Annali di Matematica 184, 315–326 (2005)

Digital Object Identifier (DOI) 10.1007/s10231-004-0116-6

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Hardy spaces H^1 for Schrödinger operators with compactly supported potentials*

Received: July 28, 2003; accepted: November 3, 2003
Published online: September 21, 2004 – © Springer-Verlag 2004

Abstract. Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where V is a non-negative compactly supported potential that belongs to L^p for some $p > d/2$. Let $\{K_t\}_{t>0}$ denote the semigroup of linear operators generated by $-L$. For a function f we define its H^1_L-norm by $\|f\|_{H^1_L} = \|\sup_{t>0}|K_t f(x)|\|_{L^1(dx)}$. It is proved that for a properly defined weight w a function f belongs to H^1_L if and only if $wf \in H^1(\mathbb{R}^d)$, where $H^1(\mathbb{R}^d)$ is the classical real Hardy space.

Mathematics Subject Classification (2000). 42B30, 35J10, 42B25

Key words. Hardy spaces – atomic decomposition – Schrödinger operators

1. Introduction

Let

$$L f(x) = -\Delta f(x) + V(x) f(x) \tag{1.1}$$

be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where V is a non-negative potential. Throughout this paper we shall assume that the potential V is compactly supported, say $\operatorname{supp} V \subset B(0,1) = \{x : |x| < 1\}$, and belongs to $L^p(\mathbb{R}^d)$ for some $p > d/2$. It is well known that $-L$ generates a semigroup $\{K_t\}_{t>0}$ of linear operators acting on $L^r(\mathbb{R}^d)$, $1 \leq r < \infty$. By the Feynman–Kac formula the integral kernels $K_t(x, y)$ of the semigroup $\{K_t\}_{t>0}$ satisfy

$$0 \leq K_t(x, y) \leq P_t(x - y), \tag{1.2}$$

where $P_t(x-y) = (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right)$ are the integral kernels of the classical heat semigroup $\{P_t\}_{t>0}$. Let

$$K^* f(x) = \sup_{t>0} |K_t f(x)|$$

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* Research partially supported by the European Commission via Harmonic Analysis and Related Problems network, Polish Grants 5P03A05020, 5P03A02821 from KBN, and Foundation for Polish Sciences, Subsidy 3/99.

be the maximal operator associated with the semigroup $\{K_t\}_{t>0}$. We say that an L^1-function f belongs to H^1_L if its norm defined by

$$\|f\|_{H^1_L} = \|K^* f\|_{L^1}$$

is finite.

Define

$$w(x) = \lim_{s\to\infty} K_s\mathbf{1}(x). \tag{1.3}$$

The limit in (1.3) exists because the function $s \mapsto K_s\mathbf{1}(x)$ is monotonic. Indeed, by (1.2),

$$0 \le K_{s+t}\mathbf{1}(x) = K_s K_t \mathbf{1}(x) \le K_s\mathbf{1}(x) \le 1. \tag{1.4}$$

We shall show that there exists a constant $0 < \delta < 1$ such that $\delta \le w(x) \le 1$. The following theorem is the main result of the present paper:

Theorem 1.5. *There exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_L} \le \|wf\|_{H^1(\mathbb{R}^d)} \le C\|f\|_{H^1_L}, \tag{1.6}$$

where $\|g\|_{H^1(\mathbb{R}^d)} = \|\sup_{t>0}|P_t g(x)|\|_{L^1(dx)}$ is the norm in the classical real Hardy space $H^1(\mathbb{R}^d)$.

We say that a function b is an H^1_L-atom if there exists a ball $B(y_0, r) = \{y \in \mathbb{R}^d : |y - y_0| < r\}$ such that $\operatorname{supp} b \subset B(y_0, r)$, $\|b\|_{L^\infty} \le |B(y_0, r)|^{-1}$, and $\int b(y)w(y)\,dy = 0$. The atomic norm $\|f\|_{H^1_L\,\text{atom}}$ is defined by

$$\|f\|_{H^1_L\,\text{atom}} = \inf \sum_j |\lambda_j|, \tag{1.7}$$

where the infimum is taken over all representations $f = \sum_j \lambda_j b_j$, where b_j are H^1_L-atoms.

