Proof The outline of the proof is the same as in that of Proposition 2.1. We provide the details, since some complications arise, owing to the use of (1.17) instead of (1.7), and to the lack of zero boundary conditions.

Let us first assume that $0 < \int_\Omega |\nabla u|^n < 1$, and proceed by contradiction. Suppose that there exists a sequence $\{u_k\}$ satisfying the assumptions, and such that

$$\int_\Omega \exp\left(n(\tfrac{1}{2}\omega_{n-1})^{\frac{1}{n-1}} p_1 |u_k|^{\frac{n}{n-1}}\right) \mathrm{d}x \to \infty \qquad \text{for some} \quad p_1 < P.$$

On passing to a subsequence, and changing the sign of the entire subsequence if necessary, we have that

$$\int_0^{\frac{\mathcal{L}_n(\Omega)}{2}} \exp\left(n(\tfrac{1}{2}\omega_{n-1})^{\frac{1}{n-1}} p_1 (u_k^\circ)^{\frac{n}{n-1}}\right) \mathrm{d}s \to \infty, \tag{3.6}$$

since u_k° is equimeasurable with u_k for $k \in \mathbb{N}$. Let λ_Ω be the isoperimetric function of Ω, and let $\lambda : (0, \mathcal{L}_n(\Omega)) \to [0, \infty)$ be the function defined in (3.2). Since $\lambda \leq \lambda_\Omega$, inequality (1.17), with u replaced with u_k, gives

$$\int_0^{\frac{\mathcal{L}_n(\Omega)}{2}} \left(\lambda(s)\left(-\frac{\mathrm{d}u_k^\circ}{\mathrm{d}s}\right)\right)^n \mathrm{d}s \leq \int_\Omega |\nabla u_k|^n \,\mathrm{d}x\,, \tag{3.7}$$

for $k \in \mathbb{N}$. Furthermore, since $u_k^\circ(\frac{1}{2}\mathcal{L}_n(\Omega)) = \mathrm{med}(u_k) = 0$,

$$0 \le u_k^\circ(s) = \int_s^{\frac{\mathcal{L}_n(\Omega)}{2}} -\frac{\mathrm{d}u_k^\circ}{\mathrm{d}r}\,\mathrm{d}r \qquad \text{for } s \in (0, \tfrac{1}{2}\mathcal{L}_n(\Omega)), \tag{3.8}$$

for $k \in \mathbb{N}$. From (3.2), (3.7), (3.8), and Hölder's inequality, we obtain that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} u_k^\circ(s) &= \int_s^{\frac{\mathcal{L}_n(\Omega)}{2}} -\frac{\mathrm{d}u_k^\circ}{\mathrm{d}r}\,\mathrm{d}r \\ &\le \left(\int_s^{\frac{\mathcal{L}_n(\Omega)}{2}} \left(\lambda(r)\left(-\frac{\mathrm{d}u_k^\circ}{\mathrm{d}r}\right)\right)^n \mathrm{d}r \right)^{\frac{1}{n}} \left(\int_s^{\frac{\mathcal{L}_n(\Omega)}{2}} \lambda(r)^{-\frac{n}{n-1}}\,\mathrm{d}r \right)^{\frac{n-1}{n}} \\ &\le \|\nabla u_k\|_{L^n(\Omega)} \left(\int_{s_\varepsilon}^{\frac{\mathcal{L}_n(\Omega)}{2}} \lambda(s_\varepsilon)^{-\frac{n}{n-1}}\,\mathrm{d}r + \int_s^{s_\varepsilon} n^{-1}(\tfrac{\omega_{n-1}}{2})^{-\frac{1}{n-1}} r^{-1}(1 - C_0 r^\beta)^{-\frac{n}{n-1}}\,\mathrm{d}r \right)^{\frac{n-1}{n}} \\ &\le C_\varepsilon + (1+\varepsilon) n^{-\frac{n-1}{n}} \left(\frac{\omega_{n-1}}{2}\right)^{-\frac{1}{n}} \log^{\frac{n-1}{n}} \left(\frac{\mathcal{L}_n(\Omega)}{s}\right) \quad \text{for } s \in (0, \tfrac{1}{2}\mathcal{L}_n(\Omega)). \end{aligned} \tag{3.9}$$

