Annali di Matematica (2009) 188:171–185
DOI 10.1007/s10231-008-0070-9

Supercritical biharmonic equations with power-type nonlinearity

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Received: 16 November 2007 / Revised: 14 January 2008 / Published online: 26 March 2008

Abstract We study two different versions of a supercritical biharmonic equation with a power-type nonlinearity. First, we focus on the equation $\Delta^2 u = |u|^{p-1}u$ over the whole space $\mathbb{R}^n$, where $n > 4$ and $p > (n+4)/(n-4)$. Assuming that $p < p_c$, where p_c is a further critical exponent, we show that all *regular* radial solutions oscillate around an explicit *singular* radial solution. As it was already known, on the other hand, no such oscillations occur in the remaining case $p \geq p_c$. We also study the Dirichlet problem for the equation $\Delta^2 u = \lambda(1+u)^p$ over the unit ball in $\mathbb{R}^n$, where $\lambda > 0$ is an eigenvalue parameter, while $n > 4$ and $p > (n+4)/(n-4)$ as before. When it comes to the extremal solution associated to this eigenvalue problem, we show that it is *regular* as long as $p < p_c$. Finally, we show that a singular solution exists for some appropriate $\lambda > 0$.

Keywords Supercritical biharmonic equation · Power-type nonlinearity · Singular solution · Oscillatory behavior · Boundedness · Extremal solution

Mathematics Subject Classification (2000) 35J60 · 35B40 · 35J30 · 35J65

1 Introduction and main results

A lot of research on elliptic reaction diffusion equations

$$-\Delta u = f(u)$$

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of second order has been done and many beautiful and important results have been proved, where it is impossible to report upon here. However, in the survey article by P.L. Lions on this subject the question is raised (see [13, Sect. 4.2(c)]) in how far these results may be generalized to systems of such equations. Accordingly, as a special case one should investigate polyharmonic reaction diffusion equations

$$(-\Delta)^m u = f(u),$$

where in the present paper we consider the biharmonic case $m = 2$ and polynomial nonlinearities. In the first part we study qualitative properties of positive entire radial solutions (defined and regular in the whole space) of

$$\Delta^2 u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n, \tag{1}$$

where at least $p > 1$ is assumed. It is well known that, if $n \geq 5$, the exponent $(n+4)/(n-4)$ plays a critical role. In the subcritical range $p \in (1, (n+4)/(n-4))$ positive entire radial solutions to (1) do not exist, see e.g. the testing-function method by Mitidieri and Pohožaev [16]. Also the case of critical growth $p = (n+4)/(n-4)$ is somehow special, see e.g. [20] and the references therein. Here, we assume supercritical growth, i.e.

$$n \geq 5 \quad \text{and} \quad p > \frac{n+4}{n-4}. \tag{2}$$

In this case, an important role is played by the explicitly known singular solution

$$u_s(r) = K_0^{1/(p-1)} r^{-4/(p-1)}, \tag{3}$$

where

$$K_0 = \frac{4}{p-1}\left(\frac{4}{p-1}+2\right)\left(n-2-\frac{4}{p-1}\right)\left(n-4-\frac{4}{p-1}\right). \tag{4}$$

It was shown in [5,19] that positive regular entire solutions to (1) exist and in [9] that asymptotically they behave like the singular solution u_{s}:

$$\lim_{r\to\infty} \frac{u(r)}{u_{\mathrm{s}}(r)} = 1.$$

In the analogous second-order problem (i.e. $m = 1$)

$$-\Delta u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n, \tag{5}$$

where critical growth is given by the exponent $(n+2)/(n-2)$, such a result is well known and goes back to fundamental work of Joseph-Lundgren [11] and Gidas-Spruck [10], see also the references therein and [17] for subsequent work. The Eq. (5) admits the singular solution $u_0(x) = C_0|x|^{-2/(p-1)}$ for some positive constant C_0. Here, more detailed information about the convergence $\frac{u(r)}{u_0(r)} \to 1$ could be obtained. In [11,21] it is proved that there exists a further critical exponent $p_\# > \frac{n+2}{n-2}$ such that if $\frac{n+2}{n-2} < p < p_\#$ then each radial entire solution u intersects u_0 infinitely many times for $r \to \infty$, while if $p \geq p_\#$ then no radial entire solution intersects u_0. Further interesting qualitative properties are proved in [8,18], where as in [21] the focus is on the corresponding parabolic problem. All these results are strongly based on the use of maximum principles, which are in general not available for higher order equations. So, it is an interesting question whether the *results* carry over to biharmonic problems while suitable new *methods* have to be found. A first step in this direction was done in [9] where, for $n > 12$ a further critical exponent $p_c \in \left(\frac{n+4}{n-4}, \infty\right)$ was introduced being in that interval the unique solution of the following polynomial equation:

$$p_c \cdot \frac{4}{p_c - 1} \cdot \left(\frac{4}{p_c - 1} + 2\right) \cdot \left(n - 2 - \frac{4}{p_c - 1}\right) \cdot \left(n - 4 - \frac{4}{p_c - 1}\right) = \frac{n^2(n-4)^2}{16}. \quad (6)$$

The third author [12] proved in particular that in the "supercritical case", i.e.

$$p \geq p_{\mathrm{c}}$$

the convergence of $\frac{u}{u_{\mathrm{s}}} \to 1$ is monotone, i.e. $\forall r:\ u(r) < u_s(r)$. Here, we study the reverse case:

Theorem 1 *Let $p_c \in ((n+4)/(n-4), \infty)$ be the number, which is defined by (6) for $n \geq 13$. We assume that*

$$\frac{n+4}{n-4} < p < p_{\mathrm{c}} \quad \textit{if}\, n \geq 13, \quad \frac{n+4}{n-4} < p < \infty \quad \textit{if}\, 5 \leq n \leq 12.$$

Let $r \mapsto u(r)$ be a positive radial entire solution to (1). Then, as $r \to \infty$, $u(r)$ oscillates infinitely many times around the singular solution $u_{\mathrm{s}}(r)$.

Together with the result of [12] this provides a biharmonic analogue of [21, Proposition 3.7].

