Calculus of Variations

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A regularity theory for intrinsic minimising fractional harmonic maps

James Roberts[1]

Abstract We define and develop an interior partial regularity theory for intrinsic energy minimising fractional harmonic maps from Euclidean space into smooth compact Riemannian manifolds for fractional powers strictly between zero and one. Intrinsic fractional harmonic maps are critical points of an energy whose first variation is a Dirichlet to Neumann map for the harmonic map problem on a half-space with a Riemannian metric which can degenerate/become singular along the boundary, depending on the fractional power. Similarly to the approach used to prove regularity for stationary intrinsic semi-harmonic maps, we take advantage of the connection between fractional harmonic maps and free boundary problems for harmonic maps in order to develop a partial regularity theory for the fractional harmonic maps we consider. In particular, we prove partial regularity for locally minimising harmonic maps with (partially) free boundary data on half-spaces with the aforementioned metrics up to the boundary; fractional harmonic maps then inherit this regularity. As a by-product of our methods we shed some new light on the monotonicity of the average energy of solutions of the degenerate linear elliptic equation related to fractional harmonic functions.

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✉ James Roberts
  j.e.roberts@bath.ac.uk

[1] Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK

1 Introduction

Harmonic maps are critical points of the Dirichlet energy for mappings of Riemannian manifolds and are non-linear analogues of solutions to Laplace's equation. They play an important role in geometry and one of the most fundamental aspects of the theory of harmonic maps, which is intimately connected to the geometry and topology of the codomain manifold N, is their regularity. For maps with domains of dimension 2, the Dirichlet energy is invariant under conformal transformations of the domain; Hélein proved that harmonic maps are smooth in this case [21]. The fact that harmonic maps are smooth on domains of dimension two is also now a consequence of a result of Rivière [34] asserting the continuity of critical points of functionals with conformally invariant Lagrangian on two dimensional domains. Motivated by Rivière's regularity result, at least partially, Da Lio and Rivière [8] introduced the notion of fractional harmonic mappings of manifolds; on a domain of dimension one these maps are critical points of functionals which satisfy a type of conformal invariance. Such maps generalise the notion of both harmonic mappings of manifolds and fractional harmonic functions (maps u with $(-\Delta)^s u = 0$) and Da Lio and Rivière showed the maps they considered are smooth on domains of one dimension.

Since the results of Da Lio and Rivière, fractional harmonic mappings of manifolds have been increasingly studied in recent years, particularly pertaining to their regularity. The purpose of this article is to extend the notion of a type of fractional harmonic map introduced by Moser [30]. In particular, we generalise the notion of intrinsic $\frac{1}{2}$-harmonic maps to all powers in $(0, 1)$ and consider the regularity of a class of such maps. In contrast to other types of fractional harmonic maps, the maps we consider are *intrinsic* meaning that they are derived from an energy which only depends on the geometry of the target N and not on the embedding of N into Euclidean space. As we will discuss in more detail subsequently, we take advantage of the connection between free boundary harmonic maps and fractional harmonic maps to establish regularity for the latter. The results we obtain and methods we use lay the foundation for future studies into the connection between the regularity and singularities of intrinsic fractional harmonic maps, free boundary harmonic maps and the geometric and topological properties of N.

In order to motivate our definition of fractional harmonic mappings of manifolds, we first outline the pertinent aspects of the theory of fractional harmonic functions from Euclidean space into $\mathbb{R}$. Let $m \in \mathbb{N}$ and $\mathbb{R}^{m+1}_+ = \mathbb{R}^m \times (0, \infty)$. Caffarelli and Silvestre [2] established that, for given boundary data $u : \partial \mathbb{R}^{m+1}_+ \to \mathbb{R}$, solutions $v : \mathbb{R}^{m+1}_+ \to \mathbb{R}$ of the Dirichlet problem:

$$\operatorname{div}(x^\beta_{m+1} \nabla v) = 0 \text{ in } \mathbb{R}^{m+1}_+ \quad \text{and} \quad v|_{\partial \mathbb{R}^{m+1}_+} = u \qquad (1.1)$$

satisfy $(-\Delta)^{\frac{1-\beta}{2}} u = \partial^\beta_{m+1} v := -(x^\beta_{m+1} \partial_{m+1} v)|_{\partial \mathbb{R}^{m+1}_+}$, where $(-\Delta)^s$ is the fractional Laplace operator of order $s \in (0, 1)$. One way to prove this result is to establish that

$$C \|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(\partial \mathbb{R}^{m+1}_+)} = \inf \left\{ E^\beta(v) : v|_{\partial \mathbb{R}^{m+1}_+} = u \right\}, \qquad (1.2)$$

where $E^\beta(v) := \frac{1}{2} \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x$, $s = \frac{1-\beta}{2}$, $C = C(m, s)$ and the infimum is taken over a suitable Sobolev space. We conclude that the first variation of the preceding functionals, respectively $u \mapsto (-\Delta)^{\frac{1-\beta}{2}} u$ and $u \mapsto \partial^\beta_{m+1} v$, must therefore coincide. Furthermore, if we wish to study u with $(-\Delta)^s u = 0$, then we may equivalently consider minimisers (or even critical points) of E^β which satisfy $v|_{\partial \mathbb{R}^{m+1}_+} = u$ and $\partial^\beta_{m+1} v = 0$. It further follows from

(1.2) that fractional harmonic functions $u \colon \mathcal{O} \to \mathbb{R}$, where $\mathcal{O} \subset \partial \mathbb{R}^{m+1}_+$ is a domain, may reasonably be defined as critical points of either $\|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(\mathcal{O})}$ or $\inf\{E^\beta(v) \colon v|_\mathcal{O} = u\}$ (which are no longer equivalent). In order to define intrinsic fractional harmonic maps, we will work with a modification of the latter energy. For $u \colon \mathcal{O} \to \mathbb{R}$, the first variation of $\inf\{E^\beta(v) \colon v|_\mathcal{O} = u\}$ is the Dirichlet to Neumann map $u \mapsto -(x^\beta_{m+1} \partial_{m+1} v)|_\mathcal{O}$ but we may no longer identify this with a fractional Laplace operator. Moreover, v satisfies (1.1) but with $v|_\mathcal{O} = u$.

We generalise the notion of fractional harmonic function to mappings $u \colon \mathcal{O} \to N$ in the following way, where $\mathcal{O} \subset \partial \mathbb{R}^{m+1}_+$ is an open domain but not necessarily all of $\mathbb{R}^{m+1}_+$ and N is a smooth compact Riemannian manifold. For technical reasons, we assume henceforth that N is isometrically embedded in $\mathbb{R}^n$ for some n, which can always be achieved for smooth compact N by the theorem of Nash [31]. The forthcoming definition will be made rigorous at the beginning of Sect. 3. Define

$$I^\beta(u) = \inf \left\{ E^\beta(v) \colon v|_\mathcal{O} = u, v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N) \right\}$$

where $\beta \in (-1, 1)$, $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N) := \{v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n) \colon v(x) \in N \text{ for a.e } x \in \mathbb{R}^{m+1}_+\}$ and $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ is the homogeneous Sobolev space with the square root of E^β as the norm. If we set $\mathcal{O} = \partial \mathbb{R}^{m+1}_+$ and N is replaced by $\mathbb{R}$ then I^β reduces to the energies considered by Caffarelli and Silvestre. Moreover, I^β is independent of the choice of embedding of N into Euclidean space since E^β is invariant under isometries of N. Hence we define critical points of I^β as intrinsic $\frac{1-\beta}{2}$-harmonic maps.

Our main result regarding fractional harmonic maps, see Theorem 3.8 in Sect. 3, asserts:

Theorem 1.1 *Fix $\mathcal{O} \subset \partial \mathbb{R}^{m+1}_+$. When $m \geq 3$ let $\beta \in (-1, 1)$ and when $m = 2$ let $\beta \in (-3^{-1}, 1)$. Suppose $u \colon \mathcal{O} \to N$ is a $\frac{1-\beta}{2}$-harmonic map that locally minimises I^β in $\mathcal{O}$. Then u is smooth with the possible exception of a set of points with vanishing $m+\beta-1$-dimensional Hausdorff measure.*

We observe that the domain of I^β plays a role in the regularity theory. We will take the domain of I^β, again see Sect. 3 for details, to be the image of $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ under some trace operator. The domain we choose does not necessarily coincide with the fractional Sobolev space $H^{\frac{1-\beta}{2}}(\mathcal{O}; N)$; there are circumstances in which it is possible to construct $H^{\frac{1-\beta}{2}}(\mathcal{O}; N)$ maps which do not have extensions to $\mathbb{R}^{m+1}_+$. This can happen for topological reasons, for example, such as when N is not simply connected [1]. We further note that the bound on the Hausdorff dimension of the singular set stated in the theorem may not be optimal; the given dimension is that which one would conjecture for u which are critical points of I^β with respect to inner and outer variations and is consistent with the scale-invariance of the problem. We expect that the dimension bound on the singular set can be improved using our constructions in Sect. 4 and similar arguments to Chapter 3 of [39] for example. It is also possible that our choice to minimise over $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ may preclude some singularities, for example the map $v = |(x', x_{m+1})|^{-1}(x', x_{m+1})$ is not an admissible extension of $u = |x'|^{-1} x'$ as it does not have finite E^β energy. However, the arguments we use are local and we expect our methods to apply to modifications of I^β which allow extensions v with only locally finite E^β energy. The restriction $\beta > -3^{-1}$ when $m = 2$ is of a technical nature and results from the use of a Sobolev embedding theorem to identify elements of $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ with functions, so that we may require they take values in N almost everywhere. It is possible that this restriction can be lifted with the use of an appropriate embedding theorem, we are not aware

of such a result. An alternative would be to replace $\dot{W}_\beta^{1,2}$ with a Sobolev space of measurable functions from the outset (as opposed to representatives of Cauchy sequences). We further note that we do not consider the case $m = 1$ and will indicate our reasons subsequently.

To prove Theorem 1.1 we will take advantage of the connection between the extensions $v : \mathbb{R}_+^{m+1} \to N$ of fractional harmonic $u : \mathcal{O} \to N$ and free boundary harmonic maps from $\mathbb{R}_+^{m+1}$ to N, similarly to Moser [30]. Moser observed that whenever I^0 is differentiable at u, its first variation is the Dirichlet to Neumann map $\Lambda : u \mapsto (-\partial_{m+1} v)|_\mathcal{O}$ where $v : \mathbb{R}_+^{m+1} \to N$ is a harmonic map with $I^0(u) = E^0(v)$. Hence, intrinsic $\frac{1}{2}$-harmonic maps satisfy $\Lambda u = (-\partial_{m+1} v)|_\mathcal{O} = 0$, which may be interpreted as a zero Neumann condition on $\mathcal{O}$ for any v which minimises E^0 among maps with u as boundary values on $\mathcal{O}$. Using this observation Moser established (partial) regularity for stationary $\frac{1}{2}$-harmonic $u : \mathcal{O} \to N$, which are critical points of I^0 with respect to inner and outer variations, using regularity theory for stationary free boundary harmonic maps $v : \mathbb{R}_+^{m+1} \to N$ up to the free boundary $\mathcal{O}$. He showed the maps he considered are smooth when $m = 1$ and smooth with the possible exception of a set of vanishing Hausdorff dimension $m - 1$ when $m \geq 2$. The following theorem is an abridged version of Theorem 4.3 in Sect. 4 which is our main theorem regarding the extensions v of fractional harmonic u.

