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A New Feature of the Efimov-Like Structure in the Hadron System: Long-Range Force as a Recoil Effect

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Abstract On the three-body kinematics, we investigate the threshold behavior which appears not only at the three-body break-up threshold (3BT), but also at the quasi two-body threshold (Q2T) for the reactions: $A + (BC) \to A + B + C$, and $(ABC) \to A + (BC)$, respectively. Recently, the author proposed a *general particle transfer* (GPT) potential which appears, not only at the 3BT, but also at the Q2T between A and (BC). The new potential indicates a Yukawa-type potential for short range, but a $1/r^n$-type potential for long range. The long range part of the GPT potential for $n = 1$ indicates an attractive Coulomb-like or a gravitation-like potential. While, $n = 2$ indicates the Efimov-like potential between A and (BC). The three-body binding energy: $E_n = \epsilon + \zeta_n$ with the two-body binding energy ϵ, and the separation energy ζ_n for $(ABC) \to A+(BC)$ satisfies $E_n/E_{n+1} = \zeta_n/\zeta_{n+1}$=const for $\epsilon = 0$ or the two-body scattering length: $a \to \infty$ (i.e. the two-body unitary limit). At the Q2T, the condensation of the three-body binding energy is given by the GPT-potential in the form of $E_n/E_{n+1} = (\zeta_n + \epsilon)/(\zeta_{n+1} + \epsilon) \to 1$ (const) for $n \to \infty$ (with $\zeta_n \to 0$) which implies the existence of Efimov-like states at the Q2T in the hadron systems, thereby the possibility of "ultra low energy nuclear transformation", where the three-body binding energies degenerate at zero energy. Finally, the origin of such a long range potential will be clarified.

1 Introduction

In 1970, Efimov proposed a theory that the three-body bound states accumulate on the zero energy level for the infinite value of the two-body scattering length where the energy ratio: $E_n/E_{n+1} = (\zeta_n + \epsilon)/(\zeta_{n+1} + \epsilon)$ becomes a constant with respect to the quantum number $n \to \infty$, and $a \to \infty$ (or $\epsilon = 0$) [1,2]. Such a phenomenon is understood by the quasi-two-body potential between A and (BC) which is a $1/r^2$-type. More than three decades after his prediction, the phenomenon was finally found in the cold atomic system [3].

Recently, we proposed the GPT-potential which is generated by a particle transfer between parent-particles [4–7] on the basis of the three-body Faddeev Equation [8] or the Alt–Grassberger–Sandhas (AGS) Equation [9].

In order to obtain the GPT-potential, Fourier–Laplace transformations are performed for "one particle transfer Feynman diagram" (PTFD) in the AGS Born-term at the 3BT ($E = 0, q = 0, p = 0$), and for the PTFD-type Born-term of the quasi two-body equation at the Q2T ($E = -\epsilon_B, q = 0, p = 0$), with the three-body energy E, a relative momentum q for the quasi two-body system, and the relative momentum p in the two-body subsystem, respectively.

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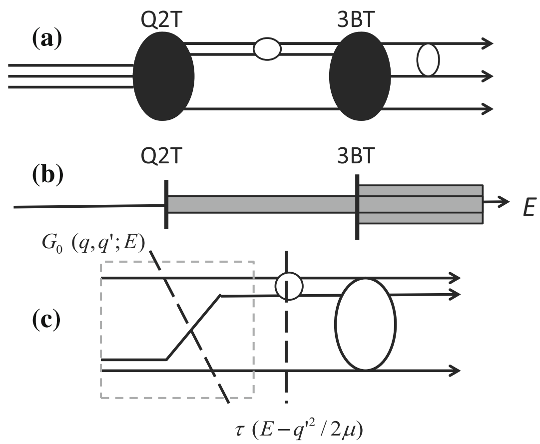

Fig. 1 **a** The 3BT and the Q2T (black ellipse) where white small ellipse represents the two-body interactions. **b** The quasi-two-body and the three-body right hand cuts (shade parts). **c** The kernel is illustrated by the PTFD (part of dashed square) and the propagator τ (small circle), and the three-body or the quasi two-body amplitude (large ellipse), where the bloken lines indicate the three-body Green's function and the propagator

The GPT-potential is given by $V(r) = V_0(a_0)^{2\gamma+2}/[r(r+a_0)^{2\gamma+2}]$ with a range $a_0(\equiv 2a)$ and a parameter γ. The potential represents a Yukawa-type potential in the short range: $r \ll a_0$, but $1/r^n$-type for the long range: $a_0 \ll r$. The Efimov's potential: $1/r^2(\gamma = -1/2)$ is included in our potential.

