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ORIGINAL PAPER

Product innovation with lumpy investment

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Abstract The paper provides a framework that enables us to analyze the important topic of capital accumulation under technological progress. We describe an algorithm to solve Impulse Control problems, based on a (multipoint) boundary value problem approach. Investment takes place in lumps and we determine the optimal timing of technology adoptions as well as the size of the corresponding investments. Our numerical approach led to some guidelines for new technology investments. First, we find that investments are larger and occur in a later stadium when more of the old capital stock needs to be scrapped. Moreover, we obtain that the size of the firm's investments increase when the technology produces more profitable products. We see that the firm

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in the beginning of the planning period adopts new technologies faster as time proceeds, but later on the opposite happens. Furthermore, we find that the firm does not invest such that marginal profit is zero, but instead marginal profit is negative.

1 Introduction

In today's knowledge economy innovation is of prime importance. Innovation has led to the extraordinary productivity gains in the 1990s . In current business practice it is felt that the heat is on and that firms must innovate faster just to stand still (The Economist, October 13th 2007, Innovation: Something new under the sun). Therefore, technological progress is a crucial input for firms in taking their investment decisions. Greenwood et al. (1997) argue that technological progress is the main driver of economic growth. They discovered that in the post-war period in the US about 60 % of labor productivity growth was investment specific. Yorokoglu (1998) notes that information technology is a prime example where embodied technological progress led to an improvement of computing technology on the order of 20 times within less than a decade in the 1980s–1990s.

This paper combines technology adoption with capital accumulation, taking into account technological progress. The aim of this paper is to study the decision of when to introduce a new product. To do so we employ the Impulse Control modeling approach that is perfectly suitable to take into account the disruptive changes caused by innovations. This also enables us to determine the length of the time interval that the firm uses a particular technology, when it is time to launch a new product generation, and how these decisions interact with the firm's capital accumulation behavior. In Kort (1989) a dynamic model of the firm is designed in which capital stock jumps upward at discrete points in time at which the firm invests. However, technological progress is not taken into account.

An example where a firm has to decide about investments in new generations of products is the LCD industry. With every new generation the size of the mother glass or substrate increases. As the LCD panels are cut out of the substrate, the substrate on the one hand determines which panel sizes can be produced and on the other hand how efficiently each possible panel size can be produced. We have a process innovation, because a larger glass area provides a more efficient solution of the "cutting problem", and thus lower costs in the production process. A product innovation arises, because the larger area of the substrate makes it possible to produce larger screens. For a firm it is important to determine when it is optimal to introduce a new product. However, since the new product will decrease the demand of the old product, the moment of introduction is crucial.

Feichtinger et al. (2006) employs a vintage capital goods structure to study the effect of embodied technological progress on the investment behavior of the firm. They show that in the case that a firm has market power a negative anticipation effect occurs, i.e. when technological progress goes faster in the future, it is optimal for the firm to decrease investments in the current generation of capital goods. However, a direct implication of the vintage capital approach is that the firm adopts an infinite amount of different technologies. Of course, in practice a firm can adopt a new technology a limited number of times.

Grass et al. (2012) also combines technology adoption with capital accumulation, while taking into account technological progress. They find that investment jumps upward right at the moment that a new technology is adopted, and that the larger the firm the later the investment in a new technology takes place. Moreover, they find that when a firm has market power, the firm cuts down on investment before a new technology is adopted. Whereas Grass et al. (2012) limits itself to process innovation, we concentrate on studying product innovation. Grass et al. (2012) uses a multi-stage optimal control approach where a firm adopts a new technology in each stage. Unlike Feichtinger et al. (2006), the number of technology adoptions is limited. However, the number of innovations is not determined by the model, but fixed exogenously instead. Unlike Feichtinger et al. (2006) and Grass et al. (2012), in this paper capital accumulation only occurs in lumps. Moreover, these lumps are determined by the model, i.e. the lumpy investments are endogenous. In Saglam (2011) a multi-stage optimal control model is studied where the number of technology adoptions is endogenous. However, unlike our paper, the model does not incorporate any (fixed) cost associated with the adoption and the considered firm has no market power. In Boucekkine et al. (2004) a two-stage optimal control model is considered, where only one adoption occurs, without adoption (fixed) cost. Both Boucekkine et al. (2004) and Saglam (2011) incorporate learning, were the firm raises productivity of a given technology over time due to learning and revenue is linear in the capital stock.

Our paper is mostly comparable to Grass et al. (2012). However, unlike Grass et al. (2012), we can endogenously determine the number of technology adoptions over the planning period and we do incorporate a fixed cost associated with a technology adoption. However, the most significant difference with Grass et al. (2012) is that in the present model investment takes place in lumps. The resulting upward jump in the capital stock approaches reality more than a continuous development of capital stock over time, the latter being the result of the analysis in Grass et al. (2012).

The method used to study firm behavior in this paper is Impulse Control. Impulse Control theory is a variant of optimal control theory where discontinuities (i.e. jumps) in the state variable are allowed. In Impulse Control the moments of these jumps as well as the sizes of the jumps are decision variables. Blaquière (1977a, b, 1979, 1985) extends the standard theory on optimal control by deriving a Maximum Principle, the so-called Impulse Control Maximum Principle, that gives necessary and sufficient optimality conditions for solving such problems. Blaquière's Impulse Control analysis is based on the present value Hamiltonian form. In this paper we apply the Impulse Control theorem in the current value Hamiltonian framework as derived in Chahim et al. (2012).

One of the striking results is that the firm does not invest such that the marginal profit is zero, but instead marginal profit is negative. Furthermore, we obtain that the firm in the beginning of the planning period adopts new technologies faster as time proceeds, but after some moment in time later technologies are used for a longer time period. This behavior is different from Grass et al. (2012), who finds that the firm adopts new technologies faster as time proceeds for the whole planning period, but this also differs from the results found in Saglam (2011), who finds that later technologies are used during a longer time period. Our results are somehow a combination of both.

This paper is organized as follows. In Sect. 2 we briefly introduce Impulse Control. Section 3 gives the general setting and builds up the product innovation Impulse Control model. Section 4 derives the necessary optimality conditions for the product innovation problem, whereas Sect. 5 gives a brief description of the algorithms present in the literature dealing with the Impulse Control Maximum Principle and describes an algorithm to solve Impulse Control problems, based on a (multipoint) boundary value problem (BVP) approach. In Sect. 6 we study the investment behavior of a product innovating firm, and in Sect. 7 we extend this analysis by adding decreasing demand, i.e. demand decreases over time due to competitors producing better products because of technological progress. Finally, in Sect. 8 we conclude and give some recommendations for future research.

2 A brief introduction to impulse control

In this section we introduce a general impulse control model and provide necessary optimality conditions.

Let us denote x as the state variable, u as an ordinary control variable and v^i as the impulse control variable, where x and u are piecewise continuous functions of time.[1] We denote r as the discount rate leading to the discount factor e^{-rt} at time t. The terminal time or horizon date of the system or process is denoted by $T > 0$, and $x(T^+)$ stands for the state value immediately after a possible jump at time T. The profit of the system between jumps is given by $F(x, u, t)$, whereas $G(x, v, t)$ is the profit function associated with a jump, and $S(x(T^+))$ is the salvage value, i.e. the total costs or profit associated with the system after time T. Finally, $f(x, u, t)$ describes the continuous change of the state variable over time between the jump points and $g(x, v, t)$ is a function that represents the instantaneous (finite) change of the state variable when there is an impulse or jump.