Here, $s_\varepsilon \in (0, \min\{s, s_1\})$, where s_1 is the number appearing in (3.2), and $s\varepsilon$ is so small that $(1 - C_0 r^\beta)^{-1} \le 1 + \varepsilon$ if $r \in (0, s_\varepsilon)$. Let us fix $p_2 \in (p_1, P)$. We claim that for every $k_0 \in \mathbb{N}$ and every $s_0 \in (0, \frac{1}{2}\mathcal{L}_n(\Omega))$ there exist $k \in \mathbb{N}$, $k > k_0$, and $s \in (0, s_0)$ such that

$$u_k^\circ(s) \ge \left(\frac{1}{p_2 n (\frac{\omega_{n-1}}{2})^{\frac{1}{n-1}}} \right)^{\frac{n-1}{n}} \log^{\frac{n-1}{n}} \left(\frac{\mathcal{L}_n(\Omega)}{s}\right).$$

We prove this claim again by contradiction. Suppose that there exist $k_0 \in \mathbb{N}$ and $s_0 \in (0, \frac{1}{2}\mathcal{L}_n(\Omega))$ satisfying

$$u_k^\circ(s) < \left(\frac{1}{p_2 n (\frac{\omega_{n-1}}{2})^{\frac{1}{n-1}}} \right)^{\frac{n-1}{n}} \log^{\frac{n-1}{n}} \left(\frac{\mathcal{L}_n(\Omega)}{s}\right) \quad \text{for every } s \in (0, s_0) \quad \text{and} \quad k \ge k_0.$$

The latter estimate and inequality (3.9) give that

$$\begin{aligned} &\int_0^{\frac{\mathcal{L}_n(\Omega)}{2}} \exp\left(n(\tfrac{1}{2}\omega_{n-1})^{\frac{1}{n-1}} p_1 (u_k^\circ)^{\frac{n}{n-1}} \right) \mathrm{d}s \\ &\quad \le \int_0^{s_0} \left(\frac{\mathcal{L}_n(\Omega)}{s}\right)^{\frac{p_1}{p_2}} \mathrm{d}s + \int_{s_0}^{\frac{\mathcal{L}_n(\Omega)}{2}} \exp\left(C + C \log\left(\frac{\mathcal{L}_n(\Omega)}{s_0}\right)\right) \mathrm{d}s < \infty, \end{aligned} \tag{3.10}$$

for some constant C, if $p_1 < p_2$ and $k \ge k_0$. Since inequality (3.10) contradicts (3.6), our claim is proved.

Thus, there exists a sequence $\{s_k\}$ such that

$$u_k^\circ(s_k) \geq \left(\frac{1}{p_2 n (\frac{\omega_{n-1}}{2})^{\frac{1}{n-1}}}\right)^{\frac{n-1}{n}} \log^{\frac{n-1}{n}}\left(\frac{\mathcal{L}_n(\Omega)}{s_k}\right) \quad \text{and} \quad s_k \leq \frac{1}{k} \quad \text{for every } k \in \mathbb{N}. \tag{3.11}$$

