Monatsh Math (2013) 172:161–166
DOI 10.1007/s00605-013-0541-y

2-Supernilpotent Mal'cev algebras

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Received: 10 September 2012 / Accepted: 1 August 2013 / Published online: 31 August 2013

Abstract In this note we prove that a Mal'cev algebra is 2-supernilpotent ($[1, 1, 1] = 0$) if and only if it is polynomially equivalent to a special expanded group. This generalizes Gumm's result that a Mal'cev algebra is abelian if and only if it is polynomially equivalent to a module over a ring.

Keywords Commutators · Nilpotent · Mal'cev algebra · Expanded group

Mathematics Subject Classification (2000) 08A05

1 Introduction

It is well known that the commutator $[a, b] = a^{-1}b^{-1}ab$ of two elements a and b of a group $\mathbf{G}$ can be seen as a "measure" how far are a and b from commuting according to the group operation of $\mathbf{G}$. Thus, the normal subgroup $[G, G]$ generated by all such commutators "measures" how far is the group $\mathbf{G}$ from an abelian group. Namely, $[G, G] = 0$ if and only if $\mathbf{G}$ is abelian. The concept of commutators of normal subgroups has been generalized to a binary operation of the congruence lattice

Communicated by A. Constantin.

The work on this paper has been supported by the Austrian research fund FWF (24077) and Research Grant 174018 of the Ministry of Science and Education of the Republic of Serbia.

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of arbitrary algebras in congruence modular varieties by Smith, Freese, McKenzie, Hagemann, Herrmann,…One can find more details in [5]. A special case of congruence modular varieties are congruence permutable varieties. In congruence permutable varieties each algebra has a Mal'cev term. A *Mal'cev term* of an algebra **A** is a ternary term operation d of **A** that satisfies $d(x, y, y) = d(y, y, x) = x$ for all $x, y \in A$. An algebra with a Mal'cev term we call a *Mal'cev algebra*. What $[1, 1] = 0$ in Mal'cev algebras means has been answered by Gumm, Hagemann and Herrmann, see [4, Theorem 13.4]. They proved that a Mal'cev algebra is *abelian* ($[1, 1] = 0$) if and only if it is polynomially equivalent to a module over a ring. Here, we call two algebras **A** and **B** on the same domain *polynomially equivalent* if they have the same set of polynomial operations. In 2001, A. Bulatov has generalized the binary commutator operations to n-ary commutator operations $[\bullet, \bullet, \ldots, \bullet]$ on congruence lattices of Mal'cev algebras, for each $n \in \mathbb{N}$, see [3]. We will restrict ourselves to the ternary commutators. The aim of this note is to answer what $[1, 1, 1] = 0$ means in Mal'cev algebras. We want to characterize all Mal'cev algebras with such a property, because it could be expected that $[1, 1, 1] = 0$ implies that the algebra is polynomially equivalent to a well studied structure. According to the property (HC3) in [2] we know that $[1, 1, 1] \leq [1, 1]$. Therefore, we have that each abelian Mal'cev algebra satisfies the condition $[1, 1, 1] = 0$. In Theorem 3.3 we prove that every Mal'cev algebra with $[1, 1, 1] = 0$ is an expanded group.

In accordance with [2, Definition 7.1], Mal'cev algebras that satisfy $[1, 1, 1] = 0$ are called *2-supernilpotent*. As it has been defined in [5] algebras that satisfy $[1, [1, 1]] = 0$ are called *2-nilpotent*. In expanded groups we deal with ideals rather then with congruences, because the corresponding commutator operations act in the same way according to [2, Corollary 6.12]. An expanded group **V** is 2-nilpotent if $[V, [V, V]] = 0$ and 2-supernilpotent if $[V, V, V] = 0$. Let us just mention that all 2-nilpotent algebras are nilpotent by definition, see [5].

2 Special expanded groups

A *polynomial* of an algebra **A** is an operation obtained by composition of projections, fundamental and constant operations of **A**, see [6, Definition 4.4]. The set of all n-ary polynomials of an algebra **A** we denote by $\mathsf{Pol}_n\mathbf{A}$ for all $n \in \mathbb{N}$.

