Annali di Matematica (2012) 191:819–844
DOI 10.1007/s10231-011-0208-z

Discrepancy of LS-sequences of partitions and points

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Received: 12 May 2010 / Accepted: 23 May 2011 / Published online: 11 June 2011

Abstract In this paper, we study a countable family of uniformly distributed sequences of partitions, called LS-*sequences of partitions*, and we give a precise estimate of their discrepancy. Among these sequences, we identify a countable class having low discrepancy (which means of order $\frac{1}{N}$). We describe an explicit algorithm that associates to each of these sequences a uniformly distributed sequence of points (we call LS-*sequences of points*). The main result of this paper says that the discrepancy of the sequences of points associated by our algorithm to the LS-sequences of partitions is of order $\alpha_N \log N$, if α_N is the discrepancy of the corresponding sequence of partitions. We obtain therefore, in particular, a countable family of low-discrepancy sequences of points.

Keywords Uniform distribution · Discrepancy

Mathematics Subject Classification (2000) 11K06 · 11K38 · 11K45

1 Introduction and preliminaries

S. Kakutani introduced in [7] the notion of *uniformly distributed sequences of partitions* of the interval $[0, 1[$. He considered the following construction. Fix a number $\alpha \in]0, 1[$. If π is any partition of $[0, 1[$, its α-refinement, denoted by $\alpha\pi$, is obtained subdividing the longest interval(s) of length ℓ into two intervals of lengths $\alpha\,\ell$ and $(1-\alpha)\ell$. By $\alpha^n\pi$, we denote the α-refinement of $\alpha^{n-1}\pi$.

Let $\omega = \{[0, 1[\}$ be the trivial partition of $[0, 1[$. The sequence $\{\alpha^n\omega\}$ will be called the Kakutani α-sequence.

Definition 1.1 Given a sequence of partitions $\{\pi_n\}$ of $[0, 1[$, with

$$\pi_n = \left\{[y_i^{(n)}, y_{i+1}^{(n)}[, \, 1 \le i \le k(n)\right\},$$

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we say that it is *uniformly distributed (u.d.)* if for any continuous function f on $[0, 1[$ we have

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f\left(y_i^{(n)}\right) = \int_0^1 f(t)\,\mathrm{d}t. \tag{1}$$

It is well known that condition (1) is equivalent to say that

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_{[a,b[}\left(y_i^{(n)}\right) = b - a \quad \text{for all } 0 \le a \le b < 1$$

or

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_{[0,b[}\left(y_i^{(n)}\right) = b \quad \text{for all } 0 < b < 1.$$

Definition 1.1 represents the natural extension to sequences of partitions of the classical definition of u.d. sequences of points (see, e.g., [8]), introduced by H. Weyl in [16]. Since it will be used later, we recall here this concept.

Definition 1.2 Given a sequence of points $\{x_n\}$ of the interval $[0, 1[$, we say that $\{x_n\}$ is *uniformly distributed (u.d.)* if for any continuous function f on $[0, 1[$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i) = \int_0^1 f(t)\,\mathrm{d}t.$$

We can now state Kakutani's result.

Theorem 1.3 *The sequence $\{\alpha^n \omega\}$ is uniformly distributed.*

This result caught the attention of several authors in the late seventies, when a different proof of the theorem was given ([AF]), and other papers were devoted to a stochastic version of it (see [9,10,12,15]).

Recently, the procedure introduced by Kakutani has been generalized in several directions.

In [1], this notion has been extended to separable metric spaces.

In [13], Kakutani's splitting procedure has been generalized, producing a new class of u.d. sequences of partitions as follows.

Definition 1.4 Consider any non-trivial finite partition ρ of $[0, 1[$. The *ρ-refinement* of a partition π of $[0, 1[$ (which will be denoted by $\rho\pi$) is obtained by subdividing all the intervals of π having maximal length positively (or directly) homothetically to ρ.

Obviously, if $\rho = \{[0, \alpha[, [\alpha, 1[\}$, then the ρ-refinement is just Kakutani's α-refinement.

As in Kakutani's case, we can iterate the splitting procedure. The ρ-refinement of $\rho\pi$ will be denoted by $\rho^2\pi$ and, for any $n \in \mathbb{N}$, the ρ-refinement of $\rho^{n-1}\pi$ will be indicated by $\rho^n\pi$. If $\{\rho^n\omega\}$ denotes the sequence of successive ρ-refinements of the trivial partition ω, the following theorem holds (see [13, Theorem 2.7]).

Theorem 1.5 *The sequence $\{\rho^n\omega\}$ is uniformly distributed.*

In [2], the splitting procedure has been generalized to higher dimensions, providing a sequence of nodes in the hypercube $[0, 1[^d$ which is uniformly distributed.

[6] studies uniform distribution on fractals.

In [3], it is presented a von Neumann type theorem (see [14]), which provides uniformly distributed sequences of partitions of the interval $[0, 1[$ rearranging sequences of partitions whose diameter tends to 0 when $n \to \infty$.

In [4], the authors give upper estimates of the discrepancy of ρ-refinements of the interval $[0, 1[$.

The theory of uniformly distributed sequences of partitions is deeply connected to the theory of uniformly distributed of sequences of points, which have an extensive application in higher dimensions in quasi-Monte Carlo methods (see [11]), but this aspect is not treated in our paper.

For a complete overview on uniformly distributed sequences of points, see [5] and [8].

In this paper, we will consider a special class of ρ-refinements of the trivial partition ω.

Definition 1.6 Let us fix two positive integers L and S and let $0 < \beta < 1$ be the real number such that $L\beta + S\beta^2 = 1$. Denote by $\rho_{L,S}$ the partition defined by L "long" intervals having length β and by S "short" intervals having length β^2. By $\left\{\rho^n_{L,S}\,\omega\right\}$ $\Big($or $\left\{\rho^n_{L,S}\right\}$ for short$\Big)$, we denote the sequence of successive $\rho_{L,S}$-refinements of the trivial partition ω. They will be called *LS-sequences of partitions*.

It is clear that the partition $\rho^n_{L,S}$ is obtained by dividing all the longest intervals of $\rho^{n-1}_{L,S}$ homothetically with respect to $\rho_{L,S}$, and that each partition $\rho^n_{L,S}$ contains only two kinds of intervals: the long intervals have length β^n while the short ones have length β^{n+1}.

In Sect. 2, we give precise estimates of the discrepancy of LS-sequences of partitions. For $L = S = 1$, we get a Kakutani sequence corresponding to $\alpha = \frac{-1+\sqrt{5}}{2}$, the positive solution if the equation $\alpha + \alpha^2 = 1$. We call it the *Kakutani-Fibonacci sequence of partitions*. We point out that this is the first time that the exact discrepancy of a Kakutani sequence is given. Its discrepancy is the best possible since it is of order. $1/N$.

In Sect. 3, we present an explicit algorithm that associates to each LS-sequence of partitions a sequence of points (called *LS-sequence of points*), and we study the discrepancy of these sequences. The countable class of sequences of partitions having low discrepancy (including the Kakutani-Fibonacci sequence) gives rise to low-discrepancy sequences of points (which means of order $\log N/N$).

2 LS-sequences of partitions

This section will be devoted to the study of LS-sequences of partitions and their discrepancy.

Denote with t_n the total number of intervals of $\rho^n_{L,S}$, with l_n the number of its long intervals and with s_n the number of its short intervals. Obviously, we have the following relations:

$$t_n = l_n + s_n, \qquad l_n = L\,l_{n-1} + s_{n-1}, \qquad s_n = S\,l_{n-1}.$$

The terms of the sequence t_n can be calculated solving the difference equation $t_n = L\,t_{n-1} + S\,t_{n-2}$ with initial conditions $t_0 = 1$ and $t_1 = L + S$.

We easily get

$$t_n = \frac{1 + S\beta}{1 + S\beta^2}\left(\frac{1}{\beta^n}\right) - \frac{S\beta - S\beta^2}{1 + S\beta^2}(-S\beta)^n. \tag{2}$$

We note that l_n and s_n satisfy the same difference equation, but the initial conditions are different since $l_0 = 1, l_1 = L$ and $s_0 = 0, s_1 = S$, respectively. Therefore, we have

$$l_n = \frac{1}{1+S\beta^2}\left(\frac{1}{\beta^n}\right) + \frac{S\beta^2}{1+S\beta^2}(-S\beta)^n. \tag{3}$$

It is convenient to introduce the constants

$$A = \frac{1+S\beta}{1+S\beta^2} \quad \text{and} \quad \mathrm{B} = \frac{\mathrm{S}\beta - \mathrm{S}\beta^2}{1+\mathrm{S}\beta^2},$$

which will frequently appear in the sequel, so that formula (2) can be rewritten in the following way

$$t_n = \frac{A - B\left(-S\beta^2\right)^n}{\beta^n}. \tag{4}$$

Since $S\beta^2 < 1$, we see that t_n has the same order as $\frac{1}{\beta^n}$ when $n \to \infty$.

Now, we recall the definitions of *discrepancy* and *star-discrepancy*.

Definition 2.1 Given a finite subset $W = \{w_1, w_2, \ldots, w_N\}$ of the interval $[0, 1[$, the *discrepancy* of W is defined as

$$D(W) = \sup_{0\le a<b<1}\left|\frac{1}{N}\sum_{j=1}^{N}\chi_{[a,\,b[}(w_j) - (b-a)\right|,$$

while the *star-discrepancy* is defined as

$$D^\star(W) = \sup_{0<b<1}\left|\frac{1}{N}\sum_{j=1}^{N}\chi_{[0,\,b[}(w_j) - b\right|.$$

If we consider a sequence of points $X = \{x_1, x_2, \ldots, x_n, \ldots\}$ in the interval $[0, 1[$, the discrepancy (respectively, star-discrepancy) of X is the sequence $\{D(X_N)\}_N$ (respectively, $\{D^*(X_N)\}_N$), where $X_N = \{x_1, x_2, \ldots, x_N\}$.

It is well known that X is uniformly distributed if and only if $D(X_N) \to 0$ when $N \to \infty$, and that $D^\star(X_N) \le D(X_N) \le 2D^\star(X_N)$.

If we consider a sequence of partitions $\{\pi_n\}$ of the interval $[0, 1[$, with $\pi_n = \left\{[y_i^{(n)}, y_{i+1}^{(n)}[, 1 \le i \le k(n)\right\}$, we put

$$D(\pi_n) = \sup_{0\le a<b<1}\left|\frac{1}{k(n)}\sum_{j=1}^{k(n)}\chi_{[a,\,b[}\left(y_j^{(n)}\right) - (b-a)\right|$$

and

$$D^\star(\pi_n) = \sup_{0<b<1}\left|\frac{1}{k(n)}\sum_{j=1}^{k(n)}\chi_{[0,\,b[}\left(y_j^{(n)}\right) - b\right|.$$

It is clear from (1) that $\{\pi_n\}$ is uniformly distributed if and only if $D(\pi_n) \to 0$ when $n \to \infty$, and of course $D^\star(\pi_n) \le D(\pi_n) \le 2D^\star(\pi_n)$.

It is also well known that the best discrepancy is $\frac{1}{k(n)}$ and it is attained, for example, by Knapowski's sequence $\left\{[\frac{i-1}{n}, \frac{i}{n}[, 1 \le i \le n\right\}$ (here $k(n) = n$).

We are now ready to determine the discrepancy of the sequence $\left\{\rho^n_{L,S}\right\}$ for every choice of the order of the long and short intervals of $\rho_{L,S}$.