As a consequence of Theorem 1.5 we have

Corollary 1.8. *There exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_L} \le \|f\|_{H^1_L\,atom} \le C\|f\|_{H^1_L}. \tag{1.9}$$

Let us mention that in contrast with the one-dimensional case or in the case of V satisfying a reverse Hölder inequality (cf. [1]–[4]) the atoms for H^1_L considered in the present paper are not variants of local atoms.

Finally we would like to remark that if $V \ne \tilde{V}$ are compactly supported L^p-potentials, $p > d/2$, then the corresponding spaces H^1_L and $H^1_{\tilde{L}}$ do not coincide. We shall discuss this property at the end of the paper.

Acknowledgement. The authors wish to thank the referee for valuable remarks.

2. Properties of the weight w

Let

$$\Gamma(x, y) = \int_0^\infty K_t(x, y)\, dt \quad \text{and} \quad \Gamma_0(x, y) = -\int_0^\infty P_t(x - y)\, dt \tag{2.1}$$

be the fundamental solutions for L and Δ, respectively. Obviously,

$$0 \leq \Gamma(x, y) = \Gamma(y, x) \leq -\Gamma_0(x, y) = -\Gamma_0(y, x).$$

The perturbation formula asserts

$$P_t = K_t + \int_0^t P_{t-s} V K_s\, ds. \tag{2.2}$$

Hence, by (1.3),

$$\begin{aligned} \int V(x)\Gamma(x, y)\, dx &= \lim_{t\to\infty} \int_0^t \int\int P_{t-s}(z - x)V(x)K_s(x, y)\, dz\, dx\, ds \\ &= \lim_{t\to\infty} \int (P_t(z - y) - K_t(z, y))\, dz \\ &= 1 - w(y). \end{aligned} \tag{2.3}$$

Lemma 2.4. *The function $w(x)$ has the following properties:*

$$\lim_{|x|\to\infty} w(x) = 1, \tag{2.5}$$

$$|w(x) - w(y)| \leq C_\gamma |x-y|^\gamma, \quad \textit{provided } 0 < \gamma < 2 - d/p, \quad \gamma \leq 1. \tag{2.6}$$

Moreover, there exists a constant δ such that

$$0 < \delta \leq w(x) \leq 1. \tag{2.7}$$

Proof. From (2.3) we conclude

$$0 \leq 1 - w(x) \leq C \int \frac{V(y)}{|x - y|^{d-2}}\, dy \leq \frac{C}{|x|^{d-2}} \|V\|_{L^1}, \tag{2.8}$$

for $|x| > 2$, which gives (2.5).

In order to prove (2.6), let us note that

$$w(x) = \lim_{s\to\infty} K_t K_s \mathbf{1}(x) = K_t w(x) \quad \text{for all } t > 0. \tag{2.9}$$

Let $q_t(x, y) = P_t(x - y) - K_t(x, y)$. Since

$$\begin{aligned} |w(x + h) - w(x)| &= \Big| \lim_{s\to\infty} \int\int \big(K_1(x + h, y) - K_1(x, y)\big)K_s(y, z)\, dz\, dy \Big| \\ &\leq \int |K_1(x + h, y) - K_1(x, y)|\, dy \end{aligned}$$

and

$$\int |P_1(x+h-y) - P_1(x,y)|\,dy \le C_\gamma |h|^\gamma,$$

it suffices to verify that

$$\int |q_1(x+h,y) - q_1(x,y)|\,dy \le C_\gamma |h|^\gamma. \tag{2.10}$$

By the perturbyation formula

$$\begin{aligned}
&\int |q_1(x+h,y) - q_1(x,y)|\,dy \\
&\quad = \int \left| \int_0^1 \int (P_s(x+h-z) - P_s(x-z)) V(z) K_{1-s}(z,y)\,dz\,ds \right| dy \\
&\quad \le \int_0^1 \int s^{-d/2} \left| P_1\left(\frac{x+h-z}{\sqrt{s}}\right) - P_1\left(\frac{x-z}{\sqrt{s}}\right) \right| V(z)\,dzds \\
&\quad \le \int_0^1 s^{-d/2+d/(2p')} \left(\int \left| P_1\left(\frac{x+h}{\sqrt{s}} - z\right) - P_1\left(\frac{x}{\sqrt{s}} - z\right) \right|^{p'} dz \right)^{1/p'} \|V\|_{L^p}\,ds \\
&\quad \le C \int_0^1 s^{-d/(2p)} \left(\frac{|h|}{\sqrt{s}}\right)^\gamma ds \\
&\quad \le C_\gamma |h|^\gamma,
\end{aligned}$$

which completes the proof of (2.6).