Definition 2.1 Let $\mathbf{V} = (V, +, -, 0, F)$ be an expanded group and let $n \in \mathbb{N}$. We call an n-ary polynomial f of **V** *absorbing* if $f(a_1, \ldots, a_n) = 0$ whenever there exists an $i \in \{1, \ldots, n\}$ such that $a_i = 0$. The set of unary absorbing polynomials of **V** we denote by $P_0(\mathbf{V})$ and the set of binary absorbing polynomials of **V** we denote by $CP(\mathbf{V})$.

More generally, for an algebra **A** and an $n \in \mathbb{N}$, $(a_1, \ldots, a_n) \in A^n$, $a \in A$ we say that an n-ary polynomial p is *absorbing at* $(a_1, \ldots, a_n)$ *with value* a if $p(x_1, \ldots, x_n) = a$ whenever there exists an $i \in \{1, \ldots, n\}$ such that $x_i = a_i$.

Lemma 2.2 *Let* **V** *be an expanded group with a group reduct* $(V, +, -, 0)$*, and let* $n \in \mathbb{N}$*. For* $i \leq n$ *let* $\pi_i^n : V^n \to V$ *be the* i*-th projection. Then the group* $(\mathsf{Pol}_n\mathbf{V}, +, -, 0)$ *is generated by constants and*

$$\bigcup_{k=1}^{n}\{f(\pi^n_{i_1},\ldots,\pi^n_{i_k}) : f \text{ is a nonzero absorbing polynomial of } \mathbf{V} \text{ of degree } k,\ i_1,\ldots,i_k \in \{1,\ldots,n\}\}.$$

Proof See [1, Lemma 3.1]. In [1, p. 260] each n-ary commutator polynomial is exactly a polynomial $f(\pi^n_{i_1},\ldots,\pi^n_{i_k})$ where $k \leq n$ and f is a nonzero k-ary absorbing polynomial defined in Definition 2.1 or constant. □

Lemma 2.3 *Let* $\mathbf{V} = (V, +, -, 0, F)$ *be an expanded group such that*

(1) *F is the set of at most binary absorbing operations on V,*
(2) *Every absorbing operation in $\mathsf{Pol}_2(\mathbf{V})$ is distributive with respect to $+$, and*
(3) *$\mathbf{V}$ is 2-nilpotent.*

Then $\mathbf{V}$ is 2-supernilpotent.

Proof First, we show that for all $k \in \mathbb{N}$ and $f \in \mathsf{Pol}_k(\mathbf{V})$ there exist $c \in V$, $p_i \in P_0(\mathbf{V})$ and $q_{ij} \in CP(\mathbf{V})$ for $1 \leq i < j \leq k$ such that

$$f(x_1,\ldots,x_k) = c + p_1(x_1) + \cdots + p_k(x_k) + \sum_{1\leq i<j\leq k} q_{ij}(x_i, x_j). \tag{2.1}$$

Let P_k be the set of operations as in (2.1). Obviously $P_k \subseteq \mathsf{Pol}_k(\mathbf{V})$. To prove $\mathsf{Pol}_k(\mathbf{V}) \subseteq P_k$, we need to show that the set P_k contains projections and constants and that is closed under the basic operations of $\mathbf{V}$. Projections are in $P_0(\mathbf{V})$ and therefore in P_k. Constants are in P_k because $P_0(\mathbf{V})$ and $CP(\mathbf{V})$ contain zero polynomials. Clearly, P_k is closed under $+$ and $-$ since $[c, p(x_i)] = r(x_i)$, for some $r \in P_0(\mathbf{V})$, $[c, q_{ij}(x_i, x_j)] = s(x_i, x_j)$ for some $s \in CP(\mathbf{V})$ and $[p(x_i), p(x_j)] = t(x_i, x_j)$ for some $t \in CP(\mathbf{V})$. Furthermore, $[q_{ij}(x_i, x_j), p_l(x_l)]$ and $[q_{ij}(x_i, x_j), q_{lt}(x_l, x_t)]$ are zero polynomials because $\mathbf{V}$ is 2-nilpotent. Therefore, in the sum of two operations in the form of (2.1) one can permute all summands to obtain again the form as in (2.1), because $P_0(\mathbf{V})$ and $CP(\mathbf{V})$ are closed under $+$ and $-$. To show that P_k is closed under all $q \in CP(\mathbf{V})$ we note that for all $q \in CP(\mathbf{V})$ and $f, f' \in P_k$ we have $q(f, f') \in P_k$ by the distributivity of q and the 2-nilpotence of $\mathbf{V}$.