Theorem 2.2 (i) *If* $S < L+1$ *there exist* $c_1 > 0$ *and* $c_2 > 0$ *such that for any* $n \in I\!N$

$$\frac{c_1}{t_n} \leq D\left(\{\rho^n_{L,S}\}\right) \leq \frac{c_2}{t_n}.$$

(ii) *If* $S = L+1$ *there exist* $c_3 > 0$ *and* $c_4 > 0$ *such that for any* $n \in I\!N$

$$c_3 \frac{\log t_n}{t_n} \leq D\left(\rho^n_{L,S}\right) \leq c_4 \frac{\log t_n}{t_n}.$$

(iii) *If* $S > L+1$ *there exist* $c_5 > 0$ *and* $c_6 > 0$ *such that for any* $n \in I\!N$

$$\frac{c_5}{(t_n)^\gamma} \leq D\left(\{\rho^n_{L,S}\}\right) \leq \frac{c_6}{(t_n)^\gamma},$$

where $\gamma = 1 + \frac{\log(S\beta)}{\log\beta} < 1$.

Proof We denote the intervals of $\rho^n_{L,S}$ as follows

$$\rho^n_{L,S} = \left\{[y_i^{(n)}, y_{i+1}^{(n)}[,\ 1 \leq i \leq t_n)\right\}$$

and we study the behavior of the star-discrepancy of $\rho^n_{L,S}$.

For simplicity, we consider the partition having L long intervals followed by S short intervals, but we point out that all the procedure described below is independent of the order of the $L+S$ intervals and, hence, points.

Denote by L_p and S_p, respectively, a long interval and a short interval of the partition $\rho^p_{L,S}$, and first of all, let us evaluate for $n \geq p$ the differences

$$\frac{1}{t_n}\sum_{j=1}^{t_n} \chi_{L_p}\left(y_j^{(n)}\right) - \lambda(L_p) \quad \text{and} \quad \frac{1}{t_n}\sum_{j=1}^{t_n} \chi_{S_p}\left(y_j^{(n)}\right) - \lambda(S_p),$$

where $\lambda(L_p) = \beta^p$ and $\lambda(S_p) = \beta^{p+1}$.

We observe that, since $\rho^n_{L,S}$ is a refinement of $\rho^p_{L,S}$, each long interval L_p of $\rho^p_{L,S}$ is the union of consecutive intervals of $\rho^n_{L,S}$ (with $n \geq p$) and, since the splitting procedure is "self-similar", this union reproduces up to a factor β^{-p} the partition $\rho^{n-p}_{L,S}$. Therefore, we have, for any $n \geq p$,

$$\sum_{j=1}^{t_n} \chi_{L_p}\left(y_j^{(n)}\right) = t_{n-p}.$$

With a simple calculation, we obtain, for $n \geq p$,

$$\frac{1}{t_n}\sum_{j=1}^{t_n} \chi_{L_p}\left(y_j^{(n)}\right) - \lambda(L_s) = \frac{t_{n-p}}{t_n} - \beta^p = -\frac{B}{t_n}\,(-S\beta)^n\left[(-S\beta)^{-p} - \beta^p\right]. \tag{5}$$

The short interval S_p of $\rho^p_{L,S}$ becomes a long one in $\rho^{p+1}_{L,S}$, which means that it is of the type L_{p+1}, so we can rewrite formula (5) in the following way:

$$\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{S_p}\left(y_j^{(n)}\right)-\lambda(S_p)=\frac{t_{n-(p+1)}}{t_n}-\beta^{p+1}$$
$$=-\frac{B}{t_n}\,(-S\beta)^n\left[(-S\beta)^{-p-1}-\beta^{p+1}\right] \tag{6}$$

for every $n \geq p+1$.

Fix now $b \in]0, 1[$.

Let $[b_1^{(n-1)}, b_2^{(n-1)}[$ be the interval of $\rho^{n-1}_{L,S}$ containing b. Since the number of points of $\rho^n_{L,S}$ contained in $[b_1^{(n-1)}, b_2^{(n-1)}[$ is 1 if this interval is a short one, and it is $L+S$ if it is a long one, we have

$$\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[0,\,b[}\left(y_j^{(n)}\right)-b \leq \frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[0,\,b_2^{(n-1)}[}\left(y_j^{(n)}\right)-b_1^{(n-1)}$$
$$=\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[0,\,b_1^{(n-1)}[}\left(y_j^{(n)}\right)-b_1^{(n-1)}+\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[b_1^{(n-1)},\,b_2^{(n-1)}[}\left(y_j^{(n)}\right)$$
$$\leq \frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[0,\,b_1^{(n-1)}[}\left(y_j^{(n)}\right)-b_1^{(n-1)}+\frac{L+S}{t_n} \tag{7}$$

and

$$\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[0,\,b[}\left(y_j^{(n)}\right)-b \geq \frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[0,\,b_1^{(n-1)}[}\left(y_j^{(n)}\right)-b_2^{(n-1)}$$
$$=\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[0,\,b_2^{(n-1)}[}\left(y_j^{(n)}\right)-b_2^{(n-1)}-\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[b_1^{(n-1)},\,b_2^{(n-1)}[}\left(y_j^{(n)}\right)$$
$$\geq \frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[0,\,b_2^{(n-1)}[}\left(y_j^{(n)}\right)-b_2^{(n-1)}-\frac{L+S}{t_n}. \tag{8}$$

We will estimate $D^*(\rho^n_{L,S})$ evaluating $\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[0,\,b_k^{(n-1)}[}\left(y_j^{(n)}\right)-b_k^{(n-1)}$ for $k=1,2$, using (7) and (8). For this purpose, it will be convenient to represent $[0,\ b_k^{(n-1)}[$ (from now on, we will write $[0,\ b^{(n-1)}[$ for short) as the union of consecutive intervals originating from the partitions $\rho^p_{L,S}$ for $p \leq n-1$.

Consider first all the consecutive intervals $I_1^1, I_2^1, \ldots, I_{m_1}^1$ of $\rho^1_{L,S}$ such that $\bigcup_{i=1}^{m_1} I_i^1 \subset [0,\ b^{(n-1)}[$ (m_1 could be zero).

Next, take all the consecutive intervals $I_1^2, I_2^2, \ldots, I_{m_2}^2$ of $\rho^2_{L,S}$ contained in $[0,\ b^{(n-1)}[\ \setminus \bigcup_{i=1}^{m_1} I_i^1$ (again m_2 could be zero).

Proceed this way taking, at the final step, all the m_{n-1} (possibly zero) consecutive intervals $I_1^{n-1}, I_2^{n-1}, \ldots, I_{m_{n-1}}^{n-1}$ of $\rho^{n-1}_{L,S}$ such that

$$\bigcup_{i=1}^{m_{n-1}} I_i^{n-1} \ \subset\ [0,\ b^{(n-1)}[\ \setminus \bigcup_{p=1}^{n-2}\left(\bigcup_{i=1}^{m_p} I_i^p\right),$$

so that at the end

$$[0,\ b^{(n-1)}[\ = \bigcup_{p=1}^{n-1}\left(\bigcup_{i=1}^{m_p} I_i^p\right). \tag{9}$$

Thus, $[0,\ b^{(n-1)}[$ is represented by the union of (at most) $n-1$ blocks of consecutive intervals of $\rho_{L,S}^p$ for $1 \le p \le n-1$.

Let l_p^b and s_p^b be, respectively, the number of long intervals (denoted by L_i^p) and short intervals (denoted by S_i^p), respectively, of $\rho_{L,S}^p$ contained in $[0,\ b[$ (obviously, $l_p^b + s_p^b = m_p$). Then, using (5, 6), and (9), in order to evaluate the star-discrepancy, we can write:

$$\begin{aligned}
&\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[0,\,b^{(n-1)}[}\left(y_j^{(n)}\right) - b^{(n-1)} \\
&= \frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{\left(\bigcup_{p=1}^{n-1}\left(\bigcup_{i=1}^{m_p} I_i^p\right)\right)}\left(y_j^{(n)}\right) - \lambda\left(\bigcup_{p=1}^{n-1}\left(\bigcup_{i=1}^{m_p} I_i^p\right)\right) \\
&= \sum_{p=1}^{n-1}\sum_{i=1}^{m_p}\left(\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{I_i^p}\left(y_j^{(n)}\right) - \lambda\left(I_i^p\right)\right) \\
&= \sum_{p=1}^{n-1}\sum_{i=1}^{l_p^b}\left(\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{L_i^p}\left(y_j^{(n)}\right) - \lambda\left(L_i^p\right)\right) + \sum_{p=1}^{n-1}\sum_{i=1}^{s_p^b}\left(\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{S_i^p}\left(y_j^{(n)}\right) - \lambda\left(S_i^p\right)\right) \\
&= \frac{B}{t_n}(-S\beta)^n\sum_{p=1}^{n-1}\left\{-l_p^b\left[(-S\beta)^{-p} - \beta^p\right] - s_p^b\left[(-S\beta)^{-p-1} - \beta^{p+1}\right]\right\} \\
&= \frac{B}{t_n}(-S\beta)^n\sum_{p=1,\,p\text{ even}}^{n-1}\left[-l_p^b\left[(-S\beta)^{-p} - \beta^p\right] - s_p^b\left[(-S\beta)^{-p-1} - \beta^{p+1}\right]\right] \\
&\quad + \frac{B}{t_n}(-S\beta)^n\sum_{p=1,\,p\text{ odd}}^{n-1}\left[-l_p^b\left[(-S\beta)^{-p} - \beta^p\right] - s_p^b\left[(-S\beta)^{-p-1} - \beta^{p+1}\right]\right] \\
&= \frac{B}{t_n}(-S\beta)^n\left\{\sum_{p=1,\,p\text{ even}}^{n-1} s_p^b\left[(S\beta)^{-p-1} + \beta^{p+1}\right] + \sum_{p=1,\,p\text{ odd}}^{n-1} l_p^b\left[(S\beta)^{-p} + \beta^p\right]\right\} \\
&\quad - \frac{B}{t_n}(-S\beta)^n\left\{\sum_{p=1,\,p\text{ even}}^{n-1} l_p^b\left[(S\beta)^{-p} - \beta^p\right] + \sum_{p=1,\,p\text{ odd}}^{n-1} s_p^b\left[(S\beta)^{-p-1} - \beta^{p+1}\right]\right\}.
\end{aligned} \tag{10}$$

It is clear from the construction that $m_p \le L+S-1$, $0 \le l_p^b \le L$ and $0 \le s_p^b \le S$ for any $1 \le p \le n-1$, otherwise at least one of the intervals of $\bigcup_{i=1}^{m_p} I_i^p$ would be already present in $\bigcup_{i=1}^{m_{p-1}} I_i^{p-1}$.

We have to distinguish the cases when $S\beta \ne 1$ and $S\beta = 1$.

(1) $S\beta \ne 1$

We will give upper and lower estimates of (10) in the case n even and in the case n odd.

Assume first that n is even and let $n = 2h$, $h \geq 1$.