If the nuclear potential by the meson transfer is interpreted as a three-body problem in the NNπ system, then the GPT-potential can represent the Yukawa potential. However, a proper long range interaction is accompanied for the Yukawa-potential. The GPT potential is independent for the two-body unitary limit, because $a \to \infty$ for "$N\pi$" system is not satisfied. However, the relation: $\lim_{n\to\infty} E_n/E_{n+1} = \lim_{n\to\infty}(\zeta_n+\epsilon)/(\zeta_{n+1}+\epsilon) \to 1$ is given for $\epsilon \neq 0$. Therefore, it should be stressed that the GPT potential can represent the rich hadron systems, although it is not very clear whether the cold atomic system could be applied to some practical problems or not.

In Sect. 2, some comments about the GPT potential, and the possibility of the "ultra low energy nuclear reaction" are given. We show how to obtain the GPT potential at the 3BT, and also at the Q2T in Sect. 3. In Sect. 4, we clarify the origin of the long range force. Some applications for the GPT-potential will be discussed in Sect. 5.

2 A Status of the GPT-Potential

A coincidence of two singularities which appears in the Born-term of AGS equations and in the propagator of the two-body sub-system brings a serious problem in the three-body scattering. This situation is very similar to the Lippmann–Schwinger equation in the two-body Coulomb scattering where the potential gives rise to a singularity at the forward scattering, and also the Green's function diverges at the on-shell threshold [10–13]. In the Coulomb problem, such as the electron–proton system, the potential is given by

$$V(r) = -e^2/r. \tag{1}$$

While for the coincident singularities at the 3BT, Efimov ponted out a different attractive long range potential,

$$V(r) = -\alpha/r^2, \tag{2}$$

with a proper constant $\alpha > 0$.

On the other hand, we have introduced a GPT [6] which is more fundamental than the Efimov's potential at the 3BT. Furthermore, the GPT potential is also satisfied below the quasi two-body threshold (Q2T) for $(ABC) \to A + (BC)$ by means of the PTFD, and given by

$$V(r) \;=\; V_0\frac{a^{2\gamma+2}}{r(r/2+a)^{2\gamma+2}} = V_0\frac{(2a)^{2\gamma+2}}{r(r+2a)^{2\gamma+2}} \tag{3}$$

Table 1 The GPT potential $V^{GPT}(r) \equiv V_0 a^{2\gamma+2}/[r(r/2+a)^{2\gamma+2}]$ is illustrated, which is given by an energy average below the 3BT ($E=0, \epsilon_B=0$) and below the Q2T ($E=-\epsilon_B, \epsilon_B \neq 0$) with two-parameters a and γ

γ	Short range potential $r \ll a$	Potential	Long range potential $a \ll r$
–1	V_0/r	V_0/r	V_0/r
–1/2	$V_0 e^{-r/2a}/r$	$V_0(2a)/[r(r+2a)]$	$V_0(2a)/r^2$
0	$V_0 e^{-2r/2a}/r$	$V_0(2a)^2/[r(r+2a)^2]$	$V_0(2a)^2/r^3$
1/2	$V_0 e^{-3r/2a}/r$	$V_0(2a)^3/[r(r+2a)^3]$	$V_0(2a)^3/r^4$
1	$V_0 e^{-4r/2a}/r$	$V_0(2a)^4/[r(r+2a)^4]$	$V_0(2a)^4/r^5$
3/2	$V_0 e^{-5r/2a}/r$	$V_0(2a)^5/[r(r+2a)^5]$	$V_0(2a)^5/r^6$
2	$V_0 e^{-6r/2a}/r$	$V_0(2a)^6/[r(r+2a)^6]$	$V_0(2a)^6/r^7$
.	...	...	...
.	...	...	...

