

# Some geometric conjectures in harmonic function theory

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**Abstract** We prove some results on the geometry of the level sets of harmonic functions, particularly regarding their ‘oscillation’ and ‘pinching’ properties. These results allow us to tackle three recent conjectures due to De Carli and Hudson (Bull London Math Soc 42:83–95, 2010). Our approach hinges on a combination of local constructions, methods from differential topology and global extension arguments.

**Keywords** Level sets · Laplace equation · Global approximation · Green’s function

**Mathematics Subject Classification (2000)** 31A05 · 35B05

## 1 Introduction and statements

The study of level sets of harmonic functions is a central topic in the geometric theory of PDEs dating back to the origins of potential theory. Topologically, a landmark result is the analysis of the connection between the families of level curves of a harmonic function in the plane and foliation theory [5, 15]. The extension of these results to higher dimensions is a long-standing open problem explicitly stated by Rubel and collected in [6, 17] (cf. also [11]). The geometric part of the problem (which is also rather subtle, as evidenced by the examples in [12, 13]) has also attracted considerable attention; in particular, an aspect that has been extensively studied is that of the curvature and shape of level sets (see e.g., [1, 9, 16, 20, 21, 24] and references therein).

Recently, De Carli and Hudson [8] analyzed the geometry of level curves of harmonic functions on the plane, deriving several interesting results and concluding their study with

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four conjectures. Our objective in this paper is to further investigate the geometric properties of these curves (e.g., ‘oscillation’, ‘concentration’, and ‘pinching’), thereby proving or disproving three of the latter authors’ conjectures. Unlike De Carli and Hudson’s, the techniques we shall employ are not restricted to the case of harmonic functions in the plane, and in fact we shall also present generalizations to higher dimensions and more general elliptic equations of second order. For the sake of brevity, we will refer to [8] for the precise statements of De Carli and Hudson’s conjectures. We will denote by  $B_r^n(x)$  the ball in  $\mathbb{R}^n$  centered at  $x$  of radius  $r$ ; when  $n = 2$  or  $x = 0$ , the superscript or the center will be respectively omitted for the ease of notation.

The first conjecture of De Carli and Hudson roughly asserts [8, Conjecture 5.1] that the zero set  $Z$  of a harmonic function  $u$  in the plane cannot contain an ‘almost closed’ curve. Moreover, in the Introduction of [8], it is also suggested that one should also check whether this conjecture holds true at least under the additional assumption that  $u$  is a harmonic polynomial. Our first theorem disproves this conjecture even in this case:

**Theorem 1.1** *For any  $\epsilon > 0$ , there exist a bounded planar domain  $D \supset B_1$  with  $C^\infty$  boundary  $\gamma$ , a connected subset  $\gamma_\epsilon \subset \gamma$  of length at most  $\epsilon$  and a harmonic polynomial in  $\mathbb{R}^2$  which is positive in the domain  $D$  and whose zero set  $Z$  contains  $\gamma \setminus \gamma_\epsilon$ .*

The second conjecture [8, Conjecture 5.3] asserts that if  $u$  is a harmonic function in  $\mathbb{R}^n$  and vanishes identically on  $B_1^{n-1} \times \{0\}$ , then  $u = 0$  on  $\mathbb{R}^{n-1} \times \{0\}$ . This is a concrete question concerning one of the basic problems addressed by the aforementioned authors, namely, under which conditions the fact that the zero set  $Z$  of  $u$  contains a subset  $S \subset \mathbb{R}^n$  implies that  $u$  must vanish on a set strictly larger than  $S$ . In a much more general context, this sort of problems was also addressed in [3]. In this direction, one can easily prove the following result, which in particular shows the validity of De Carli and Hudson’s third conjecture (and whose first part obviously holds for any elliptic equation of second order with analytic coefficients):

**Theorem 1.2** *Let  $u$  satisfy  $\Delta u = 0$  in a domain  $D \subseteq \mathbb{R}^n$ . If  $M$  is an  $m$ -dimensional real analytic submanifold of  $D$  and the zero set  $Z$  of  $u$  contains a nonempty, relatively open subset of  $M$ , then  $M \subseteq Z$ . If  $M$  is a compact, boundaryless hypersurface of  $\mathbb{R}^n$ , then  $u$  is identically 0.*

It is worth emphasizing that this is a qualitative result whose proof relies on the general theory of real analytic functions not on fine quantitative estimates. Recent related results relying on this kind of estimates (which often apply to less regular problems) are given, e.g., in [2, 4, 7] and references therein.

