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# Infinitely many small solutions for the $p(x)$-Laplacian operator with nonlinear boundary conditions

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**Abstract** In this paper, we prove the existence of infinitely many small solutions to the following quasilinear elliptic equation $-\Delta_{p(x)}u+|u|^{p(x)-2}u = f(x,u)$ in a smooth bounded domain $\Omega$ of $\mathbb{R}^N$ with nonlinear boundary conditions $|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = |u|^{q(x)-2}u$. We also assume that $\{q(x) = p^*(x)\} \neq \emptyset$, where $p^*(x) = Np(x)/(N - p(x))$ is the critical Sobolev exponent for variable exponents. The proof is based on a new version of the symmetric mountain-pass lemma due to Kajikiya, and property of these solutions is also obtained.

**Keywords** $p(x)$-Laplacian · Generalized Lebesgue-Sobolev spaces · Nonlinear boundary conditions · Concentration-compactness principle

**Mathematics Subject Classification (2000)** 35J60 · 35B33

## 1 Introduction

In this paper, we deal with quasilinear elliptic problem of the form

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = f(x,u), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = |u|^{q(x)-2}u, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary and $p(x)$, $q(x)$ are two continuous functions on $\overline{\Omega}$, $1 < p^- = \inf_{x\in\overline{\Omega}} p(x) \leq p(x) \ll q(x) < N$, where denote by $p(x) \ll q(x)$ the fact that $\inf_{x\in\Omega}(q(x) - p(x)) > 0$. $\Delta_{p(x)}u := \text{div}(|\nabla u|^{p(x)-2}\nabla u)$

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is the $p(x)$-Laplacia operator and $\frac{\partial}{\partial\nu}$ is the outer normal derivative. On the exponent $q(x)$, we assume that is the critical exponent in the sense that $\{q(x) = p^*(x)\} \neq \emptyset$, where $p^*(x) = Np(x)/(N - p(x))$ is the critical exponent according to the Sobolev embedding. In the spirit of [1–7], our goal will be to obtain infinitely many small weak solutions that tend to zero for (1.1) in the generalized Sobolev space $W^{1,p(x)}(\Omega)$ for the general nonlinearities of the type $f(x,u)$.

The study of differential equations and variational problems involving variable exponent conditions has been a very interesting and important topic. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics, image processing and so on. For example, Chen et al. [8] proposed the following model in image processing

$$F(u) = \int_{\Omega} \frac{|\nabla u(x)|^{p(x)}}{p(x)} + f(|u(x) - I(x)|)\mathrm{d}x \to \min,$$

where $p(x)$ is a function satisfies $1 \leq p(x) \leq 2$ and $f$ is a convex function. For more information on modelling physical phenomena by equations involving $p(x)$-growth condition, we refer to [9–12]. The appearance of such physical models was facilitated by the development of variable Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where $p(x)$ is a real-valued function. On the variable exponent Sobolev spaces that have been used to study $p(x)$-Laplacian problems, we refer to [13–15]. On the existence of solutions for elliptic equations with variable exponent, we refer to [16–29].

In recent years, the existence of infinitely many solutions has been obtained by many papers. When $p(x) \equiv p = 2$ (a constant) with Dirichlet boundary condition, Li and Zou [6] studied a class of elliptic problems with critical exponents, they obtained the existence theorem of infinitely many solutions under suitable hypotheses. He and Zou [4] proved that the existence infinitely many solutions under case the general nonlinearities. When $p(x) \equiv p \neq 2$. Ghoussoub and Yuan [30] obtained the existence of infinitely many nontrivial solutions for Hardy-Sobolev subcritical case and Hardy critical case by establishing Palais-Smale type conditions around appropriate chosen dual sets in bounded domain. Li and Zhang [31] studied the existence of multiple solutions for the nonlinear elliptic problems of $p\&q$-Laplacian type involving the critical Sobolev exponent, they obtained infinitely many weak solutions by using Lusternik-Schnirelman's theory for $Z_2$-invariant functional.

On the existence of infinitely many solutions for $p(x)$-Laplacian problems have been studied by [16,18,20,23], but they did not give any further information on the sequence of solutions. Moreover, these papers deal with subcritical nonlinearities. Very little is known about critical growth nonlinearities for variable exponent problems [34,35], since one of the main techniques used in order to deal with such issues is the concentration-compactness principle. This result was recently obtained for the variable exponent case independently in [1,33]. In both of these papers, the proof is similar and both relate to that of the original proof of P.L. Lions [36,37].

Recently, Kajikiya [5] established a critical point theorem related to the symmetric mountain pass lemma and applied to a sublinear elliptic equation. But there are no such results on $p(x)$-Laplacian problem with critical growth (1.1).

Motivated by reasons above, the aim of this paper is to show that the existence of infinitely many solutions of problem (1.1), and there exists a sequence of infinitely many arbitrarily small solutions converging to zero by using a new version of the symmetric mountain-pass lemma due to Kajikiya [5]. In order to use the symmetric mountain-pass lemma, there are many difficulties. The main one in solving the problem is a lack of compactness which can be illustrated by the fact that the embedding of $W^{1,p(x)}(\Omega)$ into $L^{p^*(x)}(\partial\Omega)$ is no longer compact. Hence, the concentration-compactness principle is used here to overcome the difficulty. It should be noted that the embedding of $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ can be compact even when $q = p^*$ in some points, see Kurata and Shioji [24].

The main result of this paper is as follows.

**Theorem 1.1** *Suppose that $f(x,u)$ satisfies the following conditions:*

(H$_1$) $f(x,u) \in C(\Omega \times R, R)$, $f(x,-u) = -f(x,u)$ *for all* $u \in R$;
(H$_2$) $\lim_{|u|\to\infty} \frac{f(x,u)}{|u|^{p(x)-1}} = 0$ *uniformly for* $x \in \Omega$;
(H$_3$) $\lim_{|u|\to 0^+} \frac{f(x,u)}{u^{p^- -1}} = \infty$ *uniformly for* $x \in \Omega$.