We note that the first term in (10) is positive, and the second term in (10) is negative since $B > 0$ and $(S\beta)^{-p} + \beta^p = \frac{1-(S\beta^2)^p}{S\beta^2}$. Then with simple calculations, we get the following upper and lower estimates:

$$
\begin{aligned}
&-B(L+S-1)\frac{(S\beta)^{2h}}{t_{2h}}\left\{2\left[\frac{1-(S\beta)^{2h}}{(1-S^2\beta^2)(S\beta)^{2h-2}} - \frac{1-\beta^{2h}}{1-\beta^2}\right] + \frac{1}{(S\beta)^{2h}} - \beta^{2h}\right\}\\
&\leq \frac{1}{t_{2h}}\sum_{j=1}^{t_{2h}} \chi_{[0,\, b^{(2h-1)}[}\left(y_j^{(2h)}\right) - b^{(2h-1)}\\
&\leq B(L+S-1)\frac{(S\beta)^{2h}}{t_{2h}}\left\{2\left[\frac{1-(S\beta)^{2h-2}}{(1-S^2\beta^2)(S\beta)^{2h-1}} + \beta\frac{\beta^2-\beta^{2h}}{1-\beta^2}\right] + \frac{1}{S\beta} + \beta\right\}. \qquad (11)
\end{aligned}
$$

Assume now that n is odd and let $n = 2h+1$, with $h \geq 1$. Then with the same arguments from (10), we derive the following upper and lower bounds:

$$
\begin{aligned}
&-B(L+S-1)\frac{(S\beta)^{2h+1}}{t_{2h+1}}\\
&\quad\times\left\{2\left[\frac{1-(S\beta)^{2h-2}}{(1-S^2\beta^2)(S\beta)^{2h-1}} + \beta\frac{\beta^2-\beta^{2h}}{1-\beta^2}\right] + \frac{1}{S\beta} + \beta + \frac{1}{(S\beta)^{2h+1}} + \beta^{2h+1}\right\}\\
&\leq \frac{1}{t_{2h+1}}\sum_{j=1}^{t_{2h+1}} \chi_{[0,\, b^{(2h)}[}\left(y_j^{(2h+1)}\right) - b^{(2h)}\\
&\leq B(L+S-1)\frac{(S\beta)^{2h+1}}{t_{2h+1}}\left\{2\left[\frac{1-(S\beta)^{2h+2}}{(1-S^2\beta^2)(S\beta)^{2h}} - \frac{1-\beta^{2h+2}}{1-\beta^2}\right]\right\}. \qquad (12)
\end{aligned}
$$

(2) $S\beta = 1$

Equation (10) becomes

$$
\begin{aligned}
&\frac{1}{t_n}\sum_{j=1}^{t_n} \chi_{[0,\, b^{(n-1)}[}\left(y_j^{(n)}\right) - b^{(n-1)}\\
&\quad = \frac{B}{t_n}(-1)^n\left\{\sum_{p=1,\, p \text{ even}}^{n-1} m_{p,S}\left[1+\beta^{p+1}\right] + \sum_{p=1,\, p \text{ odd}}^{n-1} m_{p,L}\left[1+\beta^{p}\right]\right\}\\
&\qquad - \frac{B}{t_n}(-1)^n\left\{\sum_{p=1,\, p \text{ even}}^{n-1} m_{p,L}\left[1-\beta^{p}\right] + \sum_{p=1,\, p \text{ odd}}^{n-1} m_{p,S}\left[1-\beta^{p+1}\right]\right\}. \qquad (13)
\end{aligned}
$$

If we assume $n = 2h$, with $h \geq 1$, from the previous identity, we derive the following upper and lower bounds

$$
\begin{aligned}
&-\frac{B(L+S-1)}{t_{2h}}\left\{2h+1-2\frac{1-\beta^{2h}}{1-\beta^2} - \beta^{2h}\right\}\\
&\quad\leq \frac{1}{t_{2h}}\sum_{j=1}^{t_{2h}} \chi_{[0,\, b^{(2h-1)}[}\left(y_j^{(2h)}\right) - b^{(2h-1)}\\
&\quad\leq \frac{B(L+S-1)}{t_{2h}}\left\{2h-1+2\beta\frac{\beta^2-\beta^{2h}}{1-\beta^2} + \beta\right\}. \qquad (14)
\end{aligned}
$$

If $n = 2h + 1$, with $h \geq 1$, from (13), we derive the following inequalities

$$
\begin{aligned}
& -\frac{B(L+S-1)}{t_{2h+1}} \left\{ 2h + 2\beta \frac{\beta^2 - \beta^{2h}}{1-\beta^2} + \beta + \beta^{2h+1} \right\} \\
& \leq \frac{1}{t_{2h+1}} \sum_{j=1}^{t_{2h+1}} \chi_{[0,\, b^{(2h)}[} \left(y_j^{(2h+1)} \right) - b^{(2h)} \\
& \leq \frac{B(L+S-1)}{t_{2h+1}} \left\{ 2h - 2\, \frac{\beta^2 - \beta^{2h+2}}{1-\beta^2} \right\}. \qquad (15)
\end{aligned}
$$

Since $b \in [b_1^{(n-1)}, b_2^{(n-1)}[$, the corresponding estimates (11, 12, 14), and (15) hold for $b \in [0,\ 1[$.

It is time now to distinguish between the three cases highlighted in the statement of our theorem, in order to prove the various asymptotic behaviors of the star-discrepancy.

(i) $S < L + 1$

Since $\beta = \frac{\sqrt{L^2+4S}-L}{2S}$, this condition is equivalent to the case $S\beta < 1$.
If n is even, from (7, 8), and (11), simple calculations give

$$
\begin{aligned}
& -L - S - B(L+S-1) \\
& \quad \times \left\{ 2 \left[\frac{S^2\beta^2 - (S\beta)^{2h+2}}{1 - S^2\beta^2} - (S\beta)^{2h} \frac{1-\beta^{2h}}{1-\beta^2} \right] + 1 - (S\beta^2)^{2h} \right\} \\
& \leq t_{2h} \left[\sup_{0<b\leq 1} \left(\frac{1}{t_{2h}} \sum_{j=1}^{t_{2h}} \chi_{[0,\, b[} \left(y_j^{(2h)} \right) - b \right) \right] \\
& \leq L + S + B(L+S-1) \\
& \quad \times \left\{ 2 \left[\frac{S\beta - (S\beta)^{2h-1}}{1 - S^2\beta^2} + \beta (S\beta)^{2h} \frac{\beta^2 - \beta^{2h}}{1-\beta^2} \right] + (S\beta)^{2h-1} + \beta (S\beta)^{2h} \right\}. \qquad (16)
\end{aligned}
$$

If n is odd, from (7, 8) and (12) we obtain

$$
\begin{aligned}
& -L - S - B(L+S-1) \left\{ 2 \left[\frac{S^2\beta^2 - (S\beta)^{2h}}{1 - S^2\beta^2} + \beta (S\beta)^{2h+1} \frac{\beta^2 - \beta^{2h}}{1-\beta^2} \right] \right. \\
& \quad \left. - (S\beta)^{2h} - \beta (S\beta)^{2h+1} - 1 - (S\beta^2)^{2h+1} \right\} \\
& \leq t_{2h+1} \left[\sup_{0<b\leq 1} \left(\frac{1}{t_{2h+1}} \sum_{j=1}^{t_{2h+1}} \chi_{[0,\, b[} \left(y_j^{(2h+1)} \right) - b \right) \right] \\
& \leq L + S + B(L+S-1) \left\{ 2 \left[\frac{S\beta - (S\beta)^{2h+3}}{1 - S^2\beta^2} - (S\beta)^{2h+1} \frac{1-\beta^{2h+2}}{1-\beta^2} \right] \right\}. \qquad (17)
\end{aligned}
$$

Since $\beta < 1$, $S\beta < 1$ and $S\beta^2 < 1$, the following limits exist:

$$
\begin{aligned}
& \lim_{h\to\infty} \left\{ 2 \left[\frac{S\beta - (S\beta)^{2h-1}}{1 - S^2\beta^2} + \beta (S\beta)^{2h} \frac{\beta^2 - \beta^{2h}}{1-\beta^2} \right] + (S\beta)^{2h-1} + \beta (S\beta)^{2h} \right\} \\
& \quad = \lim_{h\to\infty} \left\{ 2 \left[\frac{S\beta - (S\beta)^{2h+3}}{1 - S^2\beta^2} - (S\beta)^{2h+1} \frac{1-\beta^{2h+2}}{1-\beta^2} \right] \right\} = \frac{2S\beta}{1 - S^2\beta^2}
\end{aligned}
$$

and

$$\begin{aligned}
&\lim_{h\to\infty}\left\{2\left[\frac{S^2\beta^2-(S\beta)^{2h}}{1-S^2\beta^2}+\beta(S\beta)^{2h+1}\frac{\beta^2-\beta^{2h}}{1-\beta^2}\right]\right.\\
&\qquad\left.+(S\beta)^{2h}+\beta(S\beta)^{2h+1}+1+(S\beta^2)^{2h+1}\right\}\\
&=\lim_{h\to\infty}\left\{2\left[\frac{S^2\beta^2-(S\beta)^{2h+2}}{1-S^2\beta^2}-(S\beta)^{2h}\frac{1-\beta^{2h}}{1-\beta^2}\right]+1-(S\beta^2)^{2h}\right\}\\
&=\frac{2S^2\beta^2}{1-S^2\beta^2}+1,
\end{aligned}$$

so that the sequences of upper and lower bounds in (16) and (17) are bounded. Therefore, we conclude that there exist two positive constants c_1 and c_2, independent on n, depending only on L and S (since β depends on L and S), such that for any $n \in \mathbb{N}$

$$\frac{c_1}{t_n} \le D^*\left(\{\rho_{L,S}^n\}\right) \le \frac{c_2}{t_n}.$$

(ii) $S = L + 1$

First of all, we note that, since $\beta = \frac{\sqrt{L^2+4S}-L}{2S}$, we have $S\beta = 1$.
If n is even, from (7, 8), and (14), we get

$$\begin{aligned}
&-\frac{B(L+S-1)}{\log t_{2h}}\left\{2h+1-2\frac{1-\beta^2}{1-\beta^{2h}}-\beta^{2h}\right\}\\
&\le \frac{t_{2h}}{\log t_{2h}}\left[\sup_{0<b\le 1}\left(\frac{1}{t_{2h}}\sum_{j=1}^{t_{2h}}\chi_{[0,\,b[}\left(y_j^{(2h)}\right)-b\right)\right]\\
&\le \frac{B(L+S-1)}{\log t_{2h}}\left\{2h-1+2\beta\frac{\beta^2-\beta^{2h}}{1-\beta^2}+\beta\right\}.
\end{aligned}\tag{18}$$

If n is odd, we obtain from (7, 8) and (15) that

$$\begin{aligned}
&-\frac{B(L+S-1)}{\log t_{2h+1}}\left\{2h+2\beta\frac{\beta^2-\beta^{2h}}{1-\beta^2}+\beta+\beta^{2h+1}\right\}\\
&\le \frac{t_{2h+1}}{\log t_{2h+1}}\left[\sup_{0<b\le 1}\left(\frac{1}{t_{2h+1}}\sum_{j=1}^{t_{2h+1}}\chi_{[0,\,b[}\left(y_j^{(2h+1)}\right)-b\right)\right]\\
&\le \frac{B(L+S-1)}{\log t_{2h+1}}\left\{2h-2\,\frac{\beta^2-\beta^{2h+2}}{1-\beta^2}\right\}.
\end{aligned}\tag{19}$$

From formula (2) we have

$$\begin{aligned}
\log t_n &= \log\left(\frac{2}{1+\beta}\left(\frac{1}{\beta^n}\right)-\frac{1-\beta}{1+\beta}(-1)^n\right)\\
&= \log\left(\frac{2}{1+\beta}-\frac{1-\beta}{1+\beta}(-\beta)^n\right)+n\log\frac{1}{\beta}
\end{aligned}$$

for any $n \in I\!N$ so that, since $\beta < 1$ and $S\beta^2 < 1$,

$$\lim_{h\to\infty} \frac{2h - 1 + 2\beta\frac{\beta^2-\beta^{2h}}{1-\beta^2} + \beta}{\log t_{2h}} = \lim_{h\to\infty} \frac{2h + 1 - 2\frac{1-\beta^2}{1-\beta^{2h}} - \beta^{2h}}{\log t_{2h}}$$
$$= \lim_{h\to\infty} \frac{2h - 2\;\frac{\beta^2-\beta^{2h+2}}{1-\beta^2}}{\log t_{2h+1}} = \lim_{h\to\infty} \frac{2h + 2\beta\frac{\beta^2-\beta^{2h}}{1-\beta^2} + \beta + \beta^{2h}}{\log t_{2h+1}} = -\frac{1}{\log\beta}.$$

Taking the above limits into account, we conclude from (18) and (19) that there exist two positive constants c_3 and c_4, independent on n, depending only on L and S, such that

$$c_3\,\frac{\log t_n}{t_n} \le D^*\left(\left\{\rho^n_{L,S}\right\}\right) \le c_4\,\frac{\log t_n}{t_n}$$

for any $n \in I\!N$.

