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Proton Radius Puzzle in Hamiltonian Dynamics

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Abstract Relativistic lepton–proton bound-state eigenvalue equations for Hamiltonians derived from quantum field theory using second-order renormalization group procedure for effective particles, are reducible to two-body Schrödinger eigenvalue equations with the effective Coulomb potential that exhibits a tiny sensitivity to the characteristic momentum-scale of the bound system. The scale dependence is shown to be relevant to the theoretical interpretation of precisely measured lepton–proton bound-state energy levels in terms of a 4 % difference between the proton radii in muon–proton and electron–proton bound states.

1 Introduction

The size of proton charge distribution plays a relatively minor role in atomic physics since it is about five orders of magnitude smaller than the size of atoms. However, from precise measurements and calculations of atomic energy levels a puzzling situation emerges. The proton size in the muon–proton bound states appears to be about 4 % smaller than in the electron–proton bound states. Quoting Ref. [1], “... [the radius] should be a simple quantity to determine and understand, but that is not the case. Recent experimental results are not yet well understood, but future research may reveal the true value of this radius, lead to a better understanding of its structure, or demonstrate an unexpected aspect of its interactions.” It is pointed out in this article that the front form (FF) of Hamiltonian dynamics [2] equipped with the renormalization group procedure for effective particles (RGPEP) may shed a new light on the issue [3].

Briefly speaking, the proton-size puzzle is seen here as emerging from an apparent discrepancy between, on the one hand, the Standard Model (SM) assumption that the atomic systems in question are ultimately described by a local quantum field theory (QFT) and, on the other hand, the standard atomic physics assumption that in the first approximation the electron–proton and muon–proton bound states can be described using the well-known two-body Schrödinger equation with a Coulomb potential. To be specific, the QFT predicts that the physical systems are superpositions of innumerable many component states with varying and ultimately unlimited numbers of virtual field quanta, while the Schrödinger equation describes just two particles that interact through a simple potential. In essence, calculations carried out within the framework of QFT cannot be based on the simple Schrödinger picture without explanation of exactly how the latter is supposed to emerge from the former. Even if one assumes that quarks and gluons of QCD, as a part of the SM, form a proton that can be treated as nearly point-like, one is still left with a complex QED picture in which the lepton–proton bound states contain indefinite numbers of virtual photons and fermion-anti-fermion pairs on top of a very complex ground state, or vacuum. So far, the QFT vacuum complexity prevents physicists from providing for

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it any accurate construction [4,5]. Despite great progress in methodology and computational technology over more than half a century, which can be illustrated by many examples of highly advanced work of which only a small number can be quoted here [6–16], the basic question of how a complex bound-state dynamics in QFT could be systematically reduced to the Schrödinger equation for just two particles still awaits a precise answer.

The contrast between complexity of QFT and simplicity of a few-body Schrödinger equation is not specific to atomic physics. Even more striking contrast exists between the complexity of QCD and simplicity of the constituent quark model, a basis for classification of hadrons which says that mesons are made of two and baryons of three constituents. However, a full-fledged application of the RGPEP to a complete version of the Standard Model, or its potential extensions, is a long way off.

This article is very limited in scope. It only concerns corrections to lepton–proton ground-state energy levels due to the proton radius. The point is that these corrections are sensitive to the effective nature of the two-body Schrödinger equation and the sensitivity appears large enough for taking it into account in extracting the proton radius from the measured energy levels. However, it should be stressed that many other corrections due to the effective nature of the two-body Schrödinger equation need to be calculated first before one can precisely establish the magnitude of small energy terms that depend on the proton radius.

In order to outline how the FF of Hamiltonian dynamics equipped with the RGPEP may help in resolving the proton radius puzzle, the article is organized in the following way. Section 2 explains what is meant by the proton radius puzzle. The RGPEP derivation of the Schrödinger equation from QFT is outlined in Sect. 3 and key details of the derivation are described in Sect. 4. Section 5 indicates the possibility of a resolution of the proton radius puzzle using the RGPEP. Sect. 6 concludes the article with general comments concerning the proton radius puzzle, Schrödinger equation for few-body systems and QFT.

