



On $W^{2,p}$ -estimates for solutions of obstacle problems for fully nonlinear elliptic equations with oblique boundary conditions

Sun-Sig Byun¹ · Jeongmin Han² · Jehan Oh³

Received: 21 October 2021 / Accepted: 7 May 2022 / Published online: 22 June 2022
© The Author(s) 2022

Abstract

This paper concerns fully nonlinear elliptic obstacle problems with oblique boundary conditions. We investigate the existence, uniqueness and $W^{2,p}$ -regularity results by finding approximate non-obstacle problems with the same oblique boundary condition and then making a suitable limiting process.

Mathematics Subject Classification Primary: 35J25 · Secondary: 35J15 · 35J60

1 Introduction and main results

This paper concerns the existence, uniqueness and regularity for viscosity solutions to the following obstacle problem with oblique boundary data

Communicated by Y. Giga.

S.-S. Byun was supported by NRF-2021R1A4A1027378. J. Han was supported by NRF-2017R1C1B2010328. J. Oh was supported by NRF Grant (Nos. NRF-2020R1C1C1A01014904, NRF-2020R1A4A1018190).

✉ Jeongmin Han
jehan@jyu.fi

Sun-Sig Byun
byun@snu.ac.kr

Jehan Oh
jehan.oh@knu.ac.kr

¹ Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 08826, Republic of Korea

² Department of Mathematics and Statistics, University of Jyväskylä, 40014 Jyväskylä, Finland

³ Department of Mathematics, Kyungpook National University, Daegu 41566, Republic of Korea

$$\begin{cases} F(D^2u, Du, u, x) \leq f & \text{in } \Omega, \\ (F(D^2u, Du, u, x) - f)(u - \psi) = 0 & \text{in } \Omega, \\ u \geq \psi & \text{in } \Omega, \\ \beta \cdot Du = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

for a given obstacle $\psi \in W^{2,p}(\Omega)$ satisfying $\beta \cdot D\psi \geq 0$ a.e. on $\partial\Omega$. Here Ω is a bounded domain in \mathbb{R}^n with its boundary $\partial\Omega \in C^3$, F is uniformly elliptic with constants λ and Λ , i.e.,

$$\lambda \|X_2\| \leq F(X_1 + X_2, q, r, x) - F(X_1, q, r, x) \leq \Lambda \|X_2\|$$

for any $n \times n$ symmetric matrices X_1, X_2 with $X_2 \geq 0$, $q \in \mathbb{R}^n, r \in \mathbb{R}$ and $x \in \Omega$, and β is a vector-valued function with $\|\beta\|_{L^\infty(\partial\Omega)} = 1$ and $\beta \cdot \mathbf{n} \geq \delta_0$ for some positive constant δ_0 , where \mathbf{n} is the inner unit normal vector field of $\partial\Omega$.

The main purpose of this paper is to find $W^{2,p}$ -regularity theory for (1.1). More precisely, we want to identify the minimal condition of F with respect to x -variable under which the Hessian of a solution is as integrable as both the nonhomogeneous term f and the Hessian of the obstacle ψ in the setting of L^p spaces for $n < p < \infty$.

Throughout this paper, we assume that $F = F(X, q, r, x)$ is convex in X and satisfies

$$d(r_2 - r_1) \leq F(X, q, r_1, x) - F(X, q, r_2, x) \tag{1.2}$$

for any $X \in S(n), q \in \mathbb{R}^n, r_1, r_2 \in \mathbb{R}$ with $r_1 \leq r_2, x \in \Omega$, and some $d > 0$. We further assume that

$$\begin{aligned} \mathcal{M}^-(\lambda, \Lambda, X_1 - X_2) - b|q_1 - q_2| - c|r_1 - r_2| \\ \leq F(X_1, q_1, r_1, x) - F(X_2, q_2, r_2, x) \\ \leq \mathcal{M}^+(\lambda, \Lambda, X_1 - X_2) + b|q_1 - q_2| + c|r_1 - r_2| \end{aligned} \tag{1.3}$$

for $X_1, X_2 \in S(n), q_1, q_2 \in \mathbb{R}^n, r_1, r_2 \in \mathbb{R}$ and $x \in \Omega$. These assumptions are essential in order to derive our desired results for solutions of (1.1), such as the existence, uniqueness and $W^{2,p}$ -regularity.

With an oscillation function of F defined as

$$\Theta_F(x_1, x_2) := \sup_{X \in S(n) \setminus \{0\}} \frac{|F(X, 0, 0, x_1) - F(X, 0, 0, x_2)|}{\|X\|}$$

alongside a small perturbation of Θ_F from its integral average in the L^n -sense, we shall prove the $W^{2,p}$ -regularity for (1.1), as we now state the main result of the paper. We remark that this approach to derive $W^{2,p}$ -regularity was employed in [7].

Theorem 1.1 *Let $n < p < \infty$. Assume that $F = F(X, q, r, x)$ is convex in X , satisfies (1.2)-(1.3) and $F(0, 0, 0, x) \equiv 0, \partial\Omega \in C^3, f \in L^p(\Omega), \beta \in C^2(\partial\Omega)$ with $\beta \cdot \mathbf{n} \geq \delta_0$ for some $\delta_0 > 0$ and $\psi \in W^{2,p}(\Omega)$. Then there exists a small $\epsilon = \epsilon(n, \lambda, \Lambda, p, \delta_0, \|\beta\|_{C^2(\partial\Omega)}) > 0$ such that if*

$$\sup_{x_0 \in \bar{\Omega}, 0 < \rho < \rho_0} \left(\int_{B_\rho(x_0) \cap \Omega} \Theta_F(x, x_0)^n dx \right)^{1/n} \leq \epsilon \tag{1.4}$$

for some $\rho_0 > 0$, then there is a unique viscosity solution $u \in W^{2,p}(\Omega)$ of (1.1) with the following estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq c(\|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)})$$

for some constant $c = c(n, \lambda, \Lambda, p, \delta_0, b, c, \|\beta\|_{C^2(\partial\Omega)}, \|\partial\Omega\|_{C^3}, \text{diam}(\Omega), \rho_0)$.

Remark 1.2 We assumed $F(0, 0, 0, x) \equiv 0$ to derive Theorem 1.1. This allows us to neglect the oscillation of F when $X = 0$. We note that one can also obtain $W^{2,p}$ -regularity by introducing the following oscillation function

$$\tilde{\Theta}_F(x_1, x_2) := \sup_{X \in S(n)} \frac{|F(X, 0, 0, x_1) - F(X, 0, 0, x_2)|}{\|X\| + 1},$$

without the condition $F(0, 0, 0, x) \equiv 0$ (see also [35, Section 5] for the parabolic case).

