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A note on biharmonic curves in Sasakian space forms

Dorel Fetcu

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Abstract We classify the proper-biharmonic non-Legendre curves in a Sasakian space form for which the angle between the tangent vector field and the characteristic vector field is constant and then obtain explicit examples of such curves in $\mathbb{R}^{2n+1}(-3)$.

Keywords Biharmonic curves · Sasakian space forms

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1 Introduction

In 1964, Eells and Sampson [10] introduced the notion of poly-harmonic maps as a natural generalization of the well-known harmonic maps. Thus, while *harmonic maps* between Riemannian manifolds $\phi : (M, g) \to (N, h)$ are critical points of the *energy functional* $E(\phi) = \frac{1}{2}\int_M |d\phi|^2 \ v_g$, the *biharmonic maps* are critical points of the *bienergy functional* $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \ v_g$.

On the other hand, Chen [7] defined the biharmonic submanifolds in the Euclidean space as those with harmonic mean curvature vector field. If we apply the characterization formula of biharmonic maps to Riemannian immersions into Euclidean spaces, we recover Chen's notion of biharmonic submanifold.

The Euler–Lagrange equation for the energy functional is $\tau(\phi) = 0$, where $\tau(\phi) = \operatorname{trace} \nabla d\phi$ is the tension field, and the Euler–Lagrange equation for the bienergy functional

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D. Fetcu (✉)
Department of Mathematics, "Gh. Asachi" Technical University of Iasi,
Bd. Carol I no. 11, 700506 Iasi, Romania
e-mail: dfetcu@math.tuiasi.ro; dorelfetcu@yahoo.com

was derived by Jiang in [16]:

$$\begin{aligned}\tau_2(\phi) &= -\Delta\tau(\phi) - \operatorname{trace}\ R^N(d\phi, \tau(\phi))d\phi\\ &= 0.\end{aligned}$$

Since any harmonic map is biharmonic, we are interested in non-harmonic biharmonic maps, which are called *proper-biharmonic*.

Several classification results and some methods to construct biharmonic submanifolds in space forms were obtained in the last few years (see, for example, [2,4,9] and [17]) and, in a natural way, the next step was the study of biharmonic submanifolds in Sasakian space forms. Thus, all proper-biharmonic Legendre curves and Hopf cylinders in a three-dimensional Sasakian space form were classified in [15], while their explicit parametric equations were found in [11]. Furthermore, a full classification of proper-biharmonic Legendre curves, explicit examples and a method to obtain proper-biharmonic anti-invariant submanifolds in any dimensional Sasakian space form were given in [12].

Recent results on biharmonic submanifolds in Sasakian space forms and in other spaces of non-constant sectional curvature were obtained, for example, in [5,6,8,13,18] and [21].

Biharmonic submanifolds in pseudo-Euclidean spaces were also studied, and many examples and classification results were obtained (see, for example, [1,7] and [14]).

Our paper is devoted to the study of proper-biharmonic non-Legendre curves in a Sasakian space form $N^{2n+1}(c)$. We obtain some classification results which show that such curves exist whatever the φ-sectional curvature c of N^{2n+1} is, while proper-biharmonic Legendre curves exist only for $c > -3$ when $n \geq 2$ or $c > 1$ when $n = 1$ (see [12] and [15]). We illustrate these results with explicit examples in $\mathbb{R}^{2n+1}(-3)$, which is the canonical model for the Sasakian space forms with φ-sectional curvature $c = -3$ (see [3]).

For a general account of biharmonic maps see [17] and [22].

Conventions We work in the C^∞ category, that means manifolds, metrics, connections and maps are smooth. The Lie algebra of the vector fields on M is denoted by $C(TM)$.

2 Preliminaries

In this section, we briefly recall basic things from the theory of Sasakian manifolds (see, for example, [3]), which we shall use throughout the paper.

A *contact metric structure* on an odd-dimensional manifold N^{2n+1} is given by (φ, ξ, η, g), where φ is a tensor field of type $(1, 1)$ on N, ξ is a vector field, η is a 1-form and g is a Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = d\eta(X, Y), \quad \forall X, Y \in C(TN).$$

A contact metric structure (φ, ξ, η, g) is called *normal* if

$$N_\varphi + 2d\eta \otimes \xi = 0,$$

where

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y], \quad \forall X, Y \in C(TN),$$

is the Nijenhuis tensor field of φ. A contact metric manifold $(N, \varphi, \xi, \eta, g)$ is a *Sasakian manifold* if it is normal or, equivalently, if

$$(\nabla_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X, \quad \forall X, Y \in C(TN).$$

The *contact distribution* of a Sasakian manifold $(N, \varphi, \xi, \eta, g)$ is defined by $\{X \in TN : \eta(X) = 0\}$, and an integral curve of the contact distribution is called a *Legendre curve*.

Let $(N, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The sectional curvature of a 2-plane generated by X and φX, where X is a unit vector orthogonal to ξ, is called *φ-sectional curvature* determined by X. A Sasakian manifold with constant φ-sectional curvature c is called a *Sasakian space form*, and it is denoted by $N(c)$. The curvature tensor field of a Sasakian space form $N(c)$ is given by

$$\begin{aligned} R(X, Y)Z = {} & \frac{c+3}{4}\{g(Z, Y)X - g(Z, X)Y\} + \frac{c-1}{4}\{\eta(Z)\eta(X)Y \\ & - \eta(Z)\eta(Y)X + g(Z, X)\eta(Y)\xi - g(Z, Y)\eta(X)\xi \\ & + g(Z, \varphi Y)\varphi X - g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z\}. \end{aligned} \tag{2.1}$$

3 Biharmonic non-Legendre curves in Sasakian space forms

We shall work with Frenet curves of osculating order r, parametrized by arc length, which we recall here (see [3]).