Given $L > 0$, we define the truncation operators S^L and S_L acting on any function $v : \Omega \to \mathbb{R}$ as

$$S^L(v) = \min\{v, L\} \qquad \text{and} \qquad S_L(v) = v - S^L(v).$$

Observe that, since $\operatorname{med}(u_k) = 0$,

$$\operatorname{med}(S^L(u_k)) = 0 \tag{3.12}$$

as well. It is easily seen that

$$\int_\Omega |\nabla u_k|^n \,\mathrm{d}x = \int_\Omega |\nabla\big(S^L(u_k)\big)|^n \,\mathrm{d}x + \int_\Omega |\nabla\big(S_L(u_k)\big)|^n \,\mathrm{d}x,$$

$$S^L(u_k) \to S^L(u) \text{ a.e. in } \Omega \quad \text{and} \quad S_L(u_k) \to S_L(u_k) \text{ a.e. in } \Omega. \tag{3.13}$$

Moreover, $\{S^L(u_k)\}$ is a bounded sequence in $W^{1,n}(\Omega)$, since $\int_\Omega |\nabla(S^L(u_k))|^n \,\mathrm{d}x \leq 1$ and (3.12) holds. Thus, there exists a weakly convergent subsequence of $\{S^L(u_k)\}$ in $W^{1,n}(\Omega)$. Inasmuch as $\{S^L(u_k)\}$ converges almost everywhere to $S^L(u)$, it is easy to verify that

$$S^L(u_k) \rightharpoonup S^L(u) \text{ in } W^{1,n}(\Omega) \quad \text{and} \quad S_L(u_k) \rightharpoonup S_L(u_k) \text{ in } W^{1,n}(\Omega).$$

Now fix $p_3 \in (p_2, P)$, and let L be so large that

$$\frac{1 - \int_\Omega |\nabla u|^n \mathrm{d}x}{1 - \int_\Omega |\nabla\big(S^L(u)\big)|^n \mathrm{d}x} > \left(\frac{p_3}{P}\right)^{n-1}. \tag{3.14}$$

Thanks to (3.11), we may suppose that $u_k^\circ(s_k) > L$ for $k \in \mathbb{N}$ (up to subsequences). Consequently, there exists $r_k \in (s_k, \frac{1}{2}\mathcal{L}_n(\Omega))$ such that $u_k^\circ(r_k) = L$ for every $k \in \mathbb{N}$.

Owing to (3.11), Hölder's inequality, and an analogous chain as in (3.9) we obtain that

$$\begin{aligned}
&\left(\frac{1}{p_2 n \left(\frac{\omega_{n-1}}{2}\right)^{\frac{1}{n-1}}}\right)^{\frac{n-1}{n}} \log^{\frac{n-1}{n}}\left(\frac{\mathcal{L}_n(\Omega)}{s_k}\right) - L \leq u_k^\circ(s_k) - u_k^\circ(r_k) = \int_{s_k}^{r_k} -\frac{\mathrm{d}u_k^\circ}{\mathrm{d}s}\,\mathrm{d}s \\
&\leq \left\|-\frac{\mathrm{d}u_k^\circ}{\mathrm{d}s}\lambda(s)\right\|_{L^n(s_k,r_k)} \left\|\frac{1}{\lambda(s)}\right\|_{L^{\frac{n}{n-1}}(s_k,r_k)} \\
&\leq \left\|-\frac{\mathrm{d}u_k^\circ}{\mathrm{d}s}\lambda(s)\right\|_{L^n(s_k,r_k)} \left(C_\varepsilon + (1+\varepsilon) n^{-\frac{n-1}{n}} \left(\frac{\omega_{n-1}}{2}\right)^{-\frac{1}{n}} \log^{\frac{n-1}{n}}\left(\frac{\mathcal{L}_n(\Omega)}{s_k}\right)\right) \quad \text{for } k \in \mathbb{N}.
\end{aligned} \tag{3.15}$$