The above results in the following optimal control problem

$$\max_{u(\cdot), N, \tau_i, v^i} \left\{ \int_0^T e^{-rt} F(x(t), u(t), t)\, dt + \sum_{i=1}^{N} e^{-r\tau_i} G(x(\tau_i^-), v^i, \tau_i) + e^{-rT} S(x(T^+)) \right\},$$
(1a)

$$\text{s.t.} \dot{x}(t) = f(x(t), u(t), t), \qquad \text{for} \quad t \in [0, T] \setminus \{\tau_1, \ldots, \tau_N\},$$
(1b)

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[1] Note that the necessary optimality conditions presented in Theorem 1 also hold for measurable controls. Applications typically have piecewise continuous functions.

$$x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i^-), v^i, \tau_i), \quad \text{for} \ \ i \in \{1, \ldots, N\}, \tag{1c}$$

$$x(0^-) = x_0, \quad u(t) \in \mathcal{U}, \quad v^i \in \mathcal{V}, \quad i \in \{1, \ldots, N\}. \tag{1d}$$

For $N \in \mathbb{N}$ we assume the jump times to be sorted as

$$\tau_i \in [0, T] \quad \text{with} \ \ 0 \le \tau_1 < \cdots < \tau_N \le T,$$
$$x(\tau_i^+) = \lim_{t \downarrow \tau_i} x(t) \quad \text{and} \ \ x(\tau_i^-) = \lim_{t \uparrow \tau_i} x(t), \quad \text{for} \ \ i = 1, \ldots, N, \tag{1e}$$

and

$$x_0 \in \mathbb{R}^n.$$

We assume that the domains $\mathcal{U} \subset \mathbb{R}^m$ and $\mathcal{V} \subset \mathbb{R}^l$ are bounded convex sets. Further we impose that F, f, g and G are continuously differentiable in x on $\mathbb{R}^n$ and v^i on $\mathcal{V}$, $S(x)$ is continuously differentiable in x on $\mathbb{R}^n$, and that g and G are continuous in τ. Finally, when there is no jump, i.e. $v = 0$, we assume that

$$g(x, 0, t) = 0,$$

for all x and t.

2.1 Necessary optimality conditions

We apply the impulse control maximum principle in current value formulation in normal form derived in Chahim et al. (2012) to (1a)–(1e).[2] The resulting necessary optimality conditions are presented in Theorem 1.

Before we state Theorem 1, let us define the Hamiltonian $\mathcal{H}$ and the Impulse Hamiltonian $\mathcal{IH}$ by

$$\mathcal{H}(x, u, \lambda, t) := F(x, u, t) + \lambda f(x, u, t), \tag{2a}$$

$$\mathcal{IH}(x, v, \lambda, t) := G(x, v, t) + \lambda g(x, v, t), \tag{2b}$$

and define the following abbreviations

$$\mathcal{H}[s] := \mathcal{H}(x(s), u(s), \lambda(s), s), \tag{2c}$$

$$\mathcal{IH}[s, v] := \mathcal{IH}(x(s^-), v, \lambda(s^+), s), \tag{2d}$$

$$G[s, v] := G(x(s^-), v, s), \tag{2e}$$

$$g[s, v] := g(x(s^-), v, s). \tag{2f}$$

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[2] Other references deriving the necessary optimality conditions for the Impulse Control problems are Blaquière (1977a, b, 1979, 1985), Seierstad (1981) and Seierstad and Sydsæter (1987).

Theorem 1 *(Impulse control maximum principle) Let for $N \in \mathbb{N}$ with $N > 0$ $(x^*(\cdot), u^*(\cdot), N, \tau_1^*, \ldots, \tau_N^*, v^{1*}, \ldots, v^{N*})$ be an optimal solution of (1). Then there exists a (piecewise absolutely continuous) adjoint variable $\lambda(\cdot)$ such that the following conditions hold:*

$$u^*(t) \in \underset{u}{\arg\max} \mathcal{H}(x^*(t), u, \lambda(t), t), \quad t \in [0, T], \tag{3a}$$

$$\dot{\lambda}(t) = r\lambda(t) - \frac{\partial}{\partial x}\mathcal{H}(x^*(t), u^*(t), \lambda(t), t), \quad t \in [0, T] \setminus \{\tau_1^*, \ldots, \tau_N^*\}. \tag{3b}$$

For every $t = \tau_i^*$, $(i = 1, \ldots N)$, we have

$$\frac{\partial}{\partial v}\mathcal{IH}(x^*(\tau_i^{*-}), v^{i*}, \lambda(\tau_i^{*+}), \tau_i^*)(v - v^{i*}) \leq 0, \quad v \in \mathcal{V}, \tag{3c}$$

$$\lambda(\tau_i^{*+}) - \lambda(\tau_i^{*-}) = -\frac{\partial}{\partial x}\mathcal{IH}(x^*(\tau_i^{*-}), v^{i*}, \lambda(\tau_i^{*+}), \tau_i^*), \tag{3d}$$

$$\mathcal{H}[\tau_i^{*+}] - \mathcal{H}[\tau_i^{*-}] + rG[\tau_i^*, v^{i*}] - \frac{\partial}{\partial \tau}\mathcal{IH}[\tau_i^*, v^{i*}] \begin{cases} \geq 0 & \tau_i^* = 0 \\ = 0 & \tau_i^* \in (0, T) \\ \leq 0 & \tau_i^* = T. \end{cases} \tag{3e}$$

For $t \in [0, T] \setminus \{\tau_1^*, \ldots, \tau_N^*\}$ it holds that

$$\frac{\partial}{\partial v}\mathcal{IH}(x^*(t), 0, \lambda(t), t)v \leq 0, \quad v \in \mathcal{V}. \tag{3f}$$

The transversality condition is

$$\lambda(T^+) = \frac{\partial}{\partial x}S(x^*(T^+)). \tag{3g}$$

Proof See Blaquière (1977a, 1985). □

3 The product innovation model

Consider a firm that invests in lumps over time. Each time it invests it installs a production plant suitable to produce the new product. Due to product innovation the quality of the products, and thus also demand, increases over time. This implies that the later an investment takes place, the better products can be made due to these investments.

This is formalized as follows. A plant being installed at time τ will make products for which the price is given by the following inverse demand function:

$$p(t) = \theta(\tau) - q(t), \tag{4}$$

where $q(t)$ is the output at time t and $\theta(\tau) = 1 + b\tau$ is the state of technology that the firm adopts at time τ.[3] We further assume that technology within the firm does not change between two technology adoptions, i.e. $\dot{\theta} = 0$ for all $t \neq \tau$. At the moment the firm adopts a technology, the firm's technology change is given by

$$\theta(\tau_i^+) - \theta(\tau_i^-) = 1 + b\tau_i - \theta(\tau_i^-) = b(\tau_i - \tau_i^-) = b(\tau_i - \tau_{i-1}).$$

Hence, as in Feichtinger et al. (2006) and Grass et al. (2012), we impose that technological progress increases linearly over time, where b is a positive constant. In Feichtinger et al. (2006) it is argued that this holds when we consider the case that technological progress is based on Moore's law, implying that technology develops in an exponential way over time. On the other hand, a Philips manager[4] argued that utility is a logarithmic function of technology. In total this results in a linear increase of technological progress. In Saglam (2011) technology increases exponentially over time and in Boucekkine et al. (2004) there are only two different technologies available. We assume a simple production function in the sense that one capital good produces one unit of output. Denoting the stock of capital goods by $K(t)$, this gives

$$K(t) = q(t). \tag{5}$$

We impose that only the capital stock of the new plant is able to produce the new products, i.e. each plant has its own capital stock that produces the products with a quality associated with the timing of the investment in this plant. In this setting we can also model a situation where just $100\gamma\%$, where $\gamma \in [0, 1]$, of the capital stock is scrapped, while the remaining machines or tools can be reused for the new product. Hence, full scrapping corresponds to the case where $\gamma = 1$. This implies that old products, and thus also old capital goods, become worthless after the new plant is installed, implying that the old capital goods can be scrapped.