Definition 3.1 Let (N^m, g) be a Riemannian manifold and $\gamma : I \to N$ a curve parametrized by arc length, that is $|\gamma'| = 1$. Then, γ is called a *Frenet curve of osculating order* r, $1 \leq r \leq m$, if there exists orthonormal vector fields $E_1, E_2, \ldots, E_r$ along γ such that $E_1 = \gamma' = T$ and the following Frenet's equations hold

$$\nabla_T E_1 = \kappa_1 E_2, \quad \nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3, \ \ldots, \ \nabla_T E_r = -\kappa_{r-1} E_{r-1}, \tag{3.1}$$

where $\kappa_1, \ldots, \kappa_{r-1}$ are positive functions on I.

Remark 3.2 A Frenet curve of osculating order 1 is a geodesic; a Frenet curve of osculating order 2 with $\kappa_1 =$ constant is called a *circle*; a Frenet curve of osculating order r, $r \geq 3$, with $\kappa_1, \ldots, \kappa_{r-1}$ constants is called a *helix of order* r, and a helix of order 3 is simply called a *helix*.

Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian space form with constant φ-sectional curvature c and $\gamma : I \to N$ a non-Legendre Frenet curve of osculating order r with $\eta(T) = f$, where f is a function defined along γ and $f \neq 0$. Since

$$\nabla_T^3 T = (-3\kappa_1\kappa_1')E_1 + (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2)E_2 + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 + \kappa_1\kappa_2\kappa_3 E_4$$

and

$$\begin{aligned} R(T, \nabla_T T)T = {} & \left(-\frac{(c+3)\kappa_1}{4} + \frac{(c-1)\kappa_1}{4} f^2\right) E_2 - \frac{(c-1)}{4} ff'T + \frac{(c-1)}{4} f'\xi \\ & - \frac{3(c-1)\kappa_1}{4} g(E_2, \varphi T)\varphi T, \end{aligned}$$

we get the biharmonic equation of γ

$$\begin{aligned}\tau_2(\gamma) &= \nabla_T^3 T - R(T, \nabla_T T)T \\ &= \left(-3\kappa_1\kappa_1' + \frac{c-1}{4} f f'\right) E_1 + \left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + \frac{(c+3)\kappa_1}{4} - \frac{(c-1)\kappa_1}{4} f^2\right) E_2 \\ &\quad + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 + \kappa_1\kappa_2\kappa_3 E_4 - \frac{c-1}{4} f'\xi + \frac{3(c-1)\kappa_1}{4} g(E_2, \varphi T)\varphi T \\ &= 0. \end{aligned} \tag{3.2}$$

Now, if $c = 1$, the curve γ is proper-biharmonic if and only if

$$\kappa_1 = \text{constant} > 0, \quad \kappa_2 = \text{constant}, \quad \kappa_1^2 + \kappa_2^2 = 1 \quad \text{and} \quad \kappa_2\kappa_3 = 0,$$

and then we can state the following.

Theorem 3.3 *Let $N^{2n+1}(1)$ be a Sasakian space form with φ-sectional curvature $c = 1$ and $\gamma : I \to N$ a non-Legendre Frenet curve of osculating order r. Then, γ is proper-biharmonic if and only if either γ is a circle with $\kappa_1 = 1$, or γ is a helix with $\kappa_1^2 + \kappa_2^2 = 1$.*

Next, assume that $c \neq 1$. From the biharmonic equation (3.2), it follows, in this case, that the curve γ is proper-biharmonic if and only if $\kappa_1 > 0$ and

(1) $f' = 0$ or $\xi \in \text{span}\{E_1, E_2, E_3, E_4\}$ at any point of γ;
(2) $\varphi T \perp E_2$ or $\varphi T \in \text{span}\{E_2, E_3, E_4\}$;
(3) $g(\tau_2(\gamma), E_i) = 0$, for any $i = \overline{1, 4}$.

Computing $g(\tau_2(\gamma), E_i) = 0$ for all indices $i = \overline{1, 4}$, we obtain that γ is proper-biharmonic if and only if

(1) $f' = 0$ or $\xi \in \text{span}\{E_1, E_2, E_3, E_4\}$ at any point of γ;
(2) $\varphi T \perp E_2$ or $\varphi T \in \text{span}\{E_2, E_3, E_4\}$; and
(3)

$$\begin{cases} \kappa_1 = \text{constant} > 0, \\ \kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \frac{c-1}{4} f^2 - \frac{1}{\kappa_1^2}\frac{c-1}{4}(f')^2 + \frac{3(c-1)}{4}(g(E_2, \varphi T))^2 \\ \kappa_2' - \frac{1}{\kappa_1}\frac{c-1}{4} f'\eta(E_3) + \frac{3(c-1)}{4} g(E_2, \varphi T) g(E_3, \varphi T) = 0 \\ \kappa_2\kappa_3 - \frac{1}{\kappa_1}\frac{c-1}{4} f'\eta(E_4) + \frac{3(c-1)}{4} g(E_2, \varphi T) g(E_4, \varphi T) = 0. \end{cases} \tag{3.3}$$

We note that for obtaining the second equation, we used that $\eta(E_2) = g(E_2, \xi) = \frac{f'}{\kappa_1}$, which follows from $\eta(T) = g(T, \xi) = f$ and the first Frenet equation (3.1).