*Then, problem (1.1) has a sequence of nontrivial solutions $\{u_n\}$ and $u_n \to 0$ as $n \to \infty$.*

*Remark 1.1* If without the symmetry condition (i.e. $f(x,-u) = -f(x,u)$) in Theorem 1.1, we get an existence theorem of at least one nontrivial solution to problem (1.1) by the same method in this paper.

*Remark 1.2* In this paper, we use concentration-compactness principle due to [1] which is slightly more general than those in [33], since we do not require $q(x)$ to be critical everywhere.

*Remark 1.3* There exist many functions $f(x,t)$ satisfy conditions $(H_1)-(H_3)$, for example, $f(x,u) = u^{(p^- -1)/3}$, where $p^- > 1$.

*Remark 1.4* Theorem 1.1 is new as far as we know. We mainly follow the way in [7] to prove our main result.

**Definition 1.1** We say that $u_0 \in W^{1,p(x)}(\Omega)$ is a weak solution of problem (1.1) in the weak sense if for any $v \in W^{1,p(x)}(\Omega)$

$$\int_\Omega \left(|\nabla u_0|^{p(x)-2}\nabla u_0 \cdot \nabla v + |u_0|^{p(x)-2}u_0 v\right) \mathrm{d}x - \int_{\partial\Omega} |u_0|^{q(x)-2}u_0 v \mathrm{d}\sigma$$
$$- \int_\Omega f(x,u_0) v \mathrm{d}x = 0,$$

where $\mathrm{d}\sigma$ is the surface measure on the boundary.

The energy functional corresponding to problem (1.1) is defined as follows,

$$J(u) = \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} \mathrm{d}x - \int_{\partial\Omega} \frac{1}{q(x)} |u|^{q(x)} \mathrm{d}\sigma - \int_\Omega F(x,u) \mathrm{d}x,$$

where $F(x,t) = \int_0^t f(x,s)ds$. Then, it is easy to check that as arguments [38] show that $J(u)$ is well defined on $W^{1,p(x)}(\Omega)$ and $J \in C^1(W^{1,p(x)}(\Omega), \mathbb{R})$ and the weak solutions for problem (1.1) coincides with the critical points of $J$. We try to use a new version of the symmetric mountain-pass lemma due to Kajikiya [5]. But since the functional $J(u)$ is not bounded from below, we could not use the theory directly. So we follow [7] to consider a truncated functional of $J(u)$. Denote $J' : E \to E^*$ is the derivative operator of $J$ in the weak sense. Then

$$\langle J'(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \cdot \nabla v + |u|^{p(x)-2}uv\right) \mathrm{d}x - \int_{\partial\Omega} |u|^{q(x)-2}uv \mathrm{d}\sigma$$
$$- \int_\Omega f(x,u) v \mathrm{d}x, \ \forall u, v \in W^{1,p(x)}(\Omega).$$

**Definition 1.2** We say $J$ satisfies Palais-Smale condition ($(PS)$ for short) in $W^{1,p(x)}(\Omega)$, if any sequence $\{u_n\} \subset W^{1,p(x)}(\Omega)$ which satisfies that $\{J(u_n)\}$ is bounded and $\|J'(u_n)\|_{p(x)} \to 0$ as $n \to \infty$, has a convergent subsequence.

Under assumptions $(H_1)$ and $(H_2)$, we have

$$f(x,u)u = o\left(|u|^{p(x)}\right), \qquad F(x,u) = o\left(|u|^{p(x)}\right),$$

which means that, for all $\varepsilon > 0$, there exist $a(\varepsilon),\ b(\varepsilon) > 0$ such that

$$|f(x,u)u| \le a(\varepsilon) + \varepsilon|u|^{p(x)}, \tag{1.2}$$

$$|F(x,u)| \le b(\varepsilon) + \varepsilon|u|^{p(x)}. \tag{1.3}$$

Hence, for any constants $\beta$, we have

$$|F(x,u) - \beta f(x,u)u| \le c(\varepsilon) + \varepsilon|u|^{p(x)}, \tag{1.4}$$

for some $c(\varepsilon) > 0$.

The remainder of the paper is organized as follows. In Sect. 2, we shall present some basic properties of the variable exponent Sobolev spaces. In Sect. 3, we will prove the corresponding energy functional satisfies the $(PS)$ condition. In Sect. 4, we shall prove our main results.

## 2 Weighted variable exponent Lebesgue and Sobolev spaces

We recall some definitions and properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^N$. Set

$$C_+(\overline{\Omega}) = \left\{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\right\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \quad \text{and} \quad h^- = \inf_{x\in\overline{\Omega}} h(x).$$

We can introduce the variable exponent Lebesgue space as follows:

$$L^{p(\cdot)}(\Omega) = \left\{u : u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\right\},$$

for $p \in C_+(\overline{\Omega})$. Equipping with the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} \mathrm{d}x \le 1\right\},$$

which is a Banach space, we call it a generalized Lebesgue space.