Denoting investment by $I(t)$, at the moment the firm invests (adopts a new technology) capital stock changes by

$$K(\tau^+) - K(\tau^-) = I(\tau) - \gamma K(\tau^-).$$

At time zero the capital stock is equal to zero, i.e.

$$K(0) = 0.$$

For each plant it holds that capital stock depreciates with rate δ, i.e.

$$\dot{K} = -\delta K.$$

Investing in a plant implies that the firm has to pay a fixed cost, i.e. part of the cost is independent of the plant size, and a variable cost that more than proportionally

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[3] We assume that technology is continuously changing, i.e. $\theta(t) = 1 + bt$. However, the technology within the firm is the technology that the firm adopts at time τ.

[4] Theo Claassen in the Dutch magazine Elsevier (January 24, 1998).

increases with the size of the plant. In particular, we assume that the investment cost is given by

$$C(I) = \begin{cases} C + \alpha I + \beta I^2 & \text{for } I > 0, \\ 0 & \text{for } I = 0. \end{cases}$$

This type of investment cost function, but without the fixed cost, is common in the literature (e.g., among others, see Grass et al. (2012) and Sethi and Thompson (2006, pp. 83–88)), in which besides the fixed cost, the linear term consist of acquisition cost, where the unit price is equal to α and the quadratic term represents the adjustment cost. Instead of a quadratic term, any other convexly increasing function could have been imposed.

Total discounted revenue is given by

$$\int_0^T e^{-rt} \left[\theta(t) - K(t)\right] K(t) dt, \tag{6}$$

where instantaneous revenue is determined by output price times output. Concerning the objective (6) please note that $\theta(t)$ is constant in the time interval between two jumps in such a way that it equals its value obtained at the jump, when the interval starts. This means that on the time interval (τ_i, τ_{i+1}) it holds that $\theta(t) = \theta(\tau_i)$. Since we have a finite time planning period, a salvage value has to be defined. This salvage value is equal to the value of the firm at the time horizon T. We assume that this value is given by

$$+ e^{-rT} \frac{[\theta(\tau_N) - K(T^+)]K(T^+)}{r + \delta}. \tag{7}$$

The salvage value (7) is a lower bound of the discounted revenue stream of the firm after the planning period.

Total discounted investment cost is given by the sum of the cost of adopting a new technology, discounted at the time the adoption takes place. This results in

$$- \sum_{i=1}^N e^{-r\tau_i} C(I(\tau_i)). \tag{8}$$

The above gives rise to the following impulse control model:

$$\max_{I(\cdot), \tau_i, N} \left\{ \int_0^T e^{-rt} \left[\theta(t) - K(t)\right] K(t) dt - \sum_{i=1}^N e^{-r\tau_i} C(I(\tau_i)) \right.$$
$$\left. + e^{-rT} \frac{[\theta(\tau_N) - K(T^+)]K(T^+)}{r + \delta} \right\}, \tag{9}$$

subject to

$$\dot{K}(t) = -\delta K(t) \quad \text{for all} \quad t \neq \tau_1, \ldots, \tau_N, \tag{10}$$

$$\dot{\theta} = 0 \quad \text{for all} \quad t \neq \tau_1, \ldots, \tau_N, \tag{11}$$

$$K(\tau_i^+) - K(\tau_i^-) = I(\tau_i) - \gamma K(\tau_i^-) \quad \text{for all} \quad i = 1, \ldots, N, \tag{12}$$

$$\theta(\tau_i^+) - \theta(\tau_i^-) = 1 + b\tau_i - \theta(\tau_i^-) \quad \text{for all} \quad i = 1, \ldots, N, \tag{13}$$

$$K(0) = 0, \tag{14}$$

$$\theta(0) = 1. \tag{15}$$

This is an Impulse Control problem as described in Blaquière (1977a, b, 1979, 1985). Note that this innovation model only contains an impulse control variable and no ordinary control variable. This approach differs from the multi-stage approach used in Grass et al. (2012), because here investment takes place in lumps and every investment goes along with a fixed cost. As in Grass et al. (2012) we can model all situations between the extreme cases where after every new investment the old capital goods are scrapped ($\gamma = 1$) and where all the capital can be kept ($\gamma = 0$) to produce the new product. Another benefit of the above model compared to Grass et al. (2012) is that the number of technology adoptions over the planning period is endogenously determined.

4 Necessary optimality conditions for the product innovation problem

We apply the impulse control maximum principle in current value formulation derived in Chahim et al. (2012). Other good references deriving the necessary optimality conditions for the Impulse Control problems are Blaquière (1977a, b, 1979, 1985), Seierstad (1981) and Seierstad and Sydsæter (1987). We define the Hamiltonian $\mathcal{H}$ and the Impulse Hamiltonian $\mathcal{IH}$ for the product innovation problem as

$$\mathcal{H}[t] = [\theta(t) - K(t)] K(t) - \lambda_1(t) \delta K(t), \tag{16}$$

$$\mathcal{IH}[\tau_i] = -C - \alpha I(\tau_i) - \beta I(\tau_i)^2 + \lambda_1 \left(I(\tau_i) - \gamma K(\tau_i^-) \right)$$
$$+ \lambda_2 \left(1 + b\tau_i - \theta(\tau_i^-) \right), \tag{17}$$

and obtain the adjoint equations

$$\dot{\lambda}_1(t) = (r + \delta) \lambda_1(t) - \theta(t) + 2K(t), \tag{18}$$

$$\dot{\lambda}_2(t) = r\lambda_2(t) - K(t). \tag{19}$$

The jump conditions are

$$-\alpha - 2\beta I(\tau_i) + \lambda_1(\tau_i^+) = 0, \tag{20}$$

$$\lambda_1(\tau_i^+) - \lambda_1(\tau_i^-) = \gamma \lambda_1(\tau_i^+), \tag{21}$$

$$\lambda_2(\tau_i^+) - \lambda_2(\tau_i^-) = \lambda_2(\tau_i^+), \tag{22}$$

from which we conclude that

$$\lambda_1 \left(\tau_i^- \right) = (1 - \gamma) \lambda_1 \left(\tau_i^+ \right),$$

which equals zero for $\gamma = 1$, and

$$\lambda_2 \left(\tau_i^- \right) = 0.$$

The condition for determining the optimal switching time τ_i is

$$\mathcal{H}[\tau_i^+] - \mathcal{H}[\tau_i^-] - \frac{\partial G \left(x \left(\tau_i^- \right), v_i, \lambda \left(\tau_i^+ \right), \tau_i \right)}{\partial \tau} + r G \left(x \left(\tau_i^- \right), v_i, \lambda \left(\tau_i^+ \right), \tau_i \right)$$

$$- \lambda \left(\tau_i^+ \right) \frac{\partial g[\tau_i^-]}{\partial \tau} \begin{cases} \geq 0 & \text{for } \tau_i^* = 0 \\ = 0 & \text{for } \tau_i^* \in (0, T) \\ \leq 0 & \text{for } \tau_i^* = T. \end{cases}$$

Using the above specification, we get

$$\left[\theta \left(\tau_i^+ \right) - K \left(\tau_i^+ \right) \right] K \left(\tau_i^+ \right) - \left[\theta \left(\tau_i^- \right) - K \left(\tau_i^- \right) \right] K \left(\tau_i^- \right)$$

$$- \lambda_1 \left(\tau_i^+ \right) \delta K \left(\tau_i^+ \right) + \lambda_1 \left(\tau_i^- \right) \delta K \left(\tau_i^- \right) - rC - r\alpha I \left(\tau_i \right) - r\beta I \left(\tau_i \right)^2 - b \lambda_2 \left(\tau_i^+ \right)$$

$$\begin{cases} \geq 0 & \text{for } \tau_i^* = 0 \\ = 0 & \text{for } \tau_i^* \in (0, T) \\ \leq 0 & \text{for } \tau_i^* = T. \end{cases} \tag{23}$$

The transversality conditions are

$$\lambda_1 \left(T^+ \right) = \frac{\theta(\tau_N) - 2K \left(T^+ \right)}{r + \delta} \tag{24}$$

$$\lambda_2 \left(T^+ \right) = \frac{K(T^+)}{r + \delta.} \tag{25}$$

At the non-jump points $t \neq \tau_1, \ldots, \tau_N$, it holds that $\lim_{I \to 0} \frac{\partial \mathcal{IH}}{\partial I} = \infty$ due to the fixed cost. Hence, the conditions for applying the Impulse Control Maximum Principle are met, see Section 2.3 of Chahim et al. (2012).