2 Puzzle of the Proton Size in the Schrödinger Equation

The first-approximation Schrödinger eigenvalue equation for a lepton–proton system internal dynamics in momentum representation has the form (using the convention that $\hbar = c = 1$)

$$\frac{\mathbf{p}^2}{2\mu} \psi(\mathbf{p}) + \int \frac{d^3k}{(2\pi)^3} V(\mathbf{p}, \mathbf{k}) \psi(\mathbf{k}) = -E \psi(\mathbf{p}), \quad (1)$$

where \mathbf{k} and \mathbf{p} denote the relative momentum of the lepton with respect to the proton, μ is the reduced mass and E is the binding energy. The interaction kernel $V(\mathbf{p}, \mathbf{k})$ is the spin-independent Coulomb potential for point-like charges,

$$V(\mathbf{p}, \mathbf{k}) = V_C^{\text{pt}}(\mathbf{q}) = -\frac{4\pi\alpha}{\mathbf{q}^2}, \quad (2)$$

where $\mathbf{q} = \mathbf{p} - \mathbf{k}$ is called the momentum transfer and $\alpha \sim 1/137$ is the fine structure constant. The extended proton charge distribution is accounted for by replacing $V_C^{\text{pt}}(\mathbf{q})$ with

$$V_C(\mathbf{q}) = V_C^{\text{pt}}(\mathbf{q}) G_E(\mathbf{q}^2), \quad (3)$$

where $G_E(\mathbf{q}^2)$ is the proton electric form factor. For the small values of momentum transfer that characterize atomic systems,

$$G_E(\mathbf{q}^2) = 1 - \frac{1}{6} r_p^2 \mathbf{q}^2 + o(\mathbf{q}^2), \quad (4)$$

where r_p denotes the proton radius. This radius is understood to be a physical property of the proton and it is expected to be the same in the muon–proton and electron–proton bound states. Keeping only the first two terms in the expansion, one obtains the corrected Coulomb potential in the form

$$V_C(\mathbf{q}) = V_C^{\text{pt}}(\mathbf{q}) + \delta V(\mathbf{p}, \mathbf{k}), \quad (5)$$

where the correction term is

$$\delta V(\mathbf{p}, \mathbf{k}) = \frac{2\pi\alpha}{3} r_p^2. \quad (6)$$

The corresponding correction to the Coulomb potential in position representation, i.e., a correction to $-\alpha/|\mathbf{r}|$, is

$$\delta V(\mathbf{r}) = \frac{2\pi\alpha}{3} r_p^2 \delta^3(\mathbf{r}). \quad (7)$$

Thus, the first-order corrections to energy levels due to the extended nature of the proton charge distribution are described by the formula

$$\Delta E = \frac{2\pi\alpha}{3} r_p^2 |\hat{\psi}(0)|^2, \quad (8)$$

where the wave function at $\mathbf{r} = 0$ is

$$\hat{\psi}(0) = \int \frac{d^3k}{(2\pi)^3} \psi(\mathbf{k}). \quad (9)$$

For example, in the ground state described by Eq. (1), where $|\hat{\psi}(0)|^2 = (\alpha\mu)^3/\pi$, the correction due to the proton radius is

$$\Delta E = \frac{4}{3} (\alpha\mu r_p)^2 \frac{\mu\alpha^2}{2}. \quad (10)$$

The factor $\alpha\mu$ in the electron–proton bound state is practically equal to the inverse of the Bohr radius, r_B . So, the product $\alpha\mu r_p$ is the ratio of the proton and hydrogen radii, $r_p/r_B \sim 10^{-5}$. Hence, the energy correction ΔE is on the order of 10^{-10} times Rydberg. In the muon–proton bound state, the reduced mass is about 200 times greater than in the electron–proton bound state, because muons are about 200 times heavier than electrons. Therefore, the correction to energy is 200^3 times larger and precise energy measurements allow one to extract the value of the proton radius, r_p , from muon–proton bound-state data with accuracy reaching 1% [17, 18].