The second part of the present paper is devoted to positive solutions of the corresponding Dirichlet problem

$$\begin{cases} \Delta^2 u = \lambda(1+u)^p & \text{in } B, \\ u > 0 & \text{in } B, \\ u = |\nabla u| = 0 & \text{on } \partial B, \end{cases} \quad (7)$$

where $B \subset \mathbb{R}^n$ is the unit ball, $\lambda > 0$ is an eigenvalue parameter and again $n \geq 5$ and $p > \frac{n+4}{n-4}$. In [7] (see also [2]) it was proved that there exists an extremal parameter λ^* such that for $\lambda \in [0, \lambda^*)$ one has a minimal solution which is regular, while not even a weak solution does exist for $\lambda > \lambda^*$. On the extremal parameter $\lambda = \lambda^*$, an extremal solution $u^* \in H_0^2(B) \cap L^p(B)$ exists as monotone limit of the minimal solutions.

For the corresponding second order problem, such results can be found in [3,4,10,11,15, 21]. In that case, however, the starting point was an explicit singular solution for a suitable eigenvalue parameter λ which turned out to play a fundamental role for the shape of the corresponding bifurcation diagram, see in particular [4]. When turning to the biharmonic problem (7) the second boundary condition $|\nabla u| = 0$ prevents to find an explicit singular solution from the singular solution (3) to (1). It is expected that a singular (i.e. unbounded) solution u_σ for a suitable parameter λ_σ will exist also in the biharmonic Dirichlet problem (7) and that it will play an important role as far as the shape of the bifurcation diagram for (7) is concerned. However, in [7] we had to leave open even the existence of a singular solution which will be proved in the present paper:

Theorem 2 *Let $n > 4$ and $p > (n+4)/(n-4)$. Then, there exists a parameter $\lambda_\sigma > 0$ such that for $\lambda = \lambda_\sigma$, problem (7) admits a radial singular solution.*

Moreover, in [7] we left open whether the extremal solution u^* introduced above is singular (unbounded) or regular (bounded). The corresponding question has been settled for the exponential nonlinearity by Dávila et al. [6] thereby developing the previous work [1]. Here, taking advantage of an idea in [6], we prove regularity of the extremal solution of the problem with power-type nonlinearity in the "subcritical" range.

Theorem 3 *Let $p_c \in ((n+4)/(n-4), \infty)$ be the number, which is defined by (6) for $n \geq 13$. We assume that*

$$\frac{n+4}{n-4} < p < p_c \quad \text{if } n \geq 13, \quad \frac{n+4}{n-4} < p < \infty \quad \text{if } 5 \leq n \leq 12.$$

Let $u^* \in H_0^2(B) \cap L^p(B)$ be the extremal radial solution of (7) corresponding to the extremal parameter λ^*, which is obtained as monotone limit of the minimal regular solutions for $\lambda \nearrow \lambda^*$. Then, u^* is regular.

The proof of Theorem 1 is given in the following section, while Theorems 2 and 3 are proved in Sect. 3.

2 Entire solutions: the corresponding autonomous system

In this section, we give the proof of Theorem 1. When studying radial solutions to (1) a basic idea is to transform (1) into an autonomous system, where the entire singular solution transforms into an equilibrium point. While this "singular" equilibrium point is stable in the second-order situation (see [11,15]) it is hyperbolic in the biharmonic case with a three-dimensional stable and a one-dimensional unstable manifold [9]. In order to study possible oscillatory properties of entire regular solutions to (1), this stable manifold has to be analyzed more closely. In this section we will prove that with regard to entire regular solutions one may reduce to a two-dimensional submanifold of the stable manifold. Crucial for this property is a backward in time invariance property of a suitable cone in the phase space, i.e. a sort of comparison principle for a suitably written associated autonomous system, see the proof of Proposition 1. It was already observed before (see e.g. [14]) that also higher order dynamical systems may obey a certain form of a comparison principle.

Roughly speaking, this means that the dynamical properties of entire regular solutions to the biharmonic equation (1) are analogous to the second-order situation and that the additional directions around the "singular" equilibrium point may be "ignored". We remark that the "subcriticality" assumption $p < p_c$ is not needed to prove Proposition 1.

As in [9] we set

$$v(s) := e^{4s/(p-1)} u(e^s) \quad (s \in \mathbb{R}), \quad u(r) = r^{-4/(p-1)} v(\log r) \quad (r > 0). \tag{8}$$

According to [9,12], Eq. (1) is then equivalent to

$$\left(\partial_s - \frac{4}{p-1} + n - 4\right)\left(\partial_s - \frac{4}{p-1} + n - 2\right)\left(\partial_s - \frac{4}{p-1} - 2\right)\left(\partial_s - \frac{4}{p-1}\right) v(s) \\ = |v(s)|^{p-1} v(s), \tag{9}$$

where $s \in \mathbb{R}$. In order to write this as an autonomous system, we define

$$\begin{cases} w_1(s) = v(s) \\ w_2(s) = \left(\partial_s - \frac{4}{p-1}\right) w_1(s) \\ w_3(s) = \left(\partial_s - \frac{4}{p-1} - 2\right) w_2(s) \\ w_4(s) = \left(\partial_s - \frac{4}{p-1} + n - 2\right) w_3(s). \end{cases} \tag{10}$$

Equation (9) is equivalent to the following system:

$$\begin{cases} w_1'(s) = \frac{4}{p-1} w_1 + w_2, \\ w_2'(s) = \left(\frac{4}{p-1} + 2\right) w_2 + w_3, \\ w_3'(s) = \left(\frac{4}{p-1} - (n-2)\right) w_3 + w_4, \\ w_4'(s) = |w_1(s)|^{p-1} w_1(s) + \left(\frac{4}{p-1} - (n-4)\right) w_4. \end{cases} \tag{11}$$