Qualitatively, the third conjecture [8, Conjecture 5.4] asserts that as the zero set  $Z$  of a harmonic function cannot wiggle arbitrarily by the results of De Carli and Hudson, a small region of space should not contain a large concentration of arc length of  $Z$ . We shall next show, however, that this is not necessarily the case, at least when one considers a *connected component* of  $Z$  instead of the whole (possibly disconnected) set  $Z$ :

**Theorem 1.3** *For any positive reals  $r < R$  and  $\eta < 1$ , there exist a harmonic polynomial in  $\mathbb{R}^2$  and a connected component  $Z_0$  of its zero set  $Z$  such that*

$$|Z_0 \cap B_r| > \eta |Z_0 \cap B_R|.$$

Whether the above inequality also holds true for  $Z$  instead of  $Z_0$ , as in De Carli and Hudson’s statement, remains open, since the techniques we have used to construct our counterexample do not allow us to control the appearance of further connected components. Notice, moreover, that we do not have any control over the degree of the harmonic polynomial.

The proofs of Theorems 1.1–1.3 are respectively presented in Sects. 2–4. The idea of the proof of Theorems 1.1 and 1.3 is to start with a curve with the same properties as the level set we are looking for and construct a harmonic function in a neighborhood of this curve vanishing on the curve and having a suitably controlled gradient. Then, we approximate this local harmonic function by a harmonic polynomial and resort to appropriate stability arguments to derive the result. It is worth mentioning that this kind of strategy, which combines analytic approximation results and topological stability theorems, has been successfully applied to deal with periodic trajectories of solutions of the Euler equation of Fluid Mechanics [10]. On the contrary, the proof of Theorem 1.2 is elementary and follows from standard arguments in real analytic geometry.

To conclude, it is worth mentioning that the techniques we use to prove Theorems 1.1 and 1.3 do not really depend on the dimension and can be trivially adapted to construct harmonic polynomials in  $\mathbb{R}^n$  whose zero set has a connected component of prescribed geometry in a compact set (up to a small deformation). Nonetheless, in this paper, we have preferred to focus on the simpler two-dimensional statements, which are easier to visualize and suffice to prove De Carli and Hudson’s conjectures. For the benefit of the reader, we state the higher dimensional generalizations explicitly, where we are taking also into account the important Remarks 2.2 and 4.2, respectively, given in Sects. 2 and 4:

**Theorem 1.4** *Let  $D$  be a smooth bounded domain of  $\mathbb{R}^n$  whose complement  $\mathbb{R}^n \setminus D$  is connected and let us take a point  $z \in \partial D$ . Then, for any  $\epsilon > 0$ , there exists a diffeomorphism of  $\mathbb{R}^n$ , arbitrarily close to the identity in the  $C^0$ -norm, which maps the hypersurface  $\partial D \setminus B_\epsilon^n(z)$  into the zero set of a harmonic polynomial.*

**Theorem 1.5** *Let  $S$  be any smooth, properly embedded open hypersurface of  $\mathbb{R}^n$ . Then, for any  $R > 0$ , there is a diffeomorphism of  $\mathbb{R}^n$ , arbitrarily close to the identity in the  $C^1$ -norm, which transforms  $S \cap B_R^n$  into the zero set of a harmonic polynomial.*

These results also hold true for any scalar linear elliptic equation with real analytic coefficients at the expense of not having polynomial but analytic solutions. Indeed, the whole proof carries over to this case except Paramonov’s theorem on approximation by harmonic polynomials, which must be replaced by the Lax–Malgrange theorem [22, Theorem 3.10.7].

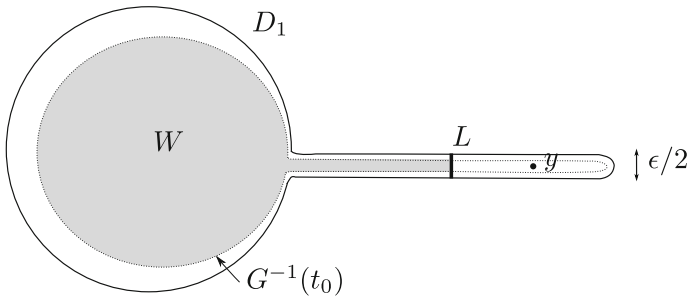
## 2 Proof of Theorem 1.1

In this section, we shall show that there exists a harmonic polynomial whose zero set has a ‘pinched’ component, in the sense specified in the statement of the theorem. As we will see, the method of proof we employ, which is of separate interest, is extremely versatile and can be used to deal with much more general geometries than that of the statement (cf. Remark 2.2 below).