Of course, many other corrections need to be included in the calculations for atomic systems before one can separate the corrections due to the small proton radius. Such calculations have a long history of extensive work using different methods [6–16]. The proton radius puzzle is that, after all the calculated terms are included, the quantity r_p extracted from the muon–proton bound-state data is smaller by about 4% than the corresponding quantity extracted from the electron–proton bound-state and scattering data (with an exception of some dispersion analysis of electron–proton scattering data, such as in Ref. [19], discussed in Ref. [1]). This article deals exclusively with the energy correction ΔE and focuses on the changes in its interpretation that follow from the effective nature of the two-body Schrödinger equation for lepton–proton bound states.

3 Outline of the RGPEP Path from QFT to the Schrödinger Equation

The RGPEP allows one to derive the Schrödinger equation for two-body bound states from relativistic QFT in several steps [3]. One begins by proposing a classical field-theoretic Lagrangian density that formally leads to an associated canonical FF Hamiltonian density, accounting for constraints. Then one quantizes the independent field degrees of freedom and integrates the resulting quantum density over the front to obtain the Hamiltonian operator. This operator is singular. One regulates it to remove the infinities in favor of the presence of cutoffs. The regulated canonical Hamiltonian provides a trial initial condition for solving the first-order differential equation of the RGPEP.

The differentiation is defined with respect to the RGPEP momentum scale parameter λ , which labels members of a family of effective Hamiltonians. The scale parameter λ varies from infinity, which labels the regulated canonical Hamiltonian, down to the characteristic momentum scale of the phenomena one is interested in describing, which labels the effective Hamiltonian that provides the simplest dynamical explanation of the phenomena of interest. The observables do not depend on λ , by construction, but the effective degrees of freedom and their dynamics do.

The generic QFT difficulty is that the initial condition provided by the regulated canonical Hamiltonian yields solutions for the effective Hamiltonians that depend on the regularization cutoff parameters in diverging ways. Therefore, the initial condition must be modified until the required effective Hamiltonian ceases to depend on the regularization. The modification is achieved by finding suitable counter-terms that are added to

the initial condition. The conceptual difficulty is that the conditions that determine the counter-terms concern the effective Hamiltonian with a finite value of λ while the counter-terms are only inserted in the initial-condition Hamiltonian at $\lambda \rightarrow \infty$. However, when the coupling constant is as small as $\alpha \sim 1/137$ and the issues of confinement are not overwhelmingly important (proton can be treated as nearly elementary particle), one can solve the differential RGPEP equation using an expansion in powers of α and find the counter-terms order by order.

The effective Hamiltonian with a suitable λ for lepton–proton bound-state calculations is relatively easy to calculate in the relevant QFT with accuracy up to terms of order α [3]. Such accuracy is sufficient to derive the effective interaction that corresponds to the Coulomb potential in the two-body Schrödinger equation, Eq. (1), with an extended proton charge. The non-local nature of the proton charge distribution can be accounted for by inserting the proton electric form factor in the initial-condition Hamiltonian at $\lambda = \infty$ and including it in all steps of the entire procedure. The eigenvalue problem for the resulting Hamiltonian still involves infinitely many Fock components because the terms of formal order $\sqrt{\alpha}$ are capable of producing photons from fermions and fermion-anti-fermion pairs from photons. But these components are now built using the effective-particle creation operators characterized by a finite scale λ , instead of the ones for bare quanta characterized by $\lambda = \infty$. Moreover, the effective-particle creation operators are now applied to a simple vacuum state.