**Proposition 2.1** [13,22]

(i) *The space* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ *is a separable, uniform convex Banach space, and its conjugate space is* $L^{p'(x)}(\Omega)$, *where* $1/p'(x) + 1/p(x) = 1$. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{p'(x)}(\Omega)$, *we have*

$$\left| \int_\Omega uv\,dx \right| \le \left( \frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(\cdot)} |v|_{p'(\cdot)}; \tag{2.1}$$

(ii) *If* $0 < |\Omega| < \infty$ *and* $p_1, p_2$ *are variable exponents in* $C_+(\overline{\Omega})$ *such that* $p_1 \le p_2$ *in* $\Omega$, *then the embedding* $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ *is continuous.*

**Proposition 2.2** [13,22] *The mapping* $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ *defined by*

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

*Then, the following relations hold:*

$$\begin{aligned}
&|u|_{p(\cdot)} < 1\ (= 1;\ > 1) \ \Leftrightarrow\ \rho_{p(\cdot)}(u) < 1\ (= 1;\ > 1),\\
&|u|_{p(\cdot)} > 1 \ \Rightarrow\ |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+},\\
&|u|_{p(\cdot)} < 1 \ \Rightarrow\ |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-},\\
&|u_n - u|_{p(\cdot)} \to 0 \ \Leftrightarrow\ \rho_{p(\cdot)}(u_n - u) \to 0.
\end{aligned}$$

Next, we define $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\}$$

and it can be equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall\, u \in W^{1,p(x)}(\Omega).$$

**Proposition 2.3** [13,22]

(i) $W^{1,p(x)}(\Omega)$ *are separable reflexive Banach spaces;*
(ii) *If* $p \in C_+(\overline{\Omega})$, *then the embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ *is compact and continuous.*

**Proposition 2.4** [32] *Let* $\Omega \subset \mathbb{R}^n$ *be an open bounded domain with Lipschitz boundary. Suppose that* $p \in C^0(\overline{\Omega})$ *and* $1 < p^- \le p^+ < N$. *If* $\sigma \in C^0(\partial\Omega)$ *satisfies the condition*

$$1 \le \sigma(x) < \frac{(N-1)p(x)}{N - p(x)}, \quad \forall x \in \partial\Omega. \tag{2.2}$$

*Then, there is a compact boundary trace embedding* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\sigma(\cdot)}(\partial\Omega)$, *we denote by* $K$ *the embedding constant.*

In this paper, we use the following equivalent norm on $W^{1,p(x)}(\Omega)$:

$$\|u\|_{p(\cdot)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{\nabla u}{\mu} \right|^{p(x)} + \left| \frac{u}{\mu} \right|^{p(x)} \mathrm{d}x \le 1 \right\}. \tag{2.3}$$

**Proposition 2.5** [14,17] *Let* $I(u)=\int_\Omega |\nabla u|^{p(x)}+|u|^{p(x)}dx$. *If* $u, u_n \in W^{1,p(x)}(\Omega)$, *then the following relations hold:*

$$\|u\|_{p(\cdot)}<1\ (=1;\ >1)\ \Leftrightarrow\ I(u)<1\ (=1;\ >1), \tag{2.4}$$

$$\|u\|_{p(\cdot)}>1\ \Rightarrow\ \|u\|_{p(\cdot)}^{p^-}\le I(u)\le \|u\|_{p(\cdot)}^{p^+}, \tag{2.5}$$

$$\|u\|_{p(\cdot)}<1\ \Rightarrow\ \|u\|_{p(\cdot)}^{p^+}\le I(u)\le \|u\|_{p(\cdot)}^{p^-}, \tag{2.6}$$

$$\|u_n-u\|_{p(\cdot)}\to 0\ \Leftrightarrow\ I(u_n-u)\to 0. \tag{2.7}$$

## 3 Preliminaries and lemmas

In the following, we always use $C$ and $c_i\,(i=1,2,\ldots)$ to denote positive constants.

To prove Theorem 1.1, since we have lost the compactness in the inclusion $W^{1,p(x)}(\Omega)\hookrightarrow L^{p^*(x)}(\partial\Omega)$, we can no longer expect the Palais-Smale condition to hold. Anyway we can prove a local Palais-Smale condition that will hold for $J(u)$ below a certain value of energy. Let $u_n$ be a bounded sequence in $W^{1,p(x)}(\Omega)$ then there exists a subsequence that we still denote $u_n$ such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{weakly in } W^{1,p(x)}(\Omega),\\ &u_n \to u \quad \text{strongly in } L^{r(x)}(\Omega),\ 1\le r(x)\ll p^*(x),\\ &|\nabla u_n|^{p(x)} \rightharpoonup d\mu, \quad |u_n|_{\partial\Omega}|^{p^*(x)} \rightharpoonup d\nu, \end{aligned}$$

weakly-* in the sense of measures. Observe that $d\nu$ is a measure supported on $\partial\Omega$.

The technical result used here, the concentration-compactness principle of the variable exponent, is mainly due to [1]. The proof is similar to the original proof of Lions [36,37]. The following lemma follows exactly as in [1] and the proof is omitted.

**Lemma 3.1** *Let* $q(x)$ *and* $p(x)$ *be two continuous functions such that*

$$1<\inf_{x\in\Omega}p(x)\le\sup_{x\in\Omega}p(x)<N \quad and \quad 1\le q(x)\le p^*(x) \quad in\ \Omega.$$

*Let* $\{u_j\}_{j\in\mathbb{N}}$ *be a weakly convergent sequence in* $W^{1,p(x)}(\Omega)$ *with weak limit u, and such that* $|\nabla u_j|^{p(x)}\rightharpoonup d\mu$ *weakly-* in the sense of measures;* $|u_j|_{\partial\Omega}|^{q(x)}\rightharpoonup d\nu$ *weakly-* in the sense of measures. Assume, moreover that* $\Gamma=\{x\in\partial\Omega: q(x)=p^*(x)\}\ne\emptyset$. *Then, for some countable index set I, there exists* $x_1,\ldots,x_l\in\Gamma$ *such that*

(i) $d\nu=|u|^{q(x)}+\sum_{i=1}^{l}\nu_i\delta_{x_i},\quad \nu_i>0;$

(ii) $d\mu\ge|\nabla u|^{p(x)}+\sum_{i=1}^{l}\mu_i\delta_{x_i},\quad \mu_i>0;$

(iii) $S\nu_i^{p(x_i)/p^*(x_i)}\le\mu_i,\quad i\in I;$

*where* $\{x_i\}_{i=1}^{l}\subset\Gamma$ *and S is the best constant in the Sobolev trace embedding theorem.*

*Remark 3.1* From Lemma 3.1, we know that if $I=\emptyset$, then $u_j\to u$ strongly in $L^{q(x)}(\partial\Omega)$.