The potential properties for the long and short ranges are shown with respect to the parameter γ. $V_0(<0)$, a potential depth which is analytically given by Eq. (13)

$$\to V_0 \frac{(2a)^{2\gamma+2}}{r^{2\gamma+3}} \qquad \text{for} \quad a \ll r, \tag{4}$$

$$\to V_0 \frac{e^{-(\gamma+1)r/a}}{r} \qquad \text{for} \quad r \ll a, \tag{5}$$

where a and γ are parameters, and $V_0(<0)$ is a proper depth parameter. Therefore, Eq. (4) indicates $1/r^n$-type attractive potential for longer range, while Eq. (5) means a Yukawa-type attractive potential for short range.

The case $\gamma = -1$ indicates an attractive Coulomb-like or the gravitation-like potential without parameter a. $\gamma = -1/2$ represents the Efimov-type potential. $\gamma = 3/2$ is the van der Waals potential of the London-type, and $\gamma = 2$ gives the Casimir-type van der Waals potential.

The interference between the attractive GPT and the repulsive Coulomb potential: $V(r) = V^{GPT}(r) + V^C(r)$ has a Coulomb barrier at short range, but a very small value at long range. If we can confine two nuclear fuels in a special material, the potential is given by $\overline{V}(r) = V(r)$ for $0 \le r \le b_B$, and $\overline{V}(r) = V(2b_B - r)$ for $b_B \le r \le 2b_B$ which illustrate a kitchen-tray-like concavity shape. If we obtain some shallow bound states for the case $b_B \ll a_B$ (a_B: the Bohr radius). Such bound states could contribute a "ultra low energy nuclear reaction or a nuclear transformation".

3 Theory of the GPT-Potential

3.1 A New Method to Obtain GPT Potential at 3BT

Let us introduce our potential by a new aspect. At the 3BT, the two-body propagator is given by using the on-shell property: $E = q^2/2\mu + z$ with the three-body energy E, a two-body energy z, the kinetic energy of the spectator particle and the reduced mass between the two-body center of mass and the spectator. The propagator τ is written with a regular numerator function $f(z)$ and the energy denominator with the two-body binding energy ϵ_B,

$$\tau(z) = \frac{f(z)}{\epsilon_B + z}. \tag{6}$$

Efimov required the unitary limit: $a \to \infty$ which means the binding energy: $\epsilon_B = 0$. At the 3BT : ($E = 0,\ q = 0$), Eq. (6) becomes,

$$\tau(z) = \frac{f(z)}{z} = \frac{f(E - q^2/\mu)}{E - q^2/2\mu} \to \infty, \tag{7}$$

$$f(0) = constant \neq 0 \tag{8}$$

or, by the effective range formula with $k = \sqrt{2\nu z}$,

$$\tau(z) \propto \frac{1}{-1/a - ik} \to i \frac{1}{\sqrt{E - q^2/2\mu}} \to i\infty. \tag{9}$$

In addition to that, the Born term of the AGS equation is given by using two-body form factors with different channels $g_\alpha(p)$, $g_\beta(p')$ and with two-body momenta p, p', and ν the reduced mass between two particles,

$$Z_{\alpha\beta}(q,q';E) = \frac{g_\alpha(p)g_\beta(p')(1-\delta_{\alpha\beta})}{E - q^2/2\mu - p^2/2\nu}. \tag{10}$$

For the 3BT, $E = 0$, $q = 0$, $p = 0$

$$Z_{\alpha\beta}(q,q';E) \to \infty. \tag{11}$$

Just below the 3BT ($E \le 0$), it gives $p = p' = 0$, the AGS-Born term becomes,

$$Z_{\alpha\beta}(q,q';E) = \frac{g_\alpha(0)g_\beta(0)(1-\delta_{\alpha\beta})}{-|E| - q^2/2\mu} = -\frac{C_{\alpha\beta}}{q^2+\sigma^2}, \tag{12}$$

where $C_{\alpha\beta} = 2\mu_\pi g_\alpha(0)g_\beta(0)(1-\delta_{\alpha\beta})$, and $\sigma = \sqrt{2\mu|E|}$. Therefore, the Fourier transformation of this energy dependent potential becomes,

$$\mathcal{F}[Z_{\alpha\beta}(q,q';E)] = -\mathcal{F}\left[\frac{C_{\alpha\beta}}{q^2+\sigma^2}\right] = V_0\frac{e^{-\sigma r}}{r}, \tag{13}$$

with $V_0 < 0$. The r-space potential is a kind of Yukawa potential, but energy dependent. For $\sigma = 0$ or $E = 0$, it becomes the Coulomb-like potential (or the gravitation-like potential), therefore, our AGS equation is essentially the same equation as the Coulomb's Lippmann-Schwinger equation such as the electron-proton scattering except for the coupling constant.