Theorem 1.2 *Suppose $u : \mathcal{O} \to N$ locally minimises I^β and suppose v satisfies $I^\beta(u) = E^\beta(v)$. There exists sets $\Sigma_{int} \subset \mathbb{R}_+^{m+1}$ and $\Sigma_{bdry} \subset \mathcal{O}$ such that the following holds. The set Σ_{int} is relatively closed in $\mathbb{R}_+^{m+1}$ and has Hausdorff dimension at most $m - 2$. The set Σ_{bdry} is relatively closed in $\mathcal{O}$ and has vanishing $m + \beta - 1$-dimensional Hausdorff measure. The set $\Sigma := \Sigma_{int} \cup \Sigma_{bdry}$ is relatively closed in $\mathbb{R}_+^{m+1} \cup \mathcal{O}$ and has vanishing $m + \beta - 1$-dimensional Hausdorff measure. Furthermore, we have $v \in C^\infty(\mathbb{R}_+^{m+1} \backslash \Sigma_{int}; N)$, $v \in C_{loc}^{0,1}((\mathbb{R}_+^{m+1} \cup \mathcal{O}) \backslash \Sigma; N)$ and for every multi-index $\alpha' \in \mathbb{N}^{m+1}$ with $\alpha'_{m+1} = 0$ we have $D^{\alpha'} v \in C_{loc}^{0,1}((\mathbb{R}_+^{m+1} \cup \mathcal{O}) \backslash \Sigma; \mathbb{R}^n)$.*

Synonymously with our discussion following Theorem 1.1, we observe that the dimension bound on the singular set of v in $\mathcal{O}$ may not be optimal. The main purpose of this article is to give a proof of Theorem 1.2 which, to our knowledge, is new for all $\beta \neq 0$. When $\beta = 0$, existing theories actually improve the bound we have given on the dimension of the singular set in $\mathcal{O}$; this improvement comes from a dimension reduction argument as mentioned previously. This translates to a lower bound on the dimension of the singular set of u. In particular, as a consequence of the results of Hardt and Lin [18] and Duzaar and Steffen [10,11], any $v : \mathbb{R}_+^{m+1} \to N$ with $I^0(u) = E^0(v)$ are smooth up to $\mathcal{O}$ away from a set in $\mathcal{O}$ of Hausdorff dimension at most $\dim\mathcal{O} - 3$. They also give examples of minimisers with singular sets in $\mathcal{O}$ which have precisely the stated Hausdorff dimensions, thus showing that the dimension bounds for the singular set are optimal.

We would like to take advantage of Moser's observation regarding the connection between fractional harmonic u and harmonic v with free boundary data in order to develop a theory for intrinsic fractional harmonic maps for powers in $(0, 1)$. In $\mathbb{R}_+^{m+1}$ we are free to utilise the theory of harmonic mappings of manifolds to obtain the stated regularity in Theorem 1.2, provided the energies E^β correspond to the energy for harmonic mappings from $(\mathbb{R}_+^{m+1}, g)$ to N for some metric g. To see that this is the case and facilitate our subsequent analysis, we first recall the definition of harmonic maps for a class of conformally Euclidean metrics on $\mathbb{R}_+^{m+1}$. Let $(\mathbb{R}_+^{m+1}, g)$ be a smooth Riemannian manifold and suppose $g = f(x_{m+1})\delta$, where δ is the Euclidean metric, $f : \mathbb{R}_+^{m+1} \to (0, \infty)$ is smooth and x_{m+1} is the $m + 1$th Euclidean coordinate. Let $W^{1,2}(\Omega; N) := \{v \in W^{1,2}(\Omega; \mathbb{R}^n) : v(x) \in N$ for almost every $x \in \Omega\}$,

where $\Omega \subset \mathbb{R}^{m+1}_+$ is open. If $x_1, \ldots, x_{m+1}$ are Euclidean coordinates on Ω, the energy density of $v \in W^{1,2}(\Omega; N)$ is given by $e(v) = f^{-1} \sum_i \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle := f^{-1}\langle \nabla v, \nabla v \rangle = f^{-1}|\nabla v|^2$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^n$. The energy of v, see [22], is defined as

$$E_g(v) = \frac{1}{2} \int_\Omega f^{-1}|\nabla v|^2 (\det(f\delta))^{\frac{1}{2}} \mathrm{d}x = \frac{1}{2} \int_\Omega f^{\frac{m-1}{2}} |\nabla v|^2 \mathrm{d}x. \qquad (1.3)$$

Observe that when $m = 1$, for this class of metric g we have $E_g \equiv E_\delta$. The Euler–Lagrange equations for E_g have the form

$$f^{-\frac{(m+1)}{2}} \mathrm{div}(f^{\frac{(m-1)}{2}} \nabla v) + f^{-1} A(v)(\nabla v, \nabla v) = 0 \text{ in } \Omega \qquad (1.4)$$

where A is the second fundamental form of N, $A(v)(\nabla v, \nabla v) := \sum_i A(v)(\partial_i v, \partial_i v)$ and div, ∇ are the Euclidean divergence and gradient operators. Critical points of E_g in $W^{1,2}(\Omega; N)$ with respect to the dependent variable (weakly) satisfy (1.4) and are called (*weakly*) *harmonic maps with respect to g*.

When $m \geq 2$ the Dirichlet energy E^β is the energy for maps $\mathbb{R}^{m+1}_+ \to N$ where $\mathbb{R}^{m+1}_+$ is equipped with a Riemannian metric $x^\alpha_{m+1}\delta$; it follows from (1.3) that $\beta = \frac{\alpha(m-1)}{2}$. When $m = 1$, any metric of the form $x^\alpha_{m+1}\delta$ gives rise to the energy E^0 in view of (1.3). It is therefore not possible that E^β corresponds to the energy of mappings from $(\mathbb{R}^{m+1}_+, x^\alpha_{m+1}\delta)$ for $\beta \neq 0$ when $m = 1$ and, in this case, we would need a different approach. We do not consider this further in this article and hence assume $m \geq 2$.

Near the boundary $\mathcal{O}$ we would like to apply the theory for free boundary harmonic mappings of Riemannian manifolds in order to obtain regularity for fractional harmonic maps defined on $\mathcal{O}$. However, for fractional powers $\frac{1-\beta}{2} \in (0, 1)$ with $\beta \neq 0$ (i.e fractional powers in $(0, 1)$ other than $\frac{1}{2}$) the metrics $x^\alpha_{m+1}\delta$ may degenerate or become singular along the boundary depending on $\beta = \frac{\alpha(m-1)}{2}$. Known theory for free boundary harmonic maps is not applicable in this case and the main focus of this article will be to develop a regularity theory in this context.

Without geometric constraints on the target manifold N (such as requiring N has non-positive sectional curvature), harmonic maps and harmonic maps with free boundary data may exhibit discontinuities. For harmonic maps defined on a domain of dimension at least 3, free boundary data can have discontinuities on the boundary [11,17,18] and there are even harmonic maps into spheres which are discontinuous everywhere [33]. However, it is possible to establish (partial) regularity for harmonic maps which are minimising (or stationary) with respect to free boundary conditions. This is the primary reason that we consider minimisers.

The methodology we use to prove our theorem is based on existing theories for minimising (free boundary) harmonic maps. Similarly to these theories, our task is divided into two parts. First we prove Hölder continuity, and then that this continuity implies higher regularity. To establish Hölder continuity our main construction will be comparison maps. Such maps were first used in the theory of harmonic maps by Morrey in order to prove interior regularity of energy minimising harmonic maps on two dimensional domains [27]. Schoen and Uhlenbeck later showed that on domains of dimension at least three, minimising harmonic maps are regular away from a set of Hausdorff dimension at most the dimension of the domain minus three [37]; they also used manifold valued comparison maps, which are more difficult to construct on domains of dimension greater than two. The aforementioned results of Hardt and Lin and Duzaar and Steffen for minimising free boundary harmonic maps are established using methods based the constructions of Schoen and Uhlenbeck. We develop a (partial)

generalisation of a construction of Luckhaus [23] as presented by Simon [39]; the advantage of this construction, at least in existing theories of harmonic maps, is that it can be used to show blow-up limits of sequences of minimisers are again locally minimising. We do not establish this fact here but expect similar results to follow from our constructions. In Sect. 4 we construct comparison maps on a half-cylinder and use these to construct comparison maps on half-balls in $\mathbb{R}^{m+1}_+$ centred on $\partial\mathbb{R}^{m+1}_+$. We also establish monotonicity of a local, re-scaled version of E^β and estimates on the mean squared oscillation of the maps we consider via a blow-up argument. The combination of these results is enough to prove continuity. In order to prove Hölder continuity implies higher regularity (away from the singular set), we use a version of harmonic replacement to show $\nabla v \in L^\infty$ similarly to [38]. We then establish energy decay for the derivatives of v via another blow-up argument which yields control of the mean squared oscillation of the derivatives of v tangential to $\mathcal{O}$.

To conclude our discussion, we compare the notions of intrinsic and extrinsic fractional harmonic maps and remark on known results regarding their regularity. The first to consider such maps were Da Lio and Rivière [8]. They analysed critical points u of the energy $||(-\Delta)^{\frac{1}{4}}u||^2_{L^2(\partial\mathbb{R}^2_+)}$ in $\dot{H}^{\frac{1}{2}}$ under the constraint that u takes values in the unit $n-1$ sphere $\mathbb{S}^{n-1}$ almost everywhere. In other words, they considered critical points of the functional

$$L(u) = \inf\left\{E^0(v) : v|_{\partial\mathbb{R}^2_+} = u, v \in \dot{W}^{1,2}(\mathbb{R}^2_+; \mathbb{R}^n), u(x) \in \mathbb{S}^{n-1} \text{ for a.e } x \in \partial\mathbb{R}^2_+\right\}.$$

The maps Da Lio and Rivière considered fall into a larger class of maps known as *extrinsic* (fractional) harmonic maps. In contrast to intrinsic fractional harmonic maps, the energies for which extrinsic fractional harmonic maps are critical points depend upon the choice of embedding of the target manifold N into some Euclidean space. Moser [30] proposed to remove this dependence in the energy L so that only the geometry of N plays a role in the theory. In particular, to define $I^0(u)$ he minimised $E^0(v)$ over extensions $v : \mathbb{R}^{m+1}_+ \to N$ with $v|_{\mathcal{O}} = u$ as opposed to minimising over extensions $v : \mathbb{R}^{m+1}_+ \to \mathbb{R}^n$ with $v|_{\mathcal{O}} = u$ as is done to define $L(u)$ (for $\mathcal{O} = \partial\mathbb{R}^2_+$). This choice results in another fundamental difference between intrinsic and extrinsic fractional harmonic maps which we illustrate for the power $\frac{1}{2}$ (when $\beta = 0$). Observe that any $v : \mathbb{R}^2_+ \to \mathbb{R}^n$ for which $L(u) = E^0(v)$ satisfies $\Delta v = 0$ in $\mathbb{R}^2_+$ and is hence smooth on $\mathbb{R}^2_+$ (Δ is the usual Laplace operator on Euclidean space). In contrast, any $v : \mathbb{R}^{m+1}_+ \to N$ for which $I^0(u) = E^0(v)$ is a (weakly) harmonic map from $\mathbb{R}^{m+1}_+$ to N and satisfies

$$\Delta v + A(v)(\nabla v, \nabla v) = 0 \text{ in } \mathbb{R}^{m+1}_+. \tag{1.5}$$

Unlike solutions of $\Delta v = 0$, solutions of (1.5) may have singularities in $\mathbb{R}^{m+1}_+$ in general. We conclude extrinsic $\frac{1}{2}$-harmonic maps always have a smooth extension to $\mathbb{R}^{m+1}_+$ but intrinsic $\frac{1}{2}$-harmonic maps need not. The same is true for all powers in $(0, 1)$.