In order to prove the functional $J$ satisfies the local $(PS)_c$ condition, we take function $\eta(x)\in C^1(\overline{\Omega})$ satisfies $p(x)\ll\eta(x)\ll q(x), \forall x\in\overline{\Omega}$. Denote

$$d_1:=\inf_{x\in\overline{\Omega}}\left(\frac{1}{p(x)}-\frac{1}{\eta(x)}\right)>0, \tag{3.1}$$

$$d_2:=\inf_{x\in\overline{\Omega}}\left(\frac{1}{\eta(x)}-\frac{1}{q(x)}\right)>0. \tag{3.2}$$

**Lemma 3.2** *Assume condition* $(H_2)$ *holds. Then, there exists positive constant* $m^* > 0$ *such that the functional* $J$ *satisfies the local* $(PS)_c$ *condition in*

$$c \in \left(-\infty,\ \ \frac{d_2}{4} \cdot S^N - m^*\right)$$

*in the following sense: if*

$$J(u_n) \to c < \frac{d_2}{4} \cdot S^N - m^*$$

*and* $J'(u_n) \to 0$ *for some sequence in* $W^{1,p(x)}(\Omega)$. *Then,* $\{u_n\}$ *contains a subsequence converging strongly in* $W^{1,p(x)}(\Omega)$.

*Proof* First, we show that $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. Indeed, assume by contradiction that $\{u_n\}$ is not bounded in $W^{1,p(x)}(\Omega)$. Then, passing eventually to a subsequence, still denoted by $\{u_n\}$, we assume that $\|u_n\|_{p(x)} \to \infty$ as $n \to \infty$. Thus, we may assume that $\|u_n\|_{p(x)} > 1$ for any integer $n$.

Then, for $n$ sufficiently large, we have

$$\begin{aligned}
&M + o(1)\|u_n\|_{p(x)} \\
&\quad\geq J(u_n) - \langle J'(u_n), \frac{u_n}{\eta}\rangle \\
&\quad= \int\limits_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{\eta(x)}\right) \cdot \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) \mathrm{d}x + \int\limits_{\partial\Omega} \left(\frac{1}{\eta(x)} - \frac{1}{q(x)}\right) \cdot |u_n|^{q(x)} \mathrm{d}\sigma \\
&\qquad - \int\limits_{\Omega} \left[F(x, u_n) - \frac{1}{\eta(x)} f(x, u_n) u_n\right] \mathrm{d}x + \int\limits_{\Omega} \frac{|\nabla u_n|^{p(x)-2} \nabla u_n u_n \nabla \eta}{\eta^2(x)} \mathrm{d}x \\
&\quad\geq d_1 \cdot \int\limits_{\Omega} \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) \mathrm{d}x + d_2 \cdot \int\limits_{\partial\Omega} |u_n|^{q(x)} \mathrm{d}\sigma \\
&\qquad - \int\limits_{\Omega} \left[F(x, u_n) - \frac{1}{\eta(x)} f(x, u_n) u_n\right] \mathrm{d}x + \int\limits_{\Omega} \frac{|\nabla u_n|^{p(x)-2} \nabla u_n u_n \nabla \eta}{\eta^2(x)} \mathrm{d}x. \qquad (3.3)
\end{aligned}$$

By (1.4), for any $(x, t) \in \Omega \times \mathbb{R}$, we have

$$\begin{aligned}
&\int\limits_{\Omega} \left[F(x, u_n) - \frac{1}{\eta(x)} f(x, u_n) u_n\right] \mathrm{d}x \\
&\quad\leq \int\limits_{\Omega} \left|F(x, u_n) - \frac{1}{\eta(x)} f(x, u_n) u_n\right| \mathrm{d}x \\
&\quad\leq \int\limits_{\Omega} \max\left\{\left|F(x, u_n) - \frac{1}{\eta^+} f(x, u_n) u_n\right|, \left|F(x, u_n) - \frac{1}{\eta^-} f(x, u_n) u_n\right|\right\} \mathrm{d}x \\
&\quad\leq c(\varepsilon_1)|\Omega| + \varepsilon_1 \int\limits_{\Omega} |u_n|^{p(x)} \mathrm{d}x. \qquad (3.4)
\end{aligned}$$

On the other hand, noting that $p(x) \ll q(x)$, by the Young inequality, for any $\varepsilon_2 \in (0,1)$, we get

$$\begin{aligned}\left|\frac{|\nabla u_n|^{p(x)-2}\nabla u_n u_n \nabla\eta}{\eta^2(x)}\right| &\leq c_1|\nabla u_n|^{p(x)-1}|u_n| \\ &\leq c_1\left(\frac{\varepsilon_2(p(x)-1)}{p(x)}|\nabla u_n|^{p(x)} + \frac{\varepsilon_2^{1-p(x)}}{p(x)}|u_n|^{p(x)}\right) \\ &\leq c_1\left(\varepsilon_2|\nabla u_n|^{p(x)} + \varepsilon_2^{1-p^+}|u_n|^{p(x)}\right), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}|u_n|^{p(x)} &\leq \frac{\varepsilon_3 p(x)}{q(x)}|u_n|^{q(x)} + \frac{q(x)-p(x)}{q(x)}\varepsilon_3^{\frac{p(x)}{p(x)-q(x)}} \\ &\leq \varepsilon_3|u_n|^{q(x)} + \varepsilon_3^{-\frac{p^+}{(q-p)^-}}. \end{aligned} \tag{3.6}$$