Since these equations are rather complicated, in the following, we will make some additional assumptions.

3.1 Biharmonic non-Legendre curves with $\eta(T) = \text{constant}$

Obviously, when the angle $\beta_1 \in (0, \pi)\setminus\left\{\frac{\pi}{2}\right\}$ between the tangent vector field T and the characteristic vector field ξ is constant, which means that $f = \eta(T) = \cos\beta_1$ is also a constant, the equations (3.3) become more handled. So, we shall study first this special case. We have

Theorem 3.4 *Let $N^{2n+1}(c)$ be a Sasakian space form with $c \neq 1$ and $\gamma : I \to N$ a non-Legendre Frenet curve of osculating order r such that $f = \eta(T) = \cos\beta_1 = \text{constant} \notin \{-1, 0, 1\}$. Then, γ is proper-biharmonic if and only if either*

(1) *γ is a circle with* $\varphi T = \pm \sin\beta_1 E_2$ *and* $\kappa_1^2 = 1 + (c-1)\sin^2\beta_1 > 0$, *or*
(2) *γ is a helix with* $\varphi T = \pm \sin\beta_1 E_2$ *and* $\kappa_1^2 + \kappa_2^2 = 1 + (c-1)\sin^2\beta_1 > 0$, *or*
(3) $n \geq 2$ *and γ is a Frenet curve of osculating order r, where $r \geq 4$, with*

$$\varphi T = \sin\beta_1 \cos\beta_2 E_2 + \sin\beta_1 \sin\beta_2 E_4$$

and

$$\begin{cases} \kappa_1 = \text{constant} > 0, \quad \kappa_2 = \text{constant} \\ \kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \frac{c-1}{4}\cos^2\beta_1 + \frac{3(c-1)}{4}\sin^2\beta_1\cos^2\beta_2 \\ \kappa_2\kappa_3 = -\frac{3(c-1)}{8}\sin^2\beta_1 \sin(2\beta_2) \end{cases}$$

where $\beta_2 \in (0, 2\pi)$ *is a constant such that*

$$c + 3 - (c-1)\cos^2\beta_1 + 3(c-1)\sin^2\beta_1\cos^2\beta_2 > 0 \quad \textit{and} \quad 3(c-1)\sin(2\beta_2) < 0.$$

Proof First, we see that $\eta(E_2) = g(E_2, \xi) = \frac{1}{\kappa_1} f' = 0$. Next, let us denote $g(E_2, \varphi T) = \alpha$, where α is a function defined along γ. Then, using the second Frenet equation (3.1), one obtains

$$\alpha' = g(\nabla_T E_2, \varphi T) + g(E_2, \nabla_T \varphi T) = \kappa_2 g(E_3, \varphi T) + g(E_2, \kappa_1 \varphi E_2 + \xi - fT)$$

and, since the second term in the right hand side vanishes, it follows that

$$\kappa_2 g(E_3, \varphi T) = \alpha'.$$

Now, assume that the curve γ is proper-biharmonic. By replacing the term $g(E_3, \varphi T)$ into the third equation of (3.3), we obtain

$$\kappa_2\kappa_2' + \frac{3(c-1)}{4}\alpha\alpha' = 0$$

and then

$$\kappa_2^2 + \frac{3(c-1)}{4}\alpha^2 + \omega_0 = 0,$$

where ω_0 is a constant. The second equation of (3.3) becomes

$$\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \frac{c-1}{4} f^2 - \kappa_2^2 - \omega_0,$$

which means that $\kappa_2 =$ constant and then $\alpha =$ constant. We also get

$$\kappa_2 g(E_3, \varphi T) = 0. \tag{3.4}$$

If $\kappa_2 = 0$ then, from the biharmonic equation (3.2), we get $E_2 \parallel \varphi T$ and, since $g(\varphi T, \varphi T) = 1 - f^2 = \sin^2\beta_1$, it follows $\varphi T = \pm\sin\beta_1 E_2$. Hence, γ is a circle with

$$\kappa_1^2 = \frac{c+3}{4} - \frac{c-1}{4}\cos^2\beta_1 + \frac{3(c-1)}{4}\sin^2\beta_1 = 1 + (c-1)\sin^2\beta_1.$$

Assume now that $\kappa_2 \neq 0$. Then, from (3.4), it results that $g(E_3, \varphi T) = 0$ and then, if $\kappa_3 = 0$, the curve γ is a helix with

$$\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \frac{c-1}{4}\cos^2\beta_1 + \frac{3(c-1)}{4}\sin^2\beta_1 = 1 + (c-1)\sin^2\beta_1,$$

since, using again the biharmonic equation (3.2), one obtains $E_2 \parallel \varphi T$ and then $\varphi T = \pm\sin\beta_1 E_2$ in this case too.