The elimination of vacuum difficulty is a characteristic feature of the regulated FF Hamiltonian dynamics, e.g., see Ref. [5]. The change from bare to effective particles is a new feature brought in by the RGPEP. The interaction vertices in the Hamiltonian for effective particles contain exponential form factors of width λ in momentum variables. These form factors are obtained by solving the RGPEP equations. They limit the allowed changes of the invariant mass of interacting effective particles. Therefore, the couplings between sectors with different numbers of particles are suppressed at large energies and cannot produce large effects if the coupling constant is small. In case of a lepton–proton bound state, one can assume that its dominant component is the state of just one effective lepton and one effective proton. The dynamical contribution of the effective lepton–proton–photon component can be calculated to order α . In summary, the great complexity of QFT is absorbed in the internal structure of the two effective constituents that interact through an effective potential.

The resulting effective two-body eigenvalue problem is nearly identical to Eq. (1), except that the Coulomb interaction is modified by the RGPEP form factor of width $\lambda \sim \sqrt{\mu m}$, where μ is the reduced and m the average mass of constituents. It turns out that this form factor is capable of generating a few-percent effect in the tiny corrections to atomic energy levels from which the proton radius is extracted.

4 Key Details of a Derivation of the Schrödinger Equation from QFT

The Lagrangian density we start from is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{n=1}^3 \bar{\psi}_n(i\partial - e_n\mathcal{A} - m_n)\psi, \quad (11)$$

where the subscripts $n = 1, 2, 3$ refer to electrons, muons and protons, correspondingly, and at this stage protons are still considered point-like. The proton form factor is inserted at a later stage, in the FF Hamiltonian that we derive starting from \mathcal{L} . All tensors are written using the same convention that is used for Minkowski's space-time coordinates, $x^\pm = x^0 \pm x^3$ and $x^\perp = (x^1, x^2)$. Variable x^+ plays the role of time and x^\perp and x^- play the roles of space co-ordinates. In gauge $A^+ = 0$, the canonical FF Hamiltonian is [20]

$$P^- = \int dx^- d^2x^\perp \left\{ \frac{1}{2}A_\mu \partial^{\perp 2} A^\mu + \sum_{n=1}^3 \left[\bar{\psi}_n \gamma^+ \frac{-\partial^{\perp 2} + m_n^2}{2i\partial^+} \psi_n + e_n \bar{\psi}_n \mathcal{A} \psi_n + e_n^2 \bar{\psi}_n \mathcal{A} \frac{\gamma^+}{2i\partial^+} \mathcal{A} \psi_n + e_n \bar{\psi}_n \gamma^+ \psi_n \frac{1}{2(i\partial^+)^2} \sum_{k=1}^3 e_k \bar{\psi}_k \gamma^+ \psi_k \right] \right\}, \quad (12)$$

where the dependent components of fields, A^- and ψ_{n-} , are solutions to the constraint equations with electric charge set to zero. Quantization is carried out by replacing the independent fields A^\perp and ψ_{n+} by operators \hat{A}^\perp and $\hat{\psi}_{n+}$ expanded into their Fourier modes that satisfy the standard FF commutation relations. The quantum Hamiltonian is regulated by limiting the range of momenta in the Fourier expansions of the fields or by limiting

the changes of momenta in the interaction terms [21]. The required counter-terms up to order α are found in the process of evaluating the Hamiltonian with a finite momentum scale λ , using the condition that its matrix elements in the basis states of small invariant mass do not depend on the regularization. The scale λ is introduced by the unitary rotation of particle creation and annihilation operators, commonly denoted below by q , from the bare ones at $\lambda = \infty$ to the effective ones at a finite λ ,

$$q_\lambda = \mathcal{U}_\lambda q_\infty \mathcal{U}_\lambda^\dagger, \quad (13)$$

and keeping the Hamiltonian operator unchanged,

$$H_\lambda(q_\lambda) = H_\infty(q_\infty). \quad (14)$$

Differentiation with respect to λ yields the RGPEP evolution equation, which includes a Hamiltonian-dependent kernel designed to decrease the range of allowed off-shellness in interaction terms as λ decreases. This equation is solved using expansion in powers of α . Here it is sufficient to include the terms of formal order 1, $\sqrt{\alpha}$ and α . The calculation follows the same steps as in the case of heavy quarkonia [22], but it is much simpler because the theory is simpler than QCD. For example, the lepton and proton mass-squared counter-terms can be adjusted to physical fermion mass values using the single fermion eigenvalue equations for H_λ , since the fermions are not confined (see [3]).