In order to perform the stability analysis around the singular solution $u_s(r) = K_0^{1/(p-1)} r^{-4/(p-1)}$, i.e. $v(s) = K_0^{1/(p-1)}$, we have to linearize (11) around the vector

$$w^{(0)} := K_0^{1/(p-1)} \left(1, -\frac{4}{p-1}, \frac{4}{p-1}\left(\frac{4}{p-1} + 2\right), \left(n - 2 - \frac{4}{p-1}\right) \frac{4}{p-1}\left(\frac{4}{p-1} + 2\right)\right).$$

This gives rise to the system $w'(s) = M \circ w(s)$, where

$$M := \begin{pmatrix} \frac{4}{p-1} & 1 & 0 & 0 \\ 0 & \frac{4}{p-1} + 2 & 1 & 0 \\ 0 & 0 & \frac{4}{p-1} - (n-2) & 1 \\ pK_0 & 0 & 0 & \frac{4}{p-1} - (n-4) \end{pmatrix},$$

and the corresponding characteristic polynomial is given by

$$P(\nu) = \left(\nu - \frac{4}{p-1} + n - 4\right)\left(\nu - \frac{4}{p-1} + n - 2\right) \times \left(\nu - \frac{4}{p-1} - 2\right)\left(\nu - \frac{4}{p-1}\right) - pK_0.$$

According to [9], the eigenvalues are given by

$$\nu_1 = \frac{N_1 + \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)}, \qquad \nu_2 = \frac{N_1 - \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)},$$

$$\nu_3 = \frac{N_1 + \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)}, \qquad \nu_4 = \frac{N_1 - \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)},$$

where

$$N_1 := -(n-4)(p-1) + 8, \quad N_2 := (n^2 - 4n + 8)(p-1)^2,$$

$$N_3 := (9n - 34)(n-2)(p-1)^4 + 8(3n-8)(n-6)(p-1)^3 + (16n^2 - 288n + 832)(p-1)^2 - 128(n-6)(p-1) + 256.$$

One has $\nu_1, \nu_2 \in \mathbb{R}$ and $\nu_2 < 0 < \nu_1$. If $n \geq 13$ and $p \geq p_c$, then the other two eigenvalues are also real and they satisfy $\nu_4 \leq \nu_3 < 0$. If either $5 \leq n \leq 12$ or $p < p_c$, on the other hand, then the other two eigenvalues are not real and they satisfy $\mathrm{Re}\, \nu_3 = \mathrm{Re}\, \nu_4 < 0$. In any case, however, one has

$$\nu_2 < \mathrm{Re}\, \nu_4 \leq \mathrm{Re}\, \nu_3 < 0 < \nu_1.$$

For a proof of these facts about the eigenvalues ν_i, we refer the reader to [9, Proposition 2]. We are now in a position to state the key ingredient in the proof of Theorem 1.

Proposition 1 *Let $w(\,.\,)$ be a solution of (11) in the stable manifold of $w^{(0)}$ being tangential to the eigenvector corresponding to ν_2. Then the corresponding solution u of (1) is singular or even not defined for all $r > 0$.*

In order to prove this proposition we need the following crucial observation regarding the sign of the components of an eigenvector corresponding to ν_2.

Lemma 1 *One eigenvector of M corresponding to ν_2 is given by $t = (t_1, t_2, t_3, t_4)$ with*

$$\begin{aligned} t_1 &= 1 > 0, \\ t_2 &= \left(\nu_2 - \frac{4}{p-1}\right) < 0, \\ t_3 &= \left(\nu_2 - 2 - \frac{4}{p-1}\right)\left(\nu_2 - \frac{4}{p-1}\right) > 0, \\ t_4 &= \left(\nu_2 + n - 2 - \frac{4}{p-1}\right)\left(\nu_2 - 2 - \frac{4}{p-1}\right)\left(\nu_2 - \frac{4}{p-1}\right) < 0. \end{aligned}$$

Proof Since $\nu_2 < 0$ we only have to show that

$$0 > \nu_2 + n - 2 - \frac{4}{p-1} = \frac{n}{2} - \frac{1}{2(p-1)}\sqrt{N_2 + 4\sqrt{N_3}} \tag{12}$$

the latter being equivalent to proving that

$$N_3 > (n-2)^2(p-1)^4.$$

Indeed, by using the supercriticality assumption $(n-4)(p-1) > 8$, we have

$$\begin{aligned} N_3 - (n-2)^2(p-1)^4 &= 8(n-2)(n-4)(p-1)^4 + 8(3n^2 - 26n + 48)(p-1)^3 \\ &\quad + 16(n^2 - 18n + 52)(p-1)^2 - 128(n-6)(p-1) + 256 \\ &= 8p(p+1)((n-2)(p-1) - 4)((n-4)(p-1) - 4) > 0. \end{aligned}$$

This proves (12) and hence the lemma. □

Proof of Proposition 1 Let $w(\,.\,)$ be a solution to (11) being tangential for $s \to \infty$ to the eigenvector t from the previous lemma. We may assume that $w(\,.\,)$ exists on the whole real line $\mathbb{R}$ because otherwise, nothing is to be proved. We put $z_1(s) = w_1(s) - w_1^{(0)}$ and further

$$\begin{aligned} z_1(s) &= w_1(s) - w_1^{(0)} = v(s) - K_0^{1/(p-1)}, \\ z_2(s) &= w_2(s) - w_2^{(0)} = \left(\partial_s - \frac{4}{p-1}\right) z_1(s), \\ z_3(s) &= w_3(s) - w_3^{(0)} = \left(\partial_s - \frac{4}{p-1} - 2\right) z_2(s), \\ z_4(s) &= w_4(s) - w_4^{(0)} = \left(\partial_s - \frac{4}{p-1} + n - 2\right) z_3(s), \end{aligned}$$

so that

$$\begin{aligned} \left(\partial_s - \frac{4}{p-1} + n - 4\right) z_4(s) &= |v(s)|^{p-1} v(s) - K_0^{p/(p-1)} \\ &= |w_1(s)|^{p-1} w_1(s) - |w_1^{(0)}|^{p-1} w_1^{(0)}. \end{aligned}$$

Writing this more systematically yields

$$\begin{cases} z_1'(s) = \frac{4}{p-1} z_1(s) + z_2(s), \\ z_2'(s) = \left(\frac{4}{p-1} + 2\right) z_2(s) + z_3(s), \\ z_3'(s) = \left(\frac{4}{p-1} - (n-2)\right) z_3(s) + z_4(s), \\ z_4'(s) = |w_1(s)|^{p-1} w_1(s) - |w_1^{(0)}|^{p-1} w_1^{(0)} + \left(\frac{4}{p-1} - (n-4)\right) z_4(s). \end{cases} \tag{13}$$

According to whether $z(\,.\,)$ approaches the origin from "above" or "below" we distinguish two cases.