From Proposition 2.1 (ii), we have

$$|u_n|_{q(x)} \leq c_2\|u_n\|_{p(x)}, \tag{3.7}$$

where $c_2 > 0$ is some positive constant. Thus, relations (3.3)–(3.7) imply that

$$\begin{aligned}&M + o(1)\|u_n\|_{p(x)} \\ &\geq \left[d_1 - c_1\varepsilon_2 - c_2\varepsilon_3(\varepsilon_1 + c_1\varepsilon_2^{1-p^+})\right]\int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)\mathrm{d}x \\ &\quad + d_2\int_{\partial\Omega}|u_n|^{q(x)}\mathrm{d}\sigma - (\varepsilon_1 + c_1\varepsilon_2^{1-p^+})\varepsilon_3^{-\frac{p^+}{(q-p)^-}}|\Omega| - c(\varepsilon_1)|\Omega| \\ &= \left[d_1 - c_1\varepsilon_2 - c_2\varepsilon_3\left(\frac{d_1}{2} + c_1\varepsilon_2^{1-p^+}\right)\right]\int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)\mathrm{d}x \\ &\quad + d_2\int_{\partial\Omega}|u_n|^{q(x)}\mathrm{d}\sigma - \left(\frac{d_1}{2} + c_1\varepsilon_2^{1-p^+}\right)\varepsilon_3^{-\frac{p^+}{(q-p)^-}}|\Omega| - c\left(\frac{d_1}{2}\right)|\Omega|, \end{aligned} \tag{3.8}$$

where $\varepsilon_1 = \frac{d_1}{2}$. Thus, we choose $\varepsilon_2, \varepsilon_3$ be so small that $d_1 - c_1\varepsilon_2 - c_2\varepsilon_3\left(\frac{d_1}{2} + c_1\varepsilon_2^{1-p^+}\right) > 0$. It follows from (2.5) and (3.8) that $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. Therefore, we can assume that by Lemma 3.1, there exists a subsequence, there exists a subsequence, that we still denote $u_n$ such that

$$\begin{aligned}&u_n \rightharpoonup u \quad \text{weakly in } W^{1,p(x)}(\Omega), \\ &u_n \to u \quad \text{strongly in } L^{r(x)}(\Omega),\ 1 \leq r \ll q(x), \text{ and a.e. in } \Omega, \\ &|u_n|_{\partial\Omega}|^{q(x)} \rightharpoonup d\nu = |u_{\partial\Omega}|^{q(x)} + \sum_{i=1}^{l}\nu_i\delta_{x_i}, \quad \nu_i > 0, \end{aligned} \tag{3.9}$$

$$|\nabla u_n|^{p(x)} \rightharpoonup d\mu \geq |\nabla u|^{p(x)} + \sum_{i=1}^{l}\mu_i\delta_{x_i}, \quad \mu_i > 0, \tag{3.10}$$

$$S\nu_i^{p(x_i)/q(x_i)} \leq \mu_i. \tag{3.11}$$

Let us show that if $c < \frac{d_2}{4} \cdot S^N - m^*$ ($m^*$ is a constant) and $\{u_n\}$ is a Palais-Smale sequence, with energy level $c$, then $I = \emptyset$.

In fact, suppose that $I \neq \emptyset$. Let $x_i$ be a singular point of the measures $\mu$ and $\nu$, define a function $\phi(x) \in C^\infty(\Omega)$ such that $\phi(x) = 1$ in $B(x_i, \varepsilon)$, $\phi(x) = 0$ in $\Omega \setminus B(x_i, 2\varepsilon)$ and $|\nabla\phi| \leq 2/\varepsilon$ in $\Omega$. As $J'(u_n) \to 0$ in $(W^{1,p(x)}(\Omega))'$, we obtain that

$$\lim_{n\to\infty} \langle J'(u_n), \phi u_n \rangle \to 0,$$

i.e.

$$\lim_{n\to\infty} \left\{ \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla(\phi u_n) \mathrm{d}x + \int_\Omega |u_n|^{p(x)-2} u_n \phi u_n \mathrm{d}x \right.$$
$$\left. - \int_{\partial\Omega} |u_n|^{q(x)-2} u_n \phi u_n \mathrm{d}\sigma - \int_\Omega f(x, u_n) \phi u_n \mathrm{d}x \right\} = 0.$$

On the other hand, by Hölder inequality and boundedness of $\{u_n\}$, we have that

$$\begin{aligned}
0 &\leq \lim_{\varepsilon\to 0} \lim_{n\to\infty} \left| \int_\Omega u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla\phi \mathrm{d}x \right| \\
&\leq C \lim_{\varepsilon\to 0} \lim_{n\to\infty} \left( \int_\Omega |u_n|^{p(x)} |\nabla\phi|^{p(x)} \mathrm{d}x \right)^{\frac{1}{p(x)}} \left( \int_\Omega |\nabla u_n|^{p(x)} \mathrm{d}x \right)^{\frac{p(x)-1}{p(x)}} \\
&\leq C \lim_{\varepsilon\to 0} \left( \int_{B(x_i,\varepsilon)} |u|^{p(x)} |\nabla\phi|^{p(x)} \mathrm{d}x \right)^{\frac{1}{p(x)}} \\
&\leq C \lim_{\varepsilon\to 0} \left( \int_{B(x_i,\varepsilon)} |\nabla\phi|^N \mathrm{d}x \right)^{\frac{1}{N}} \left( \int_{B(x_i,\varepsilon)} |u|^{p^*(x)} \mathrm{d}x \right)^{\frac{1}{p^*(x)}} \\
&\leq C \lim_{\varepsilon\to 0} \left( \int_{B(x_i,\varepsilon)} |u|^{p^*(x)} \mathrm{d}x \right)^{\frac{1}{p^*(x)}} \\
&= 0.
\end{aligned} \tag{3.12}$$

It is easy to check that

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_\Omega |u_n|^{p(x)-2} u_n \phi u_n \mathrm{d}x = 0,$$
$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_\Omega f(x, u_n) \phi u_n \mathrm{d}x = 0.$$

From above facts, we get that

$$0=\lim_{\varepsilon\to 0}\left[\int_{\Omega}\phi d\mu-\int_{\partial\Omega}\phi d\nu\right]=\mu_i-\nu_i. \tag{3.13}$$