One of the important issues regarding the obstacle problem is to study solutions near the boundary of the contact surface with the obstacle. To this end, suitable approximation methods have been used. In this regard, we revisit the argument made in [4] where the Dirichlet boundary problem was studied instead. Our main difficulty in using such an argument comes from the situation that we are treating here the oblique boundary condition. Thus, we need to modify the tools used in [4] properly to derive the desired boundary estimates in the present paper. To do this, we verify several uniform properties of a solution for the corresponding non-obstacle problem such as $W^{2,p}$ -regularity and comparison principle.

As a generalization of Neumann boundary problems, researches on oblique derivative problems have been extensively made as in [14, 24, 25, 28, 31, 33]. In particular, several notable results for fully nonlinear elliptic equations were obtained in the notion of viscosity solutions. The existence and uniqueness of fully nonlinear oblique derivative problems were proved in [15, 16, 22]. For the regularity of the associated limiting problem, there have been established $W^{2,\infty}$ -estimates (indeed, $C^{2,\alpha}$ -estimates), see [30] for the Neumann boundary condition and [29] for the oblique boundary condition, respectively. In [3] a global $W^{2,p}$ -regularity for the elliptic oblique derivative problem was proved.

On the other hand, the obstacle problem has been studied along with the free boundary problem. We refer the reader to [17, 32] for a general theory of the obstacle problem. For free boundary problems of the classical Poisson equation, we can find results about the minimal conditions to obtain regularity in [1, 20]. Meanwhile, for fully nonlinear elliptic equations, there have been a number of noteworthy preceding results, for example, [6, 23, 34]. And Indrei’s recent study [21] provided C^1 -regularity of the free boundary without density assumptions. Regularity results for the elliptic obstacle problem can be found in [2, 5, 18, 19]. We would like to point out that C^1 -regularity for the obstacle problem with the oblique boundary condition was shown in [26], while $W^{2,p}$ -regularity for the Dirichlet obstacle problem was given in [4]. Free boundary problems with oblique derivative conditions are studied in [10, 11], including applications to transonic shocks. The main purpose of this paper is to derive a $W^{2,p}$ -estimate for the oblique derivative problem with a $W^{2,p}$ -obstacle. We used the interior estimate [7, Theorem 7.1] and boundary estimate [3, Theorem 3.1] to derive our result.

The remaining part of the paper is organized in the following way. In the next section we introduce basic notation and give a brief exposition of viscosity solutions. Section 3 deals with the associated oblique derivative problem without obstacles. In particular we discuss the existence, uniqueness and $W^{2,p}$ -regularity for the non-obstacle problem. In the last section we finally give the proof of Theorem 1.1.

2 Preliminaries

2.1 Notations

We first introduce some notations which will be used in this paper.

- $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ for $x_0 \in \mathbb{R}^n, r > 0$. $B_r = B_r(0)$.
- $S(n)$ is the set of $n \times n$ symmetric matrices and $\|M\| = \sup_{|x| \leq 1} |Mx|$ for any $M \in S(n)$.
- We denote the gradient and Hessian of u by $Du = (D_1u, \dots, D_nu)$ and $D^2u = (D_{ij}u)$, respectively. Here $D_iu = \frac{\partial u}{\partial x_i}$ and $D_{ij}u = \frac{\partial^2 u}{\partial x_i \partial x_j}$ for $1 \leq i, j \leq n$.
- For any measurable set A with $|A| \neq 0$ and measurable function f , to mean the integral average of f over A ,

$$\int_A f \, dx = \frac{1}{|A|} \int_A f \, dx.$$

2.2 Basic concepts

In this subsection, we briefly present some background knowledge for our discussion. As usual, we are treating a viscosity solution. To do this, we consider the following problem with oblique boundary data

$$\begin{cases} F(D^2u, Du, u, x) = f & \text{in } \Omega, \\ \beta \cdot Du = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. There are several ways to define a viscosity solution depending on the choice of a test function. In this paper, we take a test function φ in $W^{2,p}(\Omega)$. The solution defined in this way is called an L^p -viscosity solution.

Definition 2.1 Let F be continuous in X and measurable in x . Suppose $q > n$ and $f \in L^q(\Omega)$. A continuous function u is called an L^q -viscosity solution for (2.1) if the following conditions hold:

- (a) (subsolution) For each $\varphi \in W^{2,q}(\Omega)$, whenever $\epsilon > 0$, \mathcal{O} is relatively open in $\overline{\Omega}$ and

$$F(D^2\varphi(x), D\varphi(x), \varphi(x), x) \leq f(x) - \epsilon \quad \text{a.e. in } \mathcal{O}$$

and

$$\beta \cdot D\varphi(x) \leq -\epsilon \quad \text{a.e. on } \mathcal{O} \cap \partial\Omega,$$

$u - \varphi$ cannot attain a local maximum in \mathcal{O} .

- (b) (supersolution) For each $\varphi \in W^{2,q}(\Omega)$, whenever $\epsilon > 0$, \mathcal{O} is relatively open in $\overline{\Omega}$ and

$$F(D^2\varphi(x), D\varphi(x), \varphi(x), x) \geq f(x) + \epsilon \quad \text{a.e. in } \mathcal{O}$$

and

$$\beta \cdot D\varphi(x) \geq \epsilon \quad \text{a.e. on } \mathcal{O} \cap \partial\Omega,$$

$u - \varphi$ cannot attain a local minimum in \mathcal{O} .

We remark that it is also possible to take a C^2 -function as a test function if F is continuous in each variable. In this case, the solution is called a C -viscosity solution. For a further discussion of a C -viscosity solution, we refer the reader to [8].

Next we give some tools to treat viscosity solutions. To do this, we introduce Pucci extremal operators.

Definition 2.2 Let $0 < \lambda \leq \Lambda$. For any $M \in S(n)$, the Pucci extremal operator \mathcal{M}^+ and \mathcal{M}^- are defined as follows:

$$\mathcal{M}^+(\lambda, \Lambda, X) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i$$

and

$$\mathcal{M}^-(\lambda, \Lambda, X) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i,$$

where e_i are eigenvalues of X . Moreover, for $b > 0$, we write

$$L_b^\pm(\lambda, \Lambda, u) = \mathcal{M}^\pm(\lambda, \Lambda, D^2u) \pm b|Du|,$$

respectively.