Clearly, (2.6) implies that the function w is continuous. Therefore, according to (2.5), the estimate (2.7) will be proved if we show that

$$w(x) > 0 \quad \text{for all } x. \tag{2.11}$$

On account of (2.9) and (2.5) we shall have established (2.11) if we prove that there exists $t > 0$ such that $K_t(x,y) > 0$.

Lemma 2.12. *For every $t > 0$*

$$K_t(x,y) > 0. \tag{2.13}$$

Proof. The lemma is well known. For completeness of the paper we present the proof. By (1.2), (2.2), and the Hölder inequality we have

$$\begin{aligned}
0 \le q_\varepsilon(x,y) &\le \int_0^\varepsilon \int P_{\varepsilon-s}(x-z) V(z) P_s(z-y)\,dzds \\
&\le \int_0^{\varepsilon/2} C_d \varepsilon^{-d/2} \|V\|_{L^p} \|P_s\|_{L^{p'}}\,ds \\
&\quad + \int_{\varepsilon/2}^\varepsilon \|P_{\varepsilon-s}\|_{L^{p'}} \|V\|_{L^p} C_d \varepsilon^{-d/2}\,ds \\
&= C_d' \varepsilon^{-d/2} \varepsilon^{(2p-d)/2p} \|V\|_{L^p}.
\end{aligned} \tag{2.14}$$

Clearly, there exists $c_0 > 0$ such that

$$P_\varepsilon(x-y) \geq c_0\varepsilon^{-d/2} \quad \text{for } |x-y| < \sqrt{\varepsilon}. \tag{2.15}$$

From (2.14) and (2.15) we conclude that there exists $\varepsilon_0 > 0$ such that

$$K_\varepsilon(x,y) \geq \frac{c_0}{2}\varepsilon^{-d/2} \quad \text{provided} \quad |x-y| < \sqrt{\varepsilon}, \quad 0 < \varepsilon < \varepsilon_0. \tag{2.16}$$

Applying (2.16) and the semigroup property we get

$$K_{n\varepsilon}(x,y) > 0 \ \text{ for } \ |x-y| < \sqrt{\varepsilon}n, \ \ \varepsilon < \varepsilon_0. \tag{2.17}$$

Now (2.13) follows by taking $n > \max(\frac{|x-y|^2}{t}, \frac{t}{\varepsilon_0})$ and $\varepsilon = tn^{-1}$. □

Corollary 2.18. *There exists a constant* $\delta > 0$ *such that*

$$\|VL^{-1}f\|_{L^1} \leq (1-\delta)\|f\|_{L^1}.$$

Proof. The corollary is an immediate consequence of (2.7). Indeed,

$$\|VL^{-1}\|_{L^1\to L^1} \leq \sup_y \int V(x)\Gamma(x,y)\,dx = 1 - w(y) \leq 1-\delta. \qquad \square$$

3. Estimates of maximal functions

Since the compactly supported function V belongs to $L^p(\mathbb{R}^d)$ for some $p > d/2$ and $\Delta^{-1} : L^1(\mathbb{R}^d) \to L^q_{\text{loc}}(\mathbb{R}^d)$ for every $q < d/(d-2)$, we get that $V\Delta^{-1}$ is bounded on $L^1(\mathbb{R}^d)$. Moreover, direct calculations show that

$$(I - VL^{-1})(I - V\Delta^{-1}) = (I - V\Delta^{-1})(I - VL^{-1}) = I. \tag{3.1}$$

Here and subsequently L^{-1} and Δ^{-1} are the operators with the integral kernels Γ and Γ_0, respectively (cf. (2.1)).