First case. There exists s_0 large enough such that

$$z_1(s_0) > 0, \quad z_2(s_0) < 0, \quad z_3(s_0) > 0, \quad z_4(s_0) < 0. \tag{14}$$

On any interval $[s, s_0]$ where $z_1(\,.\,) = w_1(\,.\,) - w_1^{(0)} \geq 0$, we must then have

$$\left(\partial_s + (n-4) - \frac{4}{p-1}\right) z_4(s) = |w_1(s)|^{p-1} w_1(s) - |w_1^{(0)}|^{p-1} w_1^{(0)} \geq 0.$$

This makes $e^{\left((n-4)-\frac{4}{p-1}\right)s} z_4(s)$ increasing on $[s, s_0]$, and so (14) implies that

$$e^{\left((n-4)-\frac{4}{p-1}\right)s} z_4(s) \leq e^{\left((n-4)-\frac{4}{p-1}\right)s_0} z_4(s_0) < 0$$

on $[s, s_0]$. In particular, $z_4(s) < 0$ throughout the interval, and we have

$$\left(\partial_s + (n-2) - \frac{4}{p-1}\right) z_3(s) = z_4(s) < 0.$$

This makes $e^{\left((n-2)-\frac{4}{p-1}\right)s} z_3(s)$ decreasing on $[s, s_0]$, so we similarly find that

$$e^{\left((n-2)-\frac{4}{p-1}\right)s} z_3(s) \geq e^{\left((n-2)-\frac{4}{p-1}\right)s_0} z_3(s_0) > 0$$

by (14). Since $\left(\partial_s - 2 - \frac{4}{p-1}\right) z_2(s) = z_3(s) > 0$, exactly the same argument leads us to

$$e^{\left(-2-\frac{4}{p-1}\right)s} z_2(s) \leq e^{\left(-2-\frac{4}{p-1}\right)s_0} z_2(s_0) < 0$$

by (14), hence $\left(\partial_s - \frac{4}{p-1}\right) z_1(s) = z_2(s) < 0$ and we finally get

$$e^{-\frac{4}{p-1}s} z_1(s) \geq e^{-\frac{4}{p-1}s_0} z_1(s_0) > 0.$$

That is, $z_1(s) > 0$ on any interval $[s, s_0]$ where $z_1(s) \geq 0$, so it is impossible for $z_1(s)$ to become 0 at some $s < s_0$. Hence $\forall s \leq s_0 : z_1(s) > 0$. For the original solution this means that for $r \leq r_0$, $u(\,.\,)$ lies above the singular solution. This means that $u(\,.\,)$ itself is singular at $r = 0$.

Second case. There exists s_0 large enough such that

$$z_1(s_0) < 0, \quad z_2(s_0) > 0, \quad z_3(s_0) < 0, \quad z_4(s_0) > 0. \tag{15}$$

On any interval $[s, s_0]$ where $z_1(\,.\,) = w_1(\,.\,) - w_1^{(0)} \le 0$, we must then have

$$\left(\partial_s + (n-4) - \frac{4}{p-1}\right) z_4(s) = |w_1(s)|^{p-1} w_1(s) - |w_1^{(0)}|^{p-1} w_1^{(0)} \le 0.$$

This makes $e^{\left((n-4)-\frac{4}{p-1}\right)s} z_4(s)$ decreasing on $[s, s_0]$, and so (15) implies that

$$e^{\left((n-4)-\frac{4}{p-1}\right)s} z_4(s) \ge e^{\left((n-4)-\frac{4}{p-1}\right)s_0} z_4(s_0) > 0$$

on $[s, s_0]$. In particular, $z_4(s) > 0$ throughout the interval, and we have

$$\left(\partial_s + (n-2) - \frac{4}{p-1}\right) z_3(s) = z_4(s) > 0.$$

This makes $e^{\left((n-2)-\frac{4}{p-1}\right)s} z_3(s)$ increasing on $[s, s_0]$, so we similarly find that

$$e^{\left((n-2)-\frac{4}{p-1}\right)s} z_3(s) \le e^{\left((n-2)-\frac{4}{p-1}\right)s_0} z_3(s_0) < 0 \tag{16}$$

by (15). Following this approach, as in the first case, we eventually get

$$z_4(s) > 0, \quad z_3(s) < 0, \quad z_2(s) > 0, \quad z_1(s) < 0 \tag{17}$$

on any interval $[s, s_0]$ where $z_1(s) \le 0$, so it is impossible for $z_1(s)$ to become 0 at some $s < s_0$. Hence $\forall s \le s_0 : z_1(s) < 0$, i.e. the corresponding $u(\,.\,)$ is always below the singular solution. In order to prove that $u(\,.\,)$ itself is singular also in this case, we show that $z_1(s) \to -\infty$ for $s \to -\infty$. Since $\forall s \le s_0 : z_1(s) < 0$, we have that (16) holds true for all $s \le s_0$. Referring to [7, Proposition 1] would already show that also v and so u cannot be bounded. However, here it is quite easy to show this directly. For some suitable constant $\delta_1 > 0$ one has:

$$\partial_s\left(e^{-\left(2+\frac{4}{p-1}\right)s} z_2(s)\right) = e^{-\left(2+\frac{4}{p-1}\right)s} z_3(s) \le -\delta_1 e^{-ns}$$

because of (16), and this implies that

$$\begin{aligned} e^{-\left(2+\frac{4}{p-1}\right)s} z_2(s) &\ge \frac{\delta_1}{n} e^{-ns} - \frac{\delta_1}{n} e^{-ns_0} + e^{-\left(2+\frac{4}{p-1}\right)s_0} z_2(s_0) \\ &\ge \delta_2 e^{-ns} \end{aligned}$$