Having established the effective Hamiltonian of scale λ up to order α , one poses its eigenvalue problem for lepton–proton bound states. The eigenstates are written in terms of a basis in the Fock space built using the creation operators corresponding to λ . The expansion into the effective Fock components is exponentially suppressed for invariant-mass changes greater than λ . Therefore, for very small α , one can approximate the bound state by a superposition of the effective lepton–proton and lepton–proton–photon components. This is sufficient to derive the equivalent effective eigenvalue problem of accuracy up to terms of formal order α in the effective two-body lepton–proton component, accounting for the three-body component via a one effective-photon exchange between the effective fermions (the associated self-interaction terms are accounted for in the mass-squared terms). The proton electric form factor inserted in the initial Hamiltonian appears unchanged in the resulting eigenvalue equation

$$\frac{\mathbf{p}^2}{2\mu} \psi(\mathbf{p}) + \int \frac{d^3k}{(2\pi)^3} V_\lambda(\mathbf{p}, \mathbf{k}) \psi(\mathbf{k}) = -E_B \psi(\mathbf{p}), \quad (15)$$

where

$$V_\lambda(\mathbf{p}, \mathbf{k}) = f_\lambda(\mathbf{p}, \mathbf{k}) V_C^{\text{pt}}(\mathbf{q}) G_E(\mathbf{q}^2). \quad (16)$$

The interaction differs from $V(\mathbf{p}, \mathbf{k})$ in Eq. (3) by the scale dependent RGPEP form factor

$$f_\lambda(\mathbf{p}, \mathbf{k}) = e^{-(\Delta\mathcal{M}^2/\lambda^2)^2}, \quad (17)$$

in which $\Delta\mathcal{M}^2$ is the difference of lepton–proton free invariant masses squared before and after the interaction,

$$\Delta\mathcal{M}^2 = \mathcal{M}^2(\mathbf{p}) - \mathcal{M}^2(\mathbf{k}). \quad (18)$$

For a lepton of mass m_l and proton of mass m_p , one has [3]

$$\mathcal{M}^2(\mathbf{p}) = (m_l + m_p)^2 [1 + \mathbf{p}^2/(m_l m_p)] \quad (19)$$

and the RGPEP form factor reads

$$f_\lambda(\mathbf{p}, \mathbf{k}) = e^{-(\mathbf{p}^2 - \mathbf{k}^2)^2 (m_l + m_p)^4 / (\lambda^2 m_l m_p)^2}. \quad (20)$$

To match the well-known universal scaling of the atomic Schrödinger equation with α , one needs

$$\lambda = a \sqrt{\mu (m_l + m_p)}/2, \quad (21)$$

where the number a is not expected to differ a lot from 1. For the Schrödinger equation is known to be valid in the whole range of momenta smaller than the constituent masses irrespective of the values of the masses and it does not involve important contributions from the range of momenta much larger than the masses. Thus, the form factor $f_\lambda(\mathbf{p}, \mathbf{k})$ that changes the abstract Coulomb potential for point-like particles, Eq. (3), to the effective potential that befits the effective nature of constituents in the Schrödinger equation, Eq. (16), has the form

$$f_\lambda(\mathbf{p}, \mathbf{k}) = e^{-(\mathbf{p}^2 - \mathbf{k}^2)^2 / (a\mu)^4}. \quad (22)$$