(iii) $\quad S > L + 1$

This condition is equivalent to $S\beta > 1$ since $\beta = \frac{\sqrt{L^2+4S}-L}{2S}$.
If n is even, from (7, 8) and (11) we have:

$$-\frac{B(L+S-1)}{\beta^{2h}t_{2h}}\left\{2\left[\frac{1-(S\beta)^{2h}}{(1-S^2\beta^2)(S\beta)^{2h-2}} - \frac{1-\beta^{2h}}{1-\beta^2}\right] + \frac{1}{(S\beta)^{2h}} - \beta^{2h}\right\}$$
$$\le \frac{1}{(S\beta^2)^{2h}}\left[\sup_{0<b\le 1}\left(\frac{1}{t_{2h}}\sum_{j=1}^{t_{2h}}\chi_{[0,\,b[}\left(y_j^{(2h)}\right) - b\right)\right]$$
$$\le \frac{B(L+S-1)}{\beta^{2h}t_{2h}}\left\{2\left[\frac{1-(S\beta)^{2h-2}}{(1-S^2\beta^2)(S\beta)^{2h-1}} + \beta\frac{\beta^2-\beta^{2h}}{1-\beta^2}\right] + \frac{1}{S\beta} + \beta\right\}. \tag{20}$$

If n is odd, from (7, 8), and (12), we get

$$-\frac{B(L+S-1)}{\beta^{2h+1}t_{2h+1}}$$
$$\times\left\{2\left[\frac{1-(S\beta)^{2h-2}}{(1-S^2\beta^2)(S\beta)^{2h-1}} + \beta\frac{\beta^2-\beta^{2h}}{1-\beta^2}\right] + \frac{1}{S\beta} + \beta + \frac{1}{(S\beta)^{2h+1}} + \beta^{2h+1}\right\}$$
$$\le \frac{1}{(S\beta^2)^{2h+1}}\left[\sup_{0<b\le 1}\left(\frac{1}{t_{2h+1}}\sum_{j=1}^{t_{2h+1}}\chi_{[0,\,b[}\left(y_j^{(2h+1)}\right) - b\right)\right]$$
$$\le \frac{2B(L+S-1)}{\beta^{2h+1}t_{2h+1}}\left[\frac{1-(S\beta)^{2h+2}}{(1-S^2\beta^2)(S\beta)^{2h}} - \frac{1-\beta^{2h+2}}{1-\beta^2}\right]. \tag{21}$$

Since $S\beta > 1$, $S\beta^2 < 1$ and $\beta < 1$, using formula (4) we have

$$\begin{aligned}
&\lim_{h\to\infty} \frac{1}{\beta^{2h} t_{2h}} \left\{ 2\left[\frac{1-(S\beta)^{2h-2}}{\left(1-S^2\beta^2\right)(S\beta)^{2h-1}} + \beta \frac{\beta^2-\beta^{2h}}{1-\beta^2} \right] + \frac{1}{S\beta} + \beta \right\} \\
&= \lim_{h\to\infty} \frac{1}{A - B\left(-S\beta^2\right)^{2h}} \left\{ 2\left[\frac{1-(S\beta)^{2h-2}}{\left(1-S^2\beta^2\right)(S\beta)^{2h-1}} + \beta \frac{\beta^2-\beta^{2h}}{1-\beta^2} \right] + \frac{1}{S\beta} + \beta \right\} \\
&= \lim_{h\to\infty} \frac{1}{\beta^{2h+1} t_{2h+1}} \left\{ 2\left[\frac{1-(S\beta)^{2h-2}}{\left(1-S^2\beta^2\right)(S\beta)^{2h-1}} + \beta \frac{\beta^2-\beta^{2h-1}}{1-\beta^2} \right] \right. \\
&\quad \left. + \frac{1}{S\beta} + \beta + \frac{1}{(S\beta)^{2h+1}} + \beta^{2h+1} \right\} \\
&= \frac{1}{A} \left\{ 2\left[\frac{1}{S\beta\left(S^2\beta^2-1\right)} + \frac{\beta^3}{1-\beta^2} \right] + \frac{1}{S\beta} + \beta \right\}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{h\to\infty} \frac{1}{\beta^{2h} t_{2h}} \left\{ 2\left[\frac{1-(S\beta)^{2h}}{\left(1-S^2\beta^2\right)(S\beta)^{2h-2}} - \frac{1-\beta^{2h}}{1-\beta^2} \right] + \frac{1}{(S\beta)^{2h}} - \beta^{2h} \right\} \\
&= \lim_{h\to\infty} \frac{2}{\beta^{2h+1} t_{2h+1}} \left[\frac{1-(S\beta)^{2h+2}}{\left(1-S^2\beta^2\right)(S\beta)^{2h}} - \frac{1-\beta^{2h+2}}{1-\beta^2} \right] \\
&= \frac{2}{A} \left\{ \frac{S^2\beta^2}{S^2\beta^2-1} - \frac{1}{1-\beta^2} \right\}.
\end{aligned}$$

From (20, 21) and the previous limits, we conclude that there exist two positive constants c_5' and c_6', independent on n, depending only on L and S, such that

$$c_5' \left(S\beta^2\right)^n \le D^*\left(\left\{\rho_{L,S}^n\right\}\right) \le c_6' \left(S\beta^2\right)^n$$

for any $n \in I\!N$.

It remains only to observe how $\left(S\beta^2\right)^n$ can be written in terms of t_n.

In fact, $(S\beta^2)^n = \beta^\gamma$, where $\gamma = 1 + \frac{\log(S\beta)}{\log\beta} < 1$ since $S\beta^2 = (S\beta)\beta = \beta^{1+c}$ with $c = \frac{\log(S\beta)}{\log\beta} < 0$. Consequently, $\left(S\beta^2\right)^n$ and $\frac{1}{t_n^\gamma}$ have the same order at infinity since

$$\frac{\left(S\beta^2\right)^n}{1/(t_n)^\gamma} = \frac{\left(S\beta^2\right)^n}{(\beta^n)^\gamma} \left(A + B\left(-S\beta^2\right)^n\right)^\gamma = \frac{(\beta^\gamma)^n}{(\beta^n)^\gamma} \left(A + B\left(-S\beta^2\right)^n\right)^\gamma \to A^\gamma$$

as $n \to \infty$.

Therefore, there exists $c_5 > 0$ and $c_6 > 0$ such that

$$\frac{c_5}{(t_n)^\gamma} \le D^*\left(\left\{\rho_{L,S}^n\right\}\right) \le \frac{c_6}{(t_n)^\gamma}$$

for any $n \in I\!N$, where $\gamma = 1 + \frac{\log(S\beta)}{\log\beta}$.

Since the discrepancy and the star-discrepancy are equivalent, the theorem is proved. □

Remark 2.3 In the case $S < L + 1$, we have obtained sequences of partitions with low discrepancy. Among them, we consider the simple case $L = S = 1$ $\left(\beta = \frac{\sqrt{5}-1}{2}\right)$. Of course, the sequence $\left\{\rho_{1,1}^n\right\}$ is a Kakutani sequence. Furthermore, t_n satisfies the recursive formula

$t_n = t_{n-1} + t_{n-2}$, with $t_0 = 1$ and $t_1 = 2$; therefore, the sequence $\{t_n\}$ is actually the Fibonacci sequence $1, 2, 3, 5, 8, \ldots$. For these reasons, we call it the *Kakutani-Fibonacci sequence of partitions*.

3 LS-sequences of points

In this section, we will reorder the points of the LS-sequence of partitions $\left\{\rho_{L,S}^n\right\}$ in a way that resembles the construction of the Van der Corput sequence. The aim of this section is to associate to each LS-sequence of partitions a sequence of points whose discrepancy is the best possible.

Definition 3.1 Given the sequence of partition $\left\{\rho_{L,S}^n\right\}$, we define the LS-*sequence of points* $\left\{\xi_{L,S}^n\right\}$ as follows. The first t_1 points are just the left endpoints of the intervals of $\rho_{L,S}^1$, taken in the lexicographical order, i.e. ordered by magnitude. This ordered set will be denoted by $\Lambda_{L,S}^1$, and for later convenience, its points will be denoted by $\xi_1^{(1)}, \ldots, \xi_{t_1}^{(1)}$.

For $n > 1$ and if $\Lambda_{L,S}^n = \left(\xi_1^{(n)}, \ldots, \xi_{t_n}^{(n)}\right)$ is the set of the points (written in their order) of $\rho_{L,S}^n$, then the t_{n+1} points of $\rho_{L,S}^{n+1}$ are recursively ordered as follows:

$$\begin{aligned}\Lambda_{L,S}^{n+1} = \Big(&\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{t_n}^{(n)},\\ &\varphi_1^{(n+1)}\left(\xi_1^{(n)}\right), \ldots, \varphi_1^{(n+1)}\left(\xi_{l_n}^{(n)}\right), \ldots, \varphi_L^{(n+1)}\left(\xi_1^{(n)}\right), \ldots, \varphi_L^{(n+1)}\left(\xi_{l_n}^{(n)}\right),\\ &\varphi_{L,1}^{(n+1)}\left(\xi_1^{(n)}\right), \ldots, \varphi_{L,1}^{(n+1)}\left(\xi_{l_n}^{(n)}\right), \ldots, \varphi_{L,S-1}^{(n+1)}\left(\xi_1^{(n)}\right), \ldots, \varphi_{L,S-1}^{(n+1)}\left(\xi_{l_n}^{(n)}\right)\Big). \end{aligned} \tag{22}$$

Here, l_n is the number of long intervals of $\rho_{L,S}^n$ (see 3), and the two families of functions are

$$\varphi_i^{(n+1)}(x) = x + i\beta^{n+1} \quad \text{and} \quad \varphi_{L,j}^{(n+1)}(x) = x + L\beta^{n+1} + j\beta^{n+2} \tag{23}$$

for any $1 \le i \le L$ and $1 \le j \le S-1$.