The regularity theory for extrinsic $\frac{1}{2}$-harmonic maps has been extended to a range of fractional powers for maps into general target manifolds N by Da Lio et al. [6,7,9,35,36] etc, as well as Millot and Sire [25] for the power $\frac{1}{2}$. Millot–Sire and Yu have also considered the regularity of extrinsic minimising fractional harmonic maps defined on the real line for powers in $(0, \frac{1}{2})$ [26]. The methods Da Lio, Rivière and Schikorra used to obtain regularity take advantage of compensation phenomena, namely higher than expected regularity which can be derived from the Euler–Lagrange equations for the energies they considered. The higher regularity is revealed using a well chosen gauge transformation, in analogy to techniques used in the theory of harmonic maps. One of the earliest applications of this technique

was used by Hélein to prove regularity for harmonic maps from Riemannian surfaces into spheres [20]. Millot, Sire and Yu instead consider the extensions of fractional harmonic maps and exploit their connection with free boundary problems for harmonic maps, a method first applied by Moser in the intrinsic case for $\frac{1}{2}$-harmonic maps to obtain (partial) regularity. To our knowledge, aside from the results we establish in this article, the only results regarding intrinsic fractional harmonic maps are those of Moser [30].

2 Preliminaries

The Euler–Lagrange equations for E^β at critical points $v \colon \mathbb{R}^{m+1}_+ \to N$ are semi-linear with leading order term $\operatorname{div}(x^\beta_{m+1}\nabla v)$. We will require Sobolev spaces suited to the analysis of solutions of such equations, as well as the associated linear equation $\operatorname{div}(x^\beta_{m+1}\nabla v) = 0$. For $\beta \neq 0$ the coefficient x^β_{m+1} degenerates or becomes singular on $\partial\mathbb{R}^{m+1}_+$ depending on the sign of β and the theory of uniformly elliptic second order partial differential equations does not apply on sets overlapping $\partial\mathbb{R}^{m+1}_+$. Viewing the coefficient x^β_{m+1} as a weighting (density) of the Lebesgue measure $\mathrm{d}x$, we may instead appeal to the theory of weighted second order degenerate elliptic equations. We recall and define the function spaces, and some of their analytical properties, necessary for our analysis and then record properties of solutions to $\operatorname{div}(x^\beta_{m+1}\nabla v) = 0$ which we will require for the study of solutions of the Euler–Lagrange equations of E^β.

For every $\beta \in (-1, 1)$, there exists $C > 0$ such that $\frac{1}{\int_B \mathrm{d}x}\int_B |x_{m+1}|^\beta \mathrm{d}x \frac{1}{\int_B \mathrm{d}x}\int_B |x_{m+1}|^{-\beta}$ $\mathrm{d}x \leq C$ for every ball $B \subset \mathbb{R}^{m+1}$, where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}^{m+1}$. Hence $|x_{m+1}|^\beta$ is a weight of Muckenhoupt class A_2, see [19] for an overview of these weights. Every such weight is canonically associated to corresponding Sobolev and Lebesgue spaces. Let $\Omega \subset \mathbb{R}^{m+1}$. Define

$$L^2_\beta(\Omega; \mathbb{R}^n) = \left\{ f \colon \Omega \to \mathbb{R}^n \colon f \text{ is measurable, } \int_\Omega |f|^2 |x_{m+1}|^\beta \mathrm{d}x < \infty \right\}$$

which is a Hilbert space, see [5, Theorem 3.4.1], where the inner product of $f, g \in L^2_\beta(\Omega; \mathbb{R}^n)$ is given by $\langle f, g\rangle_{L^2_\beta(\Omega;\mathbb{R}^n)} = \int_\Omega \langle f, g\rangle |x_{m+1}|^\beta \mathrm{d}x$ where $\langle f, g\rangle$ is the inner product of f and g in $\mathbb{R}^n$. Define

$$W^{1,2}_\beta(\Omega; \mathbb{R}^n) = \left\{ v \colon \Omega \to \mathbb{R}^n \colon \partial_i v \in L^2_\beta(\Omega; \mathbb{R}^n) \text{ for } i = 1, \ldots, m+1 \right\}$$

where $\partial_i v$ denotes the weak partial derivative of v with respect to x_i. Proposition 2.1.2 of [40] guarantees that $W^{1,2}_\beta(\Omega; \mathbb{R}^n)$ is a Hilbert space with the inner product $\langle v, w\rangle_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)} = \int_\Omega \langle v, w\rangle |x_{m+1}|^\beta \mathrm{d}x + \int_\Omega \langle \nabla u, \nabla v\rangle |x_{m+1}|^\beta \mathrm{d}x$ for $v, w \in W^{1,2}_\beta(\Omega; \mathbb{R}^n)$, where we write $\langle \nabla v, \nabla w\rangle = \sum_{i=1}^{m+1} \langle \partial_i v, \partial_i w\rangle$. We also define the Hilbert space $W^{1,2}_{\beta,0}(\Omega; \mathbb{R}^n)$ as the closure of $C^\infty_0(\Omega; \mathbb{R}^n)$ in $W^{1,2}_\beta(\Omega; \mathbb{R}^n)$ with respect to the norm induced by the inner product on $W^{1,2}_\beta(\Omega; \mathbb{R}^n)$. When $\beta = 0$ we omit the subscript β from the preceding notation.

It is worth noting that, for every $\beta \in (-1, 1)\setminus\{0\}$, approximation by smooth functions in $L^2_\beta(\Omega; \mathbb{R}^n)$ and $W^{1,2}_\beta(\Omega; \mathbb{R}^n)$ works in the same way as for the spaces $L^2(\Omega; \mathbb{R}^n)$ and $W^{1,2}(\Omega; \mathbb{R}^n)$. The details of this process are given in Theorem 2.1.4 and Corollary 2.1.5 in [40]. The spaces $W^{1,2}_\beta$, $W^{1,2}_{\beta,0}$ and L^2_β also have essentially the same analytical properties, such as completeness, reflexivity etc. as the unweighted spaces $W^{1,2}$, $W^{1,2}_0$ and L^2 respectively.

Moreover, since $|x_{m+1}|^\beta$ is an A_2 weight, many inequalities that hold for the spaces $W^{1,p}$, such as the Poincaré inequality, have counterparts for functions in $W^{1,2}_\beta$ [19].

We will need to refer to the relationship between $W^{1,2}_\beta$ and $W^{1,p}$ and record it in the following Lemma.

Lemma 2.1 *Let $\beta \in (-1, 1)$ and suppose $\Omega \subset \mathbb{R}^{m+1}_+$ is open and bounded.*

1. *If $\overline{\Omega} \subset \mathbb{R}^{m+1}_+$ then $W^{1,2}_\beta(\Omega; \mathbb{R}^n) = W^{1,2}(\Omega; \mathbb{R}^n)$.*
2. *If $\beta \leq 0$ then $W^{1,2}_\beta(\Omega; \mathbb{R}^n) \subset W^{1,2}(\Omega; \mathbb{R}^n)$.*
3. *If $\beta > 0$ then $W^{1,2}_\beta(\Omega; \mathbb{R}^n) \subset W^{1,p}(\Omega; \mathbb{R}^n)$ for every $1 \leq p < \frac{2}{1+\beta}$.*

Proof Part 1 follows as in this case the norms on $W^{1,2}_\beta(\Omega; \mathbb{R}^n)$ and $W^{1,2}(\Omega; \mathbb{R}^n)$ are equivalent. Part 2 follows as in this case the $W^{1,2}(\Omega; \mathbb{R}^n)$ norm is dominated by a constant multiple (depending on Ω) of the $W^{1,2}_\beta(\Omega; \mathbb{R}^n)$ norm. Part 3 follows from Hölder's inequality. □

2.1 Weighted homogeneous Sobolev spaces

Consider the Dirichlet energies

$$E^\beta(v) = \frac{1}{2} \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \tag{2.1}$$

where $\beta \in (-1, 1)$, which are well defined on

$$\mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n) := \left\{ \phi \mid \phi = f|_{\mathbb{R}^{m+1}_+} \text{ for some } f \in C^\infty_0(\mathbb{R}^{m+1}; \mathbb{R}^n) \right\}.$$

The energy E^β is naturally associated to the following Sobolev space.

Definition 2.2 Let $\beta \in (-1, 1)$. The *Weighted Homogeneous Sobolev Space* $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ is the completion of $\mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ with respect to the metric induced by the square root of E^β.

The elements of $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ are, strictly speaking, equivalence classes of Cauchy sequences and it will be necessary to have tangible representatives of these classes which may take values in N.

Lemma 2.3 *Let $m \in \mathbb{N}$ with $m \geq 2$ and Ω be an open bounded subset of $\mathbb{R}^{m+1}_+$. If $m = 2$ let $\beta \in (-3^{-1}, 1)$ and if $m \geq 3$ let $\beta \in (-1, 1)$. Then the restriction operator $I : \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n) \to W^{1,2}_\beta(\Omega; \mathbb{R}^n) : f \mapsto f|_\Omega$ extends to a bounded linear operator $I : \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n) \to W^{1,2}_\beta(\Omega; \mathbb{R}^n)$. Moreover,*

$$||Iv||_{W^{1,2}_\beta(\Omega; \mathbb{R}^n)} \leq C||v||_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)} \tag{2.2}$$

for every $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ where C is a positive constant depending on m and Ω.

Proof It suffices to establish that (2.2) holds for all $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ as this space is dense in $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ and $W^{1,2}_\beta(\Omega; \mathbb{R}^n)$ is a Banach Space. Let $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$. By definition we have $||\nabla \phi||^2_{L^2_\beta(\Omega; \mathbb{R}^n)} \leq ||\phi||^2_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)}$ so we need to estimate $||\phi||^2_{L^2_\beta(\Omega; \mathbb{R}^n)}$ in terms of $||\phi||^2_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)}$.

Suppose $m \geq 3$. We have $\phi(\cdot, x_{m+1}) \in W^{1,2}(\mathbb{R}^m; \mathbb{R}^n)$ for every $x_{m+1} \in [0, \infty)$. Let ∇' denote the derivative of ϕ with respect to x' for $(x', x_{m+1}) \in \mathbb{R}^m \times [0, \infty)$, define $a = \inf_{\Omega}(x_{m+1})$ and $b = \sup_{\Omega}(x_{m+1})$ and set $l(x_{m+1}) = \Omega \cap (\mathbb{R}^m \times \{x_{m+1}\})$. We apply Fubini's Theorem, together with Hölder's inequality, for conjugate exponents $\frac{m}{m-2}$ and $\frac{m}{2}$, and the Sobolev inequality in $\mathbb{R}^m$ to see that

$$
\begin{aligned}
\int_{\Omega} x_{m+1}^{\beta} |\phi|^2 \mathrm{d}x &= \int_a^b x_{m+1}^{\beta} \int_{l(x_{m+1})} |\phi|^2 \mathrm{d}x' \mathrm{d}x_{m+1} \\
&\leq C \, (\mathrm{diam}(\Omega))^2 \int_a^b x_{m+1}^{\beta} \left(\int_{\mathbb{R}^m} |\phi|^{\frac{2m}{m-2}} \mathrm{d}x' \right)^{\frac{m-2}{m}} \mathrm{d}x_{m+1} \\
&\leq C \, (\mathrm{diam}(\Omega))^2 \int_a^b x_{m+1}^{\beta} \int_{\mathbb{R}^m} |\nabla'\phi|^2 \mathrm{d}x' \mathrm{d}x_{m+1} \\
&\leq C \, (\mathrm{diam}(\Omega))^2 \int_{\mathbb{R}^{m+1}_+} x_{m+1}^{\beta} |\nabla'\phi|^2 \mathrm{d}x.
\end{aligned}
$$

We conclude that $||\phi||^2_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)} \leq C||\phi||^2_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)}$.