Let us now begin with the proof of the theorem. To construct the domain  $D$  appearing in the statement of the theorem, we will make use of an auxiliary domain  $D_1$ . In order to construct the latter, for any  $x < 4$  and  $\epsilon > 0$ , let us introduce the notation  $R_{x,\epsilon}$  for the open rectangle  $(x, 4) \times (-\frac{\epsilon}{4}, \frac{\epsilon}{4}) \subset \mathbb{R}^2$ . Consider the bounded domain with piecewise smooth boundary

$$D_0 := B_2 \cup R_{0,\epsilon},$$

and let us denote by  $D_1$  a  $C^\infty$  domain obtained by rounding off the corners of  $D_0$ . We can safely assume that  $\partial D_1$  coincides with  $\partial D_0$  but in a small neighborhood of the corners of



**Fig. 1** The domain  $D_1$  and the sets  $W$  and  $L$

$\partial D_0$  (say, in balls of radius at most  $\epsilon/4$ ). It is natural to assume that

$$D_0 \subset D_1 \setminus ([3, \infty) \times \mathbb{R}) \quad \text{and} \quad D_1 \setminus ((-\infty, 3] \times \mathbb{R}) \subset D_0,$$

i.e., that the corners of  $D_0$  are rounded off from the outside at  $B_2 \cap R_{0,\epsilon}$  and from the inside at the rightmost side of the rectangle  $R_{0,\epsilon}$  (See Fig. 1 above).

Now, we introduce the Green’s function of the domain  $D_1$ , which constitutes a natural means of finding a nonnegative harmonic function in “most” of  $D_1$  (that is, in  $D_1$  minus a point) having the boundary of  $D_1$  as its zero set. Therefore, let  $y = (y_1, y_2)$  be a point of  $D_1$  with  $y_1 > 3$  and denote by  $G : D_1 \setminus \{y\} \rightarrow \mathbb{R}$  the Dirichlet Green’s function of  $D_1$  with a pole at  $y$ , which satisfies the equation  $\Delta G = -\delta_y$  in  $D_1$  and the boundary condition  $G|_{\partial D_1} = 0$ . Here,  $\delta_y$  denotes the Dirac measure supported at  $y$ . Obviously  $G$  is positive in  $D_1 \setminus \{y\}$  and smooth in  $\overline{D_1} \setminus \{y\}$  by the maximum principle and standard regularity results. It is well known that the maximum principle also implies that any level set of  $G$  is necessarily connected.

By Hopf’s boundary point lemma [14, Lemma 3.4], the gradient of  $G$  does not vanish on  $\partial D_1$  (and therefore in a half neighborhood of  $\partial D_1$ , by continuity), so  $G^{-1}(t)$  is a smooth closed curve for small enough  $t$ . (Actually, one can prove that  $\nabla G$  is nonzero everywhere, but we shall not need this fact). Let us fix a small  $t_0 > 0$  and consider the open set

$$W := D_1 \setminus \overline{(R_{3,\epsilon} \cup G^{-1}((0, t_0)))}.$$

The situation is sketched in Fig. 1; one should notice that the definition of  $D_1$  ensures that  $W$  is connected and contained in  $D_0$ . We can take  $t_0$  small enough so that  $G^{-1}((0, 3t_0))$  does not intersect  $B_1$  and  $\nabla G$  is nonzero in  $G^{-1}((0, 3t_0))$ .

By construction,  $G$  is harmonic in the closure of  $W$  (that is,  $\Delta G = 0$  in a neighborhood of  $\overline{W}$ ) and  $\mathbb{R}^2 \setminus W$  does not have any compact components. Therefore, a theorem of Paramonov [23] ensures that for any  $\delta > 0$ , there exists a harmonic polynomial  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ , which approximates  $G$  in the set  $W$  in the  $C^1$  norm:

$$\|G - u\|_{C^1(W)} := \sup_W (|G - u| + |\nabla G - \nabla u|) < \delta. \tag{1}$$

We will see later on that, for  $\delta$  small enough,  $u$  is the harmonic polynomial whose existence is claimed in the statement, up to an additive constant. A key property of the level curves of this polynomial is established in the following

**Lemma 2.1** *Let  $t_0$  be defined as above and take a positive constant*

$$\delta < \min \left( t_0, \inf_{G^{-1}((0, 3t_0))} |\nabla G| \right).$$

Then,  $u^{-1}(2t_0) \cap W$  is a smooth connected curve and divides  $W$  in two connected components.

*Proof* Let us consider the set

$$A_\delta := \overline{G^{-1}((2t_0 - \delta, 2t_0 + \delta))} \cap W$$

and the line segment

$$L := \{(x_1, x_2) \in \overline{D_1} : x_1 = 3\}.$$

Clearly,  $L \cap A_\delta$  has two connected components, one in the upper half-plane and one in the lower half-plane. We recall that a function  $f$  is *transverse* to a curve  $\Gamma$  in a set  $S$  if, for any point  $x \in \Gamma \cap S$ ,  $\nabla f(x)$  is not orthogonal to the curve’s tangent vector at the point. By the definitions of  $L$  and of the domain  $D_1$  and by the fact that the normal derivative of  $G$  at  $\partial D_1$  is nonzero,  $G$  is transverse to  $L$  on  $\partial D_1$ . Hence, by continuity,  $t_0$  can be chosen small enough so that  $G$  is transverse to  $L$  in  $A_\delta$  for all  $\delta \leq t_0$ . The  $C^1$  approximation (1) ensures that  $u$  is also transverse to  $L$  in  $A_\delta$  if  $\delta$  is chosen small enough.