for some suitable constant $\delta_2 > 0$. In particular,

$$\partial_s\left(e^{-\frac{4}{p-1}s} z_1(s)\right) = e^{-\frac{4}{p-1}s} z_2(s) \ge \delta_2 e^{-(n-2)s}$$

and this implies that

$$\begin{aligned} e^{-\frac{4}{p-1}s} z_1(s) &\le \frac{\delta_2}{n-2}\left(e^{-(n-2)s_0} - e^{-(n-2)s}\right) + e^{-\frac{4}{p-1}s_0} z_1(s_0) \\ &\le -\delta_3 e^{-(n-2)s} \end{aligned}$$

for some suitable constant $\delta_3 > 0$. Thus, we end up with

$$z_1(s) \le -\delta_3 e^{-\left(n-2-\frac{4}{p-1}\right)s} \to -\infty \quad \text{as } s \to -\infty, \tag{18}$$

so that also in this case, the corresponding solution u of (1) becomes singular at $r = 0$. □

Proof of Theorem 1 Let $r \mapsto u(r)$ be a radial entire solution to (1) and let $w(.)$ be the corresponding solution to (11). Since either $5 \le n \le 12$ or $p < p_c$ by assumption, the linearized problem around the singular solution $w^{(0)}$ has two real eigenvalues ν_1, ν_2 and two nonreal eigenvalues ν_3, ν_4 with

$$\nu_2 < \mathrm{Re}\,\nu_3 = \mathrm{Re}\,\nu_4 < 0 < \nu_1.$$

By Proposition 1, all trajectories of (11) which lie in the stable manifold of $w^{(0)}$ must be tangential to the plane $w^{(0)} + OS$, where

$$OS := \{\alpha \mathbf{x} + \beta \mathbf{y} \in \mathbb{R}^4 : \alpha, \beta \in \mathbb{R}\}$$

is the plane spanned by the real vectors $\mathbf{x}, \mathbf{y}$, where $\mathbf{x} + i\mathbf{y}, \mathbf{x} - i\mathbf{y}$ are eigenvectors for the nonreal eigenvalues ν_3, ν_4, respectively. On the other hand, we know by [9, Proposition 4] that this plane intersects the hyperplane

$$H := \left\{ w = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4 : w_1 = K_0^{1/(p-1)} \right\} \tag{19}$$

transversally, i.e. $w^{(0)} + OS \not\subseteq H$. In particular, any trajectory w corresponding to an entire radial solution of (1) must intersect H an infinite number of times, hence w_1 attains the value $K_0^{1/(p-1)}$ an infinite number of times. This also means that every radial entire solution u of (1) must intersect the singular solution u_s an infinite number of times. □

3 The Dirichlet problem

If we put $r = |x|$ then the equation in (7) becomes

$$u^{(4)}(r) + \frac{2(n-1)}{r} u'''(r) + \frac{(n-1)(n-3)}{r^2} u''(r) - \frac{(n-1)(n-3)}{r^3} u'(r) = \lambda (1+u)^p, \tag{20}$$

where $r \in [0, 1]$. If we put

$$U(x) = 1 + u(x/\sqrt[4]{\lambda}) \quad \text{for } x \in B_{\sqrt[4]{\lambda}}(0) \tag{21}$$

then U solves the equation

$$\Delta^2 U = U^p \quad \text{in } B_{\sqrt[4]{\lambda}}(0). \tag{22}$$

Since Eq. (22) is invariant under the rescaling

$$U_a(x) = aU(a^{\frac{p-1}{4}} x)$$

i.e. U is a solution of (22) if and only if U_a is a solution of (22), it is not restrictive to concentrate our attention on solutions U of the Eq. (22) which satisfy the condition $U(0) = 1$.

Next we define $U_\gamma = U_\gamma(r)$ as the unique solution of the initial value problem

$$\begin{aligned} &U_\gamma^{(4)}(r) + \frac{2(n-1)}{r} U_\gamma'''(r) + \frac{(n-1)(n-3)}{r^2} U_\gamma''(r) - \frac{(n-1)(n-3)}{r^3} U_\gamma'(r) \\ &= |U_\gamma(r)|^{p-1} U_\gamma(r), \\ &U_\gamma(0) = 1, \quad U_\gamma'(0) = U_\gamma'''(0) = 0, \quad U_\gamma''(0) = \gamma < 0. \end{aligned} \tag{23}$$

We report here the following fundamental result by [9]:

Lemma 2 [9] *Let $n > 4$ and $p > (n+4)/(n-4)$.*

(i) *There exists a unique $\overline{\gamma} < 0$ such that the solution $U_{\overline{\gamma}}$ of (23) exists on the whole interval $[0, \infty)$, it is positive everywhere, it vanishes at infinity and it satisfies $U'_{\overline{\gamma}}(r) < 0$ for any $r \in (0, \infty)$.*
(ii) *If $\gamma < \overline{\gamma}$ there exist $0 < R_1 < R_2 < \infty$ such that the solution U_γ of (23) satisfies $U_\gamma(R_1) = 0$, $\lim_{r \uparrow R_2} U_\gamma(r) = -\infty$ and $U'_\gamma(r) < 0$ for any $r \in (0, R_2)$.*
(iii) *If $\gamma > \overline{\gamma}$ there exist $0 < R_1 < R_2 < \infty$ such that the solution U_γ of (23) satisfies $U'_\gamma(r) < 0$ for $r \in (0, R_1)$, $U'_\gamma(R_1) = 0$, $U'_\gamma(r) > 0$ for $r \in (R_1, R_2)$ and $\lim_{r \uparrow R_2} U_\gamma(r) = +\infty$.*
(iv) *If $\gamma_1 < \gamma_2 < 0$ then the corresponding solutions $U_{\gamma_1}, U_{\gamma_2}$ of (23) satisfy $U_{\gamma_1} < U_{\gamma_2}$ and $U'_{\gamma_1} < U'_{\gamma_2}$ as long as they both exist.*

Proof See the statement and proof of [9, Theorem 2] and also the statement of [9, Lemma 2]. □

For any $\gamma < 0$ let U_γ be the unique local solution of (23). Thanks to Lemma 2(iii), for $\gamma > \overline{\gamma}$ we may define R_γ as the unique value of $r > 0$ for which we have $U'_\gamma(R_\gamma) = 0$.