In order to solve the eigenvalue equation with the energy dependent potential of Eq. (13), we have to solve it consistently with the two energies which are seen in the potential and in the eigenvalue. However, the method is very complicated and hard to obtain with good accuracy. Therefore, we introduced in Ref [6], an energy average by using a probability density function with respect to the possible energy range, which also represents effects of the structure or the form factors of the composite particles,

$$P_\sigma = \frac{\sigma^{2\gamma+1}e^{-a\sigma}}{\rho} \tag{14}$$

with

$$\rho = \int_0^\infty \sigma^{2\gamma+1}e^{-a\sigma}d\sigma = \frac{\Gamma(2\gamma+2)}{a^{2\gamma+2}}, \tag{15}$$

where $e^{-a\sigma}$ is a damping factor with a range parameter a. By using the probability density function, the expectation value of the energy-dependent potential becomes energy independent. It means that the Laplace transformation or the Euler integral of the second kind with respect to Eq. (13).

At just below the 3BT, by using Eq. (14) and Eqs. (15), (13) becomes,

$$\mathcal{L}\left\{\mathcal{F}[Z_{\alpha,\beta}(q,q';E)]\right\} = \mathcal{L}\left\{\frac{V_0e^{-\sigma r}}{r}\right\} = \frac{V_0}{\rho}\int_0^\infty \sigma^{2\gamma+1}e^{-a\sigma}\frac{e^{-\sigma r}}{r}d\sigma$$
$$= V_0\frac{a^{2\gamma+2}}{r(r/2+a)^{2\gamma+2}}. \tag{16}$$

Therefore the predicted GPT potential is obtained.

3.2 A New Method to Obtain GPT Potential at Q2T

On the other hand, at the Q2T, the two-body bound state becomes $\epsilon_B \neq 0$, therefore, the Efimov criterion : $a \to \infty$ is not satisfied anymore.

For the propagator at the Q2T: ($E_{cm} = E + \epsilon_B = 0,\ q = 0$), by using on-shell relation $z = E - q^2/2\mu$ (≤ 0),

$$\tau_B(z) = \frac{f(z)}{\epsilon_B + z} = \frac{f(z)}{(\epsilon_B + E) - q^2/2\mu} \to \infty. \tag{17}$$

On the other hand, in the AGS Born term, the target two-body bound state cannot be broken up below the 3BT where the virtual three-body Green's function should be given by $G_0(E) = [E - q^2/2\mu + \epsilon_B]^{-1}$ which is the criterion we use for judgement. If and only if, one takes $G_0(E) = [E - q^2/2\mu - p^2/2\nu]^{-1}$, some parts of the three-body kinetic energy should be virtual and negative. If we choose a negative energy $E < 0$, and positive three-body kinetic energies $q^2/2\mu > 0$ and $p^2/2\nu > 0$, such a criterion includes a contradiction, or off-the-energy-shell for $E \leq 0$. It brings only a virtual particle transfer. Consequently, the Born term or the energy dependent two-body quasi (E2Q) potential with the two-body bound state ($\epsilon_B \neq 0$, or $a \neq \infty$) should become,

$$Z_{\alpha,\beta}(q, q'; E) = \frac{g_\alpha(p) g_\beta(p')(1 - \delta_{\alpha\beta})}{E - q^2/2\mu + \epsilon_B} \tag{18}$$

$$\to \frac{g_\alpha(0) g_\beta(0)(1 - \delta_{\alpha\beta})}{E_{cm} - q^2/2\mu} \to \infty, \tag{19}$$

with $E_{cm} = E + \epsilon_B = 0, q = 0$ and $p = p' = 0$, where we have no kinetic freedom for the two-body subsystem. For the case $E_{cm} \leq 0$, Eq. (18) becomes

$$Z_{\alpha,\beta}(q, q'; E) \to -2\mu \frac{g_\alpha(0) g_\beta(0)(1 - \delta_{\alpha\beta})}{2\mu|E_{cm}| + q^2} = -\frac{C_{\alpha\beta}}{q^2 + \sigma^2} \tag{20}$$

$$\text{with} \quad \sigma^2 = 2\mu|E_{cm}|, \quad 0 < C_{\alpha\beta}. \tag{21}$$