Since  $\nabla G$  is nonzero in  $\overline{G^{-1}((0, 3t_0))}$ , by the definition of  $\delta$  and Eq. (1), it follows that  $\nabla u$  does not vanish on  $u^{-1}(2t_0) \cap W$ , which implies that  $u^{-1}(2t_0) \cap \overline{W}$  is a smooth curve contained in  $A_\delta$ . Rolle’s theorem then ensures that  $u^{-1}(2t_0) \cap L \cap A_\delta$  consists of precisely two points, one in the upper component of  $L \cap A_\delta$  and one in the lower. From this, one can readily infer that  $u^{-1}(2t_0) \cap A_\delta$  is connected. In order to see this, notice that  $u^{-1}(2t_0) \cap A_\delta$  cannot contain a closed curve by harmonicity and cannot intersect  $G^{-1}(2t_0 + \delta) \cap W$  or  $G^{-1}(2t_0 - \delta) \cap W$  by Eq. (1), so each component must intersect  $L$  at two distinct points. As  $u^{-1}(2t_0) \cap L \cap A_\delta$  consists of exactly two points, it stems that  $u^{-1}(2t_0) \cap W$  is connected and divides  $W$  in two components. □

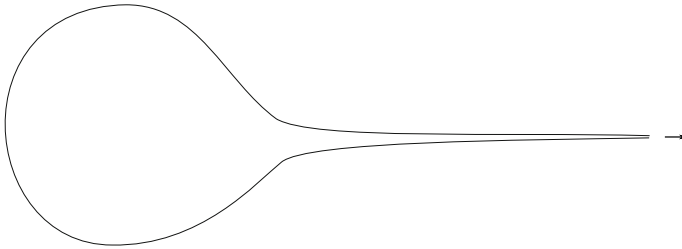
By Lemma 2.1,  $u^{-1}(2t_0)$  divides  $W$  in two connected components. Let us denote by  $D_2$  the component of  $W \setminus u^{-1}(2t_0)$  where  $v := u - 2t_0$  is positive, which by construction contains the ball  $B_1$ . Let us fix a point  $z = (z_1, z_2)$  of  $u^{-1}(2t_0) \cap W$  with  $z_1 < 0$  and let  $\Gamma$  be the component of  $\partial D_2 \setminus R_{5/2, \epsilon}$  connected with  $z$ . Since both endpoints of the smooth curve  $\Gamma$  are contained in the set

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 5/2, |x_2| < \epsilon/4\},$$

the distance between them is at most  $\epsilon/2$ , so it is standard that one can take a smooth arc  $\gamma_\epsilon \subset D_2$  of length smaller than  $\epsilon$  connecting the endpoints of  $\Gamma$  and such that  $\gamma := \overline{\Gamma \cup \gamma_\epsilon}$  is a smooth closed curve. The theorem then follows upon noticing that the harmonic polynomial  $v$  is positive in the domain  $D \supset B_1$  enclosed by  $\gamma$  and vanishes on  $\Gamma$ .

*Remark 2.2* In fact, with the above argument, we have proved a stronger result than stated in Theorem 1.1: we have proved that there exists a diffeomorphism (obtained by dragging along the curves transverse to  $\partial D_1$ ), which maps  $\gamma \setminus \gamma_\epsilon$  into the zero set of the harmonic polynomial  $v$  and is arbitrarily close to the identity in the  $C^0$ -norm. The geometry of the curve  $\gamma$  is inessential, and the method of proof can be obviously applied to any other closed curve without self-intersections.

*Remark 2.3* It is worth pointing out that De Carli and Hudson’s conjectures [8, Conjecture 5.1] are false even for quadratic harmonic polynomials if the region enclosing the ball  $B_1$ , which is determined by the zero set of the polynomial and the line segment of length at most  $\epsilon$ , is allowed to be unbounded. Indeed, the harmonic polynomial  $v(x, y) := (y - 3)^2 - x^2 + \frac{\epsilon^2}{4}$  is an explicit counterexample, since the union of the zero set of  $v$  and the segment  $\{y =$



**Fig. 2** An asymptotically pinched level set

3,  $|x| < \epsilon/2$  (which has length  $\epsilon$ ) divides the plane into four connected components, and the function  $v$  is positive in the component that contains the ball of radius 1 centered at the origin.