The idea in constructing a singular solution to (7) consists in suitably rescaling $U_\gamma(\,.\,) - U_\gamma(R_\gamma)|_{B_{R_\gamma}}$ to B and in finding a suitable subsequence for $\gamma \downarrow \overline{\gamma}$, which locally converges in $B \backslash \{0\}$ to a singular solution. A first step is proving $R_\gamma \to \infty$ for $\gamma \downarrow \overline{\gamma}$. This is done by contradiction with the help of rescaling arguments and exploiting the strict monotonicity of the entire regular solution $U_{\overline{\gamma}}$.

Lemma 3 *Let $n > 4$, $p > (n+4)/(n-4)$ and $\overline{\gamma}$ as in the statement of Lemma 2. Then the map $\gamma \mapsto R_\gamma$ is non-increasing on the interval $(\overline{\gamma}, 0)$ and*

$$\lim_{\gamma \downarrow \overline{\gamma}} R_\gamma = +\infty.$$

Proof The fact that the map $\gamma \mapsto R_\gamma$ is non-increasing on the interval $(\overline{\gamma}, 0)$ follows immediately by Lemma 2 (iii)–(iv) and the definition of R_γ. This shows that the function $\gamma \mapsto R_\gamma$ admits a limit as $\gamma \to \overline{\gamma}$. Suppose by contradiction that

$$\overline{R} := \lim_{\gamma \downarrow \overline{\gamma}} R_\gamma < +\infty.$$

Then, by Lemma 2(i), (iv) we have for all $\gamma \in (\overline{\gamma}, 0)$ that

$$U_\gamma(R_\gamma) > U_{\overline{\gamma}}(R_\gamma) \geq U_{\overline{\gamma}}(\overline{R}) > 0. \tag{24}$$

Define for any $\gamma \in (\overline{\gamma}, 0)$, $r \in [0, 1]$ the function

$$u_\gamma(r) = \frac{U_\gamma(R_\gamma r)}{U_\gamma(R_\gamma)} - 1. \tag{25}$$

Then, u_γ solves the Dirichlet problem

$$\begin{cases} \Delta^2 u_\gamma = R_\gamma^4 U_\gamma(R_\gamma)^{p-1}(1+u_\gamma)^p & \text{in } B, \\ u_\gamma = |\nabla u_\gamma| = 0 & \text{on } \partial B. \end{cases} \tag{26}$$

Since the function U_γ is decreasing on the interval $(0, R_\gamma)$ we find that

$$U_\gamma(R_\gamma) \leq U_\gamma(r) \leq U_\gamma(0) = 1 \quad \text{for all } r \in [0, R_\gamma]. \tag{27}$$

Then by (24) and (27) we obtain for all $\gamma \in (\overline{\gamma}, 0)$ and all $r \in [0, 1)$ that

$$0 \le u_\gamma(r) \le \frac{1}{U_\gamma(R_\gamma)} - 1 \le \frac{1}{U_{\overline{\gamma}}(\overline{R})} - 1. \tag{28}$$

This shows that the set $\{u_\gamma : \gamma \in (\overline{\gamma}, 0)\}$ is bounded in $L^\infty(B)$ and hence by a bootstrap argument, from (26) and the fact that $R_\gamma^4 U_\gamma(R_\gamma)^{p-1} \le \lambda^*$ (see the introduction for the definition of λ^*), we deduce that there exists a sequence $\gamma_k \downarrow \overline{\gamma}$ and a function $\overline{u} \in H_0^2(B) \cap C^\infty(\overline{B})$ such that

$$u_{\gamma_k} \to \overline{u} \quad \text{in } C^4(\overline{B}) \tag{29}$$

as $k \to \infty$. Take any $r \in [0, \overline{R})$. Since $R_{\gamma_k} \uparrow \overline{R}$, there exists $\overline{k} = \overline{k}(r)$ such that $r < R_{\gamma_k}$ for any $k > \overline{k}$. Hence, for $k > \overline{k}$, we may take r/R_{γ_k} instead of r in (25) and obtain

$$U_{\gamma_k}(r) = U_{\gamma_k}(R_{\gamma_k}) \left[u_{\gamma_k}\left(r/R_{\gamma_k}\right) + 1\right]. \tag{30}$$

Since the sequence γ_k is decreasing, by Lemma 2(iii)–(iv) we infer that $U_{\gamma_k}(R_{\gamma_k})$ is non-increasing. By (24), $U_{\gamma_k}(R_{\gamma_k})$ is also bounded from below and hence admits a finite limit. Thanks to (29) we also have $u_{\gamma_k}(r/R_{\gamma_k}) \to \overline{u}(r/\overline{R})$ as $k \to \infty$. Therefore by (30), we deduce that for any $r \in [0, \overline{R})$

$$\overline{U}(r) := \lim_{k\to\infty} U_{\gamma_k}(r) = \left[\lim_{k\to\infty} U_{\gamma_k}(R_{\gamma_k})\right] \cdot \left[\overline{u}(r/\overline{R}) + 1\right]. \tag{31}$$

In fact, from (29) and (30) we deduce that $U_{\gamma_k} \to \overline{U}$ in $C^4([0, R])$ for any $0 < R < \overline{R}$. Since $\overline{u} \in H_0^2(B)$, (31) shows that

$$\lim_{r\uparrow\overline{R}} \overline{U}'(r) = 0. \tag{32}$$

On the other hand by continuous dependence on the initial conditions it follows

$$\lim_{k\to\infty} U_{\gamma_k}(r) = U_{\overline{\gamma}}(r) \quad \text{for all } r \in [0, \overline{R})$$

and hence $\overline{U}(r) = U_{\overline{\gamma}}(r)$ for any $r \in [0, \overline{R})$. This with (32) implies

$$\lim_{r\uparrow\overline{R}} U_{\overline{\gamma}}'(r) = 0$$

which is absurd since $U_{\overline{\gamma}}'(\overline{R}) < 0$ (see Lemma 2 (i)). This completes the proof of the lemma. □