Example 3.2 (1) $L = 1, S = 1$

Let us consider the special case $L = S = 1$ mentioned in Remark 2.3. Here, we have just one generating function $\varphi_1^{(n+1)}(x) = x + \beta^{n+1}$ (see 23), and (22) becomes

$$\Lambda_{1,1}^{n+1} = \left(\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{t_n}^{(n)}, \varphi_1^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_1^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_1^{(n+1)}\left(\xi_{l_n}^{(n)}\right)\right).$$

We associate to the Kakutani-Fibonacci sequence of partitions $\left\{\rho_{1,1}^n\right\}$ the $(1, 1)$-sequence of points $\left\{\xi_{1,1}^n\right\}$ we call the *Kakutani-Fibonacci sequence of points*, proceeding as follows:

$$\begin{aligned}\Lambda_{1,1}^1 &= (0, \beta)\\ \Lambda_{1,1}^2 &= \left(0, \beta, \beta^2\right)\\ \Lambda_{1,1}^3 &= \left(0, \beta, \beta^2, \beta^3, \beta + \beta^3\right)\end{aligned}$$

$$\Lambda_{1,1}^4 = \left(0, \beta, \beta^2, \beta^3, \beta+\beta^3, \beta^4, \beta+\beta^4, \beta^2+\beta^4\right)$$
$$\Lambda_{1,1}^5 = \Big(0, \beta, \beta^2, \beta^3, \beta+\beta^3, \beta^4, \beta+\beta^4, \beta^2+\beta^4, \beta^5, \beta+\beta^5, \beta^2+\beta^5, \beta^3+\beta^5, \beta+\beta^3+\beta^5\Big)$$

and so on.

(2) $L = 1, S = 2$

Let us consider another slightly more complicated case: $L = 1$ and $S = 2$ $\left(\beta = \frac{1}{2}\right)$. The two corresponding families of functions defined in (23) are reduced to one element each:

$$\varphi_1^{(n+1)}(x) = x + \beta^{n+1} = x + \frac{1}{2^{n+1}}$$

and

$$\varphi_{1,1}^{(n+1)}(x) = x + \beta^{n+1} + \beta^{n+2} = x + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}},$$

so formula (22) becomes

$$\Lambda_{1,2}^{n+1} = \Big(\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{t_n}^{(n)}, \varphi_1^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_1^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_1^{(n+1)}\left(\xi_{l_n}^{(n)}\right), \varphi_{1,1}^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_{1,1}^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_{1,1}^{(n+1)}\left(\xi_{l_n}^{(n)}\right)\Big),$$

where as usual t_n denotes the number of intervals of $\rho_{1,2}^n$ and l_n the number of its long ones.

More precisely, we have

$$\Lambda_{1,2}^1 = \left(\xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}\right) = \left(0, \beta, \beta+\beta^2\right) = \left(0, \frac{1}{2}, \frac{3}{4}\right).$$

Since $l_1 = 1$, we apply the two functions $\varphi_1^{(2)}(x) = x + \frac{1}{2^2}$ and $\varphi_{1,1}^{(2)} = x + \frac{1}{2^2} + \frac{1}{2^3}$ only to the first point $\xi_1^{(1)} = 0$, so that $\varphi_1^{(2)}(0) = \beta^2 = \frac{1}{2^2}$ and $\varphi_{1,1}^{(2)}(0) = \beta^2 + \beta^3 = \frac{1}{2^2} + \frac{1}{2^3}$. Therefore,

$$\begin{aligned}\Lambda_{1,2}^2 &= \left(\xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}, \varphi_1^{(2)}\left(\xi_1^{(1)}\right), \varphi_{1,1}^{(2)}\left(\xi_1^{(1)}\right)\right)\\ &= \left(0, \beta, \beta+\beta^2, \varphi_1^{(2)}(0), \varphi_{1,1}^{(2)}(0)\right) = \left(0, \frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{3}{8}\right).\end{aligned}$$

Now the functions $\varphi_1^{(3)}(x) = x + \frac{1}{2^3}$ and $\varphi_{1,1}^{(3)}(x) = x + \frac{1}{2^3} + \frac{1}{2^4}$ must be applied to the first $l_2 = 3$ points of $\Lambda_{1,2}^2$, namely $\xi_1^{(2)} = 0$, $\xi_1^{(2)} = \beta$ and $\xi_1^{(2)} = \beta + \beta^2$. Therefore,

$$\begin{aligned}\Lambda_{1,2}^3 &= \Big(\xi_1^{(2)}, \xi_2^{(2)}, \xi_3^{(2)}, \xi_4^{(2)}, \xi_5^{(2)}, \ \varphi_1^{(3)}\left(\xi_1^{(2)}\right), \ \varphi_1^{(3)}\left(\xi_2^{(2)}\right), \ \varphi_1^{(3)}\left(\xi_3^{(2)}\right),\\ &\qquad \varphi_{1,1}^{(3)}\left(\xi_1^{(2)}\right), \ \varphi_{1,1}^{(3)}\left(\xi_2^{(2)}\right), \ \varphi_{1,1}^{(3)}\left(\xi_3^{(2)}\right)\Big)\\ &= \Big(0, \beta, \beta+\beta^2, \beta^2, \beta^2+\beta^3, \ \varphi_1^{(3)}(0), \ \varphi_1^{(3)}(\beta), \ \varphi_1^{(3)}(\beta+\beta^2),\\ &\qquad \varphi_{1,1}^{(3)}(0), \ \varphi_{1,1}^{(3)}(\beta), \ \varphi_{1,1}^{(3)}(\beta+\beta^2)\Big)\\ &= \left(0, \frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{3}{8}, \frac{1}{8}, \frac{5}{8}, \frac{7}{8}, \frac{3}{16}, \frac{11}{16}, \frac{15}{16}\right).\end{aligned}$$

Proceeding this way, we obtain the $(1, 2)$-sequence of points $\left\{\xi_{1,2}^n\right\}$.

In order to estimate the discrepancy of the sequence $\left\{\xi_{L,S}^n\right\}$, we need to study the discrepancy of a suitable sequence of subinterval systems of the interval $[0, 1[$, denoted by $\left\{\tilde{\rho}_{L,S}^n\right\}$, defined as follows.

Definition 3.3 Given the sequence of partitions $\left\{\rho_{L,S}^n\right\}$, we define the sequence $\left\{\tilde{\rho}_{L,S}^n\right\}$ of subinterval systems of $[0, 1[$ as the family of the l_n long intervals of $\rho_{L,S}^n$. Moreover, by $\tilde{\Lambda}_{L,S}^n$ we denote the set of the left endpoints of the intervals of $\tilde{\rho}_{L,S}^n$, reordered according to Definition 3.1.

Proposition 3.4 (i) *If* $S < L+1$, *there exist* $\tilde{c}_1 > 0$ *and* $\tilde{c}_2 > 0$ *such that for any* $n \in I\!N$

$$\frac{\tilde{c}_1}{l_n} \le D\left(\left\{\tilde{\rho}_{L,S}^n\right\}\right) \le \frac{\tilde{c}_2}{l_n}.$$

(ii) *If* $S = L+1$ *there exist* $\tilde{c}_3 > 0$ *and* $\tilde{c}_4 > 0$ *such that for any* $n \in I\!N$

$$\tilde{c}_3 \frac{\log l_n}{l_n} \le D\left(\tilde{\rho}_{L,S}^n\right) \le \tilde{c}_4 \frac{\log l_n}{l_n}.$$

(iii) *If* $S > L+1$ *there exist* $\tilde{c}_5 > 0$ *and* $\tilde{c}_6 > 0$ *such that for any* $n \in I\!N$

$$\frac{\tilde{c}_5}{(l_n)^\gamma} \le D\left(\left\{\tilde{\rho}_{L,S}^n\right\}\right) \le \frac{\tilde{c}_6}{(l_n)^\gamma},$$

where $\gamma = 1 + \frac{\log(S\beta)}{\log\beta} < 1$.

Proof It is very simple to see that the proof of Theorem 2.2 also applies to the sequence $\left\{\tilde{\rho}_{L,S}^n\right\}$ if we consider l_n instead of t_n, with the following few modifications.

Formulas (5) and (6) hold for l_n instead of t_n with constant $B' = S\beta^2/(1+S\beta^2)$ instead of $-B$.

Moreover, formulas (7) and (8) hold with l_n instead of t_n and L instead of $L+S$.

The covering in formula (9) (as the union of consecutive intervals of $\tilde{\rho}_{L,S}^p$) still holds, as well as formulas (10) and (13) with l_n instead of t_n and B' instead of $-B$. Then, it is very simple to see that analogous estimates of the upper and lower bounds in (11, 12, 14–21) can be proved with l_n instead of t_n and B' instead of $-B$.

Then the conclusions of Theorem 2.2 also hold for the sequence $\left\{\tilde{\rho}_{L,S}^n\right\}$, with suitable constants $\tilde{c}_1, \ldots, \tilde{c}_6$, independent on n, in all the three cases. □

The last ingredient we need is a suitable decomposition of the set $\left(\xi_{L,S}^1, \xi_{L,S}^2, \ldots, \xi_{L,S}^N\right)$ into subsets whose discrepancy can been easily estimated.

To this purpose, we present the decomposition in the two simple cases $L = S = 1$ and $L = 1,\ S = 2$, illustrated in Example 3.2.

Remark 3.5 First of all, we observe that, for a given the sequence $\left\{\xi_{L,S}^n\right\}$, the points of $\Lambda_{L,S}^{n+1}$ defined in (22) satisfy the following property: $\left(\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{t_n}^{(n)}\right) = \left(\xi_1^{(m)}, \xi_2^{(m)}, \ldots, \xi_{t_n}^{(m)}\right)$ for any $m > n$.

As regards formula (22), we also recall that the $t_{n+1} - t_n$ points of $\Lambda_{L,S}^{n+1} \setminus \Lambda_{L,S}^{n}$ can be obtained by applying the $L + S - 1$ recursive functions defined in (23) to the first $l_n = t_{n-1} + (L-1)l_{n-1}$ points of $\Lambda_{L,S}^{n}$.

It should be clear that, in general, the points of $\Lambda_{L,S}^{n+1}$ can be seen as subsequent blocks: the first block is made by t_n points, followed by $L + S - 1$ blocks of l_n points. Moreover, $t_n = L\,t_{n-1} + S\,t_{n-2}$ and $l_n = L\,l_{n-1} + S\,l_{n-2}$ imply that $l_n = t_{n-1} + (L-1)l_{n-1}$ and therefore, each block of l_n points can be represented as a first block of t_{n-1} points and $L - 1$ subsequent blocks, each of which consisting of l_{n-1} points. Each block of points is obtained by suitable compositions of the generating functions defined by (23).

In other words, all the $L+S$ blocks of $\Lambda_{L,S}^{n+1}$ described above represent its natural partition into $L + S$ sets of points, each of which is a ordered set, too (according to Definition 3.1).

Example 3.6 (1) $L = S = 1$

Let N be such that $t_n \le N < t_{n+1}$ and consider the family of the first N points of $\left\{\xi_{1,1}^{n}\right\}$. As we have already pointed out, these points are the first N points of $\Lambda_{1,1}^{n+1}$, i.e. $\left(\xi_{1,1}^{1}, \xi_{1,1}^{2}, \ldots, \xi_{1,1}^{N}\right) = \left(\xi_1^{(n+1)}, \xi_2^{(n+1)}, \ldots, \xi_N^{(n+1)}\right)$.

Since $\{t_n\}$ is the Fibonacci sequence, there exists $h_i \in I\!N$, with $1 \le i \le s$ and $h_1 > h_2 > \cdots > h_s \ge 0$, such that we can write $N = t_{h_1} + t_{h_2} + \cdots + t_{h_s}$ ([17]).