Next, suppose that $n \geq 2$, $\kappa_2 \neq 0$ and $\kappa_3 \neq 0$. Then, the osculating order of the curve γ is $r \geq 4$ and, from (3.4), we have $g(E_3, \varphi T) = 0$. From the biharmonic equation (3.2), it follows that $\varphi T \in \operatorname{span}\{E_2, E_4\}$. Since

$$g(\varphi T, \varphi T) = 1 - f^2 = \sin^2 \beta_1$$

it results

$$\varphi T = \sin \beta_1 \cos \beta_2 E_2 + \sin \beta_1 \sin \beta_2 E_4,$$

where

$$g(E_2, \varphi T) = \alpha = \sin \beta_1 \cos \beta_2 \quad \text{and} \quad g(E_4, \varphi T) = \sin \beta_1 \sin \beta_2,$$

with $\beta_2 = \text{constant} \in (0, 2\pi)$, and the conclusion follows from (3.3). □

Remark 3.5 If γ is a Frenet curve of osculating order $r > 1$, not necessarily biharmonic, with $\eta(T) = f = \text{constant}$, in a three-dimensional Sasakian space form, then we can consider an orthogonal system of vector fields $\{E = T - f\xi, \varphi T, \xi\}$ along γ and, using it, we easily get $E_2 \parallel \varphi T$, in this case.

3.2 Biharmonic non-Legendre curves with $E_2 \perp \varphi T$ or $E_2 \parallel \varphi T$

As a special role in the biharmonic equation (3.2) is played by the relation between E_2 and φT, it seems to be interesting to study the particular cases of the Frenet curves γ of osculating order r in a Sasakian space form $N^{2n+1}(c)$, $c \neq 1$, satisfying $E_2 \perp \varphi T$ or $E_2 \parallel \varphi T$ and with $\eta(T) = f(s) = \cos(\beta(s)) \neq 0$ not necessarily constant. However, we will prove that for such curves to be proper-biharmonic, we must have $\eta(T) = f = \text{constant}$.

Case I $\mathbf{c \neq 1}$, $\mathbf{E_2 \perp \varphi T}$.

In this case, γ is proper-biharmonic if and only if

(1) $f' = 0$ or $\xi \in \operatorname{span}\{E_1, E_2, E_3, E_4\}$ at any point of γ;
and
(2)

$$\begin{cases} \kappa_1 = \text{constant} > 0, \\ \kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \frac{c-1}{4} f^2 - \frac{1}{\kappa_1^2} \frac{c-1}{4} (f')^2 \\ \kappa_2' - \frac{1}{\kappa_1} \frac{c-1}{4} f' \eta(E_3) = 0 \\ \kappa_2 \kappa_3 - \frac{1}{\kappa_1} \frac{c-1}{4} f' \eta(E_4) = 0. \end{cases} \tag{3.5}$$

From $g(E_2, \xi) = \frac{1}{\kappa_1} f'$, one obtains $g(\nabla_T E_2, \xi) - g(E_2, \varphi T) = \frac{1}{\kappa_1} f''$ and then $\kappa_2 \eta(E_3) = \frac{1}{\kappa_1} f'' + \kappa_1 f$. By replacing into the third equation of (3.5), one obtains

$$\kappa_2 \kappa_2' - \frac{1}{\kappa_1^2} \frac{c-1}{4} (f' f'' + \kappa_1^2 f f') = 0,$$

and then

$$\kappa_2^2 - \frac{1}{\kappa_1^2} \frac{c-1}{4} ((f')^2 + \kappa_1^2 f^2) + \omega_1 = 0,$$

where ω_1 is a constant. Now, from the second equation of (3.5), we have

$$\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \kappa_2^2 - \omega_1.$$

Hence, $\kappa_2 = \text{constant}$ and $(f'' + \kappa_1^2 f) f' = 0$.

Next, using the Frenet equations (3.1), from $g(E_2, \varphi T) = 0$, one obtains

$$\kappa_2 g(E_3, \varphi T) = -\frac{1}{\kappa_1} f'$$

and then, from $\kappa_2 g(E_3, \xi) = \frac{1}{\kappa_1} f'' + \kappa_1 f$, we get

$$\kappa_2 \kappa_3 g(E_4, \xi) = \frac{1}{\kappa_1}(f''' + (\kappa_1^2 + \kappa_2^2) f').$$

Since $\tau_2(\gamma) = 0$ implies $\eta(\tau_2(\gamma)) = 0$, one obtains, after a straightforward computation, that $f' f''' = 0$. Using this result and differentiating $(f'' + \kappa_1^2 f) f' = 0$ along γ, we have

$$\kappa_1^2 (f')^2 + (f'' + \kappa_1^2 f) f'' = 0.$$

We have just obtained that $\eta(T) = f = \text{constant}$. Then, from the biharmonic equation (3.2), also using Remark 3.5, we can state

Theorem 3.6 *Let $N^{2n+1}(c)$ be a Sasakian space form with $c \neq 1$, $n \geq 2$, and $\gamma : I \to N$ a non-Legendre Frenet curve of osculating order r such that $E_2 \perp \varphi T$. Then γ is proper-biharmonic if and only if either*

(1) *γ is a circle with $\eta(T) = \cos\beta_0$ and $\kappa_1^2 = \frac{c+3}{4} - \frac{c-1}{4}\cos^2\beta_0$, or*
(2) *γ is a helix with $\eta(T) = \cos\beta_0$ and $\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} - \frac{c-1}{4}\cos^2\beta_0$, where $\beta_0 \in (0, 2\pi)\setminus\left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ is a constant such that $\frac{c+3}{4} - \frac{c-1}{4}\cos^2\beta_0 > 0$.*

Remark 3.7 We note that the biharmonic equation (3.2), of a curve γ as in Theorem 3.6, is equivalent to

$$\Delta H = \frac{1}{4}(c + 3 - (c-1)\cos^2\beta_0) H,$$

which means that H is an eigenvector of Δ, where $H = \nabla_T T = \kappa_1 E_2$ is the mean curvature vector field of γ.