5 Algorithm

In the literature three different algorithms are derived based on the Impulse Control Principle (Blaquière (1977a, b, 1979, 1985) and Chahim et al. (2012)). Luhmer (1986) derived a forward algorithm (starts at time 0) and Kort (1989, pp. 62-70) derived a backward algorithm (starts at final time horizon T). Luhmer (1986) starts at $t = 0$ and uses the costate variable, as input to initialize his algorithm. Kort (1989) implements a backward algorithm that starts at the time horizon, i.e. $t = T$, and initializes the algorithm using the values of the state variables. Finally, Grass and Chahim (2012)

designs an algorithm that is a combination of continuation techniques and a multi-point Boundary Value Problem (BVP) to solve Impulse Control problems.

The problem described by (9)–(15) has two state variables, the stock of capital $(K(t))$ and technology $(\theta(t))$. The question is which algorithm is most suitable for this model. Applying the forward algorithm to problem (9)–(15) has a drawback. Namely, we have to guess the initial values for the two costate variables, λ_1 and λ_2. A wrong guess of the costate variables at the initial time results in a solution that does not satisfy the transversality conditions (24) and (25), which implies that the necessary optimality conditions are not satisfied. For the backward algorithm we start with choosing values for the state variables at time T. The resulting solution always satisfies the necessary optimality conditions, but here the problem is that the algorithm has to end up at the right $K(0)$ and $\theta(0)$. In other words, with the backward algorithm one can apply the right necessary conditions to the wrong problem. An example where the backward algorithm is applied successfully is Chahim et al. (2013). Moreover, in Chahim et al. (2013) clear upper and lower bounds have been derived for the state variable.

In addition, the backward algorithm has another drawback. When we apply it to the problem described by (9)–(15), starting at the time horizon and going back in time requires knowledge of the technology before the investment. In particular, we obtain from equation (23) that we need to know $\theta(\tau_N^+) = 1 + b\tau_N$ and $\theta(\tau_N^-) = \theta(\tau_{N-1}) = 1 + b\tau_{N-1}$. Hence, solving this equation for τ_N requires that we know τ_{N-1}. And with the backward algorithm, this predecessor is not known. We conclude that the backward algorithm is not suitable to solve our model.

5.1 (multipoint) Boundary value approach

In this section we describe a (multipoint) boundary value problem (BVP), that is useful to solve Impulse Control problems. The idea behind the boundary value approach is that at the time interval between two jumps the system of differential equations (canonical system) combined with the boundary conditions (initial and final conditions) is solved. After each found jump the (multipoint) BVP is updated to find the next jump.

To simplify the presentation and to concentrate on the main concepts of the numerical algorithm, we make the following assumptions.

Assumption 1 For every time horizon $T \geq 0$ there exists a unique optimal solution of (1), with a finite number of jumps (which in general depends on T).

Assumption 2 Let for $T > 0$ the jump times be $(\tau_i)_{i=1}^N$ with $0 < \tau_1 < \ldots < \tau_N < T$, and $\bar{x}(T) := (x(\tau_1^-), x(\tau_1^+), v_1, \ldots, x(\tau_N^-), x(\tau_N^+), v_N)$ be the vector of left and right limits of the states together with the optimal impulse control values for the given time horizon T. Then in a neighborhood of T the solution vector $\bar{x}(T)$ is continuous.

For the algorithm T is a continuation variable. During the continuation process T is increased and the conditions for possible jumps are monitored.

Assumption 3 Condition (3c) implies

$$\frac{\partial}{\partial v}\mathcal{IH}(x^*(\tau_i^{*-}), v^{i*}, \lambda(\tau_i^{*+}), \tau_i^*) = 0, \tag{26}$$

and with $\frac{\partial^2}{\partial v^2}\mathcal{IH}(x^*(\tau_i^{*-}), v^{i*}, \lambda(\tau_i^{*+}), \tau_i^*) < 0$ this yields

$$v^{i*} = v(x^*(\tau_i^{*-}), \lambda(\tau_i^{*+}), \tau_i^*). \tag{27}$$

In general condition (3c) does not imply that the optimal impulse control value can be found as the arg max of the Impulse Hamiltonian. However, for the model in this paper it holds, since the $\mathcal{IH}$ is concave for positive I.

To formulate the (multipoint) BVP we introduce the following notation for the canonical system dynamics:

$$\dot{x}(t) = h_1(x(t), \lambda(t), t), \tag{28a}$$

$$\dot{\lambda}(t) = h_2(x(t), \lambda(t), t). \tag{28b}$$

For the conditions at a jumping time τ we define:

$$j^x(x(\tau^+), x(\tau^-), \lambda(\tau^+), \tau) := x(\tau^+) - x(\tau^-) - g[\tau, x(\tau^+) - x(\tau^-)], \tag{28c}$$

$$j^\lambda(x(\tau^-), \lambda(\tau^+), \lambda(\tau^-), \tau) := \lambda(\tau^+) - \lambda(\tau^-) + \frac{\partial}{\partial x}\mathcal{IH}[\tau, x(\tau^+) - x(\tau^-)], \tag{28d}$$

$$j^\tau(x(\tau^-), x(\tau^+), \lambda(\tau^+), \lambda(\tau^-), \tau) := \mathcal{H}[\tau^+] - \mathcal{H}[\tau^-] + rG[\tau, v]$$

$$- \frac{\partial}{\partial \tau}\mathcal{IH}[\tau, x(\tau^+) - x(\tau^-)]. \tag{28e}$$

Now let $(x^*(\cdot), u^*(\cdot), N, \tau_1^*, \ldots, \tau_N^*, v^{1*}, \ldots, v^{N*})$ be the optimal solution of (1) with $0 < \tau_1^* < \ldots < \tau_N^* < T$. Then the necessary conditions yield the following (multipoint) BVP:

$$\dot{x}_i(t) = h_1(x_i(t), \lambda_i(t), t), \quad t \in [\tau_{i-1}, \tau_i], \ i = 1, \ldots, N+1, \tag{29a}$$

$$\dot{\lambda}_i(t) = h_2(x_i(t), \lambda_i(t), t), \quad t \in [\tau_{i-1}, \tau_i], \ i = 1, \ldots, N+1, \tag{29b}$$

$$j^x(x_i(\tau_i^+), x_i(\tau_i^-), \lambda_i(\tau_i^+), \tau_i) = 0, \quad i = 1, \ldots, N, \tag{29c}$$

$$j^\lambda(x_i(\tau_i^-), \lambda_i(\tau_i^+), \lambda_i(\tau_i^-), \tau_i) = 0, \quad i = 1, \ldots, N, \tag{29d}$$

$$j^{\tau_i}(x_i(\tau_i^-), x_i(\tau_i^+), \lambda_i(\tau_i^+), \lambda_i(\tau_i^-), \tau_i) = 0, \quad i = 1, \ldots, N, \tag{29e}$$

$$\mathcal{S}(x_{N+1}(T), \lambda_{N+1}(T)) = 0, \tag{29f}$$

$$x_1(0) - x_0 = 0, \tag{29g}$$

where (29f) denotes the transversality condition (3g), $\tau_0 = 0$ and $\tau_{N+1} = T$.