This definition allows us to introduce the class \mathcal{S} . These classes can be considered as classes of viscosity solutions.

Definition 2.3 Let $0 < \lambda \leq \Lambda$. We define the class $\underline{\mathcal{S}}(\lambda, \Lambda, b, f)$ ($\overline{\mathcal{S}}(\lambda, \Lambda, b, f)$, respectively) consisting of all functions u such that

$$L_b^+(\lambda, \Lambda, u) \geq f \quad (L_b^-(\lambda, \Lambda, u) \leq f, \text{ respectively})$$

in the viscosity sense in Ω . We also define

$$S(\lambda, \Lambda, b, f) = \overline{\mathcal{S}}(\lambda, \Lambda, b, f) \cap \underline{\mathcal{S}}(\lambda, \Lambda, b, f)$$

and

$$S^*(\lambda, \Lambda, b, f) = \overline{\mathcal{S}}(\lambda, \Lambda, b, |f|) \cap \underline{\mathcal{S}}(\lambda, \Lambda, b, -|f|).$$

Remark 2.4 Let u be a viscosity subsolution (supersolution, respectively) of

$$F(D^2u, Du, u, x) = f \quad \text{in } \Omega,$$

where $F = F(X, q, r, x)$ is uniformly elliptic with constants λ, Λ satisfying the structure condition (1.3). Then we can observe that u satisfies

$$\begin{aligned} L_b^+(\lambda, \Lambda, u) + F(0, 0, u, x) &\geq f \quad \text{in } \Omega \\ (L_b^-(\lambda, \Lambda, u) + F(0, 0, u, x) &\leq f \quad \text{in } \Omega, \text{ respectively}) \end{aligned}$$

in the viscosity sense.

3 Oblique derivative problems

Before establishing $W^{2,p}$ -regularity ($p > n$) for the obstacle problem (1.1), we first discuss some issues concerning the existence, uniqueness and regularity for the oblique derivative problem (2.1).

We recall Definition 2.3 to start with an Alexandroff-Bakelman-Pucci (ABP) maximum principle for the oblique boundary problem. See [29, Theorem 2.1] for the proof.

Lemma 3.1 *Let u satisfy*

$$\begin{cases} u \in \overline{S}(\lambda, \Lambda, b, f) & \text{in } \Omega, \\ \beta \cdot Du \leq g & \text{on } \Gamma \subset \partial\Omega \end{cases}$$

in the viscosity sense for $f \in L^n(\Omega)$ and $g \in L^\infty(\Gamma)$. Suppose that there exist $\xi \in \partial B_1$ and $\delta_1 > 0$ such that $\beta(x) \cdot \xi \geq \delta_1$ for any $x \in \Gamma$. Then

$$\sup_{\Omega} u^- \leq \sup_{\partial\Omega/\Gamma} u^- + C(\|f^+\|_{L^n(\Omega)} + \max_{\Gamma} g^+),$$

where C only depends on $n, \lambda, \Lambda, b, \delta_1$ and $\text{diam}(\Omega)$.

We also give a weak Harnack’s inequality for supersolutions. The proof can be found in [36, Proposition 1.8].

Lemma 3.2 *Let $p > n$ and $f \in L^p(B_1)$. Suppose that $u \in \overline{S}(\lambda, \Lambda, b, f)$ in the viscosity sense and $u \geq 0$ in B_1 . Then there exist $p_0, C > 0$ depending only on n, λ, Λ and b such that*

$$\|u\|_{L^{p_0}(B_{1/2})} \leq C(\inf_{B_{1/2}} u + \|f\|_{L^p(B_1)}).$$

The following stability lemma, which can be found in [36, Proposition 1.5], will be used later (see also [9, Theorem 3.8]).

Proposition 3.3 *For $k \in \mathbb{N}$, let $\Omega_k \subset \Omega_{k+1}$ be an increasing sequence of bounded domains and $\Omega := \cup_{k \geq 1} \Omega_k$. Let F and F_k be measurable in x and satisfy the structure condition (1.3). Assume that for $p > n$, $f \in L^p(\Omega)$ and $f_k \in L^p(\Omega_k)$, and that $u_k \in C(\Omega_k)$ are L^p -viscosity subsolutions (supersolutions, respectively) of $F_k(D^2u_k, Du_k, u_k, x) = f_k$ in Ω_k . Suppose that $u_k \rightarrow u$ locally uniformly in Ω , and for $B_r(x_0) \subset \Omega$ and $\varphi \in W^{2,p}(B_r(x_0))$*

$$\|(s - s_k)^+\|_{L^p(B_r(x_0))} \rightarrow 0 \quad (\|(s - s_k)^-\|_{L^p(B_r(x_0))} \rightarrow 0), \tag{3.1}$$

where $s(x) = F(D^2\varphi, D\varphi, u, x) - f(x)$ and $s_k(x) = F(D^2\varphi_k, D\varphi_k, u_k, x) - f_k(x)$. Then u is an L^p -viscosity subsolution (supersolution) of

$$F(D^2u, Du, u, x) = f(x) \quad \text{in } \Omega.$$

Moreover, if F and f are continuous, then u is a C -viscosity subsolution (supersolution) provided that (3.1) holds for $\varphi \in C^2(B_r(x_0))$.

Now we return to (2.1). Here we assume that there is a continuous increasing function ω , defined on $[0, \infty)$ with $\omega(0) = 0$, such that

$$F(X_2, q, r, x_2) - F(X_1, q, r, x_1) \leq \omega(|x_1 - x_2|(|q| + 1) + \alpha|x_1 - x_2|^2) \tag{3.2}$$

holds for any $x_1, x_2 \in \Omega, q \in \mathbb{R}^n, r \in \mathbb{R}, \alpha > 0$ and $X_1, X_2 \in S(n)$ satisfying

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_2 & 0 \\ 0 & -X_1 \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.3}$$

One can find the following existence and uniqueness for the problem (2.1) in [27, Theorem 7.19]. We remark that the condition (3.2) is needed to ensure this lemma.

Lemma 3.4 *Assume that $F = F(X, q, r, x)$ is convex in X and continuous in x , satisfies (1.2)-(1.3) and (3.2), and $F(0, 0, 0, x) \equiv 0, \partial\Omega \in C^3, f \in L^p(\Omega) \cap C(\overline{\Omega})$ for $p > n, \beta \in C^2(\partial\Omega)$ with $\beta \cdot \mathbf{n} \geq \delta_0$ for some $\delta_0 > 0$. Then there exists a unique viscosity solution u of (2.1).*

We now have the following $W^{2,p}$ -estimate for the viscosity solution to (2.1).