Using (2.2) we get

$$\begin{aligned} P_t &= K_t + \int_0^t (P_{t-s} - P_t)VK_s\,ds - \int_t^\infty P_tVK_sds + \int_0^\infty P_tVK_s\,ds \\ &= K_t + \int_0^t (P_{t-s} - P_t)VK_s\,ds - \int_t^\infty P_tVK_sds + P_tVL^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} P_t(I - VL^{-1}) &= K_t - \int_0^t (P_t - P_{t-s})VK_s\,ds - \int_t^\infty P_tVK_sds \\ &= K_t - R_t - Q_t. \end{aligned} \tag{3.2}$$

We shall show that the maximal operators associated with the families $\{R_t\}_{t>0}$ and $\{Q_t\}_{t>0}$ are bounded on $L^1(\mathbb{R}^d)$.

Lemma 3.3. *There exists a constant $C > 0$ such that*

$$\|\sup_{t>2} |R_t f(x)|\|_{L^1(dx)} \leq C\|f\|_{L^1}. \tag{3.4}$$

Proof. Let $R_t(x, y)$ denote the integral kernel of the operator R_t. Put $\beta = 8/9$. Then for $t > 2$ we have

$$\begin{aligned} R_t(x, y) &= \int_0^t \int \big(P_t(x-z) - P_{t-s}(x-z)\big) V(z) K_s(z, y)\, dz\, ds = \int_0^{t^\beta} + \int_{t^\beta}^t \\ &= R_t^{[1]}(x, y) + R_t^{[2]}(x, y). \end{aligned}$$

Clearly for every $0 < c < 1$ there is a non-negative function ϕ that belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ such that

$$|P_t(x) - P_{t-s}(x)| \leq \frac{s}{t}\phi_t(x) \quad \text{for } 0 < s < ct, \tag{3.5}$$

where $\phi_t(x) = t^{-d/2}\phi(t^{-1/2}x)$. Hence there exists $\phi \in \mathcal{S}(\mathbb{R}^d)$, $\phi \geq 0$, such that for $t > 2$ we have

$$\begin{aligned} \big|R_t^{[1]}(x, y)\big| &\leq \int t^{\beta-1}\phi_t(x-z)V(z) \int_0^{t^\beta} K_s(z, y)\, ds\, dz \\ &\leq C \int t^{\beta-1}\phi_t(x-z)V(z)|z-y|^{2-d}\, dz. \end{aligned}$$

Since $\sup_{t>2} t^{\beta-1}\phi_t(x-z) \leq C(1+|x-z|)^{-d-1+\beta}$, we get

$$\begin{aligned} \int \sup_{t>2} \big|R_t^{[1]}(x, y)\big|\, dx &\leq \int \int_{|z|<1} (1+|x-z|)^{-d-1+\beta}|z-y|^{2-d}V(z)\, dz\, dx \\ &\leq C\|V\|_{L^p} \leq C. \end{aligned} \tag{3.6}$$

We now turn to estimate $R_t^{[2]}$

$$\begin{aligned} \big|R_t^{[2]}(x, y)\big| &\leq \int_{t^\beta}^t \int P_t(x-z)V(z)K_s(z, y)\, dz\, ds \\ &\quad + \int_{t^\beta}^t \int P_{t-s}(x-z)V(z)K_s(z, y)\, dz\, ds \\ &= R^{[2']}(x, y) + R^{[2'']}(x, y). \end{aligned} \tag{3.7}$$

For $s \geq t^\beta$ we have $K_s(z, y) \leq Ct^{-\beta d/2}$. Therefore

$$0 \leq R_t^{[2']}(x, y) \leq C \int P_t(x-z)V(z)t^{1-\beta d/2}\, dz$$

and, consequently,

$$\big\|\sup_{t>2} R_t^{[2']}(x, y)\big\|_{L^1(dx)} \leq C. \tag{3.8}$$

Likewise,

$$0 \le R_t^{[2'']}(x,y) \le \int_{t^\beta}^{t} \int P_{t-s}(x-z)V(z)t^{-\beta d/2}\,dz\,ds$$
$$\le Ct^{-\beta d/2} \int \int_0^t P_s(x-z)V(z)\,ds\,dz$$
$$\le C \int t^{-\beta d/2} V(z)|x-z|^{2-d} e^{-c|x-z|^2/t}\,dz.$$