Case II $\mathbf{c \neq 1}$, $\mathbf{E_2 \parallel \varphi T}$.

In this case, $g(E_2, \xi) = \frac{1}{\kappa_1} f' = 0$ and then $f = \cos\beta_0 = \text{constant}$. Since $g(\varphi T, \varphi T) = \sin^2\beta_0$, it follows $\varphi T = \pm\sin\beta_0 E_2$ and we have, using the biharmonic equation (3.2) of γ, the following

Proposition 3.8 *Let $N^{2n+1}(c)$ be a Sasakian space form with φ-sectional curvature $c \neq 1$ and $\gamma : I \to N$ a non-Legendre Frenet curve of osculating order r such that $E_2 \parallel \varphi T$. Then, γ is proper-biharmonic if and only if either*

(1) *γ is a circle with $\eta(T) = \cos\beta_0$ and $\kappa_1^2 = c - (c-1)\cos^2\beta_0$, or*
(2) *γ is a helix with $\eta(T) = \cos\beta_0$ and $\kappa_1^2 + \kappa_2^2 = c - (c-1)\cos^2\beta_0$, where $\beta_0 \in (0, 2\pi)\setminus\left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ is a constant such that $c - (c-1)\cos^2\beta_0 > 0$.*

Next, let γ be a proper-biharmonic non-Legendre curve with $E_2 \parallel \varphi T$. As $\varphi T = \pm \sin\beta_0 E_2$ one obtains, after a straightforward computation, that

$$\nabla_T E_2 = -\frac{1}{\sin\beta_0}\left(\frac{\kappa_1}{\sin\beta_0} \pm \cos\beta_0\right)T + \frac{1}{\sin\beta_0}\left(\frac{\kappa_1\cos\beta_0}{\sin\beta_0} \pm 1\right)\xi.$$

Using the second Frenet equation (3.1), we have

$$\kappa_2^2 = \frac{(\kappa_1\cos\beta_0 \pm \sin\beta_0)^2}{\sin^2\beta_0}.$$

Thus, γ is a circle if and only if $\kappa_1 = \mp\tan\beta_0 > 0$. From Proposition 3.8, we easily get that γ is a proper-biharmonic circle if and only if

$$\kappa_1^2 = \frac{c-1+\sqrt{c^2-2c+5}}{2} \quad \text{and} \quad \cos^2\beta_0 = \frac{c+1-\sqrt{c^2-2c+5}}{2(c-1)}.$$

If $\kappa_2 \neq 0$, from the expression of κ_2 and the third Frenet equation (3.1), it follows that $\kappa_3 = 0$ and then γ is a helix. Now, γ is proper-biharmonic if and only if κ_1 satisfies

$$\kappa_1^2 \pm \cos(2\beta_0)\kappa_1 + (1-c)\sin^4\beta_0 = 0$$

and $\beta_0 \in (0,2\pi)\backslash\left\{\frac{\pi}{2},\pi,\frac{3\pi}{2}\right\}$ if $c > 1$ or $\beta_0 \in (0,2\pi)\backslash\left\{\frac{\pi}{2},\pi,\frac{3\pi}{2}\right\}$ such that $\cos\beta_0 \in \left(-\sqrt{\frac{c-1}{c-2}}, \sqrt{\frac{c-1}{c-2}}\right)$ if $c < 1$.

We conclude with the following

Theorem 3.9 *Let $N^{2n+1}(c)$ be a Sasakian space form with φ-sectional curvature $c \neq 1$ and $\gamma : I \to N$ a non-Legendre Frenet curve of osculating order r such that $E_2 \parallel \varphi T$. Then, γ is proper-biharmonic if and only if either*

(1) *γ is a circle with $\eta(T) = \pm\sqrt{\frac{c+1-\sqrt{c^2-2c+5}}{2(c-1)}}$ and $\kappa_1^2 = \frac{c-1+\sqrt{c^2-2c+5}}{2}$, or*
(2) *γ is a helix with $\eta(T) = \cos\beta_0$ and κ_1 satisfies*

$$\kappa_1^2 \pm \cos(2\beta_0)\kappa_1 + (1-c)\sin^4\beta_0 = 0,$$

where $\beta_0 = \text{constant} \in (0,2\pi)\backslash\left\{\frac{\pi}{2},\pi,\frac{3\pi}{2}\right\}$ if $c > 1$ or $\beta_0 = \text{constant} \in (0,2\pi)\backslash\left\{\frac{\pi}{2},\pi,\frac{3\pi}{2}\right\}$ such that $\cos\beta_0 \in \left(-\sqrt{\frac{c-1}{c-2}}, \sqrt{\frac{c-1}{c-2}}\right)$ if $c < 1$. In this case, $\kappa_2^2 = (\kappa_1\cot\beta_0 \pm 1)^2$.

Remark 3.10 The biharmonic equation $\tau_2(\gamma) = 0$ for a curve γ as in Theorem 3.9 is equivalent to

$$\Delta H = (c - (c-1)\cos^2\beta_0)H,$$

where H is the mean curvature vector field of γ.