Lemma 3.5 *Under the assumptions and conclusion in Lemma 3.4, there exists a small $\epsilon = \epsilon(n, \lambda, \Lambda, p, \delta_0)$ such that if*

$$\sup_{x_0 \in \bar{\Omega}, 0 < \rho < \rho_0} \left(\int_{B_\rho(x_0) \cap \Omega} \Theta_F(x, x_0)^n dx \right)^{1/n} \leq \epsilon \tag{3.4}$$

for some $\rho_0 > 0$, then the unique solution u belongs to $W^{2,p}(\Omega)$ with the following estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)} \tag{3.5}$$

for some $C = C(n, \lambda, \Lambda, p, \delta_0, b, c, \|\beta\|_{C^2(\partial\Omega)}, \text{diam}(\Omega), \rho_0)$.

Proof According to [3, Theorem 4.6], u belongs to $W^{2,p}(\Omega)$ with

$$\|u\|_{W^{2,p}(\Omega)} \leq C(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)}) \tag{3.6}$$

for some $C = C(n, \lambda, \Lambda, p, \delta_0, b, c, \|\beta\|_{C^2(\partial\Omega)}, \text{diam}(\Omega), \rho_0)$.

Therefore, it suffices to obtain the estimate (3.5). To prove this, we argue by contradiction. Suppose not. Then there exist sequences $\{u_k\}$ and $\{f_k\}$ such that u_k is the viscosity solution of

$$\begin{cases} F(D^2u_k, Du_k, u_k, x) = f_k & \text{in } \Omega, \\ \beta \cdot Du_k = 0 & \text{on } \partial\Omega \end{cases} \tag{3.7}$$

with

$$\|u_k\|_{W^{2,p}(\Omega)} > k \|f_k\|_{L^p(\Omega)} \quad \text{for each } k \geq 1. \tag{3.8}$$

Consider $\tilde{u}_k = \frac{u_k}{t_k}$, $\tilde{f}_k = \frac{f_k}{t_k}$ and $\tilde{F}_k(X, q, r, x) = \frac{F(t_k X, t_k q, t_k r, x)}{t_k}$, where $t_k = \|u_k\|_{W^{2,p}(\Omega)}$. Then \tilde{u}_k is a viscosity solution of

$$\begin{cases} \tilde{F}_k(D^2\tilde{u}_k, D\tilde{u}_k, \tilde{u}_k, x) = \tilde{f}_k & \text{in } \Omega, \\ \beta \cdot D\tilde{u}_k = 0 & \text{on } \partial\Omega. \end{cases}$$

We also see that \tilde{F}_k satisfies the assumptions of Lemma 3.4 and

$$\|\tilde{u}_k\|_{W^{2,p}(\Omega)} = 1.$$

By (3.8), $\|\tilde{f}_k\|_{L^p(\Omega)} < 1/k$ and this tends to zero as $k \rightarrow \infty$. Moreover, by weak compactness theorem, we can extract a proper subsequence $\{\tilde{u}_{k_j}\} \subset \{\tilde{u}_k\}$ such that

$$\begin{cases} \tilde{u}_{k_j} \rightharpoonup \tilde{v} & \text{in } W^{2,p}(\Omega), \\ \tilde{u}_{k_j} \rightarrow \tilde{v} & \text{in } W^{1,p}(\Omega) \end{cases}$$

for some $\tilde{v} \in W^{2,p}(\Omega)$. Since $p > n$, we also observe that $W^{1,p}(\Omega) \subset\subset C(\bar{\Omega})$ and this yields $\tilde{u}_{k_j} \rightarrow \tilde{v}$ in $C(\bar{\Omega})$. Moreover, we also observe that for each j , $\tilde{u}_{k_j} \in C^{1,\alpha_0}(\partial\Omega)$ for some $0 < \alpha_0 < 1 - n/p$ and $\|D\tilde{u}_{k_j}\|_{L^\infty(\partial\Omega)} \leq C$ for some $C = C(n, p, \Omega) > 0$. Then, from Arzelá-Ascoli criterion, we get

$$D\tilde{u}_{k_j} \rightarrow D\tilde{v} \quad \text{on } \partial\Omega.$$

Hence, by using Proposition 3.3, we see that v is a viscosity solution of

$$\begin{cases} \tilde{F}(D^2\tilde{v}, D\tilde{v}, \tilde{v}, x) = 0 & \text{in } \Omega, \\ \beta \cdot D\tilde{v} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.9}$$

for some $\tilde{F} = \tilde{F}(X, q, r, x)$. Here we can check that \tilde{F} also satisfies (1.2)-(1.3) and (3.2).

Now we deduce that $\tilde{v} \equiv 0$ solves (3.9). Then \tilde{v} is the unique solution of (3.9) by Lemma 3.4. But, in this case, we get

$$1 = \|\tilde{u}_{k_j}\|_{W^{2,p}(\Omega)} \leq C(\|\tilde{u}_{k_j}\|_{L^\infty(\Omega)} + \|\tilde{f}_{k_j}\|_{L^p(\Omega)}) \rightarrow 0 \text{ as } j \rightarrow \infty,$$

which is a contradiction. □

In the above lemma, we have assumed (3.2), which says that F and f are at least continuous in x . By using mollification, we can relax this assumption.

Lemma 3.6 *Assume that $F = F(X, q, r, x)$ is convex in X and measurable in x , satisfies (1.2)-(1.3) and $F(0, 0, 0, x) \equiv 0$, $\partial\Omega \in C^3$, $f \in L^p(\Omega)$ for $p > n$, $\beta \in C^2(\partial\Omega)$ with $\beta \cdot \mathbf{n} \geq \delta_0$ for some $\delta_0 > 0$. Then there exists a unique viscosity solution u of (2.1) with the estimate (3.5).*

Proof Fix $\epsilon > 0$. With a standard mollifier φ having $\text{supp } \varphi \subset B_1$, we define $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$. Then we set $f^\epsilon(x) = (f * \varphi_\epsilon)(x)$ and

$$F^\epsilon(X, q, r, x) = F^\epsilon(X, q, r, x) = (F(X, q, r, \cdot) * \varphi_\epsilon)(x).$$

Note that we extended F and f to zero outside Ω here. Then one can check that $f^\epsilon \in L^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, F^ϵ is convex in X and $F^\epsilon(0, 0, 0, x) \equiv 0$. Furthermore, we observe that F^ϵ satisfies (1.2)-(1.3) and (3.2). By using similar arguments in the proofs of Theorem 4.3 and Theorem 4.6 in [36], we can also show that F^ϵ satisfies (3.4). Consider the following problem

$$\begin{cases} F^\epsilon(D^2u_\epsilon, Du_\epsilon, u_\epsilon, x) = f^\epsilon & \text{in } \Omega, \\ \beta \cdot Du_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Then applying Lemma 3.5 to u_ϵ , there exists the unique solution u_ϵ of (3.10) with

$$\|u_\epsilon\|_{W^{2,p}(\Omega)} \leq C\|f^\epsilon\|_{L^p(\Omega)}$$

for some $C = C(n, \lambda, \Lambda, p, \delta_0, b, c, \|\beta\|_{C^2(\partial\Omega)}, \text{diam}(\Omega), \rho_0)$ (for the existence issue, see also [36, Proposition 1.11]).