Of course, $h_1 = n$ and accordingly

$$N = t_n + t_{h_2} + \cdots + t_{h_s}. \tag{24}$$

Let us define now the following s sets of indexes

$$\begin{aligned} M_n &= (1, 2, \ldots, t_n) \\ M_{h_2} &= \left(t_n + 1, t_n + 2, \ldots, t_n + t_{h_2}\right) \\ &\ \ \vdots \\ M_{h_s} &= \left(t_n + \cdots + t_{h_{s-1}} + 1, t_n + \cdots + t_{h_{s-1}} + 2, \ldots, t_n + \cdots + t_{h_s}\right) \end{aligned}$$

and the corresponding s ordered sets of points

$$\begin{aligned} P_{h_1} &= \left(\xi_1^{(n+1)}, \xi_2^{(n+1)}, \ldots, \xi_{t_n}^{(n+1)}\right) \\ P_{h_2} &= \left(\xi_{t_n+1}^{(n+1)}, \xi_{t_n+2}^{(n+1)}, \ldots, \xi_{t_n+t_{h_2}}^{(n+1)}\right) \\ &\ \ \vdots \\ P_{h_s} &= \left(\xi_{t_n+\cdots+t_{h_{s-1}}+1}^{(n+1)}, \xi_{t_n+\cdots+t_{h_{s-1}}+2}^{(n+1)}, \ldots, \xi_{t_n+\cdots+t_{h_s}}^{(n+1)}\right). \end{aligned}$$

Then (see [KN, Chapter 2, Th. 2.6])

$$N D^*\left(\xi_{1,1}^{1}, \xi_{1,1}^{2}, \ldots, \xi_{1,1}^{N}\right) \le \sum_{j=1}^{s} t_{h_j} D^*(P_{h_j}).$$

First, we note that $P_{h_1} = \Lambda_{1,1}^{n}$ and, clearly, $D^*(P_1) = D^*\left(\Lambda_{1,1}^{n}\right) = D^*\left(\rho_{1,1}^{n}\right)$.

Furthermore, since we know from Definition 3.1 and Example 3.2.1 that the points of $\Lambda_{1,1}^{n+1}$ are obtained by applying the function $\varphi_1^{(n+1)}$ to the first l_n points of $\Lambda_{1,1}^{n}$, i.e. to the points $\left\{\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{l_n}^{(n)}\right\}$, a simple calculation shows that

$$\begin{aligned}
P_{h_2} &= \left(\varphi_1^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_1^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_1^{(n+1)}\left(\xi_{h_2}^{(n)}\right)\right) = \Lambda_{1,1}^{h_2} + \beta^{n+1} \\
P_{h_3} &= \Lambda_{1,1}^{h_3} + \beta^{n+1} + \beta^{h_2+1} \\
P_{h_4} &= \Lambda_{1,1}^{h_4} + \beta^{n+1} + \beta^{h_2+1} + \beta^{h_3+1} \\
&\ \ \vdots \\
P_{h_s} &= \Lambda_{1,1}^{h_s} + \beta^{n+1} + \beta^{h_2+1} + \beta^{h_3+1} + \cdots + \beta^{h_{s-1}+1}.
\end{aligned}$$

At this point, we introduce the notation

$$\left(\xi_{1,1}^1, \xi_{1,1}^2, \ldots, \xi_{1,1}^N\right) = \left(P_{h_1}, P_{h_2}, \ldots, P_{h_s}\right),$$

that will be used in the next example and in the subsequent proposition.

The previous construction allows to obtain the following significant estimate of the discrepancy

$$\begin{aligned}
N D^*\left(\xi_{1,1}^1, \xi_{1,1}^2, \ldots, \xi_{1,1}^N\right) &\leq \sum_{j=1}^{s} t_{h_j} D^*(P_{h_j}) = t_n D^*(P_n) + \sum_{j=2}^{s} t_{h_j} D^*(P_{h_j}) \\
&= t_n D^*\left(\Lambda_{1,1}^n\right) + \sum_{j=2}^{s} t_{h_j} D^*\left(\Lambda_{1,1}^{h_j} + \sum_{w=1}^{j} \beta^{h_w+1}\right) \\
&= t_n D^*\left(\rho_{1,1}^n\right) + \sum_{j=2}^{s} t_{h_j} D^*\left(\rho_{1,1}^{h_j} + \sum_{w=1}^{j} \beta^{h_w+1}\right).
\end{aligned}$$

This inequality, with the estimates at the end of the next example, suggests how to obtain the main result of this paper.

(2) $L=1, S=2$

We will see that also in this case a formal partition (analogous to formula 24) of the set of the first N points of the sequence $\left\{\xi_{1,2}^n\right\}$ can be obtained.

Let us preliminarly note that, since $L = 1$, then $l_n = t_{n-1}$ (see Remark 3.5).

We fix N such that $t_n \leq N < t_{n+1}$. Therefore, we can write $N = t_n + R_n$, with $0 \leq R_n < t_{n+1} - t_n = 2\, t_{n-1}$.

The first t_n points of $\left(\xi_{1,2}^1, \xi_{1,2}^2, \ldots, \xi_{1,2}^N\right)$ are the points of $\Lambda_{1,2}^{n+1} = \left(\xi_1^{(n+1)}, \xi_2^{(n+1)}, \ldots, \xi_{t_{n+1}}^{(n+1)}\right)$ taken in their order (namely, the order of Definition 3.1), i.e

$$P_n = \Lambda_{1,2}^n.$$

Due to the special nature of $\Lambda_{1,2}^{n+1}$ (see Example 3.2.2), all the remaining R_n points are strictly contained in the set

$$\left(\varphi_1^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_1^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_1^{(n+1)}\left(\xi_{t_{n-1}}^{(n)}\right), \varphi_{1,1}^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_{1,1}^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_{1,1}^{(n+1)}\left(\xi_{t_{n-1}}^{(n)}\right)\right),$$

which is made by $2\,l_n = 2\,t_{n-1}$ points. Since $\varphi_1^{(n+1)}(x) = x + \frac{1}{2^{n+1}}$ and $\varphi_{1,1}^{(n+1)}(x) = x + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}}$, it is clear that

$$\left(\varphi_1^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_1^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_1^{(n+1)}\left(\xi_{t_{n-1}}^{(n)}\right)\right) = \Lambda_{1,2}^{n-1} + \frac{1}{2^{n+1}}$$

and

$$\left(\varphi_{1,1}^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_{1,1}^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_{1,1}^{(n+1)}\left(\xi_{t_{n-1}}^{(n)}\right)\right) = \Lambda_{1,2}^{n-1} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}}.$$

As $0 \leq R_n < 2\,t_{n-1}$, we can write

$$N = t_n + \delta_1 t_{n-1} + r_1$$

for some $0 \leq \delta_1 < L + S - 1 = 2$ and $0 \leq r_1 < l_n = t_{n-1}$ (of course $\delta_1\, r_1 > 0$).

Now, we have the following two possibilities.

(a) If $r_1 = 0$, it must be $\delta_1 \neq 0$, hence

$$N = t_n + t_{n-1}$$

and the corresponding partition into two sets of points is

$$\left(\xi_{1,2}^1, \xi_{1,2}^2, \ldots, \xi_{1,2}^N\right) = (P_n, P_{n-1}),$$

where (recalling that $\beta = 1/2$)

$$P_{n-1} = \left(\varphi_1^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_1^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_1^{(n+1)}\left(\xi_{t_{n-1}}^{(n)}\right)\right) = \Lambda_{1,2}^{n-1} + \frac{1}{2^{n+1}}.$$

(b) If $r_1 \neq 0$, it is clear that δ_1 could be 0 or 1. Since $r_1 < t_{n-1}$, there exists $0 \leq q \leq n-2$ such that $t_q \leq r_1 < t_{q+1}$, and this case can be treated similarly to the initial case $N = t_n + R_n$. In fact, if $q = 0$, we write $r_1 = t_0 + R_0$ with $0 \leq R_0 < t_1 - t_0 = 2$; if $q \neq 0$ there exists $0 \leq R_q < t_{q+1} - t_q = 2\,t_{q-1}$ such that $r_1 = t_q + R_q$ and, consequently,

$$N = t_n + \delta_1\, t_{n-1} + t_q + R_q.$$

To be precise, we have to distinguish among the two possibilities $\delta_1 = 0$ or $\delta_1 = 1$.

If $\delta_1 = 0$, we write $N = t_n + t_q + R_q$. In this case, if $R_q = 0$, we have the following formal partition of N

$$N = t_n + t_q$$

to which it corresponds the following partition of sets

$$\left(\xi_{1,2}^1, \xi_{1,2}^2, \ldots, \xi_{1,2}^N\right) = \left(P_n, P_q\right),$$

where

$$P_q = \left(\varphi_1^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_1^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_1^{(n+1)}\left(\xi_{t_q}^{(n)}\right)\right) = \Lambda_{1,2}^{q} + \frac{1}{2^{n+1}}.$$

If $\delta_1 = 0$ and $R_q \neq 0$, all the subsequent points satisfy the following relation:

$$\left(\varphi_1^{(n+1)}\left(\xi_{t_q+1}^{(n)}\right), \varphi_1^{(n+1)}\left(\xi_{t_q+2}^{(n)}\right), \ldots, \varphi_1^{(n+1)}\left(\xi_{t_q+R_q}^{(n)}\right)\right) \subseteq \Lambda_{1,2}^{q-1} + \frac{1}{2^{n+1}}.$$

If $\delta_1 = 1$, we have $N = t_n + t_{n-1} + t_q + R_q$. If $R_q = 0$, we write

$$N = t_n + t_{n-1} + t_q$$

and

$$\left(\xi_{1,2}^1, \xi_{1,2}^2, \ldots, \xi_{1,2}^N\right) = \left(P_n, P_{n-1}, P_q\right),$$

where

$$P_q = \left(\varphi_{1,1}^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_{1,1}^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_{1,1}^{(n+1)}\left(\xi_{t_q}^{(n)}\right)\right) = \Lambda_{1,2}^q + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}}.$$

Suppose now $\delta_1 = 1$ and $R_q \neq 0$. We have

$$\left(\varphi_{1,1}^{(n+1)}\left(\xi_{t_q+1}^{(n)}\right), \varphi_{1,1}^{(n+1)}\left(\xi_{t_q+2}^{(n)}\right), \ldots, \varphi_{1,1}^{(n+1)}\left(\xi_{t_q+R_q}^{(n)}\right)\right) \subseteq \Lambda_{1,2}^{q-1} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}}.$$

When $R_q \neq 0$, for both choices of δ_1 we can say that there exist $0 \le \delta_{n-q+1} < 2$ and $0 \le r_{n-q+1} < l_q = t_{q-1}$ such that $R_q = \delta_{n-q+1}\, t_{q-1} + r_{n-q+1}$.

Therefore,

$$N = t_n + \delta_1\, t_{n-1} + t_q + \delta_{n-q+1}\, t_{q-1} + r_{n-q+1}$$

and we continue the procedure until we have a residual set containing less than $t_2 - t_1 = 2\, t_0 = 2$ points.