If $\varepsilon > 0$ is sufficiently small, we deduce from (3.15) that

$$\left(\frac{1}{p_3}\right)^{\frac{n-1}{n}} \leq \left\|-\frac{\mathrm{d}u_k^\circ}{\mathrm{d}s}\lambda(s)\right\|_{L^n(s_k,r_k)} \leq \left\|-\frac{\mathrm{d}u_k^\circ}{\mathrm{d}s}\lambda(s)\right\|_{L^n(0,r_k)} \tag{3.16}$$

when k is large enough. By inequality (1.17), with u replaced with $S_L(u)$,

$$\int_\Omega |\nabla(S_L(u))|^n \, dx \geq \int_0^{r_k} \Big(-\frac{du_k^\circ}{ds}\lambda(s)\Big) ds. \tag{3.17}$$

Coupling (3.16) with (3.17) tells us that

$$\left(\frac{1}{p_3}\right)^{n-1} \leq \int_\Omega |\nabla\big(S_L(u_k)\big)|^n \, dx. \tag{3.18}$$

Hence, by (3.13),

$$\int_\Omega |\nabla\big(S^L(u_k)\big)|^n \, dx \leq 1 - \left(\frac{1}{p_3}\right)^{n-1}.$$

The latter inequality, the weak lower semicontinuity of the L^n-norm of the gradient, and (3.14) yield

$$\begin{aligned} p_3 &\geq \frac{1}{(1 - \liminf_{k\to\infty} \int_\Omega |\nabla\big(S^L(u_k)\big)|^n \, dx)^{\frac{1}{n-1}}} \geq \frac{1}{(1 - \int_\Omega |\nabla\big(S^L(u)\big)|^n \, dx)^{\frac{1}{n-1}}} \\ &> \frac{p_3}{P} \frac{1}{(1 - \int_\Omega |\nabla u|^n \, dx)^{\frac{1}{n-1}}} = p_3 \,, \end{aligned}$$

a contradiction.
In the case when $\int_\Omega |\nabla u|^n \, dx = 1$, the argument is analogous. Given any $p_1 > 0$, one can just fix any $p_3 > p_2 > p_1$, and chose $L > 0$ in such a way that

$$\int_\Omega |\nabla\big(S^L(u)\big)|^n \, dx > 1 - \frac{1}{2}\left(\frac{1}{p_3}\right)^{n-1}.$$

Hence, if k is so large that (3.16) is satisfied, we obtain that

$$\int_\Omega |\nabla\big(S^L(u_k)\big)|^n \, dx \leq 1 - \left(\frac{1}{p_3}\right)^{n-1},$$

and conclude as above.

The optimality of the upper bound for p given by P can be shown by considering any bounded smooth set Ω in $\mathbb{R}^n$ such that

$$B(0, R) \cap \Omega = \{(x_1, \ldots, x_n) \in B(0, R) : x_n > 0\}$$

for some $R > 0$, and by choosing the trial sequence $\{2^{\frac{1}{n}} u_k\}$ in (3.5), where $\{u_k\}$ is the same as in the final part of the proof of Proposition 2.1.

Proof of Theorem 1.2 We first focus on cases (i), (ii). The outline of the proof of these cases is analogous to that of [11, Theorem I.6] (see also [5]), which deals with functions in $W_0^{1,n}(\Omega)$. Define

$$v_k = u_k - a \qquad \text{for } k \in \mathbb{N}.$$

Since $|\nabla v_k|^n = |\nabla u_k|^n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega})$,

$$1 \geq \int_\Omega |\nabla u_k|^n \, \mathrm{d}x = \int_\Omega |\nabla v_k|^n \, \mathrm{d}x \to \int_{\overline{\Omega}} \mathrm{d}\mu = \mu(\overline{\Omega}). \tag{3.19}$$

Next, let $\varphi \in C^1(\bar{\Omega})$ be such that $0 \leq \varphi \leq 1$. Owing to the fact that $v_k \rightharpoonup 0$ in $W^{1,n}(\Omega)$, and $v_k \to 0$ in $L^n(\Omega)$, in both cases (i) and (ii) one may infer, via the same argument as in the proof of [11, Theorem I.6], that

$$\int_\Omega |\nabla(\varphi v_k)|^n \, \mathrm{d}x = \int_\Omega |v_k \nabla\varphi + \varphi \nabla v_k|^n \, \mathrm{d}x \to \int_\Omega \varphi^n \, \mathrm{d}\mu. \tag{3.20}$$