In order to apply the preceding method of proof when $m = 2$ we would need to use the Sobolev Embedding Theorem on $\mathbb{R}$. However, in this case the theorem yields $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and our previous method of proof is no longer viable. However, provided $\beta > -3^{-1}$, Corollary 2 in Section 2.1.7 of [24] implies that for every $\phi \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^n)$, and therefore by approximation every $\phi \in W^{1,2}_\beta(\mathbb{R}^3; \mathbb{R}^n)$, we have

$$
\left(\int_{\mathbb{R}^3} |x_3|^{3\beta} |\phi|^6 \mathrm{d}x \right)^{\frac{1}{3}} \leq C \int_{\mathbb{R}^3} |x_3|^{\beta} |\nabla\phi|^2 \mathrm{d}x.
$$

The even reflection with respect to $\partial\mathbb{R}^{m+1}_+$, denoted $\tilde{\phi}$, of a $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ is in $W^{1,2}_\beta(\mathbb{R}^3; \mathbb{R}^n)$ and hence, applying Hölder's inequality, we find

$$
\begin{aligned}
\int_{\Omega} x_3^{\beta} |\phi|^2 \mathrm{d}x &\leq |\Omega|^{\frac{2}{3}} \left(\int_{\mathbb{R}^3} |x_3|^{3\beta} |\tilde{\phi}|^6 \mathrm{d}x \right)^{\frac{1}{3}} \leq C|\Omega|^{\frac{2}{3}} \int_{\mathbb{R}^3} |x_3|^{\beta} |\nabla\tilde{\phi}|^2 \mathrm{d}x \\
&= 2C|\Omega|^{\frac{2}{3}} \int_{\mathbb{R}^3_+} x_3^{\beta} |\nabla\phi|^2 \mathrm{d}x,
\end{aligned}
$$

for every $\phi \in \mathcal{D}_+(\mathbb{R}^3_+; \mathbb{R}^n)$. We again conclude $||\phi||^2_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)} \leq C||\phi||^2_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)}$. $\quad\square$

Remark 2.4 We can further embedd $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ into $W^{1,p}(\Omega; \mathbb{R}^n)$, where p depends on β, for open bounded $\Omega \subset \mathbb{R}^{m+1}_+$ using Lemma 2.1.

2.2 Compactness of the embedding $W^{1,2}_\beta \hookrightarrow L^2_\beta$

We will require an analogue of the compact embedding $W^{1,2} \hookrightarrow L^2$ in order to analyse re-scaled limits of bounded sequences of Sobolev functions. Away from the boundary, that is, for Ω with $\overline{\Omega} \subset \mathbb{R}^{m+1}_+$, the compactness of the inclusion $W^{1,2} \hookrightarrow L^2$ yields the compactness of the inclusion $W^{1,2}_\beta \hookrightarrow L^2_\beta$ in view of Lemma 2.1. We have not been able to find a proof of compactness near $\partial\mathbb{R}^{m+1}_+$ in the literature so present one for completeness. Let

$B_r(y) = \{x \in \mathbb{R}^{m+1} : |x - y| < r\}$ and $Q_r(y) = \{x \in \mathbb{R}^{m+1} : |x_i - y_i| < r, i = 1, \ldots, m+1\}$ and define $B_r^+(y) = B_r(y) \cap \mathbb{R}_+^{m+1}$ and $Q_r^+(y) = Q_r(y) \cap \mathbb{R}_+^{m+1}$ for $y \in \partial\mathbb{R}_+^{m+1}$.

Lemma 2.5 *Let $r > 0$, $y \in \partial\mathbb{R}_+^{m+1}$ and suppose $(v_j)_{j\in\mathbb{N}}$ is a sequence in $W_\beta^{1,2}(\Omega; \mathbb{R}^n)$ which satisfies $\sup_j \|v_j\|_{W_\beta^{1,2}(\Omega;\mathbb{R}^n)} < \infty$ where Ω is either $Q_r^+(y)$ or $B_r^+(y)$. Then there exists a subsequence $(v_{j_k})_{k\in\mathbb{N}}$ and a $v \in W_\beta^{1,2}(\Omega; \mathbb{R}^n)$ such that*

1. $v_{j_k} \rightharpoonup v$ in $W_\beta^{1,2}(\Omega; \mathbb{R}^n)$
2. $v_{j_k} \to v$ in $L_\beta^2(\Omega; \mathbb{R}^n)$
3. $\int_\Omega x_{m+1}^\beta |\nabla v|^2 dx \leq \liminf_{k\to\infty} \int_\Omega x_{m+1}^\beta |\nabla v_{j_k}|^2 dx$.

Proof For $\beta = 0$, a proof can be found in [39, Section 1.3, Lemma 1]. Otherwise, statement 1 follows from the weak sequential compactness of the unit ball in a Hilbert space and statement 3 follows from statement 2 and the lower semi-continuity of a Hilbert space norm. Hence, the main task is to prove statement 2. We may assume $r = 1$ and $y = 0$ since statement 2 is invariant under rescaling and translations with respect to x_i for $i = 1, \ldots, m$. We further assume that $\Omega = Q_1^+(0)$ since if $\Omega = B_1^+(0)$ and the result is true on $Q_1^+(0)$ then we may compose with the bi-Lipschitz, piecewise C^1 with piecewise C^1 inverse map $Q_1^+(0) \to B_1^+(0) : x \mapsto \max_i |x_i| |x|^{-1} x$ and deduce the statement on $B_1^+(0)$.

Suppose $(v_j)_{j\in\mathbb{N}}$ is a sequence with $v_j \in W_\beta^{1,2}(Q_1^+(0); \mathbb{R}^n)$ for every j and which satisfies

$$\sup_{j\in\mathbb{N}} \|v_j\|_{W_\beta^{1,2}(Q_1^+(0);\mathbb{R}^n)} \leq M \tag{2.3}$$

for some positive constant M. Relabelling if necessary, suppose $(v_j)_{j\in\mathbb{N}}$ is also the subsequence which satisfies $v_j \rightharpoonup v$ for $v \in W_\beta^{1,2}(Q_1^+(0); \mathbb{R}^n)$. Then

$$\|v\|_{W_\beta^{1,2}(Q_1^+(0);\mathbb{R}^n)} \leq M. \tag{2.4}$$

Let $Q' = (-1, 1)^m$, define $Q_i = \{(x', x_{m+1}) \in Q_1^+(0) : (i+1)^{-1} < x_{m+1} \leq 1\}$ for $i \in \mathbb{N}$ and let $\hat{Q}_i = Q_1^+(0) \backslash Q_i = Q' \times [0, (i+1)^{-1}]$. In view of (2.3), we have $\sup_{j\in\mathbb{N}} \|v_j\|_{W_\beta^{1,2}(Q_i;\mathbb{R}^n)} \leq M$ for each $i \in \mathbb{N}$. Hence, using the compactness of the embedding $W_\beta^{1,2}(Q_1; \mathbb{R}^n) \hookrightarrow L_\beta^2(Q_1; \mathbb{R}^n)$, we find a $\tilde{v} \in W_\beta^{1,2}(Q_1; \mathbb{R}^n)$ and a subsequence, which we denote $(v_j)_{j\in\Lambda_1}$ for an infinite set $\Lambda_1 \subset \mathbb{N}$, which satisfies $v_j \rightharpoonup \tilde{v}$ in $W_\beta^{1,2}(Q_1; \mathbb{R}^n)$, $v_j \to \tilde{v}$ in $L_\beta^2(Q_1; \mathbb{R}^n)$ and almost everywhere as $j \to \infty$ with $j \in \Lambda_1$. Notice that $(v_j)_{j\in\Lambda_1}$ converges weakly to v in $W_\beta^{1,2}(Q_1; \mathbb{R}^n)$ because $(v_j)_{j\in\mathbb{N}}$ does and so, by the uniqueness of weak limits, we deduce $\tilde{v} = v$ in Q_1. Hence $v_j \to v$ in $L_\beta^2(Q_1; \mathbb{R}^n)$ and almost everywhere as $j \to \infty$ as well. Repeating this process inductively for every $i \in \mathbb{N}$, we obtain sequences $(v_j)_{j\in\Lambda_i}$ with $\Lambda_{i+1} \subset \Lambda_i$ such that $(v_j)_{j\in\Lambda_i}$ converges to v in $L_\beta^2(Q_i; \mathbb{R}^n)$ and almost everywhere in Q_i. Hence we can choose an increasing sequence of numbers $(k_i)_{i\in\mathbb{N}}$ with $k_i \in \Lambda_i$ such that

$$\int_{Q_i} x_{m+1}^\beta |v_k - v|^2 dx < \frac{\int_{\frac{1}{i+1}}^{\frac{1}{i}} x_{m+1}^\beta dx_{m+1}}{2^i} \leq \frac{1}{i^{2+\beta}2^i} \tag{2.5}$$

for $k \geq k_i$. Then the sequence $(v_{k_i})_{i\in\mathbb{N}}$ converges to v almost everywhere in $Q_1^+(0)$ and in $L_\beta^2(Q_k; \mathbb{R}^n)$ for all $k \in \mathbb{N}$ as $i \to \infty$. Observe that

$$\int_{Q_1^+(0)} x_{m+1}^\beta |v_{k_i} - v|^2 dx = \int_{\hat{Q}_i} x_{m+1}^\beta |v_{k_i} - v|^2 dx + \int_{Q_i} x_{m+1}^\beta |v_{k_i} - v|^2 dx. \tag{2.6}$$

By (2.5) we have $\int_{Q_i} x_{m+1}^\beta |v_{k_i} - v|^2 dx < \frac{1}{i^{2+\beta}2^i} \to 0$ as $i \to \infty$ so we consider the remaining term in (2.6). Using Chebychev's inequality combined with Fubini's Theorem, we may choose $c_i \in ((i+1)^{-1}, i^{-1})$ such that

$$\int_{Q'} |v_{k_i}(x', c_i) - v(x', c_i)|^2 dx' \le \frac{1}{\left(\int_{\frac{1}{i+1}}^{\frac{1}{i}} x_{m+1}^\beta dx_{m+1}\right)} \int_{\frac{1}{i+1}}^{\frac{1}{i}} \int_{Q'} x_{m+1}^\beta |v_{k_i} - v|^2 dx. \quad (2.7)$$