Combing this with Lemma 3.1 (iii), we obtain $\mu_i^{\frac{1}{p(x_i)}}\geq S\mu_i^{\frac{1}{p^*(x_i)}}$. This result implies that

$$\mu_i=0 \quad \text{or} \quad \mu_i\geq S^N.$$

Furthermore, $I\neq\emptyset$ implies that $\mu_i\geq S^N$ for some $i\in I$, then by using Lemma 3.1 and (3.13), we have $\nu_i\geq S^N$, for some $i\in I$. Thus, we select $\varepsilon_2,\varepsilon_3$ in (3.8) such that $d_1-c_1\varepsilon_2-c_2\varepsilon_3\left(\frac{d_1}{2}+c_1\varepsilon_2^{1-p^+}\right)>\frac{d_2}{4}$, we have

$$\begin{aligned}
c&=\lim_{n\to\infty}\left(J(u_n)-\langle J'(u_n),\frac{u_n}{\eta}\rangle\right)\\
&\geq\frac{d_2}{4}\cdot\lim_{n\to\infty}\left(\int_{\Omega}\left(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\right)\mathrm{d}x\right)-\left(\frac{d_1}{2}+c_1\varepsilon_2^{1-p^+}\right)\\
&\quad\times\varepsilon_3^{-\frac{p^+}{(q-p)^-}}|\Omega|-c\left(\frac{d_1}{2}\right)|\Omega|\\
&=\frac{d_2}{4}\cdot\int_{\Omega}d\mu-\left(\frac{d_1}{2}+c_1\varepsilon_2^{1-p^+}\right)\varepsilon_3^{-\frac{p^+}{(q-p)^-}}|\Omega|-c\left(\frac{d_1}{2}\right)|\Omega|\\
&\geq\frac{d_2}{4}\cdot\int_{\Omega}|\nabla u|^{p(x)}\mathrm{d}x+\frac{d_2}{4}\cdot S^N-\left(\frac{d_1}{2}+c_1\varepsilon_2^{1-p^+}\right)\varepsilon_3^{-\frac{p^+}{(q-p)^-}}|\Omega|-c\left(\frac{d_1}{2}\right)|\Omega|\\
&\geq\frac{d_2}{4}\cdot S^N-\left(\frac{d_1}{2}+c_1\varepsilon_2^{1-p^+}\right)\varepsilon_3^{-\frac{p^+}{(q-p)^-}}|\Omega|-c\left(\frac{d_1}{2}\right)|\Omega|\\
&=\frac{d_2}{4}\cdot S^N-m^*,
\end{aligned}$$

where $m^*=\left(\frac{d_1}{2}+c_1\varepsilon_2^{1-p^+}\right)\varepsilon_3^{-\frac{p^+}{(q-p)^-}}|\Omega|-c\left(\frac{d_1}{2}\right)|\Omega|$. This is impossible. Consequently, the index set $I$ is empty. From Remark 3.1, we have

$$\int_{\partial\Omega}|u_n|^{q(x)}\mathrm{d}\sigma\to\int_{\partial\Omega}|u|^{q(x)}\mathrm{d}\sigma,\quad \text{as } n\to\infty.$$

Since $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$, we deduce that there exists a subsequence, again denoted by $\{u_n\}$, and $u_0\in W^{1,p(x)}(\Omega)$ such that $\{u_n\}$ converges weakly to $u_0$ in $W^{1,p(x)}(\Omega)$. Note that

$$\langle J'(u_n)-J'(u_0),u_n-u_0\rangle\to 0,\quad \text{as } n\to\infty.$$

On the other hand, we have

$$\int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_0|^{p(x)-2}\nabla u_0)\cdot(\nabla u_n - \nabla u_0)\mathrm{d}x$$
$$+\int_\Omega (|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0)(u_n - u_0)\mathrm{d}x$$
$$= \langle J'(u_n) - J'(u_0), u_n - u_0\rangle + \int_{\partial\Omega} (|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0)(u_n - u_0)\mathrm{d}\sigma$$
$$+\int_\Omega (f(x,u_n) - f(x,u_0))(u_n - u_0)\mathrm{d}x.$$

Using the fact that $\{u_n\}$ converges strongly to $u_0$ in $L^{q(x)}(\partial\Omega)$ and $|u_n - u_0|_{p(x)} \to 0$ as $n \to \infty$, and Proposition 2.4 implies that $W^{1,p(x)}(\Omega)$ is compactly embedded $L^{p(x)}(\partial\Omega)$ we deduce that

$$\lim_{n\to\infty}\int_\Omega (f(x,u_n) - f(x,u_0))(u_n - u_0)\mathrm{d}x = 0, \tag{3.14}$$
$$\lim_{n\to\infty}\int_{\partial\Omega} (|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0)(u_n - u_0)\mathrm{d}\sigma = 0, \tag{3.15}$$
$$\lim_{n\to\infty}\int_\Omega (|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0)(u_n - u_0)\mathrm{d}x = 0. \tag{3.16}$$

By (3.14), (3.15) and (3.16), we obtain

$$\lim_{n\to\infty}\int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_0|^{p(x)-2}\nabla u_0)\cdot(\nabla u_n - \nabla u_0)\mathrm{d}x$$
$$+\lim_{n\to\infty}\int_\Omega (|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0)(u_n - u_0)\mathrm{d}x = 0. \tag{3.17}$$

It is known that

$$\left(|s|^{p-2}s - |t|^{p-2}t, s - t\right) \ge \begin{cases} C_p|s-t|^p, & \forall\, p \ge 2,\\ C_p\frac{|s-t|^2}{(|s|+|t|)^{2-p}}, & \forall\, p \le 2, \end{cases} \quad s, t \in \mathbb{R}^N, \tag{3.18}$$

where $(\cdot\,,\ \cdot)$ is the standard scalar product in $\mathbb{R}^N$. Relations (3.17) and (3.18) yield

$$\lim_{n\to\infty}\int_\Omega \left(|\nabla u_n - \nabla u_0|^{p(x)} + |u_n - u_0|^{p(x)}\right)\mathrm{d}x = 0.$$

This fact and relation (2.7) imply $\|u_n - u_0\|_{p(x)} \to 0$ as $n \to \infty$. The proof is complete.