After defining $t(s) := \tau_i - (i - s)\Delta\tau_i$, with $\Delta\tau_i := \tau_i - \tau_{i-1}$, we rewrite (29) into

$$\dot{x}_i(s) = \Delta\tau_i h_1(x_i(s), \lambda_i(s), t(s)), \quad s \in [i-1, i], \; i = 1, \ldots, N+1, \quad (30a)$$

$$\dot{\lambda}_i(s) = \Delta\tau_i h_2(x_i(s), \lambda_i(s), t(s)), \quad s \in [i-1, i], \; i = 1, \ldots, N+1, \quad (30b)$$

$$j^x(x_i(i^+), x_i(i^-), \lambda_i(i^+), \tau_i) = 0, \quad i = 1, \ldots, N, \quad (30c)$$

$$j^\lambda(x_i(i^-), \lambda_i(i^+), \lambda_i(i^-), \tau_i) = 0, \quad i = 1, \ldots, N, \quad (30d)$$

$$j^i(x_i(i^-), x_i(i^+), \lambda_i(i^+), \lambda_i(i^-), \tau_i) = 0, \quad i = 1, \ldots, N, \quad (30e)$$

$$\mathcal{S}(x_{N+1}(N+1), \lambda_{N+1}(N+1)) = 0, \quad (30f)$$

$$x_1(0) - x_0 = 0. \quad (30g)$$

The jump times τ_i, $i = 1\ldots, N$, appear as unknown variables.

To handle the case $\tau_N = T$ we introduce the (unknown) variables

$$xT := x_{N+1}(T^+),$$
$$lT := \lambda_{N+1}(T^+),$$

together with the additional boundary conditions

$$j^x(xT, x_{N+1}(N+1), lT, T) = 0, \quad (31a)$$

$$j^\lambda(x_{N+1}(N+1), lT, \lambda_{N+1}(N+1), T) = 0, \quad (31b)$$

and replace (30f) by

$$\mathcal{S}(xT, lT) = 0. \quad (31c)$$

The case $\tau_1 = 0$ can be treated in an analogous way. We therefore set

$$x0 := x_1(0^+),$$
$$l0 := \lambda_1(0^+),$$

together with the additional boundary conditions

$$j^x(x0, x_0, l0, 0) = 0, \quad (32a)$$

$$j^\lambda(x_0, l0, \lambda_1(0), 0) = 0, \quad (32b)$$

and replace (30g) by

$$x_1(0) - x0 = 0. \quad (32c)$$

During the continuation process it may be of interest to determine the exact value of end time T where the solution jumps at the end time and additionally the condition (30e) is satisfied. In general this characterizes the crossing from a jump at the boundary

to an interior jump. For that case the time horizon T is considered as a free variable and the condition

$$j^{N+1}(x_{N+1}(N+1), xT, \lambda_{N+1}(N+1), lT, T) = 0, \tag{33}$$

is appended to (31).

Initializing the BVP To find the solution of a specific problem of type (1) we can apply a continuation strategy with respect to the time horizon T. Therefore, as a first step we have to determine an initial (optimal) solution.

Due to Assumption 1, the initial condition together with the transversality condition yield the necessary equations for $T = 0$. This solution can be used as a starting point for paths, which for a "small" time horizon do not exhibit a jump point.

6 Endogenous lumpy investments

When a firm is dealing with market power, the output price decreases with the quantity that is produced. Since it holds in this model that with one unit of capital stock one unit of output is produced, we have that the output price decreases with the amount of capital. So during the time period between two investments the output price increases, since depreciation decreases capital stock.[5] We consider no scrapping, partial scrapping and total scrapping, i.e. we consider $\gamma = 0$, $\gamma = 0.5$ and $\gamma = 1$. We provide a numerical analysis starting with the parameter values

$$b = \frac{1}{n} \log 2 = \frac{1}{2} \log 2, \quad \alpha = 0, \quad \beta = 0.2, \quad C = 2 \quad r = 0.04, \quad \delta = 0.2,$$

which we adopt as the benchmark throughout this paper. As in Grass et al. (2012), we base our value for b on Moore's law,[6] where the value for b is such that the efficiency of the technology doubles every n years where we put $n = 2$. The results of the first ten investments, are presented in Table 1 for $T = 100$. It turns out that the number of investments, N, undertaken by the firm is $N = 40$ for $\gamma = 1$, $N = 49$ for $\gamma = 0.5$ and $N = 59$ for $\gamma = 0$.

Ignoring the first and last investment, we see that the better the technology is, the larger the investment becomes. It seems as if the firm delays the first investment (compared to the others) to start production of a new good. In Fig. 1a this is clearly seen (also see Figs. 4a, 6a in "Appendix"). To understand what happens with the first investment we have to distinguish between $\gamma < 1$ and $\gamma = 1$. When $\gamma < 1$ capital growth is increased with each investment without fully scrapping the old capital stock. Because there is only limited scrapping, at an early stage the firm undertakes a relatively high investment to start production. A firm only undertakes this relatively

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[5] Capital decreases due to depreciation via (5) this causes output to decrease and finally (4) implies that output price increases.

[6] Moore's law still holds, The Economist, July 14th 2012, Chipping in: A deal to keep Moore's law alive.

Table 1 First ten investments of Impulse Control solutions for γ. $T = 100$ and parameter values $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$. $C = 2$, $K_0 = 0$ and $\theta(0) = 1$

|	$\gamma = 0$	$\gamma = 0.5$	$\gamma = 1$
$(\tau_i . I(\tau_i))$	4.1651 : 1.4877	4.1462 : 1.4682	3.8509 : 1.3689
	7.3464 : 1.3571	7.4147 : 1.7204	7.1308 : 1.9589
	10.0022 : 1.4032	10.1649 : 2.0101	9.9511 : 2.4614
	12.3693 : 1.4610	12.6433 : 2.2785	12.5559 : 2.9262
	14.5474 : 1.5188	14.9499 : 2.5312	15.0389 : 3.3716
	16.5895 : 1.5751	17.1370 : 2.7731	17.4476 : 3.8067
	18.5276 : 1.6299	19.2361 : 3.0070	19.8100 : 4.2365
	20.3835 : 1.6837	21.2682 : 3.2353	22.1437 : 4.6639
	22.1724 : 1.7365	23.2479 : 3.4594	24.4606 : 5.0910
	23.9056 : 1.7887	25.1861 : 3.6805	26.7688 : 5.5191
Revenue (discounted)	802.4809	790.1920	771.3955
Investment cost (discounted)	35.3109	67.8103	97.6050
Total profit (discounted)	767.1700	722.3817	673.7904
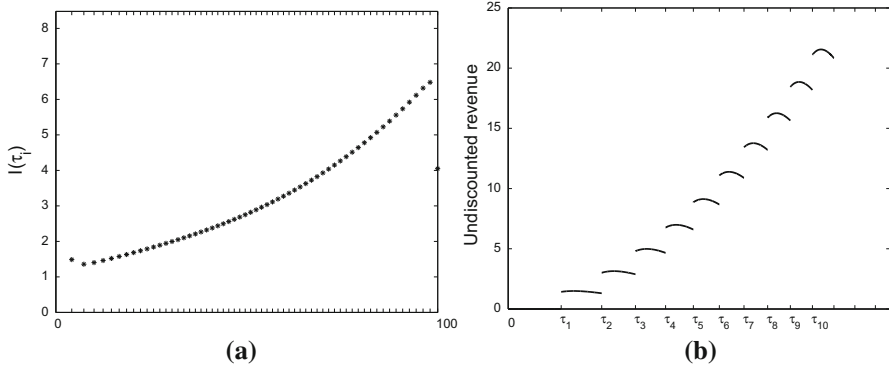
Fig. 1 For $T = 100$ and parameter values $r = 0.04$, $\delta = 0.2$, $\gamma = 0$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$. **a** Lumpy investments, $I(\tau_i)$. **b** Undiscounted revenue for the first ten investments

high investment if there is limited scrapping, because the investments help to increase the capital stock in the future.