Now for each $i \in \mathbb{N}$, we calculate

$$\int_{\hat{Q}_i} x_{m+1}^\beta |v_{k_i} - v|^2 dx \le 4 \int_{\hat{Q}_i} x_{m+1}^\beta |v_{k_i} - v_{k_i}(x', c_i)|^2 dx + 4 \int_{\hat{Q}_i} x_{m+1}^\beta |v - v(x', c_i)|^2 dx$$
$$+ 4 \int_{\hat{Q}_i} x_{m+1}^\beta |v_{k_i}(x', c_i) - v(x', c_i)|^2 dx. \quad (2.8)$$

We apply Hölder's inequality and (2.4) to see that

$$\int_{\hat{Q}_i} x_{m+1}^\beta |v(x', x_{m+1}) - v(x', c_i)|^2 dx = \int_{\hat{Q}_i} x_{m+1}^\beta \left| \int_{x_{m+1}}^{c_i} s^{-\frac{\beta}{2}} s^{\frac{\beta}{2}} \partial_{m+1} v(x', s) ds \right|^2 dx$$
$$\le \frac{c_i^{1-\beta}}{1 - \beta^2} c_i^{1+\beta} \int_0^{c_i} \int_{Q'} x_{m+1}^\beta |\partial_{m+1} v|^2 dx$$
$$\le \frac{1}{1 - \beta^2} \frac{1}{i^2} M^2. \quad (2.9)$$

The bound for the integral on the left hand side of (2.9) but with v replaced by v_{k_i} is identical. We apply (2.7) followed by (2.5) to see that

$$\int_{\hat{Q}_i} x_{m+1}^\beta |v_{k_i}(x', c_i) - v(x', c_i)|^2 dx \le \frac{\int_0^{\frac{1}{i+1}} x_{m+1}^\beta dx_{m+1}}{\int_{\frac{1}{i+1}}^{\frac{1}{i}} x_{m+1}^\beta dx_{m+1}} \int_{\frac{1}{i+1}}^{\frac{1}{i}} \int_{Q'} x_{m+1}^\beta |v_{k_i} - v|^2 dx$$
$$\le \frac{1}{(1+\beta)i^{1+\beta}2^i}. \quad (2.10)$$

Finally we combine (2.5), (2.6), (2.8), (2.9) and (2.10) and let $i \to \infty$ to conclude the proof. $\square$

Remark 2.6 The method of proof of Lemma 2.5 is also valid for Ω of the form $\Omega = \mathcal{O} \times [0, r]$ for $r > 0$ and $\mathcal{O} \subset \partial \mathbb{R}_+^{m+1}$.

2.3 Energy decay for a linear Neumann-type problem

When examining the limit of re-scaled sequences of Sobolev functions, as in the proof of Lemma 4.18, we will obtain a weak solution of the Neumann-type problem

$$\text{div}(x_{m+1}^\beta \nabla v) = 0 \text{ in } B_R^+(x_0) \quad \text{and} \quad x_{m+1}^\beta \partial_{m+1} v = 0 \text{ in } \partial B_R^+(x_0) \cap \partial \mathbb{R}_+^{m+1}, \quad (2.11)$$

for some $R > 0$ and $x_0 \in \partial \mathbb{R}_+^{m+1}$. The rate of decay of the re-scaled energy of such solutions on concentric half-balls centred at x_0 will play a role in the proof of the aforementioned lemma and we estimate this decay here.

Lemma 2.7 *Let $\beta \in (-1, 1)$, $x_0 \in \partial\mathbb{R}^{m+1}_+$ and suppose $v \in W^{1,2}_\beta(B^+_R(x_0); \mathbb{R}^n)$ satisfies*

$$\int_{B^+_R(x_0)} x^\beta_{m+1}\langle\nabla v, \nabla\psi\rangle\mathrm{d}x = 0 \tag{2.12}$$

for every $\psi \in C^\infty_0(B_R(x_0); \mathbb{R}^n)$. There exists a $\gamma = \gamma(m, \beta) \in (0, 1)$ and a positive constant C such that

$$\left(\frac{r}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{r}{2}}(x_0)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \le Cr^{2\gamma} \tag{2.13}$$

for every $r \le \frac{R}{2}$.

Proof The even reflection of v in $\partial\mathbb{R}^{m+1}_+$, which we do not relabel, belongs to $W^{1,2}_\beta(B_R(x_0); \mathbb{R}^n)$ and satisfies

$$\int_{B_R(x_0)} |x_{m+1}|^\beta\langle\nabla v, \nabla\psi\rangle\mathrm{d}x = 0 \tag{2.14}$$

for every $\psi \in C^\infty_0(B_R(x_0); \mathbb{R}^n)$. A result of Fabes et al., see [13, Theorem 2.3.12], implies the local Hölder continuity of v in $B_R(x_0)$. In particular, there exists a constant C such that

$$|v(x) - v(y)| \le C|x - y|^\gamma \tag{2.15}$$

for some $\gamma \in (0, 1)$ and every $x, y \in B_{\frac{R}{2}}(x_0)$.

By approximation, (2.14) holds for every $\psi \in W^{1,2}_{\beta,0}(B_R(x_0); \mathbb{R}^n)$. Let $\eta \in C^\infty_0(B_r(x_0); [0, 1])$ be a cutoff function with $\eta \equiv 1$ in $B_{\frac{r}{2}}(x_0)$ and $|\nabla\eta| \le \frac{C}{r}$ for a fixed positive $C \ge 2$. We observe that $\phi = \eta^2(v - \lambda)$ is an admissible test function for every $\lambda \in \mathbb{R}^n$. Testing (2.14) against ϕ and using Young's inequality, $ab \le \delta\frac{a^2}{2} + \frac{b^2}{2\delta}$ for $a, b \ge 0$ and $\delta > 0$, we see that

$$\int_{B_r(x_0)} |x_{m+1}|^\beta\eta^2|\nabla v|^2\mathrm{d}x \le \delta\int_{B_r(x_0)} |x_{m+1}|^\beta\eta^2|\nabla v|^2\mathrm{d}x$$
$$+ \frac{C}{\delta}\int_{B_r(x_0)} |x_{m+1}|^\beta|\nabla\eta|^2|v - \lambda|^2\mathrm{d}x.$$

We choose $\delta = \frac{1}{2}$, set $\lambda = v(x_0)$, recall $|\nabla\eta| \le \frac{C}{r}$ and apply (2.15) to see that

$$\int_{B_r(x_0)} |x_{m+1}|^\beta\eta^2|\nabla v|^2\mathrm{d}x \le Cr^{m+\beta-1+2\gamma} \tag{2.16}$$

for another positive $C > 0$, independent of $r \le \frac{R}{2}$. Multiplying (2.16) by $(2^{-1}r)^{1-m-\beta}$ concludes the proof. $\square$

2.4 Boundary monotonicity formula for the average energy of solutions to the degenerate linear equation

In order to utilise a version of the method of harmonic replacement, see Lemma 4.23 in Sect. 4.12, we need to know how the average energy of solutions to $\mathrm{div}(|x_{m+1}|^\beta\nabla v) = 0$ behaves on concentric balls. When these balls are centred on $\partial\mathbb{R}^{m+1}_+$, we have the following.

Lemma 2.8 *Let* $B_R(x_0) \subset \mathbb{R}^{m+1}$ *with* $x_0 \in \partial\mathbb{R}^{m+1}_+$ *and* $R \leq 1$. *Suppose* $v \in W^{1,2}_\beta(B_R(x_0); \mathbb{R}^n)$ *is a weak solution of* $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ *in* $B_R(x_0)$. *If* $\beta \in (-1, 0]$, *or if* $\beta \in (0, 1)$ *and* v *is symmetric with respect to* $\partial\mathbb{R}^{m+1}_+$ *in* $B_R(x_0)$, *then*

$$s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x \leq r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x$$

for every $0 < s \leq r \leq R$.

Remark 2.9 It appears that Lemma 2.8 is a particular case of Theorem 2.6 of [3]. The function

$$\tilde{v}(x', x_{m+1}) = \begin{cases} \frac{1}{1-\beta} x_{m+1}^{1-\beta} & \text{if } x_{m+1} \geq 0 \\ -\frac{1}{1-\beta}(-x_{m+1})^{1-\beta} & \text{if } x_{m+1} < 0 \end{cases} = (1-\beta)^{-1} x_{m+1} |x_{m+1}|^{-\beta}$$

fulfils the hypothesis of Lemma 2.8, with the exception that it is not symmetric with respect to $\partial\mathbb{R}^{m+1}_+$ when $\beta \in (0, 1)$. Furthermore, $s^{-(1+m+\beta)} \int_{B_s(0)} |x_{m+1}|^\beta |\nabla\tilde{v}|^2 \mathrm{d}x = C(m, \beta)s^{-2\beta}$. It follows that $\tilde{v}$ does not satisfy the conclusion of the lemma (e.g on $B_1(0)$) when $\beta \in (0, 1)$. Hence Lemma 2.8 does not hold in general when $\beta \in (0, 1)$. This also potentially affects the aforementioned theorem.

The remainder of this section is predominantly devoted to a proof of Lemma 2.8. We also establish a similar monotonicity formula for balls $B_R(x_0)$ with $(x_0)_{m+1} \geq \theta R$ for $\theta \geq 2$ and give explicit dependence on θ. Our method is essentially that used to show the well known monotonicity results for the average energy of sub-harmonic functions, which is based on the observation $\Delta|\nabla v|^2 \geq 0$ if $\Delta v = 0$. When $\beta = 0$ our result reduces to the usual growth formula for the average energy of sub-harmonic functions.

Define $v^* := |x_{m+1}|^\beta \partial_{m+1} v$. This function will be integral to our proof of Lemma 2.8 because, as we will see in more detail later, it satisfies $\mathrm{div}(|x_{m+1}|^{-\beta} \nabla v^*) = 0$ when $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$. The fact that if $\mathrm{div}(x^\beta_{m+1} \nabla v) = 0$ in $\mathbb{R}^{m+1}_+$ then $\mathrm{div}(x^{-\beta}_{m+1} \nabla v^*) = 0$ was first observed by Caffarelli and Silvestre [2].

First, we consider the regularity of solutions of $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$; the following regularity results are known and have essentially been obtained in, for example [4]. We give a proof regardless, as an illustration that the method of difference quotients works essentially unchanged in the directions $x_1, \ldots, x_m$, which we will take advantage of when considering the regularity of manifold valued minimisers of E^β.

Let $i = 1, \ldots, m$, $\Omega \subset \mathbb{R}^{m+1}$ and let $h \in \mathbb{R}\backslash\{0\}$. Define the difference quotient of $v : \Omega \to \mathbb{R}^n$ by $\Delta^h_i v(x) = h^{-1}(v(x + he_i) - v(x))$ where e_i denotes the ith basis vector in $\mathbb{R}^{m+1}$ and $\mathrm{dist}(x, \partial\Omega) < |h|$.