Lemma 4 *Let $n > 4$ and $p > (n+4)/(n-4)$ and let u be a regular solution of (7). Then*

$$u(x) \le \left(\frac{\lambda^*}{\lambda}\right)^{1/(p-1)} |x|^{-4/(p-1)} - 1 \quad \textit{for all } x \in B\backslash\{0\}.$$

Proof Let u be a regular solution of (7) for some $\lambda > 0$ and define the rescaled function

$$U(x) = \frac{1}{1+u(0)} \left[1 + u\left(\frac{x}{\sqrt[4]{\lambda}(1+u(0))^{\frac{p-1}{4}}}\right)\right] \tag{33}$$

so that U satisfies

$$\Delta^2 U = U^p \quad \text{in } B_R(0) \quad \text{and} \quad U(0) = 1 \tag{34}$$

where we put $R = \sqrt[4]{\lambda}(1+u(0))^{\frac{p-1}{4}}$.

Define

$$M = \max_{r\in[0,R]} r^{4/(p-1)} U(r)$$

and let $\overline{R} \in (0, R]$ be such that $\overline{R}^{4/(p-1)} U(\overline{R}) = M$. If we define

$$w(r) = \frac{U(\overline{R}r)}{U(\overline{R})} - 1$$

then w solves the problem

$$\begin{cases} \Delta^2 w = \overline{R}^4 U(\overline{R})^{p-1}(1+w)^p & \text{in } B \\ w = 0 & \text{on } \partial B \\ w' \le 0 & \text{on } \partial B. \end{cases}$$

This proves that $M^{p-1} = \overline{R}^4 U(\overline{R})^{p-1} \le \lambda^*$ since otherwise by the super-subsolution method (see [2, Lemma 3.3] for more details) we would obtain a solution of (7) for $\lambda = \overline{R}^4 U(\overline{R})^{p-1} > \lambda^*$. This yields for all $r \in [0, R]$ that

$$U(r) \le M r^{-4/(p-1)} \le (\lambda^*)^{1/(p-1)} r^{-4/(p-1)}. \tag{35}$$

Then reversing the identity (33), by (35) we obtain

$$u(r) = \lambda^{-1/(p-1)} R^{4/(p-1)} U(Rr) - 1 \le \left(\frac{\lambda^*}{\lambda}\right)^{1/(p-1)} r^{-4/(p-1)} - 1$$

which completes the proof of the lemma. □

Proof of Theorem 2 For $\gamma \in (\overline{\gamma}, 0)$ consider the corresponding solution U_γ of the Cauchy problem (23) and the function u_γ introduced in (25). If we put $\lambda_\gamma = R_\gamma^4 U_\gamma(R_\gamma)^{p-1}$ then by (26) we have that u_γ solves

$$\begin{cases} \Delta^2 u_\gamma = \lambda_\gamma (1+u_\gamma)^p & \text{in } \ B, \\ u_\gamma = |\nabla u_\gamma| = 0 & \text{on } \ \partial B. \end{cases} \tag{36}$$

We show that λ_γ remains bounded away from zero for $\gamma > \overline{\gamma}$ sufficiently close to $\overline{\gamma}$, which is defined in Lemma 2. By [9, Theorem 3] we infer that for a fixed $\varepsilon \in (0, K_0^{1/(p-1)})$ there exists a corresponding $r_\varepsilon > 0$ such that

$$U_{\overline{\gamma}}(r) > (K_0^{1/(p-1)} - \varepsilon) r^{-4/(p-1)} \quad \text{for all } r > r_\varepsilon. \tag{37}$$

On the other hand, by Lemma 3, we deduce that there exists $\gamma_0 \in (\overline{\gamma}, 0)$ such that for any $\gamma \in (\overline{\gamma}, \gamma_0)$ then $R_\gamma > r_\varepsilon$. Therefore by Lemma 2(iv) we obtain for all $\gamma \in (\overline{\gamma}, \gamma_0)$

$$U_\gamma(R_\gamma) > U_{\overline{\gamma}}(R_\gamma) > (K_0^{1/(p-1)} - \varepsilon) R_\gamma^{-4/(p-1)}$$

and this yields

$$\forall \gamma \in (\overline{\gamma}, \gamma_0): \quad \lambda_\gamma > (K_0^{1/(p-1)} - \varepsilon)^{p-1} =: C. \tag{38}$$

Combining (38) and Lemma 4 we obtain for all $\gamma \in (\overline{\gamma}, \gamma_0)$, $x \in B\backslash\{0\}$

$$u_\gamma(x) \le \left(\frac{\lambda^*}{C}\right)^{1/(p-1)} |x|^{-4/(p-1)} - 1. \tag{39}$$

Since u_γ solves (36), by (39) we obtain

$$\int_B |\Delta u_\gamma|^2 dx = \lambda_\gamma \int_B (1+u_\gamma)^p u_\gamma dx \le \lambda^* \int_B (1+u_\gamma)^{p+1} dx$$
$$\le \frac{(\lambda^*)^{\frac{2p}{p-1}}}{C^{\frac{p+1}{p-1}}} \int_B |x|^{-\frac{4(p+1)}{p-1}} dx < +\infty$$

since $p > (n+4)/(n-4)$. This proves that the set $\{u_\gamma : \gamma \in (\overline{\gamma}, \gamma_0)\}$ is bounded in $H_0^2(B)$ and hence there exists a sequence $\gamma_k \downarrow \overline{\gamma}$ and a function $u \in H_0^2(B)$ such that $u_{\gamma_k} \rightharpoonup u$ in $H_0^2(B)$. Moreover, by (39) and applying Lebesgue's theorem, u weakly solves (7) for a suitable $\widetilde{\lambda} \ge C$.