This behavior is illustrated in Fig. 1a. Drawing a line in the point of Fig. 1a ignoring the first and last investment not only tells us that the first investment is relatively large, but also that the last investment is small. This last investment being small occurs due to the fact that the salvage value of the problem is (too) low, because it does not take into account technological improvement after time T.

Table 1 shows that the higher the scrapping percentage the larger the investments become. This makes sense because a firm that wants similar production has to invest extra to replace the scrapped parts. This scrapping increases the investment cost and at the same time, due to the quadratic term in the investment cost function, investing

Table 2 Technology level and capital for $T = 100$ and parameter values $\gamma = 0$, $r = 0.04$, $\delta = 0$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$

τ_i	$\theta(\tau_i^+)$	$K(\tau_i^+)$	$\frac{\theta}{K}$
19.6234	7.8009	3.8574	2.0224
34.5329	12.9682	6.4650	2.0059
50.7184	18.5777	9.2706	2.0039
70.6244	25.4766	12.7165	2.0034
99.7453	35.5691	17.7443	2.0045

such that the same level of capital is reached as in the case of no scrapping, is too expensive. Hence, the optimal level of capital stock in the case of scrapping is lower than under no scrapping, which explains the lower revenue. It turns out that a higher scrapping percentage decreases the number of investments during the planning period.

Another striking effect can be noticed when looking at Fig. 1b. We see that the firm invests in a new product such that marginal revenue is negative. In a "static" model (i.e. a model that does not depend on time) we know that the firms optimize profit and hence invest at the moment that marginal cost is equal to marginal revenue. Since we did not include any operation cost, we know that marginal cost is equal to zero. Hence, when marginal revenue is equal to zero, [i.e. $K(t) = \theta(\tau_i)/2$] investment would be optimal according to this rule. In our dynamic setting it is impossible to stay at the point where marginal revenue is equal to zero, due to depreciation. In Table 2 we show the results for a case where we have no depreciation. We see that indeed the investments are such that the level of capital is set to $K(t) = \theta(\tau_i)/2$. In the case that we have depreciation, the firm overinvests, i.e., invests such that marginal revenue is negative. Then up until the next investment, marginal revenue increases, becomes zero after some time, and then turns positive.

In Fig. 2 we have plotted the length of the time interval between two investments. We see that in the beginning of the planning period the firm adopts new technologies
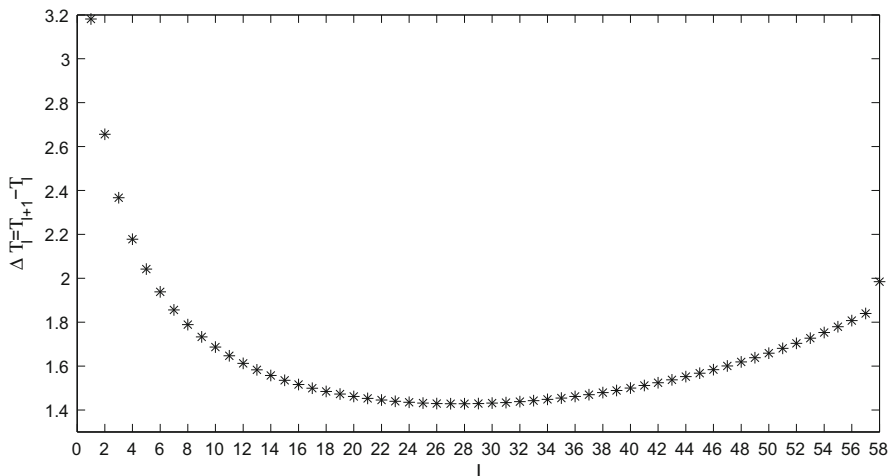

Fig. 2 The length between two investments for $T = 100$ and parameter values $r = 0.04$, $\delta = 0.2$, $\gamma = 0$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$

faster as time proceeds and after some moment it uses later technologies for a longer time period. This behavior is different from Grass et al. (2012), who finds that the firm adopts new technologies faster as time proceeds for the whole planning period. It is also different from the results found in Saglam (2011), who finds that later technologies are used during a longer time period. Our results are somehow a combination of both. An explanation for this could be that in the beginning of the planning period the firm does not invest much since productivity is low. After some time technological progress is such that each investment is more profitable, which makes that the corresponding capital goods are used for a longer time. For this reason the time between investments increases. Also for higher T a similar effect is found.

6.1 Sensitivity analysis with respect to the rate of technology change

Here we study how the rate of technological progress affects the investment behavior of a firm. Remember that we have assumed, using Moore's law, that the efficiency of a technology doubles every n years, setting $n = 2$ for our benchmark case. Table 3 shows the first ten investments for different values of the technology rate b. In turns out that the number of investments, N, undertaken by the firm is $N = 45$ for $b = \frac{1}{3} \log 2$, $N = 41$ for $b = \frac{1}{4} \log 2$, $N = 38$ for $b = \frac{1}{5} \log 2$, $N = 36$ for $b = \frac{1}{6} \log 2$, $N = 29$ for $b = \frac{1}{10} \log 2$. When $n > 5$ an investment takes place at $t = 0$. The explanation behind this is that for $n > 5$ we have, under Moore's law, that it takes more than five years for the efficiency of a technology to double. Since we have a depreciation rate of 20 %, this means that the firm's capital stock is (almost) depreciated before the efficiency of a technology doubles. So the firm has no incentive to wait and invests at $t = 0$.

6.2 Sensitivity analysis with respect to the fixed cost

One of the main differences between Grass et al. (2012), Boucekkine et al. (2004) and Saglam (2011) is that they do not incorporate any (fixed) cost, whereas this paper assumes that a fixed cost is included for each investment. Here we study how increasing this fixed cost affects the investment behavior of the firm. Table 4 shows the first ten investments for different sizes of fixed cost. In turns out that the number of investments, N, undertaken by the firm is $N = 44$ for $C = 4$, $N = 39$ for $C = 8$, $N = 33$ for $C = 16$ and $N = 27$ for $C = 32$. It is easily seen, that if we increase the fixed cost, the first investment is delayed and at the same time the time period between two investments increases. Hence, the number of investments decreases if the fixed cost increases. Comparing the results more carefully, we see that the size of the lumpy investments (i.e. jumps) increases, when C increases.

7 Lumpy investments under decreasing demand

In this section we consider the case where the demand for an existing product decreases over time. A main reason could be that the competitors' products become better due to their product innovations.

Table 3 First ten investments of Impulse Control solutions for b, $T = 100$ and parameter values $\gamma = 0.5$, $r = 0.04$. $\delta = 0.2$, $\beta = 0.2$, $\alpha = 0$. $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