It is now clear that in all the above cases, we may partition the set of the first N points of the sequence $\left\{\xi_{1,2}^n\right\}$ into a suitable number of sets of right shifts of some $\rho_{1,2}^q$. Therefore, we can write the formal partition of N as follows:

$$N = t_{h_1} + t_{h_2} + \cdots + t_{h_s} + r_0, \tag{25}$$

where $n = h_1 > h_2 > \cdots > h_s \ge 0$ and $0 \le r_0 < 2$. Correspondingly, we have the family of $s + 1$ sets of indexes M_j, with $1 \le j \le s + 1$, where

$$\begin{aligned} M_1 &= (1, 2, \ldots, t_n) \\ M_j &= \left(t_n + \cdots + t_{h_{j-1}} + 1, t_n + \cdots + t_{h_{j-1}} + 2, \ldots, t_n + \cdots + t_{h_j}\right) \end{aligned}$$

for any $2 \le j \le s$ and

$$M_{s+1} = \left(t_n + \cdots + t_{h_s} + 1, t_n + \cdots + t_{h_s} + r_0\right)$$

(we put $M_{s+1} = \emptyset$ if $r_0 = 0$).

To this family, it corresponds the family of $s + 1$ ordered sets of points P_{r_0} and P_{h_j}, with $1 \le j \le s$, where

$$\begin{aligned} P_{h_1} &= \left(\xi_1^{(n+1)}, \xi_2^{(n+1)}, \ldots, \xi_{t_n}^{(n+1)}\right) \\ P_{h_j} &= \left(\xi_{t_n+\cdots+t_{h_{j-1}}+1}^{(n+1)}, \xi_{t_n+\cdots+t_{h_{j-1}}+2}^{(n+1)}, \ldots, \xi_{t_n+\cdots+t_{h_j}}^{(n+1)}\right) \end{aligned}$$

for any $2 \le j \le s$ and

$$P_{r_0} = \left(\xi_{t_n+\cdots+t_{h_s}+1}^{(n+1)}, \xi_{t_n+\cdots+t_{h_s}+r_0}^{(n+1)}\right)$$

(with $P_{r_0} = \emptyset$ if $r_0 = 0$).

According to the notation introduced at the end of the previous example, we write

$$\left(\xi_{1,2}^1, \xi_{1,2}^2, \ldots, \xi_{1,2}^N\right) = \left(P_{h_1}, P_{h_2}, \ldots, P_{h_s}, P_{r_0}\right).$$

As we have already seen, there exists a constant c_j such that

$$P_{h_j} = \Lambda_{1,2}^{h_j} + c_j$$

for any $1 \le j \le s$ (with $c_1 = 0$).

Consequently,

$$\begin{aligned} N D^*\left(\xi_{1,2}^1, \xi_{1,2}^2, \ldots, \xi_{1,2}^N\right) &\le \sum_{j=1}^{s+1} t_{h_j} D^*(P_{h_j}) \\ &= t_n D^*\left(\Lambda_{1,2}^n\right) + \sum_{j=2}^{s} t_{h_j} D^*\left(\Lambda_{1,2}^{h_j} + c_j\right) + r_0\, D^*(P_{r_0}) \\ &\le t_n D^*\left(\rho_{1,2}^n\right) + \sum_{j=2}^{s} t_{h_j} D^*(\rho_{1,2}^{h_j} + c_j) + 2. \end{aligned}$$

It is worth observing that the above construction can be easily extended to the cases $L = 1$ and $S > 2$, because of the identity $l_n = t_{n-1}$. Just to give the idea, when we write, at the beginning of the procedure, $N = t_n + R_n$, we have $0 < R_n < t_{n+1} - t_n = S\, t_{n-1}$, i.e. the remaining points lie in one of the S blocks of length $l_n = t_{n-1}$.

However, in the general case, the construction requires more attention, as we will see in the next proposition, even if the above two examples indicate the way for proving the main results of this paper.

Proposition 3.7 *Given the sequence of points $\left\{\xi_{L,S}^n\right\}$, for any $N \in \mathbb{N}$ such that $t_n \le N < t_{n+1}$ there exist $s, h_j, \delta_j, r_0 \in \mathbb{N}$ with $1 \le s < n$, $n = h_1 > h_2 > \cdots > h_s \ge 0$, $0 \le \delta_j < L + S - 1$ for every $1 \le j \le s$ and $0 \le r_0 < L + S - 1$ such that the following formal partition of N holds:*

$$N = \sum_{j=1}^{s} (t_{h_j} + \delta_j\, l_{h_j}) + r_0, \tag{26}$$

to which it corresponds the following partition of the set of the first N points of $\left\{\xi_{L,S}^n\right\}$:

$$\begin{aligned} \left(\xi_{L,S}^1, \xi_{L,S}^2, \ldots, \xi_{L,S}^N\right) = \Big(P_{h_1}, \tilde{P}_{h_1,1}, \ldots, \tilde{P}_{h_1,\delta_1}, \ldots, P_{h_s}, \\ \tilde{P}_{h_s,1}, \ldots, \tilde{P}_{h_1,\delta_s}, P_{r_0}\Big), \end{aligned} \tag{27}$$

where P_{h_j} and $\tilde{P}_{h_j,i}$ (for any $1 \le j \le s$ and $1 \le i \le \delta_j$) are suitable right shifts of $\Lambda_{L,S}^{h_i}$ and $\tilde{\Lambda}_{L,S}^{h_i}$, respectively, and P_{r_0} contains r_0 points.

Proof We prove (26) generalizing the arguments used in the proof of formula (25) in Example 3.6.2.

First, we note that if $N = t_n$, we have $\left(\xi_{L,S}^1, \xi_{L,S}^2, \ldots, \xi_{L,S}^N\right) = \Lambda_{L,S}^n$ and therefore, (26) and (27) hold true.

Suppose now $t_n < N < t_{n+1}$ and let us follow the scheme outlined in Example 3.6.

According to Definition 3.3 and Remark 3.5, we partition the points of $\Lambda_{L,S}^{n+1}$ as follows:

$$
\begin{aligned}
P_n &= \left(\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{t_n}^{(n)}\right) = \Lambda_{L,S}^n \\
\tilde{P}_{l_n,1} &= \left(\varphi_1^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_1^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_1^{(n+1)}\left(\xi_{l_n}^{(n)}\right)\right) = \tilde{\Lambda}_{L,S}^n + \beta^{n+1} \\
&\vdots \\
\tilde{P}_{l_n,L} &= \left(\varphi_L^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_L^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_L^{(n+1)}\left(\xi_{l_n}^{(n)}\right)\right) = \tilde{\Lambda}_{L,S}^n + L\beta^{n+1} \\
\tilde{P}_{l_n,L+1} &= \left(\varphi_{L,1}^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_{L,1}^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_{L,1}^{(n+1)}\left(\xi_{l_n}^{(n)}\right)\right) = \tilde{\Lambda}_{L,S}^n + L\beta^{n+1} + \beta^{n+2} \\
&\vdots \\
\tilde{P}_{l_n,L+S-1} &= \left(\varphi_{L,S-1}^{(n+1)}\left(\xi_1^{(n)}\right), \varphi_{L,S-1}^{(n+1)}\left(\xi_2^{(n)}\right), \ldots, \varphi_{L,S-1}^{(n+1)}\left(\xi_{l_n}^{(n)}\right)\right) \\
&= \tilde{\Lambda}_{L,S}^n + L\beta^{n+1} + (S-1)\beta^{n+2}.
\end{aligned}
$$

Taking the above notation into account (and according to the notation introduced in Example 3.6), for later convenience, we shall write

$$\Lambda_{L,S}^{n+1} = \left(P_n, \tilde{P}_{l_n,1}, \ldots, \tilde{P}_{l_n,L+1}, \tilde{P}_{l_n,L+1}, \ldots, \tilde{P}_{l_n,L+S-1}\right).$$

We shall prove that the analogous partition (27) into ordered sets of points also holds for the first N points of $\left\{\xi_{L,S}^n\right\}$.

Since $t_n < N < t_{n+1}$, it must be $N = t_n + R_n$, with $0 < R_n < t_{n+1} - t_n = (L+S-1)l_n$.

As regards the first t_n points, we have already noticed that $P_n = \Lambda_{L,S}^n$.

The remaining R_n points lie in one of the sets described above. Therefore, we can write

$$N = t_n + \delta_1 l_n + r_1$$

for some $0 \le \delta_1 < L + S - 1$ and $0 \le r_1 < l_n$ (of course $\delta_1 r_1 > 0$).

Now, we have the following possible situations.

(a) If $r_1 = 0$, we obtain

$$N = t_n + \delta_1 l_n$$

for some $0 < \delta_1 < L + S - 1$ and the following partition into $\delta_1 + 1$ sets of points

$$\left(\xi_{L,S}^1, \xi_{L,S}^2, \ldots, \xi_{L,S}^N\right) = \left(P_n, \tilde{P}_{n,1}, \ldots, \tilde{P}_{n,\delta_1}\right),$$

where

$$\tilde{P}_{n,j} = \tilde{\Lambda}_{L,S}^n + c_{n,j}$$

for any $1 \le j \le \delta_1$ and the constants $c_{n,j}$ (depending on β and n) are the linear combinations of β^{n+1} and β^{n+2} written above.

(b) If $r_1 \ne 0$, it is clear that δ_1 could also be 0 and, since $r_1 < l_n$ and $l_n = t_{n-1} + (L-1)l_{n-1}$, we have to consider the following three cases:

(b_1) $r_1 = t_{n-1}$,
(b_2) $t_{n-1} < r_1 < l_n$,
(b_3) $r_1 < t_{n-1}$.

(b_1) If $r_1 = t_{n-1}$, we have

$$N = t_n + \delta_1 \, l_n + t_{n-1}$$

and the following partition into $\delta_1 + 2$ sets of points

$$\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) = \left(P_n, \tilde{P}_{n,1}, \ldots, \tilde{P}_{n,\delta_1}, P_{n-1}\right),$$

where

$$P_{n-1} = \Lambda^{n-1}_{L,S} + c_{n-1}$$

and the constant c_{n-1} depends on β and n.

(b_2) If $t_{n-1} < r_1 < l_n$, there exist $0 \leq \delta_2 \leq L - 1$ and $0 \leq r_2 < l_{n-1}$ such that $r_1 = t_{n-1} + \delta_2 \, l_{n-1} + r_2$ (of course $\delta_2 \, r_2 > 0$, otherwise we fall into the case b_1).

If $r_2 = 0$, and therefore, $\delta_2 \neq 0$, we can write

$$N = t_n + \delta_1 \, l_n + t_{n-1} + \delta_2 \, l_{n-1}$$

and the corresponding set partition

$$\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) = \left(P_n, \tilde{P}_{n,1}, \ldots, \tilde{P}_{n,\delta_1}, P_{n-1}, \tilde{P}_{n-1,1}, \ldots, \tilde{P}_{n-1,\delta_2}\right),$$

where

$$\tilde{P}_{n-1,j} = \tilde{\Lambda}^{n-1}_{L,S} + c_{n-1,j}$$

for any $1 \leq j \leq \delta_2$, and the constant $c_{n-1,j}$ depends on β and n.

If $r_2 \neq 0$, since $l_{n-1} = t_{n-2} + (L - 1)l_{n-2}$, it is clear that if $t_{n-2} \leq r_2 < l_{n-1}$, we are back to the case b_1 and b_2; the case $r_2 < t_{n-2}$ can be treated as the next and last case.

(b_3) If $r_1 < t_{n-1}$, there exists $0 \leq q \leq n - 2$ such that $t_q \leq r_1 < t_{q+1}$, and this case is quite similar to the initial case $N = t_n + R_n$. In fact, there exists $0 \leq R_q < t_{q+1} - t_q = (L + S - 1)l_q$ such that $r_1 = t_q + R_q$ and, consequently,

$$N = t_n + \delta_1 \, l_n + t_q + R_q.$$

If $R_q = 0$, we have

$$N = t_n + \delta_1 \, l_n + t_q$$

and

$$\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) = \left(P_n, \tilde{P}_{n,1}, \ldots, \tilde{P}_{n,\delta_1}, P_q\right),$$

where

$$P_q = \Lambda^q_{L,S} + c_q,$$

with the constant c_q depending on β and n.