5 Toward a Solution of the Proton Radius Puzzle Using the RGPEP

Using the input from Sects. 2, 3 and 4, one arrives at the conclusion that the small two-body interaction Hamiltonian term due to the physical proton radius is not given by Eq. (6), but by the formula

$$\delta V(\mathbf{p}, \mathbf{k}) = f_\lambda(\mathbf{p}, \mathbf{k}) \frac{2\pi\alpha}{3} r_p^2. \quad (23)$$

Consequently, the associated energy corrections are described not by Eq. (8), but by the formula

$$\Delta E = c_\lambda \frac{2\pi\alpha}{3} r_p^2 |\hat{\psi}(0)|^2 \quad (24)$$

where the coefficient c_λ is given by

$$c_\lambda = \frac{1}{|\hat{\psi}(0)|^2} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \psi(\mathbf{p}) f_\lambda(\mathbf{p}, \mathbf{k}) \psi(\mathbf{k}), \quad (25)$$

for the wave functions normalized to 1. Since $f_\lambda < 1$ for most values of the momenta, the result for c_λ is smaller than 1.

The magnitude of deviation of c_λ from 1 can be estimated using the momentum variables in units of the Bohr momentum, or $\mathbf{p} = \alpha\mu\mathbf{u}$ and $\mathbf{k} = \alpha\mu\mathbf{v}$. In these variables, Eq. (22) reads

$$f_\lambda(\mathbf{p}, \mathbf{k}) = e^{-(\alpha/a)^4 (\mathbf{u}^2 - \mathbf{v}^2)^2}. \quad (26)$$

The exponential contains $(\alpha/a)^4$ and one might think that c_λ differs from 1 by terms order $(\alpha/a)^4 \sim 3 \times 10^{-9}/a^4$, which would be a negligible effect. Thus, if one assumed that an expansion in powers of α should apply, the effective nature of the Schrödinger equation could be ignored. However, the expansion in powers of α does not apply. For example, the unperturbed ground-state wave functions are $N/(1+\mathbf{u}^2)^2$ and $N/(1+\mathbf{v}^2)^2$, where N is the normalization constant. Therefore, the term order α^4 in expansion of c_λ is divergent. One has to evaluate c_λ numerically. Since muons are about 200 times heavier than electrons, the same value of λ in one and the same effective theory for both electron–proton and muon–proton systems implies that the factor a in Eq. (21) is about 14 times smaller in the muon–proton system than in the electron–proton system. Such large change of a is capable of changing c_λ from nearly 1 in the electron–proton system to about 0.92 in the muon–proton system. The associated 8% reduction in the radius squared means 4% reduction in the radius itself, which is about the magnitude of the difference that causes the proton radius puzzle.

6 Conclusion

According to the RGPEP, the proton radius puzzle stems from the difference between Eqs. (8) and (24). If one employs the usual Eq. (8), the proton radius is extracted from small corrections to energy levels as the quantity which, according to Eq. (24), is $c_\lambda r_p^2$, instead of r_p^2 itself. The coefficient c_λ accounts for the effective nature of constituents that appear in the two-body Schrödinger equation for lepton–proton bound states. Being smaller than 1, c_λ implies that the usual way of interpreting the standard two-body Schrödinger equation leads to extraction of a slightly smaller quantity for the proton radius than its value in the electric form factor; the heavier the lepton the smaller the extracted quantity. The numerical deviation of c_λ from 1 is of the order required for solving the puzzle.

However, it would be premature to conclude that the RGPEP factor c_λ has already solved the proton radius puzzle. The reason is that there are many other corrections to the energy levels known in other approaches that need to be calculated using the RGPEP before it could reliably help in isolating the size of corrections due to the proton charge radius and in extracting the latter from atomic and scattering data. Carrying out such calculations constitutes, on the one hand, a major research program and, on the other hand, presents itself as an opportunity for the theory of few-body systems to assert its position in the field of particle dynamics, not only in the atomic physics but in all areas of physics where equations of the Schrödinger type apply as approximate representations of the underlying theory.

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