Lemma 2.10 *Let* $\Omega \subset \mathbb{R}^{m+1}$ *be open and* $v \in W^{1,2}_\beta(\Omega; \mathbb{R}^n)$. *Then for any* $i = 1, \ldots, m$ *we have* $\Delta^h_i v \in L^2_\beta(K; \mathbb{R}^n)$ *for any compact* $K \subset \Omega$, *provided* $|h| < \mathrm{dist}(K, \partial\Omega)$. *In particular,*

$$\int_K |x_{m+1}|^\beta |\Delta^h_i v|^2 \mathrm{d}x \leq \int_\Omega |x_{m+1}|^\beta |\partial_i v|^2 \, \mathrm{d}x.$$

Proof This proof follows the proof of Lemma 7.23 in [15]. We assume that $h \geq 0$, the argument for negative h is analogous. Let $v \in C^1(\Omega; \mathbb{R}^n) \cap W^{1,2}_\beta(\Omega; \mathbb{R}^n)$. Using the notation $K_h = \{x \in \mathbb{R}^{m+1} : \mathrm{dist}(x, K) \leq h\}$ and noting $K_h \subset \Omega$, by Fubini's Theorem and the compactness of K, for h with $|h| < \mathrm{dist}(K, \partial\Omega)$ and any $i = 1, \ldots, m$, we calculate

$$\int_K |x_{m+1}|^\beta |\Delta^h_i v|^2 \mathrm{d}x \leq \int_K |x_{m+1}|^\beta \frac{1}{h} \int_0^h |\partial_i v(x + te_i)|^2 \, \mathrm{d}t \mathrm{d}x$$

$$\leq \frac{1}{h} \int_0^h \int_{K_h} |x_{m+1}|^\beta |\partial_i v|^2 \, dx dt$$

$$\leq \int_\Omega |x_{m+1}|^\beta |\partial_i v|^2 \, dx.$$

We deduce the result for $v \in W_\beta^{1,2}(\Omega; \mathbb{R}^n)$ by approximation. □

Next we prove a criterion for the existence of weak derivatives in the directions x_i for $i = 1, \ldots, m$.

Lemma 2.11 *Suppose $\Omega \subset \mathbb{R}^{m+1}$ is open and bounded and let $v \in L_\beta^2(\Omega; \mathbb{R}^n)$. For any $i = 1, \ldots, m$, suppose there exist constants $M > 0$ and $\tilde{h} > 0$ such that*

$$\int_K |x_{m+1}|^\beta |\Delta_i^h v|^2 dx \leq M$$

for every $h \neq 0$ with $|h| < \tilde{h}$ and compact $K \subset \Omega$ with $\mathrm{dist}(K, \partial\Omega) > |h|$. Then the weak derivative $\partial_i v$ exists in Ω and satisfies

$$\int_\Omega |x_{m+1}|^\beta |\partial_i v|^2 \, dx \leq M.$$

Proof First choose a sequence $(h_k)_{k \in \mathbb{N}}$ with $h_k \to 0$. We discard h_k with $|h_k| \geq \tilde{h}$ and re-index to $k \in \mathbb{N}$. Define $v_i^{h_k}(x) = \Delta_i^{h_k} v(x)$ when $x \in \Omega$ and $\mathrm{dist}(x, \partial\Omega) \geq 2|h_k|$ and $v_i^{h_k} = 0$ otherwise. It follows that $\{v_i^{h_k}\}_{k \in \mathbb{N}}$ is a bounded sequence in $L_\beta^2(\Omega; \mathbb{R}^n)$. Hence there is a subsequence, which we index again by $k \in \mathbb{N}$, such that $h_k \to 0$ and $v_i^{h_k} \to \tilde{v}_i$ weakly in $L_\beta^2(\Omega; \mathbb{R}^n)$. Furthermore, this convergence, together with the weak lower semi-continuity of a Hilbert space norm, guarantees that $\int_\Omega x_{m+1}^\beta |\tilde{v}_i|^2 dx \leq M$. Note that Hölder's inequality implies $L_\beta^2(\Omega; \mathbb{R}^n) \subset L^p(\Omega; \mathbb{R}^n)$ for some $p \in (1, 2]$ depending on β. Thus each linear functional on $L^p(\Omega; \mathbb{R}^n)$ restricts to a linear functional on $L_\beta^2(\Omega; \mathbb{R}^n)$ and $v_i^{h_k}$ converges to $\tilde{v}_i$ weakly in $L^p(\Omega; \mathbb{R}^n)$. As in the proof of Lemma 7.24 in [15], it follows that $\tilde{v}_i$ is the weak derivative $\partial_i v$. □

We now establish the regularity properties of solutions of $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ needed for the proof of Lemma 2.8.

Lemma 2.12 *Let $v \in W_\beta^{1,2}(B_R(x_0); \mathbb{R}^n)$ and suppose v is a weak solution of $\mathrm{div}(|x_{m+1}|^\beta \nabla v)$ $= 0$ in $B_R(x_0)$. For every $r < R$ and $i = 1, \ldots, m$ it follows that $\partial_i v \in W_\beta^{1,2}(B_r(x_0); \mathbb{R}^n)$, $\partial_i v$ is a weak solution of $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_r(x_0)$ and $\partial_i v$ is locally Hölder continuous in $B_R(x_0)$. In addition, $v^* \in W_{-\beta}^{1,2}(B_r(x_0); \mathbb{R}^n)$ and v^* is a weak solution of $\mathrm{div}(|x_{m+1}|^{-\beta} \nabla v^*) = 0$ in $B_r(x_0)$ and v^* is locally Hölder continuous in $B_R(x_0)$.*

Proof Elliptic regularity theory shows that v is smooth in $B_R(x_0) \backslash \partial \mathbb{R}_+^{m+1}$ [15]. Observe that v satisfies

$$\int_{B_R(x_0)} |x_{m+1}|^\beta \langle \nabla v, \nabla \phi \rangle \, dx = 0 \tag{2.17}$$

for every $\phi \in W_{\beta,0}^{1,2}(B_R(x_0); \mathbb{R}^n)$ by approximation. Let $r < R$ and choose $\eta \in C_0^\infty(B_R(x_0))$ with $\eta \equiv 1$ in $B_r(x_0)$, $\eta \equiv 0$ in $B_R(x_0) \backslash B_{r+\frac{R-r}{2}}(x_0)$, $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq \frac{C}{R-r}$. Let $\Delta_i^h v$

be the difference quotient of v for some $i = 1, \ldots, m$ and suppose $|h| < \frac{R-r}{4}$. Then $\phi = -\Delta_i^{-h}(\eta^2 \Delta_i^h v) \in W_{\beta,0}^{1,2}(B_R(x_0); \mathbb{R}^n)$ is an admissible test function for (2.17) and an application of Young's inequality, $ab \leq \delta \frac{a^2}{2} + \delta^{-1} \frac{b^2}{2}$ for $a, b \geq 0$ and $\delta > 0$, together with an integration by parts and Lemma 2.10 implies

$$\int_{B_R(x_0)} \eta^2 |x_{m+1}|^\beta |\nabla \Delta_i^h v|^2 dx = -\int_{B_R(x_0)} 2\eta |x_{m+1}|^\beta \left\langle \nabla \Delta_i^h v \cdot \nabla \eta, \Delta_i^h v \right\rangle dx$$

$$\leq \frac{C}{R-r} \delta \int_{B_R(x_0)} \eta^2 |x_{m+1}|^\beta |\nabla \Delta_i^h v|^2 dx$$

$$+ \frac{C}{R-r} \delta^{-1} \int_{B_R(x_0)} |x_{m+1}|^\beta |\partial_i v|^2 dx.$$

Since $\eta \equiv 1$ in $B_r(x_0)$, choosing $\delta = \frac{R-r}{2C}$ we deduce that

$$\int_{B_r(x_0)} |x_{m+1}|^\beta |\nabla \Delta_i^h v|^2 dx \leq \frac{C}{(R-r)^2} \int_{B_R(x_0)} |x_{m+1}|^\beta |\partial_i v|^2 dx.$$

The right hand side above is independent of h and thus Lemma 2.11 implies the weak derivative $\nabla \partial_i v$ exists and is in $L_\beta^2(B_r(x_0); \mathbb{R}^{(m+1)n})$. Hence $\partial_i v \in W_\beta^{1,2}(B_r(x_0); \mathbb{R}^n)$ for every $r < R$. We integrate by parts in (2.17) to see that $\partial_i v$ is a weak solution of $\text{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_r(x_0)$ for every $r < R$. It follows from [13, Theorem 2.3.12] that each $\partial_i v$ is locally Hölder continuous in $B_R(x_0)$. We inductively deduce that for any multi-index α' with $\alpha'_{m+1} = 0$, $D^{\alpha'} v \in W_\beta^{1,2}(B_r(x_0); \mathbb{R}^n)$ is a weak solution of $\text{div}(|x_{m+1}|^\beta \nabla D^{\alpha'} v) = 0$ in $B_r(x_0)$ for every $r < R$ and $D^{\alpha'} v$ is locally Hölder continuous in $B_R(x_0)$.

Since $\partial_i v \in W_\beta^{1,2}(B_r(x_0); \mathbb{R}^n)$ for every $r < R$ and $i = 1, \ldots, m$, we have

$$\int_{B_r(x_0)} |x_{m+1}|^{-\beta} |\partial_i v^*|^2 dx = \int_{B_r(x_0)} |x_{m+1}|^\beta |\partial_i \partial_{m+1} v|^2 dx < \infty. \tag{2.18}$$

We also have $\Delta' v \in W_\beta^{1,2}(B_r(x_0); \mathbb{R}^n)$, where Δ' is the Laplace operator with respect to the variables $x_1, \ldots, x_m$. Furthermore, as v solves $\text{div}(|x_{m+1}|^\beta \nabla v) = 0$ classically in $B_R(x_0) \backslash \partial \mathbb{R}_+^{m+1}$, we have $|x_{m+1}|^{-\beta} \partial_{m+1} v^* = -\Delta' v \in W_\beta^{1,2}(B_r(x_0); \mathbb{R}^n)$ for every $r < R$. Hence

$$\int_{B_r(x_0)} |x_{m+1}|^{-\beta} |\partial_{m+1} v^*|^2 dx = \int_{B_r(x_0)} |x_{m+1}|^\beta |\Delta' v|^2 dx < \infty. \tag{2.19}$$

Together, (2.18) and (2.19) imply $\int_{B_r(x_0)} |x_{m+1}|^{-\beta} |\nabla v^*|^2 dx < \infty$ for every $r < R$. Moreover, we have

$$\int_{B_r(x_0)} |x_{m+1}|^{-\beta} |v^*|^2 dx = \int_{B_r(x_0)} |x_{m+1}|^\beta |\partial_{m+1} v|^2 dx < \infty, \tag{2.20}$$

since $v \in W_\beta^{1,2}(B_R(x_0); \mathbb{R}^n)$. We can directly verify that $\partial_i v^*$, $i = 1, \ldots, m$, are the weak derivatives of v^* and omit the details. Now consider $\partial_{m+1} v^*$. Let ∇' denote the gradient operator with respect to the variables $x_1, \ldots, x_m$ and let $\psi \in C_0^\infty(B_R(x_0); \mathbb{R}^n)$. Since $|x_{m+1}|^{-\beta} \partial_{m+1} v^* = -\Delta' v \in W_\beta^{1,2}(B_r(x_0); \mathbb{R}^n)$ and v is a weak solution of $\text{div}(|x_{m+1}|^\beta \nabla v) = 0$, we see that

$$\int_{B_R(x_0)} \left\langle \partial_{m+1} v^*, \psi \right\rangle dx = -\int_{B_R(x_0)} |x_{m+1}|^\beta \left\langle \Delta' v, \psi \right\rangle dx$$

$$= \int_{B_R(x_0)} |x_{m+1}|^\beta \left\langle \nabla' v, \nabla' \psi \right\rangle dx$$

$$= -\int_{B_R(x_0)} \left\langle v^*, \partial_{m+1} \psi \right\rangle dx.$$

It follows that $v^* \in W^{1,2}_{-\beta}(B_r(x_0); \mathbb{R}^n)$ for every $r < R$. In a similar manner we calculate

$$\int_{B_R(x_0)} |x_{m+1}|^{-\beta} \left\langle \nabla v^*, \nabla \psi \right\rangle dx = \int_{B_R(x_0)} |x_{m+1}|^{-\beta} \left\langle \partial_{m+1} v^*, \partial_{m+1} \psi \right\rangle dx$$

$$+ \int_{B_R(x_0)} |x_{m+1}|^{-\beta} \left\langle \nabla' v^*, \nabla' \psi \right\rangle dx$$

$$= -\int_{B_R(x_0)} \left\langle \Delta' v, \partial_{m+1} \psi \right\rangle dx + \int_{B_R(x_0)} \left\langle \Delta' v, \partial_{m+1} \psi \right\rangle dx$$

$$= 0.$$

Hence v^* is a weak solution of $\mathrm{div}(|x_{m+1}|^\beta \nabla v^*) = 0$ in $B_r(x_0)$ for every $r < R$. It follows from [13, Theorem 2.3.12] that v^* is locally Hölder continuous in $B_R(x_0)$. $\qquad\square$