Since $\sup_{t>2} t^{-\beta d/2} e^{-c|x-z|^2/t} \le C(1+|x-z|)^{-\beta d}$, we obtain

$$\| \sup_{t>2} R_t^{[2'']}(x,y) \|_{L^1(dx)} \le C \int \int (1+|x-z|)^{-\beta d} |x-z|^{2-d} V(z)\,dz\,dx \le C\|V\|_{L^1}. \tag{3.9}$$

The lemma follows from (3.6)–(3.9). □

Lemma 3.10. *There exists a constant $C > 0$ such that*

$$\| \sup_{0<t\le 2} |R_t f(x)| \|_{L^1(dx)} \le C\|f\|_{L^1}. \tag{3.11}$$

Proof. Fix $y \in \mathbb{R}^d$.

$$R_t(x,y) = \int_0^{t/2} \int \big(P_t(x-z) - P_{t-s}(x-z)\big) V(z) K_s(z,y)\,dz\,ds$$
$$+ \int_{t/2}^{t} \int \big(P_t(x-z) - P_{t-s}(x-z)\big) V(z) K_s(z,y)\,dz\,ds$$
$$= R_t^{[3]}(x,y) + R_t^{[4]}(x,y).$$

Similarly to that what was done above there exists $\psi \in \mathcal{S}(\mathbb{R}^d)$, $\psi \ge 0$, such that

$$\big|R_t^{[3]}(x,y)\big| \le \int_0^1 \int \psi_t(x-z)V(z)K_s(z,y)\,ds\,dz \le \int \psi_t(x-z)V(z)|z-y|^{2-d}\,dz. \tag{3.12}$$

Let us note that for any fixed r, $1 < r < dp/(dp+d-2p)$, the function $z \mapsto |z-y|^{2-d}V(z)$ is supported by the unit ball and its L^r-norm is bounded by a constant independent of y. Then, by using standard methods, we obtain

$$\| \sup_{t<2} \big|R_t^{[3]}(x,y)\big| \|_{L^1(dx)} \le C. \tag{3.13}$$

To estimate $R_t^{[4]}$, we observe that for $2^{-m-1} < t \le 2^{-m}$ we have

$$\begin{aligned} |R_t^{[4]}(x,y)| &\le \int_{t/2}^{t}\int \big(P_t(x-z)V(z)\phi_t(z-y) + P_{t-s}(x-z)V(z)\phi_t(z-y)\big)\,dz\,ds \\ &\le C\int \big(tP_t(x-z) + |x-z|^{2-d}e^{-c|x-z|^2/t}\big)V(z)\phi_t(z-y)\,dz \\ &\le C\int \left(2^{-m}P_{2^{-m}}(x-z) + 2^{-m(2-d)/2}\left(\frac{|x-z|}{2^{-m/2}}\right)^{2-d} e^{-c2^m|x-z|^2}\right) \\ &\quad \times V(z)\phi_{2^{-m}}(z-y)\,dz, \end{aligned} \tag{3.14}$$

where ϕ is a non-negative Schwartz class function. Therefore applying (3.14) and the Hölder inequality, we obtain

$$\begin{aligned} \int \sup_{0<t\le 2}\left|R_t^{[4]}(x,y)\right| dx &\le \sum_{m\ge -1}\int \sup_{2^{-m-1}<t\le 2^{-m}}\left|R_t^{[4]}(x,y)\right| dx \\ &\le C\sum_{m\ge -1} 2^{-m}\|V\|_{L^p}\|\|\phi_{2^{-m}}(z,y)\|_{L^{p'}(dz)} \\ &\le C\|V\|_{L^p}\sum_{m\ge -1} 2^{-m+md/(2p)} \\ &\le C\|V\|_{L^p}. \end{aligned}$$

This ends the proof of the lemma. □

Lemma 3.15. *There exists a constant $C > 0$ such that*

$$\|\sup_{t>0}|Q_t f(x)|\|_{L^1(dx)} \le C\|f\|_{L^1}.$$

Proof. We shall show that $\int \sup_{t>0} Q_t(x,y)\,dx \le C$ with C independent of y, where $Q_t(x,y)$ denotes the integral kernel of Q_t. Let us note that

$$\begin{aligned} 0 \le Q_t(x,y) &\le \int\int_t^\infty P_t(x-z)V(z)P_s(z-y)\,ds\,dz \\ &\le C\int P_t(x-z)\frac{V(z)}{(t^{1/2}+|z-y|)^{d-2}}\,dz. \end{aligned} \tag{3.16}$$