The Fourier transformation of Eq. (18) at just below the Q2T is given by

$$\mathcal{F}[Z_{\alpha,\beta}(q, q'; E)] = V_0 \frac{e^{-\sigma r}}{r}, \tag{22}$$

and the energy average below the Q2T becomes

$$\mathcal{L}\left\{\mathcal{F}[Z_{\alpha,\beta}(q, q'; E)]\right\} = \mathcal{L}\left\{\frac{V_0 e^{-\sigma r}}{r}\right\} = V_0 \frac{a^{2\gamma+2}}{r(r/2 + a)^{2\gamma+2}}. \tag{23}$$

The result has the same form as Eq. (16). This is a proof to show that the GPT potential for the case below Q2T has the same form as that for 3BT.

In the NNπ system, Eq. (5) becomes

$$V(r) = V_0 \exp\left[-\left(\frac{\gamma+1}{a}\right)r\right]/r = V_0 \mu_\pi \frac{e^{-\mu_\pi r}}{\mu_\pi r}, \tag{24}$$

with $\gamma = a\mu_\pi - 1$. For $\gamma = -1/2$ taking $\hbar = c = 1$ unit, it gives $a = 0.5/\mu_\pi$ where the GPT-potential becomes $1/r^2$-type potential with the Efimov-like energy levels. We estimated the deuteron's first excited state which will appear at 13 keV [6]. If not, take $\gamma = 0$, it gives $a = 1/\mu_\pi$ which means $1/r^3$-potential. Such a potential could still have an excited state.

We have used the quasi two-body Green's function of Eq. (18) instead of the three-body free Green's function Eq. (10) which is seen in the usual three-body Faddeev equation. Here, it should be remembered at "below the 3BT", that the Green's function could be constructed not by the two-body free but by a bound or loosely bound pair plus the spectator kinetic energies. Therefore, if we insist on the free three-particle Green's function as in the Faddeev's form, we have to adopt an energy-momentum "scale-translation" [13] to preserve the on-shell condition in the quasi two-body Green's function apart from the original Faddeev equation.

4 One Pion Transfer NN Interaction and Recoil Effect

One may feel a curiosity about the existence of the long range NN interaction in NNπ system, because the hadron problems have been investigated by using some short range nuclear potential where the $r \sim 1/\mu_\pi$ is a longest range. This kind of annexed long range effect could not be found in the *static*-NN interpretation where the initial and the intermediate nucleon kinetic energies are not considered in the meson theory . However, modern relativistic three-body calculations are intensively investigated by some authors [14,15].

Let us imagine that two nucleons rest at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ where no meson exists which is written by $|0>$. By using the second order perturbation formula with a Hamiltonian $H'_{N\pi}$, we can calculate the energy difference between the $|0>$ state with the kinetic energy $E_0 = 0$ and the intermediate state $|m>$ which is given by two nucleons and one meson with the energy: $E_m = E'(q_1, q_2) + \omega_{\boldsymbol{k}}$. Therefore, the perturbation formula indicates that a nucleon "1" produces a meson by an interaction $H'_{N_1 \cdot \pi}$ which is absorbed by another nucleon "2" by $H'_{N_2 \cdot \pi}$. The Hamiltonian $H'_{N_n \cdot \pi}$ is defined for the pseudo-scalar meson by using the meson creation $a^*_{\boldsymbol{k}}$, annihilation $a_{\boldsymbol{k}}$ operators, and $\hbar = c = 1$ units,

$$H'_{N_n \cdot \pi} = \frac{i f_0}{\mu_\pi} \sum_{\boldsymbol{k}} \left(\frac{2\pi}{V \omega_k} \right)^{1/2} \times (\boldsymbol{\sigma}_n \cdot \boldsymbol{k}) \{ a_{\boldsymbol{k}} \exp(i \boldsymbol{k} \cdot \boldsymbol{r}_n) - a^*_{\boldsymbol{k}} \exp(-i \boldsymbol{k} \cdot \boldsymbol{r}_n) \} \tag{25}$$