Since $W^{2,p}(\Omega) \subset\subset C^{1,\alpha}(\overline{\Omega})$ with $0 < \alpha < 1 - n/p$ by Sobolev imbedding, we have $\{u_\epsilon\}_{\epsilon>0}$ is uniformly bounded in $C^{1,\alpha_0}(\overline{\Omega})$ for any small $\epsilon > 0$ and some $0 < \alpha_0 < 1 - n/p$. Thus, by using Arzelá-Ascoli criterion, we can obtain that there exists a function v with

$$u_{\epsilon_j} \rightarrow v \text{ uniformly in } \overline{\Omega}$$

for some subsequence $\{u_{\epsilon_j}\} \subset \{u_\epsilon\}$. Again, applying Proposition 3.3 to v , we can derive that v is a viscosity solution of (2.1) with

$$\|v\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}$$

for some $C = C(n, \lambda, \Lambda, p, \delta_0, b, c, \|\beta\|_{C^2(\partial\Omega)}, \text{diam}(\Omega), \rho_0)$. The uniqueness can be deduced by Lemma 3.7 below. □

Meanwhile, a comparison principle for (2.1) can be also obtained as in the case of Dirichlet problems, see [9, Theorem 2.10]. This will be used to prove Theorem 1.1 in the next section.

Lemma 3.7 *Let $\Omega_0 \subset \Omega$, and let $\Gamma \in C^2$ be relatively open in $\partial\Omega$. Suppose that $F = F(X, q, r, x)$ is convex in X and continuous in x , satisfies (1.2)-(3.2) and $F(0, 0, 0, x) \equiv 0$, $\beta \in C^2(\bar{\Gamma})$ with $\beta \cdot \mathbf{n} \geq \delta_0$ for some $\delta_0 > 0$, $\psi \in C(\partial\Omega_0/\Gamma)$ and $f \in L^p(\Omega_0)$ for $n < p < \infty$.*

Let $u_1, u_2 \in W^{2,p}(\Omega) \cap C(\bar{\Omega})$ satisfy

$$\begin{cases} F(D^2u_1, Du_1, u_1, x) \leq f & \text{in } \Omega_0, \\ u_1 \geq \psi & \text{on } \partial\Omega_0/\Gamma, \\ \beta \cdot Du_1 \leq 0 & \text{on } \Gamma \end{cases}$$

and

$$\begin{cases} F(D^2u_2, Du_2, u_2, x) \geq f & \text{in } \Omega_0, \\ u_2 \leq \psi & \text{on } \partial\Omega_0/\Gamma, \\ \beta \cdot Du_2 \geq 0 & \text{on } \Gamma \end{cases}$$

in the viscosity sense. Then we have $u_1 \geq u_2$ in Ω_0 .

Proof First we set

$$G(X, q, r, x) = F(X + D^2u_2, q + Du_2, r + u_2, x) - F(D^2u_2, Du_2, u_2, x).$$

One can see that G satisfies (1.2)-(3.2), $w := u_1 - u_2$ solves

$$G(D^2w, Dw, w, x) = F(D^2u_1, Du_1, u_1, x) - F(D^2u_2, Du_2, u_2, x) \leq 0$$

in the viscosity sense, and that $w \in \bar{S}(\lambda, \Lambda, b, -g)$ for

$$g(x) = G(0, 0, w, x) = F(D^2u_2, Du_2, u_1, x) - F(D^2u_2, Du_2, u_2, x).$$

Set

$$V := \{x \in \bar{\Omega}_0 : w(x) < 0\} = \{x \in \bar{\Omega}_0 : u_1(x) < u_2(x)\}.$$

We want to claim that $V = \emptyset$. Suppose not. Then we have $\inf_V w < 0$. Since

$$g(x) > d(u_2 - u_1)(x) = -dw(x) > 0 \quad \text{for } x \in V,$$

w satisfies

$$\begin{cases} w \in \bar{S}(\lambda, \Lambda, b, 0) & \text{in } V, \\ w \geq 0 & \text{on } \partial V/\Gamma, \\ \beta \cdot Dw \leq 0 & \text{on } \Gamma \end{cases}$$

in the viscosity sense, according to [29, Theorem 3.1].

We first consider the case $\partial V/\Gamma \neq \emptyset$. Observe that $w \equiv 0$ on $\partial V \cap \Omega_0$ and $w \geq 0$ on $(\partial V \cap \partial\Omega_0)/\Gamma$. From ABP maximum principle (see [9, Proposition 3.3]), we can deduce that

$$\sup_V w^- = \sup_{\partial V} w^-.$$

Thus, w attains a minimum point x_0 on $\partial V \cap \Gamma$.

Now define $\tilde{w}(x) = w(x) - \inf_V w$ for $x \in V$. Then we see that $\tilde{w} \geq 0$ in \bar{V} with $\tilde{w}(x_0) = 0$. Consider a small neighborhood $N(x_0) \subset \Omega \cup \Gamma$ of x_0 . According to [12, Proposition 11], we can deduce that there exists a point $x_1 \in N(x_0) \cap \Omega$ with $\tilde{w}(x_1) = 0$ since $\beta \cdot Dw \leq 0$. Then, by Lemma 3.2, there exists a small number $\rho > 0$ with $B_\rho(x_1) \subset\subset \Omega$ such that

$$\|\tilde{w}\|_{L^{p_0}(B_{\rho/2}(x_1))} = 0.$$

That is,

$$\tilde{w} \equiv 0 \quad \text{in } B_{\rho/2}(x_1).$$

Repeating the above procedure, we can deduce that

$$\tilde{w} \equiv 0 \quad \text{in } V.$$

Since \tilde{w} is continuous in \overline{V} , we have

$$\tilde{w} \equiv 0 \quad \text{in } \overline{V},$$

and this implies $w = \tilde{w}$. However, it is a contradiction, as we have assumed that $\inf_V w < 0$. Hence, $V = \emptyset$ and we conclude that

$$w \geq 0 \quad \text{in } \Omega_0,$$

and this implies

$$u_1 \geq u_2 \quad \text{in } \Omega_0.$$

On the other hand, if $\Gamma = \partial V$, we have

$$\begin{cases} w \in \overline{S}(\lambda, \Lambda, b, 0) & \text{in } V, \\ \beta \cdot Dw \leq 0 & \text{on } \Gamma. \end{cases}$$