Fix $y \in \mathbb{R}^d$. Then

$$\begin{aligned} \int \sup_{t>1} Q_t(x,y)\,dx &\le C\int\int \sup_{t>1} t^{(2-d)/2}P_t(x-z)V(z)\,dz\,dx \\ &\le C\int\int (1+|x-z|)^{-d-d+2}V(z)dz\,dx \\ &\le C\|V\|_{L^1}. \end{aligned}$$

To deal with $\sup_{0<t\le 1} Q_t(x,y)$, let us observe that for $0 < t \le 1$ we have

$$Q_t(x,y) \le C\int P_t(x,y)V(z)|z-y|^{2-d}\,dz.$$

Thus we can repeat the same arguments as we used for the estimation of $R_t^{[3]}$ in the proof of Lemma 3.10 and obtain

$$\| \sup_{t\le 1} Q_t(x,y)\|_{L^1(dx)} \le C.$$

This completes the proof of the lemma. □

As a consequence of (3.2), Lemmata 3.3, 3.10, 3.15, and Corollary 2.18, we get:

Corollary 3.17. *A function f belongs to H^1_L if and only if $(I - VL^{-1})f \in H^1(\mathbb{R}^d)$. Moreover,*

$$\|f\|_{H^1_L} \sim \|(I - VL^{-1})f\|_{H^1(\mathbb{R}^d)}.$$

4. Proof of Theorem 1.5

Before we start the proof of Theorem 1.5 we recall some basic facts from the theory of the classical Hardy spaces $H^1(\mathbb{R}^d)$ (see [6]). Let $1 < q \le \infty$. We say that that a function a is a $(1,q)$-atom if there exists a ball $B(y_0, r)$ such that

$$\operatorname{supp} a \subset B(y_0, r), \tag{4.1}$$

$$\int a(x)\,dx = 0, \tag{4.2}$$

$$\|a\|_{L^q} \le |B(y_0, r)|^{\frac{1}{q}-1}. \tag{4.3}$$

The atomic $\|f\|_{H^1_{(q)}}$-norm is defined by

$$\|f\|_{H^1_{(q)}} = \inf \sum_j |\lambda_j|, \tag{4.4}$$

where the infimum is taken over all representations $f = \sum_j \lambda_j a_j$, where a_j are $(1,q)$-atoms. This is well known that for every $1 < q \le \infty$ there exists a constant $C_q > 0$ such that

$$C_q^{-1}\|f\|_{H^1_{(q)}} \le \|f\|_{H^1(\mathbb{R}^d)} \le C_q \|f\|_{H^1_{(q)}}. \tag{4.5}$$

Proof of the second inequality in (1.6). In order to prove the second inequality in (1.6) it suffices to show, by (3.1) and Corollary 3.17, that for every a being a classical $(1,\infty)$-atom the function wb belongs to $H^1(\mathbb{R}^d)$, where $b = (I - V\Delta^{-1})a$, and

$$\|wb\|_{H^1(\mathbb{R}^d)} = \|wa - wV\Delta^{-1}a\|_{H^1(\mathbb{R}^d)} \le C \tag{4.6}$$

with a constant C independent of a. Let a be a $(1,\infty)$-atom associated with a ball $B(y_0, r)$. Then, by (2.3) and (3.1),

$$\int w(x)b(x)\,dx = 0. \tag{4.7}$$

We shall consider three cases. We would like to remark that we shall use only the estimate $|\Gamma_0(x,y)| \le C|x-y|^{2-d}$ for the fundamental solution Γ_0 of Δ.

Case 1. $r \geq 1$, $|y_0| \leq 4r$.

Then $\operatorname{supp} b \subset B(y_0, 5r)$. Moreover,

$$|\Delta^{-1}a(x)| \leq Cr^{-d} \int_{|y|\leq 5r} |x-y|^{2-d}\, dy \leq Cr^{2-d}. \tag{4.8}$$

Therefore, setting $q = d/2$ and using (2.7), we get

$$\|wb\|_{L^q} = \|w(I - V\Delta^{-1})a\|_{L^q} \leq Cr^{-d+2} = C|B(y_0, 5r)|^{\frac{1}{q}-1}. \tag{4.9}$$

We conclude from (4.7) and (4.9) that wb is a multiple of a $(1, q)$-atom associated with the ball $B(y_0, 5r)$. Hence (4.6) holds.