It remains to show that P_k is closed under any unary operation $g \in F$. Note that $g'(x, y) := g(x + y) - g(x) - g(y)$ is in $CP(\mathbf{V})$, hence distributive. By induction it follows that for every n there exist $c_{ij} \in CP(\mathbf{V})$ such that

$$g(x_1 + \cdots + x_n) = g(x_1) + \cdots + g(x_n) + \sum_{1\leq i<j\leq n} c_{ij}(x_i, x_j). \tag{2.2}$$

For f as in (2.1) this yields

$$\begin{aligned} gf(x_1,\ldots,x_k) &= g(c) + gp_1(x_1) + \cdots + gp_k(x_k) + \sum_{1\leq i<j\leq k} gq_{ij}(x_i, x_j)\\ &\quad + \sum_{1\leq i\leq k} c_i(c, p_i(x_i)) + \sum_{1\leq i<j\leq k} c_{ij}(p(x_i), p(x_j)) \end{aligned} \tag{2.3}$$

for some $c_i, c_{ij} \in CP(\mathbf{V})$. Note that the other terms vanish by $[V, [V, V]] = 0$. Thus $gf \in P_k$. We have proved $P_k = \mathsf{Pol}_k(\mathbf{V})$.

Now we know that for every ternary polynomial operation p on $\mathbf{V}$ there exist a $c \in V$, $f, g, h \in P_0(\mathbf{V})$ and $r, s, t \in CP(\mathbf{V})$ such that

$$p(x, y, z) = c + f(x) + g(y) + h(z) + r(x, y) + s(x, z) + t(y, z) \tag{2.4}$$

for all $x, y, z \in V$. Using the absorbing property of p, f, g, h, r, s and t we obtain the following. First, $c = 0$ because $p(0, 0, 0) = 0$ from (2.4). Then, we substitute $y = z = 0$ in (2.4) and obtain $f(x) = 0$. Analogously, we have $g(y) = h(z) = 0$. It remains $p(x, y, z) = r(x, y) + s(x, z) + t(y, z)$. We have $t(y, z) = 0$, because $0 = p(0, y, z)$. Analogously, we obtain $r(x, y) = s(x, z) = 0$. Hence, every ternary absorbing polynomial operation is constant 0. Whence, $\mathbf{V}$ is 2-supernilpotent, by [2, Corollary 6.12]. □

3 Mal'cev algebras

Proposition 3.1 (cf. [5, Corollary 7.4]) *Let $\mathbf{A}$ be a nilpotent Mal'cev algebra with a Mal'cev term d and let $o \in A$. Then for all $a_1, a_2, b_1, b_2 \in A$ there exist $x, y \in A$ such that $d(x, o, a_1) = b_1$ and $d(a_2, o, y) = b_2$.*

Proof The function $x \mapsto d(x, o, a_1)$ is bijective for all $a_1 \in A$, by [5, Corollary 7.4]. Hence, the equation $d(x, o, a_1) = b_1$ has a unique solution for all $a_1, b_1 \in A$. We know that $D(x, y, z) := d(z, y, x)$ for all $x, y, z \in A$ is also a Mal'cev term of $\mathbf{A}$. Therefore, $y \mapsto D(y, o, a_2)$ is bijective for all $a_2 \in A$ by [5, Corollary 7.4]. Hence, $y \mapsto d(a_2, o, y)$ is bijective for all $a_2 \in A$. Whence, the equation $d(a_2, o, y) = b_2$ has a unique solution for all $a_2, b_2 \in A$. □

Lemma 3.2 (cf. [7, Theorem 1.2]) *Every semigroup $(G, +)$ such that the equations $a_1 + x = b_1$ and $y + a_2 = b_2$ are solvable for all $a_1, a_2, b_1, b_2 \in A$, is a group.*

Proof See [7, Definition 1.1, Theorem 1.2]. □

Theorem 3.3 *For a Mal'cev algebra $\mathbf{A}$ the following are equivalent:*

(1) *$\mathbf{A}$ is 2-supernilpotent* ($[1, 1, 1] = 0$)
(2) *$\mathbf{A}$ is polynomially equivalent to an expanded group $\mathbf{V} = (A, +, -, 0, F)$ such that*
  (a) *F is a set of at most binary absorbing operations on $\mathbf{V}$,*
  (b) *Every absorbing operation in $\mathsf{Pol}_2(\mathbf{V})$ is distributive with respect to $+$ on both arguments, and*
  (c) *$\mathbf{V}$ is 2-nilpotent.*