If $R_q \neq 0$ and $q \neq 0$, there exist $0 \leq \delta_{n-q+1} < L + S - 1$ and $0 \leq r_{n-q+1} < l_q$ such that $R_q = \delta_{n-q+1} \, l_q + r_{n-q+1}$. Therefore,

$$N = t_n + \delta_1 \, l_n + t_q + \delta_{n-q+1} \, l_q + r_{n-q+1}$$

and we continue the procedure until we have a residual set P_{r_0} of r_0 points, with $0 \leq r_0 < t_1 - t_0 = L + S - 1$.

The above discussion takes care of all the possible cases. We point out that the coefficients δ_j for any $2 \leq j \leq s$ are sometimes bounded by $L - 1$ and sometimes by $L + S - 1$, but it will be sufficient to consider the bound $\delta_j < L + S - 1$, that is good enough to get the conclusions, even if it is not the sharpest estimate.

Therefore, in the most general situation, we have the formal partition

$$N = t_n + \delta_1 \, l_n + \sum_{j=2}^{s} (t_{h_j} + \delta_j \, l_{h_j}) + r_0 = \sum_{j=1}^{s} (t_{h_j} + \delta_j \, l_{h_j}) + r_0,$$

where $n = h_1 > h_2 > \cdots > h_s \geq 0, 0 \leq \delta_j < L + S - 1$ for any $1 \leq j \leq s$ and $0 \leq r_0 < L + S - 1$.

To this formal partition, it corresponds the following partition into ordered sets of points

$$\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) = \Big(P_{h_1}, \tilde{P}_{h_1}, \ldots, \tilde{P}_{h_1,\delta_1}, P_{h_2}, \tilde{P}_{h_2}, \ldots, \tilde{P}_{h_2,\delta_2}, \ldots, P_{h_s}, \tilde{P}_{h_s,1}, \ldots, \tilde{P}_{h_s,\delta_s}, P_{r_0}\Big).$$

Moreover, all the above sets are of the kind

$$P_{h_j} = \Lambda^{h_j}_{L,S} + c_{h_j} \quad \text{and} \quad \tilde{P}_{h_j,i} = \tilde{\Lambda}^{h_j}_{L,S} + c_{h_j,i} \tag{28}$$

for any $1 \leq j \leq s$, $1 \leq i \leq h_j$ (with $c_{h_1} = c_n = 0$), P_{r_0} contains r_0 points and the constants c_{h_j} and $c_{h_j,i}$ depend on β and n.

Thus, the result is completely proved. □

In the proof of the main theorem, we need the following technical but elementary result.

Lemma 3.8 *If $X = \{x_n\}$ is a sequence of points in the interval $[0, 1[$, then for every $t > 0$ such that $X + t \subset [0, 1[$ we have*

$$D^*(X_N + t) \leq 2 \, D^*(X_N),$$

where $X_N = \{x_1, x_2, \ldots, x_N\}$.

Now we are ready to prove the main theorem of this paper.

Theorem 3.9 (i) *If $S < L + 1$ there exists $k_1 > 0$ such that for any $N \in \mathbb{N}$*

$$D\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) \leq k_1 \frac{\log N}{N}.$$

(ii) *If $S = L + 1$ there exists $k_2 > 0$ such that for any $N \in \mathbb{N}$*

$$D\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) \leq k_2 \frac{\log^2 N}{N}.$$

(iii) *If $S > L + 1$ there exists $k_3 > 0$ such that for any $N \in \mathbb{N}$*

$$D\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) \leq k_3 \frac{\log N}{N^\gamma},$$

where $\gamma = 1 + \frac{\log(S\beta)}{\log \beta} < 1$.

Proof If $N = t_n$ for some $n \geq 1$, we note that $D^*\left(\Lambda^n_{L,S}\right) = D^*\left(\rho^n_{L,S}\right)$ and therefore, the conclusion follows directly from Theorem 2.2.

Let us consider $t_n < N < t_{n+1}$ and let us utilize the same notation introduced in Proposition 3.7. From formulas (26–28) and Theorem 2.43, Chapter 2 of [8], it follows that

$$\begin{aligned} N D^*\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) &\leq \sum_{j=1}^{s}\left[t_{h_j} D^*(P_{h_j}) + \delta_j l_{h_j} D^*\left(\tilde{P}_{h_j,i}\right)\right] \\ &\leq t_n D^*(P_n) + \sum_{i=1}^{\delta_1} l_n D^*\left(\tilde{P}_{n,i}\right) \\ &\quad + \sum_{j=2}^{s}\left[t_{h_j}\, D^*(P_{h_j}) + \sum_{i=1}^{\delta_j} l_{h_j} D^*\left(\tilde{P}_{h_j,i}\right)\right] + r_0\, D^*(P_{r_0}) \\ &= t_n D^*\left(\Lambda^n_{L,S}\right) + \sum_{i=1}^{\delta_1} l_n D^*\left(\tilde{\Lambda}^n_{L,S} + c_{n,i}\right) \\ &\quad + \sum_{j=2}^{s}\left[t_{h_j}\, D^*\left(\Lambda^{h_j}_{L,S} + c_{h_j}\right) + \sum_{i=1}^{\delta_j} l_{h_j} D^*\left(\tilde{\Lambda}^{h_j}_{L,S} + c_{h_j,i}\right)\right] \\ &\quad + r_0\, D^*(P_{r_0}). \end{aligned}$$

Now, we apply Lemma 3.8 to $\Lambda^{h_j}_{L,S}$ and $\Lambda^{h_j}_{L,S}$ and we get

$$D^*\left(\Lambda^{h_j}_{L,S} + c_{h_j}\right) \leq 2\, D^*\left(\Lambda^{h_j}_{L,S}\right)$$

and

$$D^*\left(\tilde{\Lambda}^{h_j}_{L,S} + c_{h_j,i}\right) \leq 2\, D^*\left(\tilde{\Lambda}^{h_j}_{L,S}\right).$$

At this point, we recall that $D^*\left(\Lambda^{h_j}_{L,S}\right) = D^*\left(\rho^j_{L,S}\right)$ and $D^*\left(\tilde{\Lambda}^{h_j}_{L,S}\right) = D^*\left(\tilde{\rho}^j_{L,S}\right)$. Accordingly, we obtain the following estimate:

$$\begin{aligned} & N D^*\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) \\ &\leq t_n D^*\left(\rho^n_{L,S}\right) + \sum_{i=1}^{\delta_1} l_n D^*\left(\tilde{\rho}^n_{L,S}\right) + 2\sum_{j=2}^{s}\left[t_{h_j}\, D^*\left(\rho^j_{L,S}\right) + \sum_{i=1}^{\delta_j} l_{h_j} D^*\left(\tilde{\rho}^j_{L,S}\right)\right] + r_0 \\ &\leq 2\sum_{j=1}^{n}\left[t_j\, D^*\left(\rho^j_{L,S}\right) + \delta_j l_j\, D^*\left(\tilde{\rho}^i_{L,S}\right)\right] + r_0 \\ &\leq 2\sum_{j=1}^{n}\left[t_j\, D^*\left(\rho^j_{L,S}\right) + (L+S-2) l_j\, D^*\left(\tilde{\rho}^i_{L,S}\right)\right] + L + S - 2. \end{aligned}$$

It is now time to use Theorem 2.2 and Proposition 3.4 and to distinguish among the three cases presented in those results.

(i) If $S < L + 1$, from Theorem 2.2, (i) and Proposition 3.4, (i) we obtain that

$$N D^*\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) \leq 2n\left[c_2 + (L+S-2)\tilde{c}_2\right] + L + S - 2. \tag{29}$$

(ii) If $S = L+1$, from (2) and (3) it follows that $\log t_j \leq j \log 1/\beta$ and $\log l_j \leq j \log 1/\beta$, hence from Theorem 2.2, (ii) and Proposition 3.4, (ii) we get

$$\begin{aligned} &ND^*\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) \\ &\leq 2\sum_{j=1}^{n}\left[c_4 + (L+S-2)\tilde{c}_4\right] j \log 1/\beta + L + S - 2 \\ &= \left[c_4 + (L+S-2)\tilde{c}_4\right]\left(n \log 1/\beta + n^2 \log 1/\beta\right) + L + S - 2. \end{aligned} \tag{30}$$

(iii) If $S > L+1$, from Theorem 2.2, (iii) and Proposition 3.4, (iii) it follows that

$$\begin{aligned} &ND^*\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) \\ &\leq 2\sum_{j=1}^{n}\left[c_6 \frac{1}{t_j^{\gamma-1}} + \tilde{c}_6\,(L+S-2)\frac{1}{l_j^{\gamma-1}}\right] + L + S - 2 \\ &\leq 2\,n\,\frac{c_6 + \tilde{c}_6\,(L+S-2)}{N^{\gamma-1}} + L + S - 2. \end{aligned} \tag{31}$$

It remains only to express n in terms of N. Since $N \geq t_n$ we have

$$N \geq \frac{1+S\beta}{1+S\beta^2}\left(\frac{1}{\beta^n}\right) - \frac{S\beta - S\beta^2}{1+S\beta^2}(-S\beta)^n = \frac{A - B(-S\beta^2)^n}{\beta^n} \geq \frac{A-B}{\beta^n} = \frac{1}{\beta^n}.$$

This is equivalent to say that

$$n \leq \frac{\log N}{\log 1/\beta}.$$

Therefore, in the case $L < S+1$ from (29), we conclude that

$$\begin{aligned} &ND^*\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) \\ &\leq 2\,\frac{c_2 + (L+S-2)\tilde{c}_2}{\log 1/\beta} \log N + L + S - 2 \leq k_1 \log N, \end{aligned}$$

with a constant $k_1 > 0$ independent on n.

If $L = S+1$, because of (30) we have

$$\begin{aligned} &ND^*\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) \\ &\leq \left[c_4 + (L+S-2)\tilde{c}_4\right]\left(\log N + \frac{\log^2 N}{\log 1/\beta}\right) + L + S - 2 \leq k_2 \log^2 N, \end{aligned}$$

where $k_2 > 0$ is a constant independent on n.

If $L > S+1$, from (31) it follows that

$$\begin{aligned} &ND^*\left(\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^N_{L,S}\right) \\ &\leq 2\,\frac{\log N}{N^{\gamma-1}}\,\frac{c_6 + \tilde{c}_6\,(L+S-2)}{\log 1/\beta} + L + S - 2 \leq k_3\,\frac{\log N}{N^{\gamma-1}}, \end{aligned}$$

where $k_3 > 0$ is a constant independent on n. $\square$

Remark 3.10 All the LS-sequences of points for which $S < L+1$ have low discrepancy. One of them is the Kakutani-Fibonacci sequence for which, as we have seen in Example 3.6.1, the algorithm is particularly simple, since the only function in action is $\varphi_1^n(x) = x + \beta^n$ and (24) is, actually, a particular case of formula (26) with $r_0 = 0$ and $\delta_j = 0$ for every $1 \leq j \leq s$.

Acknowledgments The author wishes to express her gratitude to Aljoša Volčič for his critical reading of the manuscript and to the referee for helpful remarks.

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