4 Biharmonic curves in $\mathbb{R}^{2n+1}(-3)$

While proper-biharmonic Legendre curves exist only in Sasakian space forms $N^{2n+1}(c)$ with constant φ-sectional curvature $c > 1$ if $n = 1$ or $c > -3$ if $n > 1$ (see [12,15]), proper-biharmonic non-Legendre curves can be found in Sasakian space forms with any φ-sectional curvature.

We mention that, in the case when $c = -3$, Sasahara studied in [19] the submanifolds in the Sasakian space form $\mathbb{R}^{2n+1}(-3)$ whose φ-mean curvature vectors are eigenvectors of the Laplacian and in [20] the Legendre surfaces in $\mathbb{R}^5(-3)$ for which mean curvature vectors field are eigenvectors of the Laplacian.

In this section, we obtain the explicit equations for proper-biharmonic circles with $E_2 \perp \varphi T$ and for all proper-biharmonic curves with $E_2 \parallel \varphi T$ in $\mathbb{R}^{2n+1}(-3)$.

First, let us briefly recall some notions and results about the structure of the Sasakian space form $\mathbb{R}^{2n+1}(-3)$ as they are presented in [3].

Consider on $\mathbb{R}^{2n+1}(-3)$, with elements of the form $(x^1, \dots, x^n, y^1, \dots, y^n, z)$, its standard contact structure defined by the 1-form $\eta = \frac{1}{2}(dz - \sum_{i=1}^n y^i dx^i)$, the characteristic vector field $\xi = 2\frac{\partial}{\partial z}$ and the tensor field φ given by the matrix

$$\begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}.$$

Then $g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^n ((dx^i)^2 + (dy^i)^2)$ is an associated Riemannian metric and $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$ is a Sasakian space form with constant φ-sectional curvature equal to -3, denoted $\mathbb{R}^{2n+1}(-3)$.

The vector fields $X_i = 2\frac{\partial}{\partial y^i}$, $X_{n+i} = \varphi X_i = 2(\frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z})$, $i = \overline{1, n}$ and $\xi = 2\frac{\partial}{\partial z}$ form an orthonormal basis in $\mathbb{R}^{2n+1}(-3)$ and, after some straightforward computations, one obtains

$$[X_i, X_j] = [X_{n+i}, X_{n+j}] = [X_i, \xi] = [X_{n+i}, \xi] = 0, \quad [X_i, X_{n+j}] = 2\delta_{ij}\xi$$

and

$$\nabla_{X_i} X_j = \nabla_{X_{n+i}} X_{n+j} = 0, \quad \nabla_{X_i} X_{n+j} = \delta_{ij}\xi, \quad \nabla_{X_{n+i}} X_j = -\delta_{ij}\xi,$$
$$\nabla_{X_i}\xi = \nabla_\xi X_i = -X_{n+i}, \quad \nabla_{X_{n+i}}\xi = \nabla_\xi X_{n+i} = X_i$$

for any $i, j = \overline{1, n}$.

Now, let $\gamma : I \to \mathbb{R}^{2n+1}(-3)$ be a Frenet curve of osculating order $r > 1$, parametrized by arc length, with the tangent vector field $T = \gamma'$ given by

$$T = \sum_{i=1}^n (T_i X_i + T_{n+i} X_{n+i}) + \cos\beta_0 \xi, \tag{4.1}$$

where $\cos\beta_0$ is a constant. Using the above formulas for the Levi-Civita connection, we have

$$\nabla_T T = \sum_{i=1}^n ((T_i' + 2\cos\beta_0 T_{n+i}) X_i + (T_{n+i}' - 2\cos\beta_0 T_i) X_{n+i}). \tag{4.2}$$

From Theorems 3.6 and 3.9, using the same techniques as in [5,6] and [8], we get

Theorem 4.1 *The parametric equations of proper-biharmonic circles parametrized by arc length in* $\mathbb{R}^{2n+1}(-3)$*,* $n \geq 2$*, with* $E_2 \perp \varphi T$*, are*

$$\begin{cases} x^i(s) = \pm\frac{1}{\kappa_1}\{2\sin(\kappa_1 s)c_1^i \mp 2\cos(\kappa_1 s)c_2^i - \cos(2\kappa_1 s)d_1^i - \sin(2\kappa_1 s)d_2^i\} + a^i \\ y^i(s) = \frac{1}{\kappa_1}\{2\cos(\kappa_1 s)c_1^i \pm 2\sin(\kappa_1 s)c_2^i + \sin(2\kappa_1 s)d_1^i - \cos(2\kappa_1 s)d_2^i\} + b^i \\ z(s) = \pm\frac{2}{\kappa_1}\{1 + \sum_{i=1}^n((c_1^i)^2 + (c_2^i)^2)\}s \\ \quad + \frac{1}{2\kappa_1^2}\sum_{i=1}^n\{\pm\cos(4\kappa_1 s)d_1^i d_2^i - 2\cos(2\kappa_1 s)c_1^i c_2^i \\ \quad + 4\cos(3\kappa_1 s)c_2^i d_2^i - 4\sin(3\kappa_1 s)c_1^i d_2^i\} \\ \quad \mp\frac{1}{\kappa_1}\sum_{i=1}^n b^i\{-2\sin(\kappa_1 s)c_1^i \pm 2\cos(\kappa_1 s)c_2^i \\ \quad + \cos(2\kappa_1 s)d_1^i + \sin(2\kappa_1 s)d_2^i\} + e \end{cases} \tag{4.3}$$

where $\kappa_1^2 = \cos^2\beta_0$, $\beta_0 \in (0, 2\pi)\backslash\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ *is a constant and* a^i, b^i, c_1^i, c_2^i, d_1^i, d_2^i *and* e *are constants such that the* n*-dimensional constant vectors* $c_j = (c_j^1, \dots, c_j^n)$ *and* $d_j = (d_j^1, \dots, d_j^n)$, $j = \overline{1, 2}$, *satisfy*

$$\begin{cases} |c_1|^2 + |c_2|^2 + |d_1|^2 + |d_2|^2 = \sin^2\beta_0 \\ \langle c_1, d_1\rangle \pm \langle c_2, d_2\rangle = 0, \quad \langle c_1, d_2\rangle \mp \langle c_2, d_1\rangle = 0. \end{cases}$$