It remains to prove that the function u is unbounded. For simplicity, in the rest of the proof $u_{\gamma_k}, U_{\gamma_k}, R_{\gamma_k}, \lambda_{\gamma_k}$ will be denoted, respectively, by u_k, U_k, R_k, λ_k.

By compact embedding we have that $u_k \to u$ in $L^1(B)$ and hence we have

$$\lim_{r\downarrow 0} \frac{1}{|B_r(0)|} \int_{B_r(0)} u(x)dx = \lim_{r\downarrow 0} \left[\frac{1}{r^n |B|} \lim_{k\to\infty} \int_{B_r(0)} u_k(x)dx \right]$$

and passing to radial coordinates, by (25) and Lemma 2(iv), we obtain

$$\lim_{r\downarrow 0} \frac{1}{|B_r(0)|} \int_{B_r(0)} u(x)dx = \lim_{r\downarrow 0} \left[-1 + \frac{n}{r^n} \lim_{k\to\infty} \int_0^r \frac{U_k(R_k\rho)}{U_k(R_k)} \rho^{n-1} d\rho \right]$$
$$= \lim_{r\downarrow 0} \left[-1 + \frac{n}{r^n} \lim_{k\to\infty} \frac{1}{R_k^n U_k(R_k)} \int_0^{R_k r} U_k(\rho)\rho^{n-1} d\rho \right]$$
$$\ge \lim_{r\downarrow 0} \left[-1 + \frac{n}{r^n} \lim_{k\to\infty} \frac{1}{R_k^n U_k(R_k)} \int_0^{R_k r} U_{\overline{\gamma}}(\rho)\rho^{n-1} d\rho \right]. \tag{40}$$

By (37) we have that there exist $C, R_0 > 0$ such that

$$\forall \rho \in (R_0, \infty): \quad U_{\overline{\gamma}}(\rho) > C\rho^{-4/(p-1)}. \tag{41}$$

Hence we have for $k > \overline{k} = \overline{k}(r)$

$$\int_0^{R_k r} U_{\overline{\gamma}}(\rho)\rho^{n-1} d\rho \ge \int_0^{R_0} U_{\overline{\gamma}}(\rho)\rho^{n-1} d\rho + C\left(n - \frac{4}{p-1}\right)^{-1} \left((R_k r)^{n-\frac{4}{p-1}} - R_0^{n-\frac{4}{p-1}} \right). \tag{42}$$

Since $p > (n+4)/(n-4) > (n+4)/n$ and since by (38), λ_k is bounded away from zero as $k \to \infty$ then

$$\lim_{k\to\infty} R_k^n U_k(R_k) = \lim_{k\to\infty} R_k^{n-\frac{4}{p-1}} \lambda_k^{\frac{1}{p-1}} = +\infty$$

and hence by (42) we obtain

$$\begin{aligned}
&\lim_{k\to\infty} \frac{1}{R_k^n U_k(R_k)} \int_0^{R_k r} U_{\overline{\gamma}}(\rho)\rho^{n-1} d\rho \\
&\geq \liminf_{k\to\infty} \frac{C}{\left(n-\frac{4}{p-1}\right) R_k^n U_k(R_k)} \left(R_k^{n-\frac{4}{p-1}} r^{n-\frac{4}{p-1}} - R_0^{n-\frac{4}{p-1}}\right) \\
&= \liminf_{k\to\infty} \frac{C r^{n-\frac{4}{p-1}}}{\left(n-\frac{4}{p-1}\right) \lambda_k^{1/(p-1)}} \\
&\geq \frac{C r^{n-4/(p-1)}}{\left(n-\frac{4}{p-1}\right) (\lambda^*)^{1/(p-1)}} =: \widetilde{C} r^{n-4/(p-1)}.
\end{aligned} \tag{43}$$

Inserting (43) in (40) we obtain

$$\lim_{r\downarrow 0} \frac{1}{|B_r(0)|} \int_{B_r(0)} u(x)dx \geq \lim_{r\downarrow 0}(-1 + n\widetilde{C} r^{-4/(p-1)}) = +\infty.$$

This proves that $u \notin L^\infty(B)$. □

Proof of Theorem 3 We make use of an idea from [6]. Let u_λ denote the positive minimal regular solution of (7) for $0 \leq \lambda < \lambda^*$. According to [7, Theorem 2], these are stable so that one has in particular:

$$\forall \varphi \in C_0^\infty(B): \quad \int_B (\Delta\varphi(x))^2 \, dx - p\lambda \int_B (1 + u_\lambda(x))^{p-1} \varphi(x)^2 \, dx \geq 0.$$

By taking the monotone limit we obtain that

$$\forall \varphi \in C_0^\infty(B): \quad \int_B (\Delta\varphi(x))^2 \, dx - p\lambda^* \int_B (1 + u^*(x))^{p-1} \varphi(x)^2 \, dx \geq 0. \tag{44}$$

We assume now for contradiction that u^* is singular. Then, according to [7, Theorem 5] we have the following estimate from below:

$$u^*(x) > \left(\frac{K_0}{\lambda^*}\right)^{1/(p-1)} |x|^{-4/(p-1)} - 1. \tag{45}$$

Combining this with (44) yields

$$\forall \varphi \in C_0^\infty(B): \quad \int_B (\Delta\varphi(x))^2 \, dx \geq pK_0 \int_B |x|^{-4} \varphi(x)^2 \, dx. \tag{46}$$

However, under the subcriticality assumptions made we have that $pK_0 > n^2(n-4)^2/16$. This contradicts the optimality of the constant in Hardy's inequality

$$\forall \varphi \in C_0^\infty(B): \quad \int_B (\Delta\varphi(x))^2 \, dx \geq \frac{n^2(n-4)^2}{16} \int_B |x|^{-4} \varphi(x)^2 \, dx,$$

so that u^* has indeed to be regular. □

Acknowledgments We are grateful to Gianni Arioli and Filippo Gazzola for fruitful discussions and interesting numerical experiments. We thank the referee for useful suggestions on how to improve the presentation of the present paper.

Note added in proof After this article was accepted we learnt that Z. Guo and J. Wei simultaneously and independently found different proofs for Theorem 1 and the nonoscillation result in [12].

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