where $\boldsymbol{\sigma}_n$ is the spin vector for the n-th nucleon, $\boldsymbol{k}$ the meson momentum and $\omega_k = \sqrt{k^2 + \mu_\pi^2}$ is the meson energy. Therefore, the second order perturbation formula becomes

$$W_2 = \sum_{\boldsymbol{q}_1, \boldsymbol{q}_2} \sum_{\boldsymbol{k}} \left\{ \frac{< 0|H'_{N_2 \cdot \pi}|m >< m|H'_{N_1 \cdot \pi}|0 >}{E_0 - (E'(q_1, q_2) + \omega_k)} + (1 \leftrightarrow 2) \right\}. \tag{26}$$

If we assume a *static* approximation $0 = E_0 \approx E'(q_1, q_2)$ by the reason that the nucleon mass is much larger than the meson mass, then the nucleon-recoils by the meson creation and annihilation are neglected, although $\Delta = \mu_\pi / M_N = 0.14703$ is not very small.

Therefore, by using Eqs. (25), (26) becomes

$$W_2 \approx -\frac{4\pi f_0^2}{\mu_\pi^2 V} \sum_{\boldsymbol{k}} \frac{(\boldsymbol{\sigma}_1 \cdot \boldsymbol{k})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{k}) \cos\{\boldsymbol{k} \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_2)\}}{\omega_k^2}. \tag{27}$$

The nuclear potential is obtained by putting $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, in Eq. (27), and using well-known relation: $\sum_{\boldsymbol{k}} = \int V d\boldsymbol{k}/(2\pi)^3$,

$$U_0 = -\frac{4\pi f_0^2}{(2\pi)^3 \mu_\pi^2} \int \frac{(\boldsymbol{\sigma}_1 \cdot \boldsymbol{k})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{k}) \cos(\boldsymbol{k} \cdot \boldsymbol{r})}{k^2 + \mu_\pi^2} d\boldsymbol{k} \tag{28}$$

$$= \frac{1}{3} \mu_\pi f_0^2 \left\{ (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) + \left(1 + \frac{3}{\mu_\pi r} + \frac{3}{\mu_\pi^2 r^2} \right) S_{1,2} \right\} \frac{e^{-\mu_\pi r}}{\mu_\pi r}, \tag{29}$$

$$S_{1,2} = 3 \frac{(\boldsymbol{\sigma}_1 \cdot \mathbf{r})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{r})}{r^2} - (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2), \tag{30}$$

where $S_{1,2}$ is the tensor operator. This is the well-known Yukawa-potential for π-meson transfer where the recoil effect is neglected.

However, in the three-body scattering problem, Eq. (26) can be rewritten by taking the initial and the intermediate energies as,

$$E_0 = \omega_{01} + \omega_{02} = \sqrt{q_{01}^2 + m_1^2} + \sqrt{q_{02}^2 + m_2^2}, \tag{31}$$

$$E'(q_1, q_2) = \omega_1 + \omega_2 = \sqrt{q_1^2 + m_1^2} + \sqrt{q_2^2 + m_2^2}, \tag{32}$$

$$\omega_3 = \sqrt{q_3^2 + m_3^2} \equiv \sqrt{k^2 + \mu_\pi^2} = \omega_k, \tag{33}$$

where nucleon masses m_1, m_2, and pion mass $m_3 = \mu_\pi$. q_{01}, q_{02}, and q_1, q_2, k are the initial nucleon momenta and the intermediate nucleons and pion momenta, respectively. That is, we have

$$W_2 = \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2} \sum_{\boldsymbol{k}} \frac{<0|H'_{N_2\cdot\pi}|m><m|H'_{N_1\cdot\pi}|0>}{E_0 - \omega_1 - \omega_2 - \omega_3} + (1 \leftrightarrow 2). \tag{34}$$

If we adopt the non-relativistic approximation $\omega_j \approx q_j^2/2m_j + m_j$, then we can reduce Eq. (34), by putting $E \equiv E_0 - m_1 - m_2 - m_3 \approx q_{01}^2/2m_1 + q_{02}^2/2m_2 - m_3$, which leads on $E_{cm} \equiv E + m_3 = q_{01}^2/2m_1 + q_{02}^2/2m_2$ in Eq. (19).