What is true, as suspected by De Carli and Hudson, is that a harmonic polynomial  $v(x, y)$  cannot have an ‘asymptotically pinched’ level set, as the one displayed in Fig. 2. This can be readily proved using that, up to a multiplicative constant and a rigid motion, the highest order homogeneous term of  $v(x, y)$  is the real part of the complex polynomial  $z^p$ , where  $p$  is the degree of  $v$ . By looking at the auxiliary polynomial

$$\tilde{v}(x, y) := (x^2 + y^2)^p v\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right),$$

which by the latter observation has the asymptotics

$$\tilde{v}(r \cos \theta, r \sin \theta) = r^p \cos p\theta + \mathcal{O}(r^{p+1})$$

in polar coordinates, it is not difficult to prove that, sufficiently far away from the origin, the zero set of  $v$  consists of  $2p$  branches asymptotic to the straight lines of angles  $\theta = \pi k/p$ , with  $0 \leq k \leq 2p - 1$ . On the contrary, non-polynomial harmonic functions can have asymptotically pinched level sets, as evidenced by the function  $u(z) := \text{Im } e^{e^z} - 1$ .

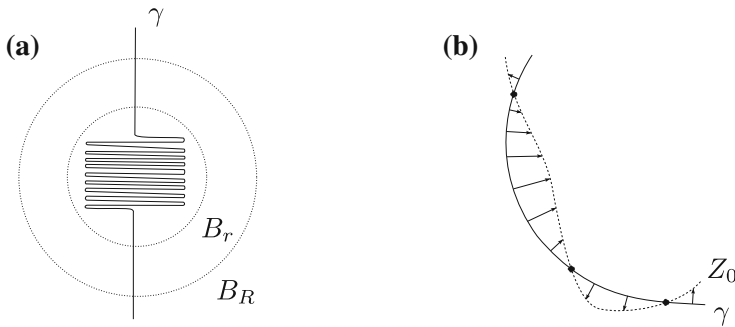
### 3 Proof of Theorem 1.2

Since  $u$  is harmonic in  $D$ , it is well known that  $u$  is real analytic in  $D$  [14], which is the only property of  $u$  we need to prove the theorem. The argument is totally standard but we include it here for completeness; notice that this result also follows after an easy discussion from [2, Remark 2.1], where maximal analytic extensions are considered.

Before providing the proof, let us recall that the fact that  $M$  is an  $m$ -dimensional real analytic submanifold of a domain  $D \subseteq \mathbb{R}^n$  means that  $M$  is a connected, locally arc-connected subset of  $D$ , which is a real analytic manifold with the induced topology, and that the inclusion map  $j : M \rightarrow D$  is a real analytic embedding. For simplicity of notation, we will always identify a set  $U \subset M$  with its embedded image  $j(U)$ .

By the hypotheses of the theorem, there is a set  $U$  relatively open in  $M$  that is contained in  $Z$  but  $M \setminus Z$  is nonempty (and relatively open in  $M$ , since  $Z$  is closed). There is no loss of generality in assuming that  $U$  is connected and maximal in the sense that there is no open connected subset of  $M$  properly containing  $U$  and contained in  $Z$ .

Let us take a point  $y \in \partial U$ . As  $M$  is a real analytic  $m$ -manifold, there are a relatively open neighborhood  $B$  of  $y$  in  $M$ , a ball  $B_r^m$  in  $\mathbb{R}^m$  and a real analytic diffeomorphism  $\varphi : B \rightarrow B_r^m$



**Fig. 3** Construction of the curve  $\gamma$  and of the normal deformation

mapping  $y$  to 0. Consider the real analytic function  $v : B_r^m \rightarrow \mathbb{R}$  given by

$$v := u \circ \varphi^{-1}.$$

As  $v$  is real analytic, the radius of convergence  $\rho_v(x)$  of the Taylor expansion of  $v$  at a point  $x$  is positive. Let us set

$$\rho := \inf_{x \in B_{r/2}^m} \rho_v(x) \in (0, r)$$

and let  $z$  be a point in  $B_{\rho/2}^m \cap \varphi(U)$ . The Taylor expansion of  $v$  at  $z$  is obviously identically zero because  $v$  vanishes identically in a neighborhood of  $z$ . However, since  $|z| < \rho/2$  and the radius of convergence of the aforementioned Taylor series is at least  $\rho$ ,  $v$  is guaranteed to vanish in  $\varphi(U) \cup B_{\rho/2}^m$ . This means that  $u$  vanishes in  $U \cup \varphi^{-1}(B_{\rho/2}^m)$ , which is strictly larger than  $U$ , thereby contradicting the maximality of  $U$  and completing the proof of the theorem.