Case (i): $u = a$ **and** $\mu = \delta_{x_0}$**.**

To simplify notations, set $K = n(\frac{1}{2}\omega_{n-1})^{\frac{1}{n-1}}$. Let φ be a test function such that $\varphi = 1$ on $\Omega \setminus B(x_0, \eta)$ and $\varphi = 0$ on $B(x_0, \eta/2)$ for some $\eta > 0$. On making use of (3.20), we obtain that

$$\int_\Omega |\nabla(\varphi v_k)|^n \, \mathrm{d}x \to 0. \tag{3.21}$$

By inequality (1.8), for every $K' < K$ there exists a constant C such that

$$\int_\Omega \exp\left(\frac{K' |\varphi v_k|^{\frac{n}{n-1}}}{\|\nabla(\varphi v_k)\|_{L^n(\Omega)}} \right) \mathrm{d}x \leq C.$$

Hence, by (3.21), there exists $\delta > 0$ such that

$$\int_{\Omega \setminus B(x_0,\eta)} \exp(K(1+\delta)|v_k|^{\frac{n}{n-1}}) \, \mathrm{d}x \leq \int_\Omega \exp(K(1+\delta)|\varphi v_k|^{\frac{n}{n-1}}) \, \mathrm{d}x \leq C. \tag{3.22}$$

From (3.22) and Vitali's convergence theorem for equi-integrable sequences of functions, one has that

$$\int_{\Omega \setminus B(x_0,\eta)} \left(\exp(K|v_k|^{\frac{n}{n-1}}) - 1 \right) \mathrm{d}x \to 0. \tag{3.23}$$

Equation (3.23) and assumption (1.11) imply that

$$\int_{B(x_0,\eta)} \left(\exp(K|v_k|^{\frac{n}{n-1}}) - 1 \right) \mathrm{d}x \to c. \tag{3.24}$$

Now, fix an arbitrary test function $\psi \in C(\overline{\Omega})$ and let $\varepsilon > 0$. There exists $\eta > 0$ such that

$$|\psi(x) - \psi(x_0)| < \frac{\varepsilon}{2\max(c, 1)} \qquad \text{whenever } x \in \overline{\Omega} \text{ and } |x - x_0| < \eta. \tag{3.25}$$

We have that

$$\left| \int_{\overline{\Omega}} \psi \, \mathrm{d}(c\delta_{x_0}) - \int_\Omega \psi \left(\exp(K|v_k|^{\frac{n}{n-1}}) - 1 \right) \mathrm{d}x \right|$$

$$
\begin{aligned}
&= \left| c\psi(x_0) - \int_{\Omega} \psi \left(\exp(K|v_k|^{\frac{n}{n-1}}) - 1 \right) \mathrm{d}x \right| \\
&\leq \int_{\Omega \setminus B(x_0,\eta)} |\psi| \left(\exp(K|v_k|^{\frac{n}{n-1}}) - 1 \right) \mathrm{d}x \\
&\quad + \int_{B(x_0,\eta)} |\psi - \psi(x_0)| \left(\exp(K|v_k|^{\frac{n}{n-1}}) - 1 \right) \mathrm{d}x + \\
&\quad + |\psi(x_0)| \cdot \left| c - \int_{B(x_0,\eta)} \left(\exp(K|v_k|^{\frac{n}{n-1}}) - 1 \right) \mathrm{d}x \right|. \qquad (3.26)
\end{aligned}
$$