Proof Let $\gamma : I \to \mathbb{R}^{2n+1}(-3)$ be a circle parametrized by arc length, with the tangent vector field $T = \gamma'$ given by (4.1) and $E_2 \perp \varphi T$. From the equation (4.2), one obtains

$$E_2 = \frac{1}{\kappa_1}\sum_{i=1}^n((T_i' + 2\cos\beta_0 T_{n+i})X_i + (T_{n+i}' - 2\cos\beta_0 T_i)X_{n+i})$$

and, using $g(E_2, \varphi T) = 0$, a direct computation shows that

$$\nabla_T E_2 = \frac{1}{\kappa_1}\left(\sum_{i=1}^n((T_i' + 2\cos\beta_0 T_{n+i})' + (T_{n+i}' - 2\cos\beta_0 T_i)\cos\beta_0)X_i \right. \\ \left. + ((T_{n+i}' - 2\cos\beta_0 T_i)' - (T_i' + 2\cos\beta_0 T_{n+i})\cos\beta_0)X_{n+i}\right)$$

and, since γ is a circle, it follows

$$\begin{cases} A_i' + B_i\cos\beta_0 = 0 \\ B_i' - A_i\cos\beta_0 = 0 \end{cases} \tag{4.4}$$

where $A_i = \frac{1}{\kappa_1}(T_i' + 2\cos\beta_0 T_{n+i})$ and $B_i = \frac{1}{\kappa_1}(T_{n+i}' - 2\cos\beta_0 T_i)$.

Solving system (4.4) and imposing for γ to be proper-biharmonic (according to Theorem 3.6, this means $\kappa_1 = \pm\cos\beta_0 > 0$), we get the following equations

$$\begin{cases} T_i' \pm 2\kappa_1 T_{n+i} = \kappa_1\cos(\kappa_1 s)c_1^i \pm \kappa_1\sin(\kappa_1 s)c_2^i \\ T_{n+i}' \mp 2\kappa_1 T_i = \pm\kappa_1\sin(\kappa_1 s)c_1^i - \kappa_1\cos(\kappa_1 s)c_2^i \end{cases}$$

whose general solutions are

$$\begin{cases} T_i = -\sin(\kappa_1 s)c_1^i \pm \cos(\kappa_1 s)c_2^i + \cos(2\kappa_1 s)d_1^i + \sin(2\kappa_1 s)d_2^i \\ T_{n+i} = \pm\cos(\kappa_1 s)c_1^i + \sin(\kappa_1 s)c_2^i \pm \sin(2\kappa_1 s)d_1^i \mp \cos(2\kappa_1 s)d_2^i \end{cases}$$

where c_1^i, c_2^i, d_1^i and d_2^i are constants, such that

$$\begin{cases} \sum_{i=1}^n ((c_1^i)^2 + (c_2^i)^2 + (d_1^i)^2 + (d_2^i)^2) = \sin^2\beta_0 \\ \sum_{i=1}^n ((c_1^i)(d_1^i) \pm (c_2^i)(d_2^i)) = 0 \\ \sum_{i=1}^n ((c_1^i)(d_2^i) \mp (c_2^i)(d_1^i)) = 0 \end{cases}$$

since $g(T, T) = 1$.

Finally, replacing into expression of γ' and integrating, we get (4.3). □

Remark 4.2 In order to find explicit examples of proper-biharmonic curves with $E_2 \perp \varphi T$ in $\mathbb{R}^{2n+1}(-3)$, we restrict our study only at proper-biharmonic circles since the computations in the case of helices are rather complicated.

Now, concerning the proper-biharmonic curves in $\mathbb{R}^{2n+1}(-3)$ with $E_2 \parallel \varphi T$ we can state

Theorem 4.3 *Proper-biharmonic curves parametrized by arc length in $\mathbb{R}^{2n+1}(-3)$ with $E_2 \parallel \varphi T$, are either*