Again, by ABP maximum principle, we also have

$$\sup_V w^- = \sup_{\partial V} w^- < 0.$$

Set $\tilde{w} = w - \inf_V w$ in V . Then there is a point $y_0 \in \partial V$ with $\tilde{w}(y_0) = 0$. From Lemma 3.1, we can see that there is an interior point y_1 such that $\tilde{w}(y_1) = 0$. Now we derive that $\tilde{w} \equiv 0$ in V by using a similar argument as above. This yields that $w \equiv c_1$ in V for some $c_1 < 0$.

By the definition of w , we have $u_1 \equiv u_2 + c_1$ in V . Then we can observe that

$$\begin{aligned} F(D^2u_1, Du_1, u_1, x) &= F(D^2(u_2 + c_1), D(u_2 + c_1), u_2 + c_1, x) \\ &= F(D^2u_2, Du_2, u_2 + c_1, x) \\ &\geq F(D^2u_2, Du_2, u_2, x) - dc_1 \\ &> f \end{aligned}$$

in V . But it is a contradiction because $F(D^2u_1, Du_1, u_1, x) \leq f$. Therefore, we conclude that $V = \emptyset$. This completes the proof. □

4 Proof of Theorem 1.1

In this section we establish our main result, Theorem 1.1. Our strategy is to construct a sequence of approximating oblique derivative problems to (1.1). This construction makes it possible to utilize those results obtained in the previous section for (2.1).

In the process of the proof, we are going to use the following Schauder’s fixed point theorem (see [13, Theorem V.9.5]).

Lemma 4.1 (Schauder’s fixed point theorem) Assume that X is a Banach space, $K \subset X$ is closed, bounded and convex, and suppose that $S : K \rightarrow K$ is compact. Then S has a fixed point in K .

We now prove the main result of this paper.

Proof of Theorem 1.1 Fix $\epsilon > 0$ and choose a non-decreasing function $\Phi_\epsilon \in C^\infty(\mathbb{R})$ such that

$$\Phi_\epsilon(s) \equiv 0 \text{ if } s \leq 0; \quad \Phi_\epsilon(s) \equiv 1 \text{ if } s \geq \epsilon, \tag{4.1}$$

and

$$0 \leq \Phi_\epsilon(s) \leq 1 \text{ for any } s \in \mathbb{R}. \tag{4.2}$$

Set

$$g(x) := f(x) - F(D^2\psi, D\psi, \psi, x).$$

Then we have $g \in L^p(\Omega)$ with the estimate

$$\begin{aligned} \|g\|_{L^p(\Omega)} &\leq \|f\|_{L^p(\Omega)} + \|F(D^2\psi, D\psi, \psi, x)\|_{L^p(\Omega)} \\ &\leq C(\|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)}) \end{aligned} \tag{4.3}$$

for some $C = C(n, \lambda, \Lambda, b, c) > 0$, since $f, D^2\psi \in L^p(\Omega)$ and F satisfies (1.3).

We now consider the following oblique derivative problem without obstacles

$$\begin{cases} F(D^2u_\epsilon, Du_\epsilon, u_\epsilon, x) = g^+ \Phi_\epsilon(u_\epsilon - \psi) + f - g^+ & \text{in } \Omega, \\ \beta \cdot Du_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

We want to show that (4.4) has a unique viscosity solution. For this, fix a function $v_0 \in L^p(\Omega)$. Then according to Lemma 3.6, we know that there exists a unique viscosity solution $v_\epsilon \in W^{2,p}(\Omega)$ of

$$\begin{cases} F(D^2v_\epsilon, Dv_\epsilon, v_\epsilon, x) = g^+ \Phi_\epsilon(v_0 - \psi) + f - g^+ & \text{in } \Omega, \\ \beta \cdot Dv_\epsilon = 0 & \text{on } \partial\Omega \end{cases}$$

with the estimate

$$\begin{aligned} \|v_\epsilon\|_{W^{2,p}(\Omega)} &\leq C\|g^+ \Phi_\epsilon(v_0 - \psi) + f - g^+\|_{L^p(\Omega)} \\ &\leq C(\|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}) \\ &\leq C(\|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)}) \end{aligned}$$

for some $C = C(n, \lambda, \Lambda, p, \delta_0, b, c, \|\beta\|_{C^2(\partial\Omega)}, \rho_0, \text{diam}(\Omega)) > 0$, where we have used (4.1) and (4.2). Thus,

$$\|v_\epsilon\|_{W^{2,p}(\Omega)} \leq C_0$$

for some

$$C_0 = C_0(n, \lambda, \Lambda, p, \delta_0, b, c, \|\beta\|_{C^2(\partial\Omega)} \text{diam}(\Omega), \|f\|_{L^p(\Omega)}, \|\psi\|_{L^p(\Omega)}, \rho_0).$$

Note that C_0 is independent of v_0 . Now we can define a nonlinear operator $S_\epsilon : L^p(\Omega) \rightarrow W^{2,p}(\Omega) \subset L^p(\Omega)$ such that $S_\epsilon v_0 = v_\epsilon$ with (4.4). Write

$$K := \{h \in L^p(\Omega) : \|h\|_{L^p(\Omega)} \leq C_0\}.$$

Note that K is a closed convex subset of $L^p(\Omega)$. On the other hand, by Rellich-Kondrachov compactness theorem, we observe that $W^{2,p}(\Omega)$ is compactly imbedded in $W^{1,p}(\Omega)$ and so is in $L^p(\Omega)$. Hence, the closure of $S_\epsilon(A)$ is compact for every $A \subset K$. Meanwhile, by Proposition 3.3, we can also conclude that S_ϵ is a continuous operator.