Let us denote by I_1, I_2 and I_3, respectively, the addends on the rightmost side of (3.26). By (3.23) and the fact that $\sup_{\Omega} |\psi| < \infty$, there exists $k_1 \in \mathbb{N}$ such that $I_1 < \varepsilon$ for $k > k_1$. Furthermore, on making use of (3.24) and (3.25), we obtain

$$
\begin{aligned}
I_2 &= \int_{B(x_0,\eta)} |\psi - \psi(x_0)| (\exp(K|v_k|^{\frac{n}{n-1}}) - 1) \, \mathrm{d}x \\
&\leq \frac{\varepsilon}{2\max(c,1)} \int_{B(x_0,\eta)} (\exp(K|v_k|^{\frac{n}{n-1}}) - 1) \, \mathrm{d}x \to \frac{\varepsilon}{2} \frac{c}{\max(c,1)}.
\end{aligned}
$$

Therefore we can find $k_2 > k_1$ such that $I_2 < \varepsilon$ for $k > k_2$. Finally, owing to (3.24), there exists $k_3 > k_2$ such that $I_3 < \varepsilon$ for $k > k_3$. Thus,

$$
\lim_{k\to\infty} \int_{\overline{\Omega}} \psi \, \mathrm{d}(c\delta_{x_0}) - \int_{\Omega} \psi \left(\exp(K|v_k|^{\frac{n}{n-1}}) - 1 \right) \mathrm{d}x = 0,
$$

and (1.12) follows.

Case (ii): $u = a$ **and** μ **is not a Dirac mass at a single point.**

By (3.19), $\mu(\overline{\Omega}) \leq 1$. Let us distinguish two subcases. If $\mu(\overline{\Omega}) < 1$, then, by (3.19) again, there exists $\delta > 0$ such that $\|\nabla((1+\delta)^{\frac{n-1}{n}} u_k)\|_{L^n(\Omega)} < 1$ if k is sufficiently large, and thus (1.13) follows via the Moser–Trudinger inequality applied to the sequence $\{(1+\delta)^{\frac{n-1}{n}} u_k\}$.

Next, assume that $\mu(\Omega) = 1$. Since μ is not a Dirac mass at one point, there exists a compact set $A \subset \overline{\Omega}$ such that $0 < \mu(A) < 1$. Moreover, inasmuch as μ is a Radon measure, $0 < \mu(A + B(0, 2\eta)) < 1$ if η is a sufficiently small positive number. Via a standard argument, one can find two smooth test functions $0 \leq \varphi_1 \leq 1$ and $0 \leq \varphi_2 \leq 1$ such that

$$
\begin{aligned}
&\varphi_1 \equiv 1 \text{ on } A + B(0,\eta), \quad \varphi_1 \equiv 0 \text{ on } \Omega \setminus (A + B(0, 2\eta)), \\
&\varphi_2 \equiv 1 \text{ on } \Omega \setminus (A + B(0,\eta)) \quad \text{and} \quad \varphi_2 \equiv 0 \text{ on } A.
\end{aligned}
$$

From (3.20) we obtain

$$
\int_{\Omega} |\nabla(\varphi_1 v_k)|^n \, \mathrm{d}x \to \int_{\Omega} \varphi_1^n \, \mathrm{d}\mu \leq \mu(A + B(0, 2\eta)) < 1
$$

and

$$
\int_{\Omega} |\nabla(\varphi_2 v_k)|^n \, \mathrm{d}x \to \int_{\Omega} \varphi_2^n \, \mathrm{d}\mu \leq \mu(\Omega \setminus A) < 1.
$$

Thus, there exists $\delta > 0$ such that, if k is sufficiently large, then

$$\left\|\nabla((1+\delta)^{\frac{n-1}{n}}\varphi_1 v_k)\right\|_{L^n(\Omega)} < 1 \quad \text{and} \quad \left\|\nabla((1+\delta)^{\frac{n-1}{n}}\varphi_2 v_k)\right\|_{L^n(\Omega)} < 1.$$