Case 2. $r \geq 1$, $|y_0| > 4r$.

We write

$$\begin{aligned} w(x)b(x) &= \big(w(x)a(x) - c_0 w(x)\chi_{B(y_0,r)}(x)\big) \\ &\quad + \big(c_0 w(x)\chi_{B(y_0,r)}(x) - w(x)V(x)\Delta^{-1}a(x)\big) \\ &= a_1(x) + a_2(x), \end{aligned} \tag{4.10}$$

where $c_0 = \big(\int_{B(y_0,r)} w(x)\, dx\big)^{-1}\big(\int w(x)a(x)\, dx\big)$.

Obviously,

$$|\Delta^{-1}a(x)| \leq C|y_0|^{2-d} \quad \text{for } |x| \leq 1.$$

Since, by (4.7) and (2.7),

$$\Big|\int w(x)a(x)\, dx\Big| = \Big|\int w(x)V(x)\Delta^{-1}a(x)\, dx\Big| \leq C|y_0|^{2-d},$$

we have $|c_0| \leq C|y_0|^{2-d}|B(y_0, r)|^{-1}$. Therefore a_1 is a multiple of a $(1, \infty)$-atom associated with the ball $B(y_0, r)$.

Next observe that a_2 is a multiple of a $(1, \frac{d}{2})$-atom associated with $B(y_0, 2|y_0|)$. Indeed, $\operatorname{supp} a_2 \subset B(y_0, 2|y_0|)$,

$$\int a_2(x)\, dx = 0$$

and

$$\|a_2\|_{L^{d/2}} \leq C|c_0||B(y_0, r)|^{2/d} + C|y_0|^{2-d} \leq C|y_0|^{2-d}.$$

Thus (4.6) is satisfied.

Case 3. $r \leq 1$.

Let us note that

$$|w(x)V(x)\Delta^{-1}a(x)| \leq CV(x)|x - y_0|^{2-d}. \tag{4.11}$$

Set $s = [\log_2 r^{-1}]$. Then

$$w(x)a(x) = \big(w(x)a(x) - c_0\chi_{B(y_0,r)}(x)\big) + \sum_{j=0}^{s} \big(c_j\chi_{B(y_0,2^j r)}(x) - c_{j+1}\chi_{B(y_0,2^{j+1}r)}(x)\big) + c_{s+1}\chi_{B(y_0,2^{s+1}r)}(x), \tag{4.12}$$

where

$$c_0 = |B(y_0,r)|^{-1}\int w(x)a(x)\,dx, \quad c_j = c_0|B(y_0,r)|\big|B\big(y_0,2^j r\big)\big|^{-1}.$$

Applying (2.6), we obtain

$$|c_0| \le Cr^\gamma |B(y_0,r)|^{-1}, \quad |c_j| \le Cr^\gamma \big|B\big(y_0,2^j r\big)\big|^{-1}. \tag{4.13}$$

We check at once that

$$\|wa - c_0\chi_{B(y_0,r)}\|_{H^1(\mathbb{R}^d)} \le C, \tag{4.14}$$

$$\sum_{j=0}^{s} \|c_j\chi_{B(y_0,2^j r)} - c_{j+1}\chi_{B(y_0,2^{j+1}r)}\|_{H^1(\mathbb{R}^d)} \le Csr^\gamma \le C. \tag{4.15}$$

Moreover,

$$\int \big(w(x)V(x)\Delta^{-1}a(x) - c_{s+1}\chi_{B(y_0,2^{s+1}r)}\big)\,dx = 0. \tag{4.16}$$