Therefore, by Lemma 4.1, there exists a function $u_\epsilon \in K$ satisfying $S_\epsilon u_\epsilon = u_\epsilon$, and this implies that u_ϵ is a viscosity solution of (4.4). Furthermore, from (4.3) and Lemma 3.6, we also observe that

$$\|u_\epsilon\|_{W^{2,p}(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)}) \tag{4.5}$$

for some $C = C(n, \lambda, \Lambda, p, \delta_0, b, c, \|\beta\|_{C^2(\partial\Omega)}, \text{diam}(\Omega), \rho_0)$. This shows that $\{u_\epsilon\}_{\epsilon>0}$ is uniformly bounded in $W^{2,p}(\Omega)$.

Recall that $p > n$. Then we observe that $W^{2,p}(\Omega) \subset\subset C^{1,\alpha_0}(\overline{\Omega})$ for some $0 < \alpha_0 < 1 - n/p$ by Morrey imbedding. Therefore, we can find a subsequence $\{u_{\epsilon_j}\}$ with $\epsilon_j \searrow 0$ and a function $u \in W^{2,p}(\Omega)$ such that

$$\begin{cases} u_{\epsilon_j} \rightharpoonup u & \text{in } W^{2,p}(\Omega), \\ u_{\epsilon_j} \rightarrow u & \text{in } W^{1,p}(\Omega) \subset C^{0,\alpha_0}(\overline{\Omega}) \end{cases} \tag{4.6}$$

as $j \rightarrow \infty$.

Now we claim that u is indeed the unique viscosity solution of (1.1). We first see that u is uniformly bounded and equicontinuous on $\partial\Omega$ from (4.5) and Morrey imbedding. Thus, by using Arzelá-Ascoli criterion, we have

$$\beta \cdot Du = 0 \quad \text{on } \partial\Omega.$$

On the other hand, from (4.4), we observe that

$$F(D^2u_{\epsilon_j}, Du_{\epsilon_j}, u_{\epsilon_j}, x) = g^+ \Phi_{\epsilon_j}(u_{\epsilon_j} - \psi) + f - g^+ \leq f \quad \text{in } \Omega$$

in the viscosity sense for each j . Recall again (4.6). We see that $u_{\epsilon_j} \rightarrow u$ uniformly, and then we can use the result of [9, Theorem 3.8]. Thus, we have $F(D^2u, Du, u, x) \leq f$ in Ω in the viscosity sense.

Next, we show that

$$u \geq \psi \quad \text{in } \overline{\Omega}.$$

We first see that $\Phi_{\epsilon_j}(u_{\epsilon_j} - \psi) \equiv 0$ on the set

$$V_j = \{x \in \overline{\Omega} : u_{\epsilon_j}(x) < \psi(x)\}.$$

If $V_j = \emptyset$, we have $u_{\epsilon_j} \geq \psi$ in $\overline{\Omega}$, and so we are done. Now suppose that $V_j \neq \emptyset$. Then,

$$F(D^2u_{\epsilon_j}, Du_{\epsilon_j}, u_{\epsilon_j}, x) = f(x) - g^+(x) \quad \text{for } x \in V_j.$$

We note that V_j is relatively open in $\overline{\Omega}$ for each j since $u_{\epsilon_j} \in C(\overline{\Omega})$.

Recall that

$$F(D^2\psi, D\psi, \psi, x) = f - g \geq F(D^2u_{\epsilon_j}, Du_{\epsilon_j}, u_{\epsilon_j}, x) \quad \text{in } V_j.$$

And we also have $u_{\epsilon_j} = \psi$ on $\partial V_j / \partial\Omega$.

Now we can apply Lemma 3.7 to obtain $u_{\epsilon_j} \geq \psi$ in V_j , which is a contradiction to the definition of V_j and thus $V_j = \emptyset$ for each j . Therefore, we can obtain $u \geq \psi$ in $\overline{\Omega}$.

We next claim that

$$F(D^2u, Du, u, x) = f \quad \text{in } V := \{x \in \Omega : u(x) > \psi(x)\}.$$

For each $m \in \mathbb{N}$, we have

$$\Phi_{\epsilon_j}(u_{\epsilon_j} - \psi) \rightarrow 1 \quad \text{a.e. in } \left\{x \in \Omega : u(x) > \psi(x) + \frac{1}{m}\right\}$$

as $j \rightarrow \infty$. Thus, for

$$V = \{x \in \Omega : u(x) > \psi(x)\} = \bigcup_{m=1}^{\infty} \left\{x \in \Omega : u(x) > \psi(x) + \frac{1}{m}\right\},$$

we derive

$$g^+ \Phi_{\epsilon_j}(u_{\epsilon_j} - \psi) + f - g^+ \rightarrow f \quad \text{a.e. in } V$$

as $j \rightarrow \infty$. Thus, we deduce that

$$F(D^2u, Du, u, x) = g^+ + f - g^+ = f \quad \text{in } V$$

in the viscosity sense.

Therefore, we can conclude that u is a viscosity solution of (1.1). Moreover, from (4.5) and (4.6), we have

$$\|u\|_{W^{2,p}(\Omega)} \leq \liminf_{j \rightarrow \infty} \|u_{\epsilon_j}\|_{W^{2,p}(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)})$$

for some constant $C = C(n, \lambda, \Lambda, p, \delta_0, b, c, \|\beta\|_{C^2(\partial\Omega)}, \text{diam}(\Omega), \rho_0)$.

For the uniqueness, let u_1 and u_2 be two viscosity solutions of (1.1). Suppose that $u_1 \not\equiv u_2$. Then we can assume without loss of generality that

$$G = \{u_2 > u_1\} \neq \emptyset.$$

Since $u_2 > u_1 \geq \psi$ in G , we see that $F(D^2u_2, Du_2, u_2, x) = f$ in G in the viscosity sense. Then we have

$$\begin{cases} F(D^2u_1, Du_1, u_1, x) \leq F(D^2u_2, Du_2, u_2, x) = f & \text{in } G, \\ u_1 = u_2 & \text{on } \partial G / \partial \Omega, \\ \beta \cdot Du_1 = \beta \cdot Du_2 = 0 & \text{on } \partial G \cap \partial \Omega. \end{cases}$$

Now applying [9, Theorem 2.10] or Lemma 3.7 to $u_1 - u_2$, we deduce that $u_1 \geq u_2$ in G whether $\partial G \cap \partial \Omega = \emptyset$ or not. This contradicts the definition of the set G , and hence $u_1 = u_2$. \square

Acknowledgements The authors thank the referees for careful reading of the early version of this manuscript and offering valuable suggestions and comments

Funding Open Access funding provided by University of Jyväskylä (JYU).