### 4 Proof of Theorem 1.3

Let us start by constructing a curve  $\gamma$  having the same properties as the level curve we are looking for. To this end, we take some  $\eta' \in (\eta, 1)$  and an open, connected, real analytic curve  $\gamma \subset \mathbb{R}^2$  without self-intersections such that

$$|\gamma \cap B_r| > \eta' |\gamma \cap B_R|. \tag{2}$$

We can assume that  $\gamma$  looks approximately as in Fig. 3a above. It is standard that such a curve can be constructed from a piecewise linear curve satisfying an inequality analogous to (2) using Whitney’s approximation theorem [19].

Let us choose an orientation of  $\gamma$  and denote by  $\nu(x)$ , the corresponding unit normal at a point  $x \in \gamma$ . A natural way to define a harmonic function associated with  $\gamma$  and having some control on its zero set and on its gradient is via the following Cauchy problem:

$$\Delta v = 0, \quad v|_{\gamma \cap B_{5R}} = 0, \quad \frac{\partial v}{\partial \nu} \Big|_{\gamma \cap B_{5R}} = 1. \tag{3}$$

The intersection with the ball  $B_{5R}$  has been introduced for convenience. The Cauchy–Kowalewski theorem ensures the existence of a solution  $v$  to the above problem in a neighborhood of  $\gamma \cap B_{5R}$ ; by the boundedness of  $B_{5R}$ , one can therefore assume that  $v$  is defined

in the closure of

$$U := \{x \in B_{4R} : |\sigma(x)| < c\}$$

for sufficiently small  $c$ . Here,  $\sigma$  denotes the signed distance function to  $\gamma$ , which is real analytic if  $c$  is small enough [18] and satisfies  $|\nabla\sigma| = 1$ . We can also assume that  $U$  contains the component of  $v^{-1}([-a, a]) \cap B_{4R}$  connected with  $\gamma \cap B_{4R}$  for some  $a > 0$ . It should be noticed that as the gradient of  $v$  does not vanish on  $\gamma \cap B_{5R}$ , by the implicit function theorem

$$v^{-1}(0) \cap U = \gamma \cap U \tag{4}$$

provided  $c$  is sufficiently small.

Since obviously  $\mathbb{R}^2 \setminus U$  does not have any compact components, a theorem of Paramonov [23] yields a harmonic polynomial  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$\|u - v\|_{C^2(U)} := \sum_{|\alpha| \leq 2} \sup_U |D^\alpha(u - v)| < \delta, \tag{5}$$

where  $\delta$  is a small quantity to be specified later. Setting  $Z := u^{-1}(0)$ , Eq. (5) implies that  $Z \cap U \subset v^{-1}((-\delta, \delta))$ , so  $Z \cap U$  does not intersect the curves  $\sigma^{-1}(\pm c) \cap \bar{U}$  if  $\delta < a$ .

As  $\nabla\sigma(x) = v(x)$  for each  $x \in \gamma$ , it stems from (3) that

$$\nabla\sigma \cdot \nabla v \neq 0 \tag{6}$$

on  $\gamma \cap B_{5R}$ . By continuity, the inequality (6) will also hold true in  $\bar{U}$  for small enough  $c$ , so we can assume that

$$\inf_U |\nabla\sigma \cdot \nabla u| > 0 \tag{7}$$

for  $\delta < \inf_U |\nabla\sigma \cdot \nabla v|$ .

We shall next show that  $u$  satisfies the properties we claimed in the statement of the theorem by using a diffeomorphism, constructed as the time-1 flow of a vector field, mapping the intersection of the curve  $\gamma$  with the ball  $B_R$  into the zero set of  $u$ . In order to do so, for each  $z \in \gamma \cap B_{2R}$  let us define a parameterized normal curve  $\xi_z : (-c, c) \rightarrow U$  by

$$\xi_z(s) := z + s v(z).$$

As  $\nabla\sigma(\xi_z(s)) = v(z)$  for all  $s$ , it is clear that

$$\frac{d}{ds} u(\xi_z(s)) = \nabla u(\xi_z(s)) \cdot \nabla\sigma(\xi_z(s)) \neq 0 \tag{8}$$

by (7) for each fixed  $z \in \gamma \cap B_{2R}$  and all  $s \in (-c, c)$ . Since  $[-a, a] \subset v(\xi_z(-c, c))$  for all  $z \in \gamma \cap B_{2R}$  and  $\delta < a$ , we can ensure that  $u \circ \xi_z$  has a zero in  $(-c, c)$  by (5). This zero, which we shall call  $\bar{\sigma}(z)$ , is necessarily unique by Eq. (8) and Rolle's theorem, which means that  $Z \cap U \cap B_{2R}$  is connected. We can then denote by  $Z_0$  the unique component of  $Z$  connected with  $U \cap B_{2R}$ .