We easily observe, using (4.11) and (4.16), that

$$|c_{s+1}| \le C(1+|y_0|)^{2-d}.$$

Therefore if $|y_0| \le 3$, then $\operatorname{supp}\big(wV\Delta^{-1}a - c_{s+1}\chi_{B(y_0,2^{s+1}r)}\big) \subset B(0,5)$ and

$$\big\|wV\Delta^{-1}a - c_{s+1}\chi_{B(y_0,2^{s+1}r)}\big\|_{L^q} \le C_q \tag{4.17}$$

provided $1 < q < \frac{dp}{d+(d-2)p}$. From (4.16) and (4.17) we see that

$$\big\|wV\Delta^{-1}a - c_{s+1}\chi_{B(y_0,2^{s+1}r)}\big\|_{H^1(\mathbb{R}^d)} \le C. \tag{4.18}$$

If $|y_0| > 3$, then $\operatorname{supp}\big(wV\Delta^{-1}a - c_{s+1}\chi_{B(y_0,2^{s+1}r)}\big) \subset B(0,2|y_0|)$. Moreover,

$$\big\|wV\Delta^{-1}a - c_{s+1}\chi_{B(y_0,2^{s+1}r)}\big\|_{L^q} \le C_q|y_0|^{2-d}, \tag{4.19}$$

with $q = \frac{d}{2}$. The estimate (4.19) together with (4.16) imply that the function $wV\Delta^{-1}a - c_{s+1}\chi_{B(y_0,2^{s+1}r)}$ is a multiple of a $(1,\frac{d}{2})$-atom. Therefore (4.6) is proved. The proof of the second inequality in (1.6) is complete.

Our task is now to prove the first inequality in (1.6). Assume that $wf \in H^1(\mathbb{R}^d)$. Then $wf = \sum_j \lambda_j a_j$, $\sum |\lambda_j| \le C\|wf\|_{H^1(\mathbb{R}^d)}$, where a_j are classical $(1, \infty)$-atoms. Thus $f = \sum \lambda_j b_j$, where $b_j = a_j/w$ are H^1_L-atoms. Hence it suffices to verify that

$$\|b\|_{H^1_L} \le C, \tag{4.20}$$

for every b being an H^1_L-atom. By Corollary 3.17 the proof of (4.20) reduces to proving that

$$\|b - VL^{-1}b\|_{H^1(\mathbb{R}^d)} \le C. \tag{4.21}$$

Let us observe that

$$\int \big(b(x) - V(x)L^{-1}b(x)\big)\, dx = 0. \tag{4.22}$$

Since $|\Gamma(x, y)| \le c|x - y|^{2-d}$ and the function $w_1(x) = w(x)^{-1}$ is Hölder and satisfies $1 \le w_1(x) \le C$, the proof of (4.21) goes by the same analysis as the proof of (4.6). The details are omitted. □

We now turn to show that if $V \ne \tilde{V}$ are compactly supported L^p-potentials, then the corresponding H^1_L and $H^1_{\tilde{L}}$ spaces do not coincide. Assume that $H^1_L = H^1_{\tilde{L}}$ for some compactly supported non-negative function V and $\tilde{V}$ that belong to $L^p(\mathbb{R}^d)$ for some $p > d/2$. Then Theorem 1.5 combined with (2.5) imply $w = \tilde{w}$. Using (3.1) and (2.3) we obtain

$$1 = (I - \Delta^{-1}V)w,$$
$$1 = (I - \Delta^{-1}\tilde{V})w,$$

and, consequently,

$$0 = \Delta^{-1}\big((V - \tilde{V})w\big).$$

Since $(V - \tilde{V})w$ is a compactly supported L^p-function, we get $V = \tilde{V}$, by (2.7).

References

1. Dziubański, J., Zienkiewicz, J.: Hardy spaces associated with some Schrödinger operators. Stud. Math. **125**, 149–160 (1998)
2. Dziubański, J., Zienkiewicz, J.: Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality. Rev. Mat. Iberoam. **15**, 279–296 (1999)
3. Dziubański, J., Zienkiewicz, J.: H^p spaces for Schrödinger operators. Fourier Analysis and Related Topics, Banach Cent. Publ. **56**, 45–53 (2002)
4. Dziubański, J., Zienkiewicz, J.: H^p spaces associated with Schrödinger operators with potentials from reverse Hölder classes. Colloq. Math. **98**, 5–38 (2003)
5. Goldberg, D.: A local version of real Hardy spaces. Duke Math. J. **46**, 27–42 (1979)
6. Stein, E.: Harmonic analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals. Princeton, NJ: Princeton University Press 1993