□

## 4 Existence of a sequence of arbitrarily small solutions

In this section, we prove the existence of infinitely many solutions of (1.1) which tend to zero. Let $X$ be a Banach space and denote

$$\Sigma := \{A \subset X \setminus \{0\} :\ A \text{ is closed in } X \text{ and symmetric with respect to the orgin}\}.$$

For $A \in \Sigma$, we define genus $\gamma(A)$ as

$$\gamma(A) := \inf\{m \in N :\ \exists\, \varphi \in C(A, R^m \setminus \{0\}), -\varphi(x) = \varphi(-x)\}.$$

If there is no mapping $\varphi$ as above for any $m \in N$, then $\gamma(A) = +\infty$. Let $\Sigma_k$ denote the family of closed symmetric subsets $A$ of $X$ such that $0 \notin A$ and $\gamma(A) \geq k$. We list some properties of the genus (see [5]).

**Proposition 4.1** *Let A and B be closed symmetric subsets of X which do not contain the origin. Then the following hold.*

(1) *If there exists an odd continuous mapping from A to B, then* $\gamma(A) \leq \gamma(B)$*;*
(2) *If there exists an odd homeomorphism from A to B, then* $\gamma(A) = \gamma(B)$*;*
(3) *If* $\gamma(B) < \infty$*, then* $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$*;*
(4) *Then n-dimensional sphere* $S^n$ *has a genus of* $n+1$ *by the Borsuk-Ulam Theorem;*
(5) *If A is compact, then* $\gamma(A) < +\infty$ *and there exists* $\delta > 0$ *such that* $U_\delta(A) \in \Sigma$ *and* $\gamma(U_\delta(A)) = \gamma(A)$*, where* $U_\delta(A) = \{x \in X :\ \|x - A\| \leq \delta\}$.

The following version of the symmetric mountain-pass lemma is due to Kajikiya [5].

**Lemma 4.1** *Let E be an infinite-dimensional space and* $J \in C^1(E, R)$ *and suppose the following conditions hold.*

($C_1$) $J(u)$ *is even, bounded from below,* $J(0) = 0$ *and* $J(u)$ *satisfies the Palais-Smale condition;*
($C_2$) *For each* $k \in N$*, there exists an* $A_k \in \Sigma_k$ *such that* $\sup_{u \in A_k} J(u) < 0$.

*Then either* ($R_1$) *or* ($R_2$) *below holds.*

($R_1$) *There exists a sequence* $\{u_k\}$ *such that* $J'(u_k) = 0$, $J(u_k) < 0$ *and* $\{u_k\}$ *converges to zero.*
($R_2$) *There exist two sequences* $\{u_k\}$ *and* $\{v_k\}$ *such that* $J'(u_k) = 0$, $J(u_k) < 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $J'(v_k) = 0$, $J(v_k) < 0$, $\lim_{k\to\infty} v_k = 0$*, and* $\{v_k\}$ *converges to a non-zero limit.*

*Remark 4.1* From Lemma 4.1, we have a sequence $\{u_k\}$ of critical points such that $J(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

In order to get infinitely many solutions, we need some lemmas. we focus our attention on the case when $u \in W^{1,p(x)}(\Omega)$ with $\|u\|_{p(x)} < 1$. For such a $u$ by relation (2.6), we obtain

$$\int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)} \mathrm{d}x \geq \|u\|_{p(x)}^{p^+}. \tag{4.1}$$

Using (1.3), we deduce that

$$
\begin{aligned}
J(u) &= \int\limits_{\Omega} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} \mathrm{d}x - \int\limits_{\partial\Omega} \frac{1}{q(x)} |u|^{q(x)} \mathrm{d}\sigma - \int\limits_{\Omega} F(x,u) \mathrm{d}x \\
&\geq \left( \frac{1}{p^+} - \varepsilon \right) \cdot \|u\|_{p(x)}^{p^+} - \frac{S^{q^-} + S^{q^+}}{q^-} \cdot \left( \|u\|_{p(x)}^{q^+} + \|u\|_{p(x)}^{q^-} \right) - b(\varepsilon)|\Omega| \\
&\geq \frac{1}{2p^+} \cdot \|u\|_{p(x)}^{p^+} - \frac{S^{q^-} + S^{q^+}}{q^-} \cdot \|u\|_{p(x)}^{q^+} - \frac{S^{q^-} + S^{q^+}}{q^-} \cdot \|u\|_{p(x)}^{q^-} - b\left( \frac{1}{2p^+} \right) |\Omega| \\
&\geq A\|u\|_{p(x)}^{p^+} - B\|u\|_{p(x)}^{q^+} - C, \qquad (4.2)
\end{aligned}
$$

where $\varepsilon = \frac{1}{2p^+}$ and

$$
A = \frac{1}{2p^+}, \quad B = \frac{S^{q^-} + S^{q^+}}{q^-}, \quad C = \frac{S^{q^-} + S^{q^+}}{q^-} + b\left( \frac{1}{2p^+} \right) |\Omega|,
$$

for any $u \in W^{1,p(x)}(\Omega)$ with $\|u\|_{p(x)} < 1$. If we define

$$
Q(s) = As^{p^+} - Bs^{q^+} - C.
$$

As $Q(s)$ attains a local but not a global minimum ($Q$ is not bounded below), we have to perform some sort of truncation. To this end, let $R_0$, $R_1$ be such that $m < R_0 < M < R_1$, where $m$ is the local minimum of $Q(s)$ and $M$ is the local maximum and $Q(R_0) > Q(m)$. For these values $R_1$ and $R_0$, we can choose a smooth function $\chi(t)$ defined as follows

$$
\chi(t) = \begin{cases} 1, & 0 \leq t \leq R_0, \\ 0, & t \geq R_1, \\ C^\infty, \quad \chi(t) \in [0,1], & R_0 \leq t \leq R_1. \end{cases}
$$