Hence, via inequality (1.9), we deduce that there exists $\delta > 0$ such that

$$\int\limits_{A+B(0,\eta)} \exp\left(K(1+\delta)|v_k|^{\frac{n}{n-1}}\right) \mathrm{d}x \leq \int\limits_{\Omega} \exp\left(K(1+\delta)|\varphi_1 v_k|^{\frac{n}{n-1}}\right) \mathrm{d}x \leq C \tag{3.27}$$

and

$$\int\limits_{\Omega\setminus(A+B(0,\eta))} \exp\left(K(1+\delta)|v_k|^{\frac{n}{n-1}}\right) \mathrm{d}x \leq \int\limits_{\Omega} \exp\left(K(1+\delta)|\varphi_2 v_k|^{\frac{n}{n-1}}\right) \mathrm{d}x \leq C. \tag{3.28}$$

Combining (3.27) and (3.28) tells us that

$$\int\limits_{\Omega} \exp\left(K(1+\delta)|u_k - a|^{\frac{n}{n-1}}\right) \mathrm{d}x = \int\limits_{\Omega} \exp\left(K(1+\delta)|v_k|^{\frac{n}{n-1}}\right) \mathrm{d}x \leq C \tag{3.29}$$

for some constant C. From (3.29), inequality (1.13) easily follows for every $p < 1 + \delta$.

Case (iii): u is not constant.

The sequence $\{u_k\}$ is bounded in $W^{1,n}(\Omega)$, since it is weakly convergent in $W^{1,n}(\Omega)$. Inasmuch as

$$|\mathrm{med}(u_k)|^n \frac{\mathcal{L}_n(\Omega)}{2} \leq \int\limits_{\Omega} |u_k|^n \,\mathrm{d}x$$

for every $k \in \mathbb{N}$, the sequence $\{\mathrm{med}(u_k)\}$ is also bounded. Let C be any constant such that $|\mathrm{med}(u_k)| \leq C$ for $k \in \mathbb{N}$. Assume, by contradiction, that (1.14) fails for some $p_1 < P$, namely

$$\int\limits_{\Omega} \exp\left(n(\tfrac{1}{2}\omega_{n-1})^{\frac{1}{n-1}} p_1 |u_k|^{\frac{n}{n-1}}\right) \mathrm{d}x \to \infty. \tag{3.30}$$

Next fix $p_2 \in (p_1, P)$. Since the sequence $\{u_k - \mathrm{med}(u_k)\}$ is bounded in $W^{1,n}(\Omega)$, there exists a weakly convergent subsequence, still denoted by $\{u_k - \mathrm{med}(u_k)\}$. The sequence $\{u_k - \mathrm{med}(u_k)\}$ satisfies the assumptions of Proposition 3.1. Thus, there exists a constant C' such that

$$\int\limits_{\Omega} \exp\left(n(\tfrac{1}{2}\omega_{n-1})^{\frac{1}{n-1}} p_2 |u_k - \mathrm{med}(u_k)|^{\frac{n}{n-1}}\right) \mathrm{d}x \leq C' \quad \text{for } k \in \mathbb{N}. \tag{3.31}$$

On the set $\{x \in \Omega : (\frac{p_2}{p_1} - 1)|u_k(x) - \mathrm{med}(u_k)| \geq C\}$ we have that

$$p_1|u_k| \leq p_1(|u_k - \mathrm{med}(u_k)| + C) \leq p_2|u_k - \mathrm{med}(u_k)|.$$

On the other hand,

$$|u_k| < \frac{p_2}{p_2 - p_1} C$$

on the set $\{x \in \Omega : (\frac{p_2}{p_1} - 1)|u_k(x) - \mathrm{med}(u_k)| < C\}$. Hence, by inequality (3.31), we derive a contradiction for (3.30). Equation (1.14) is fully proved.

Finally, assertion (1.15) follows in both cases (ii) and (iii) from Vitali convergence theorem on equiintegrable functions. □

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