Corollary 2.13 *Suppose* $v \in W^{1,2}_\beta(B_R(x_0); \mathbb{R}^n)$ *and assume* v *weakly satisfies* $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ *in* $B_R(x_0)$. *Then the derivatives* $D^{\alpha'} v$, *where* α' *is a multi-index with* $(\alpha')_{m+1} = 0$, *are elements of* $W^{1,2}_\beta(B_r(x_0); \mathbb{R}^n)$ *and weak solutions of* $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ *in* $B_r(x_0)$ *for every* $r < R$ *and are locally Hölder continuous in* $B_R(x_0)$. *Furthermore, the functions* $(D^{\alpha'} v)^* := |x_{m+1}|^\beta \partial_{m+1} D^{\alpha'} v$ *are elements of* $W^{1,2}_{-\beta}(B_r(x_0); \mathbb{R}^n)$ *and weak solutions of* $\mathrm{div}(|x_{m+1}|^{-\beta} \nabla (D^{\alpha'} v)^*) = 0$ *in* $B_r(x_0)$ *for every* $r < R$ *and are locally Hölder continuous in* $B_R(x_0)$.

Proof This follows from a direct application of Lemma 2.12. $\qquad\square$

Next we record a condition on the integral of the normal derivative of Sobolev functions which implies the type of monotonicity we want to establish in Lemma 2.8.

Lemma 2.14 *Suppose that* $v \in W^{1,2}_\beta(B_R(x_0))$, *where* $x_0 \in \partial \mathbb{R}^{m+1}_+$, *and that*

$$\int_{\partial B_\rho(x_0)} v \cdot |x_{m+1}|^\beta \nabla v \, dS(x) \geq 0$$

for almost every $\rho \in (0, R)$, *where* v *is the outward pointing unit normal on* $\partial B_\rho(x_0)$, *then*

$$s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta v dx \leq r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta v dx \qquad (2.21)$$

for every $0 < s \leq r \leq R$.

Proof This proof follows the proof of Theorem 2.1 in Section 2 of [15] and the proof of Proposition 2.2 in Section III of [14]. For almost every $0 < s \leq r < R$, using Fubini's Theorem, we see that

$$r^{-(m+\beta)} \int_{\partial B_r(x_0)} |x_{m+1}|^\beta v dS(x) - s^{-(m+\beta)} \int_{\partial B_s(x_0)} |x_{m+1}|^\beta v dS(x)$$

$$= \int_{\partial B_1(0)} |\omega_{m+1}|^\beta \int_s^r \frac{\partial}{\partial t} v(t\omega + x_0) dt d\omega$$

$$= \int_s^r t^{-(m+\beta)} \int_{\partial B_t(x_0)} |x_{m+1}|^\beta \nu \cdot \nabla v \, dS(x) \, dt$$

$$\geq 0. \tag{2.22}$$

Define the absolutely continuous function $f(r) = \int_{B_r(x_0)} |x_{m+1}|^\beta v \, dx$ for $0 \leq r \leq R$. Using (2.22) we calculate

$$f(r) = \int_0^r f'(\rho) \, d\rho = \int_0^r \rho^{m+\beta} \rho^{-(m+\beta)} f'(\rho) \, d\rho$$

$$\leq \int_0^r \rho^{m+\beta} r^{-(m+\beta)} f'(r) \, d\rho = \frac{r}{1+m+\beta} f'(r)$$

for $0 < r < R$. It follows that $(r^{-(1+m+\beta)} f(r))' \geq 0$ and integrating between $s \leq r \leq R$ completes the proof. □

Proof of Lemma 2.8 Note that v is smooth in $B_R(x_0) \backslash \partial \mathbb{R}^{m+1}_+$. Hence $\operatorname{div}(|x_{m+1}|^\beta \nabla \partial_i v) = 0$ and $\operatorname{div}(|x_{m+1}|^{-\beta} \nabla v^*) = 0$ classically in this set. Furthermore, we have

$$0 = \partial_{m+1} \operatorname{div}(|x_{m+1}|^\beta \nabla v) = \operatorname{div}(|x_{m+1}|^\beta \nabla \partial_{m+1} v) + \operatorname{sgn}(x_{m+1}) \frac{\beta}{|x_{m+1}|} \operatorname{div}(|x_{m+1}|^\beta \nabla v)$$

$$- \beta |x_{m+1}|^{\beta-2} \partial_{m+1} v$$

$$= \operatorname{div}(|x_{m+1}|^\beta \nabla \partial_{m+1} v) - \beta |x_{m+1}|^{\beta-2} \partial_{m+1} v,$$

so that $\operatorname{div}(|x_{m+1}|^\beta \nabla \partial_{m+1} v) = \beta |x_{m+1}|^{\beta-2} \partial_{m+1} v$ in $B_R(x_0) \backslash \partial \mathbb{R}^{m+1}_+$. Hence on $B_R(x_0) \backslash \partial \mathbb{R}^{m+1}_+$ we have

$$\operatorname{div}\left(|x_{m+1}|^\beta \nabla |\partial_i v|^2\right) = 2|x_{m+1}|^\beta |\nabla \partial_i v|^2 + 2\langle \partial_i v, \operatorname{div}(|x_{m+1}|^\beta \nabla \partial_i v)\rangle \geq 0,$$

$$\operatorname{div}\left(|x_{m+1}|^{-\beta} \nabla |v^*|^2\right) = 2|x_{m+1}|^\beta |\nabla v^*|^2 + 2\langle v^*, \operatorname{div}(|x_{m+1}|^{-\beta} \nabla v^*)\rangle \geq 0,$$

and, when $\beta \in (0, 1)$,

$$\operatorname{div}\left(|x_{m+1}|^\beta \nabla |\partial_{m+1} v|^2\right) = 2|x_{m+1}|^\beta |\nabla \partial_{m+1} v|^2 + 2\beta |x_{m+1}|^{\beta-2} |\partial_{m+1} v|^2 \geq 0,$$

classically where $i = 1, \ldots, m$.

Fix $R > r > \varepsilon > 0$ and let $B_r^\varepsilon(x_0) = B_r(x_0) \cap \{x \in \mathbb{R}^{m+1} : |x_{m+1}| \geq \varepsilon\}$. Using the divergence theorem, we calculate

$$0 \leq \int_{\partial B_r(x_0)} \mathbb{1}_{\overline{B_r^\varepsilon(x_0)}} |x_{m+1}|^\beta \nu \cdot \nabla |\partial_i v|^2 \, dS(x)$$

$$- \int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^\beta e_{m+1} \cdot \left(\nabla |\partial_i v|^2(x', \varepsilon) - \nabla |\partial_i v|^2(x', -\varepsilon)\right) dx', \tag{2.23}$$

$$0 \leq \int_{\partial B_r(x_0)} \mathbb{1}_{\overline{B_r^\varepsilon(x_0)}} |x_{m+1}|^{-\beta} \nu \cdot \nabla |v^*|^2 \, dS(x)$$

$$- \int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^{-\beta} e_{m+1} \cdot \left(\nabla |v^*|^2(x', \varepsilon) - \nabla |v^*|^2(x', -\varepsilon)\right) dx' \tag{2.24}$$

and, when $\beta \in (0, 1)$,

$$0 \leq \int_{\partial B_r(x_0)} \mathbb{1}_{\overline{B_r^\varepsilon(x_0)}} |x_{m+1}|^\beta \nu \cdot \nabla |\partial_{m+1} v|^2 \, dS(x)$$

$$- \int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^\beta e_{m+1} \cdot \left(\nabla |\partial_{m+1} v|^2 (x', \varepsilon) - \nabla |\partial_{m+1} v|^2 (x', -\varepsilon) \right) dx' \quad (2.25)$$

where $\mathbb{1}_{\overline{B^\varepsilon_r(x_0)}}$ is the indicator function of $\overline{B^\varepsilon_r(x_0)}$, ν is the outward unit normal on $\partial B_r(x_0)$, dS is the Lebesgue measure on $\partial B_r(x_0)$ and $B^m_s(x_0) = \{x \in \mathbb{R}^m : |x - x_0| < s\}$.

We consider the terms on the right hand side of (2.23)–(2.25) separately with a view to taking the limit as $\varepsilon \to 0^+$. Lemma 2.12 and Corollary 2.13 imply that $\partial_i v, \partial_j \partial_i v \in W^{1,2}_\beta(B_r(x_0); \mathbb{R}^n)$ and are locally Hölder continuous in $B_R(x_0)$ for $i, j = 1, \ldots, m$. The lemma and corollary further imply that $v^*, (\partial_i v)^* \in W^{1,2}_{-\beta}(B_r(x_0); \mathbb{R}^n)$, where $i = 1, \ldots, m$, are locally Hölder continuous in $B_R(x_0)$ and hence uniformly continuous in $\overline{B_r(x_0)}$. We can therefore check that $|\partial_i v|^2 \in W^{1,2}_\beta(B_r(x_0))$, integrating over $B^\varepsilon_r(x_0)$ and letting $\varepsilon \to 0^+$ for the $m + 1$th derivative. It follows that

$$||x_{m+1}|^\beta \nu \cdot \nabla |\partial_i v|^2| \leq 2 \left| \langle \nu_{m+1} (\partial_i v)^*, \partial_i v \rangle \right| + 2 \sum_{j=1}^m \left| |x_{m+1}|^\beta \langle \nu_j \partial_j \partial_i v, \partial_i v \rangle \right|$$

$$\leq C(1 + |x_{m+1}|^\beta), \quad (2.26)$$

where C is a positive constant that may depend on r but is independent of ε. Furthermore, since $\partial_i v$ and $(\partial_i v)^*$ are uniformly continuous in $\overline{B_r(x_0)}$, we see that

$$\mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} e_{m+1} \varepsilon^\beta \cdot (\nabla |\partial_i v|^2 (x', \varepsilon) - \nabla |\partial_i v|^2 (x', -\varepsilon)) \to 0 \text{ uniformly as } \varepsilon \to 0^+$$

$$(2.27)$$

for $(x', 0) \in \overline{B_r(x_0)}$.