	$b = \frac{1}{3}\log 2$	$b = \frac{1}{4}\log 2$	$b = \frac{1}{5}\log 2$	$b = \frac{1}{6}\log 2$	$b = \frac{1}{10}\log 2$
$(\tau_i, I(\tau_i))$	4.6759 : 1.3116	5.1658 : 1.2381	5.5832 : 1.1914	0 : 0.7418	0 : 0.7752
	8.6561 : 1.5392	9.7814 : 1.4539	10.7534 : 1.3980	7.7656 : 1.2080	9.7219 : 1.1432
	11.9662 : 1.7807	13.5911 : 1.6692	14.9977 : 1.5949	13.0448 : 1.4152	16.3705 : 1.3132
	14.9229 : 1.9995	16.9755 : 1.8614	18.7535 : 1.7685	17.4932 : 1.5907	21.9534 : 1.4524
	17.6530 : 2.2025	20.0857 : 2.0378	22.1932 : 1.9266	21.4683 : 1.7459	26.9204 : 1.5730
	20.2231 : 2.3943	23.0005 : 2.2031	25.4066 : 2.0736	25.1251 : 1.8873	31.4676 : 1.6810
	22.6732 : 2.5779	25.7678 : 2.3601	28.4478 : 2.2125	28.5485 : 2.0188	35.7025 : 1.7797
	25.0300 : 2.7553	28.4191 : 2.5108	31.3530 : 2.3450	31.7914 : 2.1429	39.6921 : 1.8712
	27.3121 : 2.9280	30.9766 : 2.6566	34.1472 : 2.4725	34.8894 : 2.2610	43.4813 : 1.9569
	29.5335 : 3.0970	33.4569 : 2.7986	36.8494 : 2.5960	37.8681 : 2.3745	47.1023 : 2.0376
Revenue (discounted)	371.5616	220.0775	148.0959	108.6965	47.4170
Investment cost (discounted)	39.2258	27.6829	21.7123	19.5772	12.9673
Total profit (discounted)	332.3358	192.3946	126.3837	89.1193	34.4497

Table 4 Impulse Control solutions for C, $T = 100$ and parameter values $\gamma = 0.5$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $K_0 = 0$ and $\theta(0) = 1$

|	$C = 4$	$C = 8$	$C = 16$	$C = 32$
$(\tau_i . I(\tau_i))$	5.7915 : 1.8832	8.0844 : 2.4856	11.1517 : 3.3199	15.2866 : 4.4754
	9.6593 : 2.2099	12.7147 : 2.9206	16.6712 : 3.8947	21.8148 : 5.2241
	12.8816 : 2.5607	16.5386 : 3.3546	21.1933 : 4.4297	27.1293 : 5.8789
	15.7638 : 2.8797	19.9372 : 3.7422	25.1901 : 4.8993	31.8052 : 6.4443
	18.4283 : 3.1763	23.0621 : 4.0984	28.8471 : 5.3256	36.0657 : 6.9513
	20.9394 : 3.4571	25.9923 : 4.4325	32.2606 : 5.7215	40.0266 : 7.4169
	23.3358 : 3.7265	28.7755 : 4.7502	35.4889 : 6.0947	43.7577 : 7.8515
	25.6433 : 3.9871	31.4435 : 5.0556	38.5705 : 6.4506	47.3050 : 8.2618
	27.8799 : 4.2412	34.0186 : 5.3513	41.5327 : 6.7926	50.7012 : 8.6525
	30.0590 : 4.4903	36.5173 : 5.6394	44.3957 : 7.1237	53.9701 : 9.0270
Revenue (discounted)	780.7835	769.1875	747.0746	712.6433
Investment cost (discounted)	79.5936	96.8939	120.5584	150.9987
Total profit (discounted)	701.1899	672.2936	626.5162	561.6447
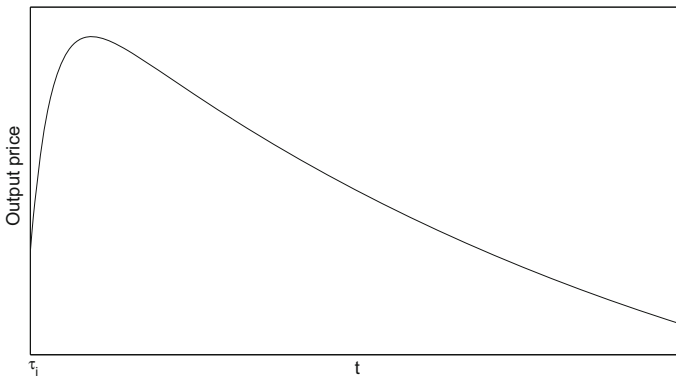
Fig. 3 output price as a function in time for $\delta > \eta$

We incorporate decreasing demand by setting $\dot{\theta} = -\eta\theta(t)$, where η is some positive constant. Since it is reasonable to assume $\delta > \eta > 0$,[7] the output price after investment is first increasing and then decreasing, see Fig. 3. Hence, after a firm invests, capital stock depreciates and the output price first increases, and after some time this output price is decreasing due to this decreasing demand. Then the model becomes

$$\max_{i,I,\tau_i,N} \left\{ \int_0^T e^{-rt} \left[\theta(t) - K(t) \right] K(t) dt \right.$$

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[7] Since we are dealing with product innovation and assume a depreciation rate of 20% it is unlikely that demand decreases by more than (or equal to) 20% and hence we do not consider the case that $\eta \geq \delta > 0$.

$$-\sum_{i=1}^{N} e^{-r\tau_i} \left(C + \alpha I\left(\tau_i\right) + \beta I\left(\tau_i\right)^2 \right)$$

$$+ e^{-rT} \frac{[\theta\left(T^+\right) - K(T^+)]K(T^+)}{r + \delta + \eta} \Bigg\}, \tag{34}$$

subject to

$$\dot{K}\left(t\right) = -\delta K\left(t\right) \quad \text{for all } t \neq \tau_1, \ldots, \tau_N, \tag{35}$$

$$\dot{\theta}\left(t\right) = -\eta\theta\left(t\right) \quad \text{for all } t \neq \tau_1, \ldots, \tau_N, \tag{36}$$

$$K(\tau_i^+) - K(\tau_i^-) = I\left(\tau_i\right) - \gamma K(\tau_i^-) \quad \text{for all } i = 1, \ldots, N, \tag{37}$$

$$\theta(\tau_i^+) - \theta(\tau_i^-) = 1 + b\tau_i - \theta(\tau_i^-) \quad \text{for all } i = 1, \ldots, N, \tag{38}$$

$$K(0) = 0, \tag{39}$$

$$\theta\left(0\right) = 1. \tag{40}$$

Remember that in Sect. 6 the output price was decreasing in capital stock. Hence, due to depreciation the output price is increasing in the time period between two investments. Since we are considering product innovation, it makes more sense that demand of a given product during the time period decreases. This is because over time new products are invented by other firms, which reduce demand of the current product. This demand decrease has a negative effect on output price and hence the firm has even a greater incentive to invest in a new technology.

In turns out that the number of investments, N, undertaken by the firm is $N = 40$ for $\eta = 0.01$, $N = 35$ for $\eta = 0.02$ and $N = 31$ for $\eta = 0.03$. Looking at the results of Table 5 we can see that a change in the decrease of demand directly affects

Table 5 First ten investments of Impulse Control solutions for η, $T = 100$ and parameter values $\gamma = 0.5$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$

	$\eta = 0.01$	$\eta = 0.02$	$\eta = 0.03$
$(\tau_i . I(\tau_i))$	5.2730 : 1.7250	6.3504 : 1.9594	7.5126 : 2.2042
	8.9696 : 2.0366	10.4003 : 2.3175	11.902 : 2.6060
	12.0850 : 2.3821	13.8098 : 2.7062	15.5932 : 3.0359
	14.9011 : 2.7029	16.8941 : 3.0676	18.9345 : 3.4366
	17.5308 : 3.0067	19.7779 : 3.4110	22.0629 : 3.8188
	20.0327 : 3.2991	22.5261 : 3.7427	25.0493 : 4.1897
	22.4425 : 3.5837	25.1779 : 4.0670	27.9368 : 4.5542
	24.7835 : 3.8631	27.7594 : 4.3869	30.7539 : 4.9156
	27.0723 : 4.1393	30.2889 : 4.7047	33.5212 : 5.2765
	29.3212 : 4.4136	32.7803 : 5.0219	36.2541 : 5.6390
Revenue (discounted)	762.5966	733.2291	701.2148
Investment cost (discounted)	61.1145	56.6083	52.6074
Total profit (discounted)	701.4821	676.6208	648.6074

the investment behavior. It is clear to see, that if we increase η the first investment is delayed (compared to a smaller η) and at the same time the time period between two investments also increases. Hence, the number of investments decreases if the decay rate of the demand increases. This makes sense, since less demand makes investing less attractive. This results in a lower investment cost for higher η. Moreover, the larger η the lower the output price (compared to a lower η) and hence the lower the revenue.