(1) *proper-biharmonic circles given by*

$$\begin{cases} x^i(s) = (\sqrt{5}+1)\left(\cos\left(\frac{\sqrt{5}-1}{2}s\right)c_1^i + \sin\left(\frac{\sqrt{5}-1}{2}s\right)c_2^i\right) + a^i \\ y^i(s) = (\sqrt{5}+1)\left(\sin\left(\frac{\sqrt{5}-1}{2}s\right)c_1^i - \cos\left(\frac{\sqrt{5}-1}{2}s\right)c_2^i\right) + b^i \\ z(s) = \frac{1-\sqrt{5}\pm 2\sqrt{1+\sqrt{5}}}{2}s + \frac{3+\sqrt{5}}{2}\sum_{i=1}^n \left\{\left(\left(c_1^i\right)^2 - \left(c_2^i\right)^2\right)\sin((\sqrt{5}-1)s)\right. \\ \quad \left. -2\cos\left(\left(\sqrt{5}-1\right)s\right)c_1^i c_2^i\right\} + (1+\sqrt{5})\sum_{i=1}^n b_i\left\{\sin\left(\frac{\sqrt{5}-1}{2}s\right)c_2^i\right. \\ \quad \left.+\cos\left(\frac{\sqrt{5}-1}{2}s\right)c_1^i\right\} + d \end{cases} \tag{4.5}$$

where a^i, b^i, c_1^i, c_2^i and d are constants, and the n-dimensional constant vectors $c_j = (c_j^1, \ldots, c_j^n)$, $j = \overline{1,2}$, satisfy $|c_1|^2 + |c_2|^2 = \frac{3-\sqrt{5}}{4}$,
or

(2) *proper-biharmonic helices given by*

$$\begin{cases} x^i(s) = -\frac{2\kappa_1}{\kappa_1 \pm \sin(2\beta_0)}\left(\cos\left(\frac{\kappa_1\pm\sin(2\beta_0)}{\kappa_1}s\right)c_1^i + \sin\left(\frac{\kappa_1\pm\sin(2\beta_0)}{\kappa_1}s\right)c_2^i\right) + a^i \\ y^i(s) = \frac{2\kappa_1}{\kappa_1 \pm \sin(2\beta_0)}\left(\sin\left(\frac{\kappa_1\pm\sin(2\beta_0)}{\kappa_1}s\right)c_1^i - \cos\left(\frac{\kappa_1\pm\sin(2\beta_0)}{\kappa_1}s\right)c_2^i\right) + b^i \\ z(s) = 2\left(\cos\beta_0 + \frac{\kappa_1\sin^2\beta_0}{\kappa_1\pm\sin(2\beta_0)}\right)s + \frac{\kappa_1^2}{(\kappa_1\pm\sin(2\beta_0))^2} \\ \quad \cdot\left\{\sin\left(\frac{2(\kappa_1\pm\sin(2\beta_0))}{\kappa_1}s\right)\sum_{i=1}^n((c_1^i)^2 - (c_2^i)^2)\right. \\ \quad \left.+\cos\left(\frac{2(\kappa_1\pm\sin(2\beta_0))}{\kappa_1}s\right)\sum_{i=1}^n(c_1^i c_2^i)\right\} \\ \quad -\frac{2\kappa_1}{\kappa_1\pm\sin(2\beta_0)}\sum_{i=1}^n b^i\left\{\cos\left(\frac{\kappa_1\pm\sin(2\beta_0)}{\kappa_1}s\right)c_1^i + \sin\left(\frac{\kappa_1\pm\sin(2\beta_0)}{\kappa_1}s\right)c_2^i\right\} + d \end{cases} \tag{4.6}$$

where $\beta_0 \in (0, 2\pi)\setminus\left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ is a constant such that $\cos\beta_0 \in \left(-1, -\frac{2\sqrt{5}}{5}\right) \cup \left(\frac{2\sqrt{5}}{5}, 1\right)$, κ_1 is a positive solution of the equation

$$\kappa_1^2 \pm \sin(2\beta_0)\kappa_1 + 4\sin^4\beta_0 = 0,$$

a^i, b^i, c_1^i, c_2^i and d are constants, and the n-dimensional constant vectors $c_j = (c_j^1, \ldots, c_j^n)$, $j = \overline{1,2}$, satisfy $|c_1|^2 + |c_2|^2 = \sin^2\beta_0$.

Proof We have to prove only the first statement because the second one can be obtained in a similar way by the meaning of Theorem 3.9.

Assume that γ is a proper-biharmonic circle in $\mathbb{R}^{2n+1}(-3)$ parametrized by arc length, such that $E_2 \parallel \varphi T$. Then, from (4.2) and since $\varphi T = \sum_{i=1}^{n}(-T_{n+i}X_i + T_i X_{n+i})$ and $g(\varphi T, \varphi T) = \sin^2\beta_0$, where $\eta(T) = \cos\beta_0$, one obtains

$$T_i' = \left(\mp\frac{\sin(2\beta_0)}{\kappa_1} - 1\right)T_{n+i}, \quad T_{n+i}' = \left(\pm\frac{\sin(2\beta_0)}{\kappa_1} + 1\right)T_i.$$

Now, since γ is a proper-biharmonic circle we get, from Theorem 3.9, $\kappa_1 = \mp\tan\beta_0 > 0$ and $\cos^2\beta_0 = \frac{1+\sqrt{5}}{4}$, and then the above equations become

$$T_i' = \frac{\sqrt{5}-1}{2}T_{n+i}, \quad T_{n+i}' = \frac{1-\sqrt{5}}{2}T_i,$$

whose general solutions are

$$T_i = \cos\left(\frac{\sqrt{5}-1}{2}\right)c_1^i + \sin\left(\frac{\sqrt{5}-1}{2}\right)c_2^i, \quad T_{n+i} = \cos\left(\frac{\sqrt{5}-1}{2}\right)c_2^i - \sin\left(\frac{\sqrt{5}-1}{2}\right)c_1^i,$$

where c_1^i and c_2^i, $i = \overline{1,n}$, are constants.

By replacing in the expression of $T = \gamma'$, integrating and imposing $g(T, T) = 1$, we come to the conclusion. □

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