Finally, we obtain,

$$W_2 \approx \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2} \sum_{\boldsymbol{k}(\boldsymbol{q}_3)} \frac{<0|H'_{N_2\cdot\pi}|m><m|H'_{N_1\cdot\pi}|0>}{E - q_1^2/2m_1 - q_2^2/2m_2 - q_3^2/2m_3} + (1 \leftrightarrow 2) \tag{35}$$

$$\Rightarrow \frac{g_\alpha(p)[\bar{\delta}_{\alpha\beta}]g_\beta(p')}{E - q^2/2\mu - p^2/2\nu} + (\alpha \leftrightarrow \beta), \quad \text{with} \quad \bar{\delta}_{\alpha\beta} \equiv (1 - \delta_{\alpha\beta}), \tag{36}$$

where $\boldsymbol{p}$ is the relative momentum of any two-body couple, and $\boldsymbol{q}$ is the relative momentum between the center of mass and the spectator particle with respect to α, β and γ channels. In Eq. (36), the first and the second terms are separately calculated in the AGS equation by removing $\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2} \sum_{\boldsymbol{k}}$ from Eq. (35). The first term of Eqs. (36) or (10) is the Born terms of AGS equations where the recoil effect is completely taken into account. Only if, we integrate Eq. (35) by $\boldsymbol{k}(\equiv \boldsymbol{q}_3)$, the Yukawa range $[\mu_\pi]^{-1}$ becomes a function of q_1 and q_2 by $\Delta(q_1, q_2) \equiv E_0 - E'(q_1, q_2) \neq 0$. A special case: $E_0 - E'(q_1, q_2) - m_3 = 0$ gives a potential of $W_2 \propto 1/r$. Therefore, one could conclude that the NN interaction "accompanied with the long range effect" is rather natural in the few-body formulation which is seen in the Efimov-like potential.

5 Conclusion and Discussion

We investigated the quasi-two-body potential below the 3BT. As a result, we obtained the GPT-potential: $1/r^n$ for long range in which the $n = 1$ indicates an attractive Coulomb-like potential or the gravitation-like potential where the transfer-particle should be massless. The second is the Efimov's potential with $n = 2$. This fact seems to suggest that the transfer-particle must be very small compared to the parent-particle's masses. By this speculation, one could imagine what kind of three-body system causes the Efimov's phenomena?

We imagine that the next Efimov-like potential could be given by the mass ratio between a transfer particle m_3 and the remaining masses: $\xi = m_3/(M_1 + M_2) \ll 1$. Two-heavy and one-light particles system is one of the promising system, if our GPT potential is used. However, such a system is historically investigated by the Born-Oppenheimer approximation, therefore the method includes a static approximation where the recoil effect is partly neglected and the potential becomes a Yukawa-type at long range potential tail [16]. Pioneering work in 1961 indicated that the size parameter between α-α in ^{9}Be nucleus with $\alpha - n - \alpha$ cluster system is larger than that in ^{8}Be nucleus [17]. However, both articles do not represent the long range tail with $1/r^n$.

The Efimov-like phenomena in the hadron systems are only obtained by introducing "the quasi-two-body Green's (Q2G) function" in Eq. (18) at the Q2T. However, the Q2G-function can not be represented by the original three-body Faddeev equation where the "on-shell condition of the Q2G-function" is not satisfied for the three-body free Green's function below the 3BT.

Finally, we raise an important question: "Where the long range potential comes from?" [6]. One of the reasons which we point out, is that the short range Yukawa potential was introduced by a *static* approximation neglecting the *recoil effect* in the meson transfer. However, the "exact" three-body treatment for the NNπ-system can overcome the *recoilless-shortage*, and brings the long range potential.

It should be stressed that the "static" approximation in the "field theory" is generally used for the *many-body* system. However, we have to look at the validity of the static approximation in the *few-body* system. The recoil effect seems to be usually minor for the light-particle transfer, but massless photon transfer could be sometimes very important as shown in the Mössbauer effect.

In the end, the most interesting and plausible application of this work is the ultra low energy nuclear transformation which seems to be hopeful rather than the Efimov effect for the cold atomic science.

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