8 Conclusions

This paper employs an Impulse Control modeling approach that is perfectly suitable to take into account the disruptive changes caused by innovations. We describe and implement an algorithm based on current value necessary optimality conditions. The necessary conditions are solved using a multi-point Boundary Value Problem (BVP) combined with some continuation techniques.

Based on our numerical analysis we have derived some guidelines for lumpy investments in new technology:

- A striking result is that the firm does not invest such that marginal profit is zero, but instead marginal profit is negative. Indeed, due to depreciation capital stock decreases in between two investments, implying that marginal profit goes up there due to the decreasing returns to scale assumption. The implication is that during such an interval first marginal profit is negative, but then after a while it turns positive and this stays that way until it is time for the next investment.
- We find that investments are larger and the time between investments is larger when more of the old capital stock needs to be scrapped. If a change in technology permits the firm to keep, update and reuse part of its capital stock, the investments are smaller.
- A nontrivial result is the optimal timing of investments. We see that the firm in the beginning of the planning period adopts new technologies faster as time proceeds, but later on the opposite happens. Moreover, we obtain that the firm's investments increase when the technology produces more profitable products.
- Numerical experiments show that if the time it takes to double the efficiency of a technology is larger than the time it takes for the capital stock to depreciate to half of its original level, the firm undertakes an initial investment.
- Further sensitivity results were provided for a scenario of decreasing demand. We find that when demand decreases over time and when fixed investment cost is higher, then the firm invests less throughout the planning period, the time between two investments increases and the first investment is delayed.

Interesting directions for further work would be to consider running cost in the model or to introduce a learning effect. Another possible extension would be to let the scrapping percentage depend on time.

Appendix: Figures for all cases
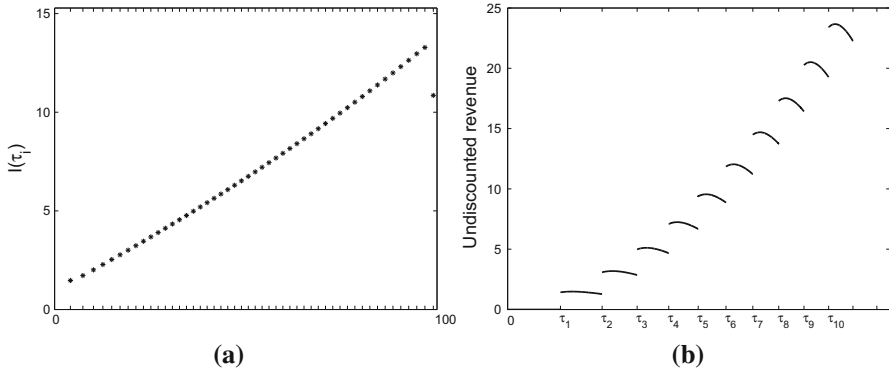
See Figs. 4, 5, 6 and 7.

Fig. 4 For $T = 100$ and parameter values $r = 0.04$, $\delta = 0.05$, $\gamma = 0.5$, $b = \frac{1}{2} \log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$. **a** Lumpy investments, $I(\tau_i)$, **b** Undiscounted revenue for the first ten investments
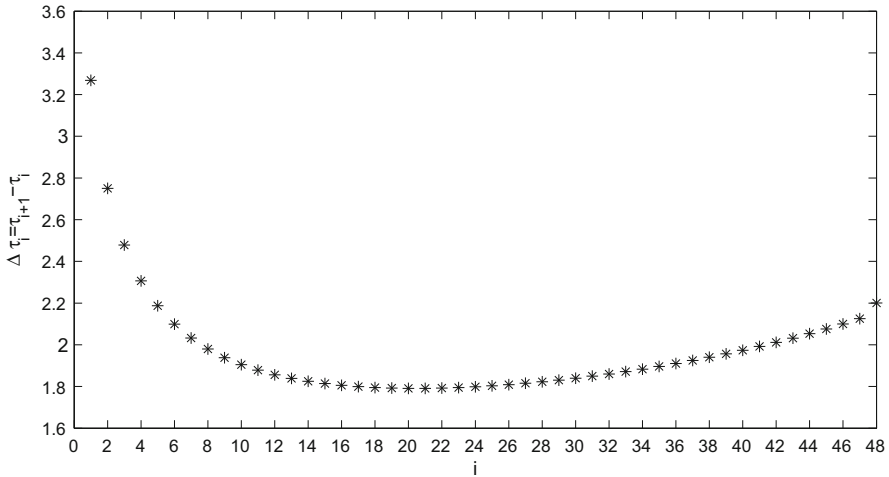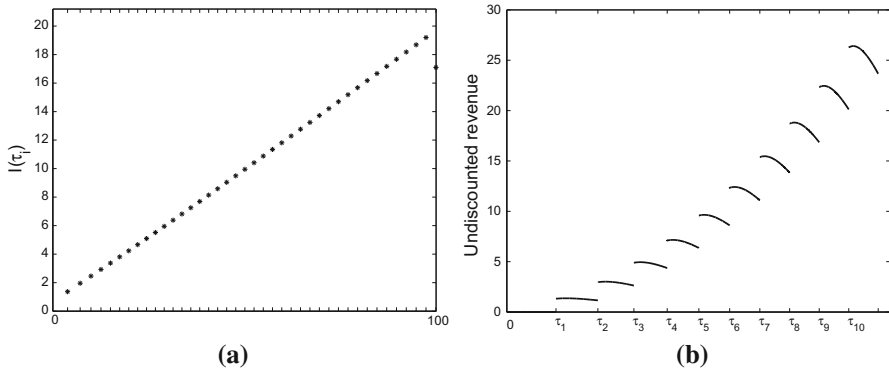

Fig. 5 For $T = 100$ and parameter values $r = 0.04$, $\delta = 0.2$, $\gamma = 0.5$, $b = \frac{1}{2} \log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$

Fig. 6 For $T = 100$ and parameter values $r = 0.04$, $\delta = 0.05$, $\gamma = 1$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$. **a** Lumpy investments, $I(\tau_i)$. **b** Undiscounted revenue for the first ten investments
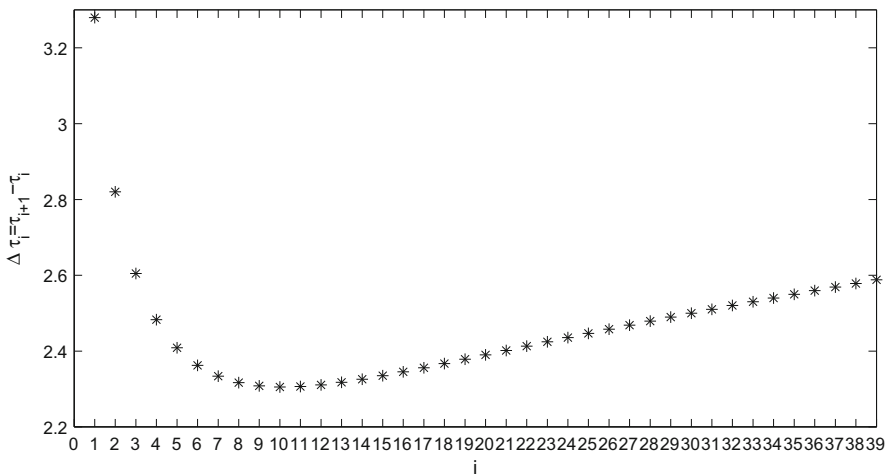

Fig. 7 For $T = 100$ and parameter values $r = 0.04$, $\delta = 0.2$, $\gamma = 1$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$

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