Proof (1) $\Rightarrow$ (2) Using [2, (HC8)] we obtain $[1, [1, 1]] = 0$. Therefore, $\mathbf{A}$ is a 2-nilpotent Mal'cev algebra. Let $o \in A$ and let d be a Mal'cev term. We define $+ : A^2 \to A$ by

$$x + y := d(x, o, y)$$

for all $x, y \in A$. From Proposition 3.1 we know that the equations $a_1 + x = b_1$ and $y + a_2 = b_2$ are solvable for all $a_1, a_2, b_1, b_2 \in A$. Let us show that $+$ is associative. We observe that the polynomial

$$p(x, y, z) := d(d(d(x, o, y), o, z), d(x, o, d(y, o, z)), o)$$

for all $x, y, z \in A$ is an absorbing polynomial at (o, o, o) with value o. Therefore, $(p(a, b, c), o) \in [1, 1, 1]$ for all $a, b, c \in A$ by [2, Lemma 6.9]. Using the assumption $[1, 1, 1] = 0$, we obtain $p(a, b, c) = o$. Equivalently,

$$d(d(d(a, o, b), o, c), d(a, o, d(b, o, c)), o) = o.$$

By [5, Corollary 7.4] we have that $x \mapsto d(x, d(a, o, d(b, o, c)), o)$ is a bijective mapping of A. Hence, $d(d(a, o, b), o, c) = d(a, o, d(b, o, c))$, because $d(d(d(a, o, b), o, c), d(a, o, d(b, o, c)), o) = d(d(a, o, d(b, o, c)), d(a, o, d(b, o, c)), o)$. Using the recently introduced notation we have proved $(a+b)+c = a+(b+c)$. Now using Lemma 3.2 we obtain that $+$ is a group operation with the neutral element o. We denote the inverse of an element $a \in A$ by $-a$. This is a polynomial operation given by

$$-x := d(o, d(d(o, x, o), o, x), d(o, x, o))$$

for all $x \in A$, by [5, Lemma 7.3]. We have proved that $\mathbf{A}$ is polynomially equivalent to an expanded group $\mathbf{V}$ with the group reduct $(A, +, -, o)$. Now, we shall prove that $\mathbf{A}$ is polynomially equivalent to the expanded group $\mathbf{V} := (A, +, -, o, F)$, where $F := P_o(\mathbf{A}) \cup CP(\mathbf{A})$. We have already that all fundamental operations of $\mathbf{V}$ are polynomials of $\mathbf{A}$.

By [2, (HC3)], we know that $[\underbrace{1, \ldots, 1}_{n}] = 0$ for all $n \geq 3$. Hence, $f = 0$ for all n-ary absorbing polynomial operations f of $\mathbf{A}$ if $n \geq 3$, by [2, Corollary 6.12]. Hence, the set of all non-constant absorbing polynomial operations of $\mathbf{A}$ is $F := P_o(\mathbf{A}) \cup CP(\mathbf{A})$. Using Lemma 2.2 we obtain that each polynomial of $\mathbf{A}$ is also a polynomial of $\mathbf{V}$.

Let $f \in CP(\mathbf{V})$. One can easily see that the polynomial q defined by

$$q(x, y, z) := f(x, y + z) - f(x, z) - f(x, y)$$

for all $x, y, z \in A$ is absorbing at (o, o, o) with value o. Therefore, $(q(a, b, c), o) \in [1, 1, 1]$ by [2, Corollary 6.9] for all $a, b, c \in A$. Hence, $q(a, b, c) = o$ or equivalently,

$$f(a, b + c) = f(a, b) + f(a, c)$$

for all $a, b, c \in A$, by the assumption $[1, 1, 1] = 0$. The proof for the distributivity on the first argument is analogous.

(2) $\Rightarrow$ (1) Polynomially equivalent algebras have the same congruence lattice and the same commutator operations acting on the congruence lattice, by [2, Corollary 6.11]. Therefore, $\mathbf{A}$ is 2-supernilpotent by Lemma 2.3. $\square$

Acknowledgments The author wants to thank Erhard Aichinger for helpful discussions during the work on this paper and to anonymous referee who suggested the shorter version of the proof for the part (2) $\Rightarrow$ (1) in the main theorem. The work on this paper has been supported by Austrian research fund FWF 24077 and Research Grant 174018 of the Ministry of Science and Education of the Republic of Serbia.

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