Let  $\chi : \mathbb{R}^2 \rightarrow [0, 1]$  be a  $C^\infty$  function equal to 1 in  $v^{-1}((-\delta, \delta)) \cap U \cap B_{3R}$  and equal to 0 in

$$(\mathbb{R}^2 \setminus U) \cup \{x \in U \cap B_{3R} : |v(x)| > \delta^{1/2} + 2\delta\}. \tag{9}$$

We assume  $\delta < a/3$  so that

$$\overline{v^{-1}((-\delta, \delta)) \cap B_{3R} \cap U} \subset U.$$



From (7), it stems that

$$\text{dist}(x, \gamma) < C \delta' \tag{10}$$

for all  $x \in U$  with  $|v(x)| < \delta'$  and all  $\delta' < a$ , so we can safely assume that

$$\sup_{\mathbb{R}^2} |\nabla \chi| < C_1 \delta^{-1/2} \tag{11}$$

for some constant independent of  $\delta$ . Consider the set

$$V := \{\xi_z(s) : z \in \gamma \cap B_{2R}, |s| < c\},$$

which is contained in  $U$  and define a  $C^\infty$  vector field  $X$  in  $V$  by setting

$$X(\xi_z(s)) := \bar{\sigma}(z) \chi(\xi_z(s)) \nabla \sigma(\xi_z(s)) \tag{12}$$

for all  $z \in \gamma \cap B_{2R}$  and all  $|s| < c$ .

Our next objective is to prove that the time-1 flow of the vector field  $X$  is  $C^1$ -close to the identity. For this purpose, we find it convenient to introduce coordinates  $(\zeta, \sigma)$  in  $V$  by choosing a parametrization  $\pi : \gamma \cap B_{3R} \rightarrow I$  that is a diffeomorphism onto some interval  $I \subseteq \mathbb{R}$  and setting

$$\zeta(\xi_z(s)) := \pi(z)$$

for all  $z \in \gamma \cap B_{2R}$  and  $s \in (-c, c)$ . In view of this formula and Eq. (12), it is clear that estimating the derivatives of  $\tilde{\sigma} \circ \zeta$ , with  $\tilde{\sigma} := \bar{\sigma} \circ \pi^{-1}$ , is crucial in order to control the derivatives of  $X$ , which in turn are needed to control its time-1 flow. In the following lemma, we show that the  $C^1$  norm of the vector field  $X$  is small. Before stating this result, we recall that the function  $\sigma$  is the signed distance to  $\gamma$ ; it should be noticed that, by construction,  $\nabla \zeta$  and  $\nabla \sigma$  are orthogonal in  $V$ .

**Lemma 4.1** *There is a constant  $C' > 0$  such that  $\|X\|_{C^1(\mathbb{R}^2)} < C' \delta^{1/2}$ .*

*Proof* A first observation is that

$$|\tilde{\sigma} \circ \zeta| < C_2 \delta \tag{13}$$

in  $V$ , where by each  $C_j$  we will henceforth denote a constant independent of  $\delta$ . In order to see this, it should be noticed that, for any  $x \in \xi_z((-c, c))$ , we defined  $\tilde{\sigma}(\zeta(x))$  as the value for which

$$(u \circ \xi_z)(\tilde{\sigma}(\zeta(x))) = 0,$$

which by (5) implies that

$$|(v \circ \xi_z)(\tilde{\sigma}(\zeta(x)))| < \delta.$$

As  $v(z) = 0$  and

$$\left| \frac{d}{ds} (v \circ \xi_z)(s) \right| = |\nabla v(\xi_z(s)) \cdot \nabla \sigma(\xi_z(s))| > C_3$$

in  $U$  by (6), Eq. (13) readily follows.

Similarly, and writing  $u(\zeta, \sigma)$  for the expression of  $u$  in the coordinates  $(\zeta, \sigma)$  with a slight abuse of notation, we can use the implicit function theorem and the fact that  $u(\zeta, \tilde{\sigma}(\zeta)) = 0$  to estimate its gradient as

$$|\nabla(\tilde{\sigma} \circ \zeta)| = |\nabla \zeta| \left| \frac{d\tilde{\sigma}}{d\zeta} \right| = |\nabla \zeta| \left| \frac{\frac{\partial u}{\partial \zeta}}{\frac{\partial u}{\partial \sigma}} \right| = \frac{|\nabla \zeta \cdot \nabla u|}{|\nabla \zeta| |\nabla \sigma \cdot \nabla u|} \leq C_4 |\nabla \zeta \cdot \nabla u| \tag{14}$$