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Andersson, J., Lindgren, E., Shahgholian, H.: Optimal regularity for the no-sign obstacle problem. *Comm. Pure Appl. Math.* **66**(2), 245–262 (2013)
2. Andersson, J., Lindgren, E., Shahgholian, H.: Optimal regularity for the obstacle problem for the p -Laplacian. *J. Differ. Equ.* **259**(6), 2167–2179 (2015)

3. Byun, S.-S., Han, J.: $W^{2,p}$ -estimates for fully nonlinear elliptic equations with oblique boundary conditions. *J. Differ. Equ.* **268**(5), 2125–2150 (2020)
4. Byun, S.S., Lee, K.A., Oh, J., Park, J.: Nondivergence elliptic and parabolic problems with irregular obstacles. *Math. Z.* **290**(3–4), 973–990 (2018)
5. Byun, S.S., Lee, K.A., Oh, J., Park, J.: Regularity results of the thin obstacle problem for the $p(x)$ -Laplacian. *J. Funct. Anal.* **276**(2), 496–519 (2019)
6. Caffarelli, L.A.: The regularity of free boundaries in higher dimensions. *Acta Math.* **139**(3–4), 155–184 (1977)
7. Caffarelli, L.A.: Interior a priori estimates for solutions of fully nonlinear equations. *Ann. of Math.* **130**(1), 189–213 (1989)
8. Caffarelli, L.A., Cabré, X.: Fully nonlinear elliptic equations, vol. 43. American Mathematical Society, Providence, RI (1995)
9. Caffarelli, L.A., Crandall, M.G., Kocan, M., Świąch, A.: On viscosity solutions of fully nonlinear equations with measurable ingredients. *Comm. Pure Appl. Math.* **49**(4), 365–397 (1996)
10. Čanić, S., Keyfitz, B.L., Kim, E.H.: A free boundary problem for a quasi-linear degenerate elliptic equation: regular reflection of weak shocks. *Comm. Pure Appl. Math.* **55**(1), 71–92 (2002)
11. Čanić, S., Keyfitz, B.L., Lieberman, G.M.: A proof of existence of perturbed steady transonic shocks via a free boundary problem. *Comm. Pure Appl. Math.* **53**(4), 484–511 (2000)
12. Charro, F., Montoro, L., Sciuza, B.: Monotonicity of solutions of fully nonlinear uniformly elliptic equations in the half-plane. *J. Differ. Equ.* **251**(6), 1562–1579 (2011)
13. Conway, J.B.: A course in functional analysis, vol. 96. Springer-Verlag, New York (1985)
14. Di Fazio, G., Palagachev, D.K.: Oblique derivative problem for elliptic equations in non-divergence form with VMO coefficients. *Comment. Math. Univ. Carolin.* **37**(3), 537–556 (1996)
15. Dupuis, P., Ishii, H.: On oblique derivative problems for fully nonlinear second-order elliptic partial differential equations on nonsmooth domains. *Nonlinear Anal.* **15**(12), 1123–1138 (1990)
16. Dupuis, P., Ishii, H.: On oblique derivative problems for fully nonlinear second-order elliptic PDEs on domains with corners. *Hokkaido Math. J.* **20**(1), 135–164 (1991)
17. Friedman, A.: Variational principles and free-boundary problems. Pure and Applied Mathematics. John Wiley & Sons Inc, New York (1982)
18. Figalli, A., Shahgholian, H.: A general class of free boundary problems for fully nonlinear elliptic equations. *Arch. Ration. Mech. Anal.* **213**(1), 269–286 (2014)
19. Indrei, E., Minne, A.: Regularity of solutions to fully nonlinear elliptic and parabolic free boundary problems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33**(5), 1259–1277 (2016)
20. Indrei, E., Minne, A., Nurbekyan, L.: Regularity of solutions in semilinear elliptic theory. *Bull. Math. Sci.* **7**(1), 177–200 (2017)
21. Indrei, E.: Boundary regularity and nontransversal intersection for the fully nonlinear obstacle problem. *Comm. Pure Appl. Math.* **72**(7), 1459–1473 (2019)
22. Ishii, H.: Fully nonlinear oblique derivative problems for nonlinear second-order elliptic PDEs. *Duke Math. J.* **62**(3), 633–661 (1991)
23. Kinderlehrer, D., Nirenberg, L.: Regularity in free boundary problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **4**(2), 373–391 (1977)
24. Lieberman, G.M.: Solvability of quasilinear elliptic equations with nonlinear boundary conditions. II. *J. Funct. Anal.* **56**(2), 210–219 (1984)
25. Lieberman, G.M.: Oblique derivative problems in Lipschitz domains. I. continuous boundary data. *Boll. Un. Mat. Ital. B* **1**(4), 1185–1210 (1987)
26. Lieberman, G.M.: Regularity of solutions of obstacle problems for elliptic equations with oblique boundary conditions. *Pacific J. Math.* **201**(2), 389–419 (2001)
27. Lieberman, G.M.: Oblique derivative problems for elliptic equations. Pte. Ltd., Hackensack, NJ (2013)
28. Lieberman, G.M., Trudinger, N.S.: Nonlinear oblique boundary value problems for nonlinear elliptic equations. *Trans. Amer. Math. Soc.* **295**(2), 509–546 (1986)
29. Li, D., Zhang, K.: Regularity for fully nonlinear elliptic equations with oblique boundary conditions. *Arch. Ration. Mech. Anal.* **228**(3), 923–967 (2018)
30. Milakis, E., Silvestre, L.E.: Regularity for fully nonlinear elliptic equations with Neumann boundary data. *Comm. Partial. Differ. Equ.* **31**(7–9), 1227–1252 (2006)
31. Palagachev, D.K., Ragusa, M.A., Softova, L.G.: Regular oblique derivative problem in Morrey spaces. *Electron. J. Differential Equations* (2000)
32. Petrosyan, A., Shahgholian, H., Uraltseva, N.: Regularity of free boundaries in obstacle-type problems, vol. 136. American Mathematical Society, Providence, RI (2012)
33. Safonov, M.V.: On the oblique derivative problem for second order elliptic equations. *Comm. Partial. Differ. Equ.* **20**(7–8), 1349–1367 (1995)

34. Schaeffer, D.G.: Some examples of singularities in a free boundary. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **4**(1), 133–144 (1977)
35. Wang, L.: On the regularity theory of fully nonlinear parabolic equations. I. *Comm. Pure Appl. Math.* **45**(1), 27–76 (1992)
36. Winter, N.: $W^{2,p}$ and $W^{1,p}$ -estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations. *Z. Anal. Anwend.* **28**(2), 129–164 (2009)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.