Then, it is easy to see $\chi(t) \in [0,1]$ and $\chi(t)$ is $C^\infty$. Let $\varphi(u) = \chi(\|u\|_{p(x)})$ and consider the perturbation of $J(u)$:

$$
G(u) = \int\limits_{\Omega} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} \mathrm{d}x - \varphi(u) \int\limits_{\partial\Omega} \frac{1}{q(x)} |u|^{q(x)} \mathrm{d}\sigma - \int\limits_{\Omega} F(x,u) \mathrm{d}x. \qquad (4.3)
$$

Then

$$
\begin{aligned}
G(u) &\geq A\|u\|_{p(x)}^{p^+} - B\varphi(u)\|u\|_{p(x)}^{q^+} - C \\
&= \overline{Q}(\|u\|_{p(x)}),
\end{aligned}
$$

where $\overline{Q}(t) = At^{p^+} - B\chi(t)t^{q^+} - C$ and

$$
\overline{Q}(t) = \begin{cases} Q(t), & t \leq R_0, \\ At^{p^+} - C, & t \geq R_1. \end{cases}
$$

From the above arguments, we have the following:

**Lemma 4.2** *Let $G(u)$ is defined as in (4.3). Then*

(i) *$G \in C^1(E, R)$ and $G$ is even and bounded from below;*
(ii) *If $G(t) \leq 0$, then $\overline{Q}(\|u_{p(x)}\|) \leq 0$, consequently, $\|u\|_{p(x)} < R_0$ and $J(u) = G(u)$;*
(iii) *$G$ satisfies a local $(PS)_c$ condition for $c < \frac{d_2}{4} \cdot S^N - m^*$, where $m^*$ is given by Lemma 3.2.*

*Proof* Items (i) and (ii) are immediate. Item (iii) is a consequence of item (ii) and Lemma 3.2. □

**Lemma 4.3** *Assume that* $(H_3)$ *of Theorem* 1.1 *holds. Then for any* $k \in N$*, there exists* $\delta = \delta(k) > 0$ *such that* $\gamma(\{u \in W^{1,p(x)}(\Omega) : G(u) \leq -\delta(k)\} \setminus \{0\}) \geq k$.

*Proof* First, by $(H_3)$ of Theorem 1.1, for any fixed $u \in W^{1,p(x)}(\Omega)$, $u \neq 0$, we have

$$F(x, \rho u) \geq M(\rho)(\rho u)^{p^-} \quad \text{with} \quad M(\rho) \to \infty \quad \text{as} \quad \rho \to 0. \tag{4.4}$$

Next, given any $k \in N$, let $E_k$ be a $k$-dimensional subspace of $W^{1,p(x)}(\Omega)$. We take $u \in E_k$ with norm $\|u\|_{p(x)} = 1$, for $0 < \rho < \min\{R_0, 1\}$, we get

$$\begin{aligned} G(\rho u) &= J(\rho u) \\ &= \int_\Omega \rho^{p(x)} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} \mathrm{d}x - \int_{\partial\Omega} \rho^{q(x)} \frac{1}{q(x)} |u|^{q(x)} \mathrm{d}x - \int_\Omega F(x, \rho u) \mathrm{d}x \\ &\leq \frac{1}{p^-} \rho^{p^-} \int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)} \mathrm{d}x - \frac{1}{q^+} \rho^{q^+} \int_{\partial\Omega} |u|^{q(x)} \mathrm{d}\sigma - M(\rho)\rho^{p^-} \int_\Omega |u|^{p^-} \mathrm{d}x. \end{aligned}$$

Since $E_k$ is a space of finite dimension, all the norms in $E_k$ are equivalent. If we define

$$A_k = \inf \left\{ \int_{\partial\Omega} |u|^{q(x)} \mathrm{d}\sigma : \ u \in E_k, \ \|u\|_{p(x)} = 1 \right\} > 0,$$

$$B_k = \inf \left\{ \int_\Omega |u|^{p^-} \mathrm{d}x : \ u \in E_k, \ \|u\|_{p(x)} = 1 \right\} > 0.$$

It follows from (4.4) that

$$\begin{aligned} G(\rho u) &\leq \frac{1}{p^-} \rho^{p^-} - \frac{1}{q^+} \rho^{q^+} A_k - M(\rho)\rho^{p^-} B_k \\ &\leq \rho^{p^-} \left( \frac{1}{p^-} - M(\rho) B_k \right) - \frac{1}{q^+} \rho^{q^+} A_k \\ &= -\delta(k)\rho^{p^-} < 0, \ \text{as} \ \ \rho \to 0, \end{aligned}$$

since $\lim_{|\rho| \to 0} M(\rho) = +\infty$. That is,

$$\{u \in E_k : \ \|u\|_{p(x)} = \rho\} \subset \{u \in W^{1,p(x)}(\Omega) : \ G(u) \leq -\delta(k)\} \setminus \{0\}.$$

This completes the proof. □

Now, we give the proof of Theorem 1.1 as following.

*Proof of Theorem* 1.1 Recall that

$$\Sigma_k = \{A \in E \setminus \{0\} : A \text{ is closed and } A = -A, \gamma(A) \geq k\}$$

and define

$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} G(u).$$

By Lemmas 4.2 (i) and 4.3, we know that $-\infty < c_k < 0$. Therefore, assumptions $(C_1)$ and $(C_2)$ of Lemma 4.1 are satisfied. This means that $G$ has a sequence of solutions $\{u_n\}$ converging to zero. Hence, Theorem 1.1 follows by Lemma 4.2 (ii). □

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