in  $V$ . In the last two steps, we have used (7) and the fact that  $(\zeta, \sigma)$  is an orthogonal coordinate system. Let us now consider the function  $f := |\nabla\zeta \cdot \nabla u|$ . As  $\nabla\zeta$  is tangent to  $\gamma$ , Eq. (3) implies that  $\nabla v \cdot \nabla\zeta = 0$  on  $\gamma \cap B_{2R}$ , so that  $|f| < C_5 \delta$  on  $\gamma \cap B_{2R}$  by (5). Moreover,  $|\nabla f| < C_6$  in  $V$  with  $C_6$  independent of  $\delta$  by the  $C^2$  estimate (5), which in view of (9) and (10) clearly guarantees that

$$|f| < C_5 \delta + C C_6 (\delta^{1/2} + 2\delta)$$

in  $V \cap \text{supp } \chi$ . Hence, (14) yields

$$|\nabla(\tilde{\sigma} \circ \zeta)| < C_7 \delta^{1/2} \quad \text{in } V \cap \text{supp } \chi. \tag{15}$$

Using the estimates (11), (13) and (15) and denoting by  $\nabla X$  the Jacobian of  $X$ , one then obtains that

$$\begin{aligned} |X| &\leq \chi |\tilde{\sigma} \circ \zeta| < C_2 \delta, \\ |\nabla X| &\leq \chi |\nabla(\tilde{\sigma} \circ \zeta)| + |\nabla\chi| |(\tilde{\sigma} \circ \zeta)| + \chi |(\tilde{\sigma} \circ \zeta)| |\nabla^2\sigma| \\ &< C_7 \delta^{1/2} + C_1 C_2 \delta^{1/2} + C_8 \delta < C_9 \delta^{1/2} \end{aligned}$$

in  $V$ , so it is standard that one can extend  $X$  to a vector field  $\tilde{X}$  in  $\mathbb{R}^2$  that coincides with  $X$  in  $V$  and satisfies

$$|\tilde{X}| < 2C_2 \delta, \quad |\nabla\tilde{X}| < 2C_9 \delta^{1/2}$$

in  $\mathbb{R}^2$ . This proves the lemma. □

Let us now denote by  $\varphi_t$  the flow defined by the bounded vector field  $\tilde{X}$ . Since

$$\xi_z^{-1}([0, \tilde{\sigma}(z)]) \subset v^{-1}((-\delta, \delta)) \cap V \cap B_{3R}$$

for all  $z \in \gamma \cap B_{2R}$ , one can easily compute the integral curves of  $X$  to show that

$$\varphi_1(z) = \xi_z(\tilde{\sigma}(z))$$

for all  $z \in \gamma \cap B_{2R}$  by the definition of  $\chi$  and of  $X$  (cf. Fig. 3b). This implies that the time-1 flow  $\varphi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a smooth diffeomorphism mapping  $\gamma \cap B_{2R}$  into  $Z_0 \cap B_{3R}$ . Furthermore, the bound for the vector field  $X$  proved in Lemma 4.1 immediately yields the  $C^1$  estimate

$$\|\varphi_1 - \text{id}\|_{C^1(\mathbb{R}^2)} < C_{10} \delta^{1/2}. \tag{16}$$

To conclude, we shall use the above estimate to show that  $|Z_0 \cap B_r| > \eta |Z_0 \cap B_R|$  if  $\delta$  is sufficiently small, which completes the proof of the theorem. Denoting by  $\nabla\varphi_1(x) w$  the action of the differential of  $\varphi_1$  at  $x$  on a vector  $w \in \mathbb{R}^2$  and defining

$$\lambda_{\max}(x) := \max_{|w|=1} |\nabla\varphi_1(x) w|$$

(and analogously  $\lambda_{\min}(x)$ ), it is standard that the length of a curve  $\Gamma$  and its image by  $\varphi_1$  are related by

$$\text{length } \Gamma \cdot \inf_{\Gamma} \lambda_{\min} \leq \text{length } \varphi_1(\Gamma) \leq \text{length } \Gamma \cdot \sup_{\Gamma} \lambda_{\max}.$$

Therefore, since  $\varphi_1(\gamma) \cap B_R = Z_0 \cap B_R$ , using the estimate (16) for  $\nabla\varphi_1$  and the inequality (2), one can readily check that

$$|Z_0 \cap B_r| > (\eta' - C_{11} \delta^{1/2}) |Z_0 \cap B_R|,$$

so the theorem follows provided that  $\delta^{1/2} < (\eta' - \eta)/C_{11}$ .

*Remark 4.2* The proof we have given actually yields a stronger result than Theorem 1.3: we have showed that, given a smooth open curve  $\gamma$  and a positive number  $R$ , there exists a harmonic polynomial  $u$  and a smooth diffeomorphism  $\Phi_1$ , arbitrarily close to the identity in the  $C^1$  norm, which maps  $\gamma \cap B_R$  into a connected component of the zero set of  $u$ .

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