Since each $\partial_j \partial_i v \in W^{1,2}_\beta(B_r(x_0); \mathbb{R}^n)$ is locally Hölder continuous in $B_R(x_0)$ for $i, j = 1, \ldots, m$ and $\text{div}(|x_{m+1}|^\beta \nabla v) = 0$ classically in $B_R(x_0) \backslash \partial \mathbb{R}^{m+1}_+$, we have $|x_{m+1}|^{-\beta} \partial_{m+1} v^* = -\Delta' v \in W^{1,2}_\beta(B_r(x_0); \mathbb{R}^n)$ is uniformly continuous in $\overline{B_r(x_0)}$ for every $r < R$. We can hence check $|v^*|^2 \in W^{1,2}_{-\beta}(B_r(x_0))$. We also have

$$||x_{m+1}|^{-\beta} \nu \cdot \nabla |v^*|^2| \leq 2 \left| |x_{m+1}|^{-\beta} \langle \nu_{m+1} \partial_{m+1} v^*, v^* \rangle \right| + 2 \sum_{i=1}^m \left| |x_{m+1}|^{-\beta} \langle \nu_i \partial_i v^*, v^* \rangle \right|$$

$$\leq C(1 + |x_{m+1}|^{-\beta}) \quad (2.28)$$

and

$$\mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} e_{m+1} \varepsilon^{-\beta} \cdot \left(\nabla |v^*|^2 (x', \varepsilon) - \nabla |v^*|^2 (x', -\varepsilon) \right) \to 0 \text{ uniformly as } \varepsilon \to 0^+$$

$$(2.29)$$

for $(x', 0) \in \overline{B_r(x_0)}$.

In order to derive similar conclusions to (2.26) and (2.27) for the constituent integrands of (2.25), we assume that $\beta \in (0, 1)$ and that v is *symmetric* with respect to $\partial \mathbb{R}^{m+1}_+$, namely $v(x', x_{m+1}) = v(x', -x_{m+1})$ for every $(x', x_{m+1}) \in B_R(x_0)$. The symmetry of v implies v^* must be odd with respect to $\partial \mathbb{R}^{m+1}_+$, that is $v^*(x', x_{m+1}) = -v^*(x', -x_{m+1})$ for every $(x', x_{m+1}) \in B_R(x_0)$ and hence, as it is also continuous in $B_R(x_0)$ we have $v^*(x', 0) = 0$ for every $(x', 0) \in B_R(x_0)$.

Fix $(x', 0) \in B_R(x_0)$ and note that $(x', 0) \in B_r(x_0)$ for some $r < R$ and choose h with $|h|$ sufficiently small as to ensure $(x', h) \in B_r(x_0)$. We see that

$$|h|^{-1} |v(x', h) - v(x', 0)| = |\partial_{m+1} v(x', x_{m+1})|$$

$$= |x_{m+1}|^{-\beta}|v^*(x', x_{m+1}) - 0|$$
$$= |x_{m+1}|^{-\beta}|\partial_{m+1}v^*(x', \xi)||x_{m+1}|$$
$$\leq |x_{m+1}|||\xi|^{-\beta}\partial_{m+1}v^*(x', \xi)|$$
$$\leq C|h| \to 0 \text{ as } h \to 0,$$

where x_{m+1} with $|x_{m+1}| \in (0, |h|)$ and ξ with $|\xi| \in (0, |x_{m+1}|)$ are chosen such that the Mean Value Theorem holds. Thus we see that $\partial_{m+1}v(x', 0) = 0$ classically for $(x', 0) \in B_R(x_0)$. Analogous calculations to those on the right hand side above show that $\partial_{m+1}v$ is continuous at $(x', 0)$ and hence continuous in $B_R(x_0)$. We also have

$$|x_{m+1}|^\beta \partial_{m+1}^2 v = \partial_{m+1}v^* - \beta x_{m+1}^{-1}v^*$$

for $(x', x_{m+1}) \in \overline{B_r(x_0)} \setminus \partial\mathbb{R}_+^{m+1}$ and hence

$$||x_{m+1}|^\beta \partial_{m+1}^2 v(x', x_{m+1})| \leq |\partial_{m+1}v^*| + \beta|x_{m+1}^{-1}v^*|$$
$$\leq C + \beta|\partial_{m+1}v^*(x', \xi)|$$
$$\leq C,$$

in $\overline{B_r(x_0)} \setminus \partial\mathbb{R}_+^{m+1}$, where ξ is chosen with $|\xi| \in (0, |x_{m+1}|)$ such that the Mean Value Theorem holds. It follows that $|x_{m+1}|^\beta \partial_{m+1}^2 v$ is essentially bounded in $\overline{B_r(x_0)}$ and $\partial B_r(x_0)$.

The preceding discussion implies $|\partial_{m+1}v|^2 \in W_\beta^{1,2}(B_r(x_0))$ and

$$||x_{m+1}|^\beta v \cdot \nabla|\partial_{m+1}v|^2| \leq C \tag{2.30}$$

on $\overline{B_r(x_0)} \setminus \partial\mathbb{R}_+^{m+1}$. Furthermore, using the symmetry of v, we see that

$$\mathbb{1}_{B_{\sqrt{r^2-\varepsilon^2}}^m(x_0)} e_{m+1} \varepsilon^\beta \cdot (\nabla|\partial_{m+1}v|^2(x', \varepsilon) - \nabla|\partial_{m+1}v|^2(x', -\varepsilon)) \to 0 \text{ uniformly as } \varepsilon \to 0^+ \tag{2.31}$$

for $(x', 0) \in \overline{B_r(x_0)}$.

Using Lebesgue's Dominated Convergence Theorem, we combine (2.23) with (2.26) and (2.27), (2.24) with (2.28) and (2.29) and (2.25) with (2.30) and (2.31) to see that

$$0 \leq \int_{\partial B_r(x_0)} |x_{m+1}|^\beta v \cdot \nabla|\partial_i v|^2 \mathrm{d}S(x), \quad 0 \leq \int_{\partial B_r(x_0)} |x_{m+1}|^{-\beta} v \cdot \nabla|v^*|^2 \mathrm{d}S(x)$$

and, when $\beta \in (0, 1)$ and v is symmetric with respect to $\partial\mathbb{R}_+^{m+1}$,

$$0 \leq \int_{\partial B_r(x_0)} |x_{m+1}|^\beta v \cdot \nabla|\partial_{m+1}v|^2 \mathrm{d}S(x)$$

respectively. Noting that $|x_{m+1}|^{-\beta}|v^*|^2 = |x_{m+1}|^\beta|\partial_{m+1}v|^2$, we apply Lemma 2.14, to see that

$$s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\nabla'v|^2 \mathrm{d}x \leq r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\nabla'v|^2 \mathrm{d}x, \tag{2.32}$$

$$s^{-(1+m-\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\partial_{m+1}v|^2 \mathrm{d}x \leq r^{-(1+m-\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\partial_{m+1}v|^2 \mathrm{d}x \tag{2.33}$$

and, when $\beta \in (0, 1)$ and v is symmetric with respect to $\partial\mathbb{R}_+^{m+1}$,

$$s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\partial_{m+1}v|^2 \mathrm{d}x \leq r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\partial_{m+1}v|^2 \mathrm{d}x \tag{2.34}$$

for every $0 < s \leq r < R$. We apply Lebesgue's Dominated Convergence Theorem, sending $r \to R^-$ and combine (2.32), (2.33) and (2.34) to see the Lemma holds for $0 < s \leq r \leq R$. $\square$

Remark 2.15 We observe from (2.33) and (2.34) that without the symmetry condition on v, the monotonicity of the average energy of $|\partial_{m+1}v|^2$ is better than required if $\beta \in (-1, 0]$ and may be worse than required if $\beta \in (0, 1)$; the symmetry of v resolves this issue by implying continuity of $\partial_{m+1}v$ on $\partial\mathbb{R}^{m+1}_+$ which, as the function given in Remark 2.9 illustrates, cannot be expected in general.

2.5 Interior monotonicity formula

We need a counterpart to Lemma 2.8 for balls in the interior of $\mathbb{R}^{m+1}_+$. Since the weight x^β_{m+1} is not scale invariant on a (Euclidean) ball $B_\rho(y)$ with $y_{m+1} \geq 2\rho$, we only expect monotonicity of the average energy (with respect to the Lebesgue measure) of solutions to $\text{div}(x^\beta_{m+1}\nabla v) = 0$ up to a correcting factor. As the radius tends to zero, the coefficients of uniform ellipticity for the preceding equation tend to constants. Accordingly, the correcting factor becomes smaller with the radius. General monotonicity-type formulas for linear uniformly elliptic equations are available in [14] for example. We perform the following calculations in order to determine how the correcting factor behaves explicitly as the radius decays geometrically. We first establish monotonicity on the boundary of concentric balls.

Lemma 2.16 *Let $\beta \in (-1, 1)$, $B_R(y) \subset \mathbb{R}^{m+1}_+$ with $\overline{B_R(y)} \subset \mathbb{R}^{m+1}_+$ and suppose $v \in C^2(B_R(y); \mathbb{R}^n)$ satisfies $\text{div}(x^\beta_{m+1}\nabla|v|^2) \geq 0$ classically in $B_R(y)$. Then, letting $\mathrm{d}S$ denote Lebesgue surface measure and sgn denote the sign function, for $0 < s \leq r < R$ we have*

$$s^{-m}\frac{1}{(y_{m+1} - sgn(\beta)s)^\beta}\int_{\partial B_s(y)} x^\beta_{m+1}|v|^2\mathrm{d}S(x)$$

$$\leq r^{-m}\frac{1}{(y_{m+1} - sgn(\beta)r)^\beta}\int_{\partial B_r(y)} x^\beta_{m+1}|v|^2\mathrm{d}S(x).$$

Proof Let $\rho < R$. We calculate $0 \leq \int_{\partial B_\rho(y)} x^\beta_{m+1}\nu \cdot \nabla|v|^2\mathrm{d}S(x)$, where ν is the unit outward normal on $\partial B_\rho(y)$. Now using variables $\rho = |x - y|$ and $\omega = \frac{x-y}{\rho}$ we have

$$0 \leq \rho^m \int_{\mathbb{S}^m} (\rho\omega_{m+1} + y_{m+1})^\beta \frac{\partial}{\partial\rho}(|v(\rho\omega + y)|^2)\mathrm{d}\omega$$

$$= \rho^m \int_{\mathbb{S}^m} \frac{\partial}{\partial\rho}\left((\rho\omega_{m+1} + y_{m+1})^\beta|v(\rho\omega + y)|^2\right)\mathrm{d}\omega$$

$$- \rho^m \int_{\mathbb{S}^m} \beta\omega_{m+1}(\rho\omega_{m+1} + y_{m+1})^{\beta-1}|v(\rho\omega + y)|^2\mathrm{d}\omega$$

$$= \rho^m \frac{\partial}{\partial\rho}\left(\rho^{-m}\int_{\partial B_\rho(y)} x^\beta_{m+1}|v|^2\mathrm{d}S(x)\right) - \int_{\partial B_\rho(y)} \frac{\beta}{\rho}x^{\beta-1}_{m+1}(x_{m+1} - y_{m+1})|v|^2\mathrm{d}S(x).$$

We define $f(\rho) = \rho^{-m}\int_{\partial B_\rho(y)} x^\beta_{m+1}|v|^2\mathrm{d}S(x)$ and divide by ρ^m to see that

$$0 \leq f'(\rho) - \frac{\beta}{\rho}f(\rho) + \frac{y_{m+1}\beta}{\rho^{m+1}}\int_{\partial B_\rho(y)} x^{\beta-1}_{m+1}|v|^2\mathrm{d}S(x)$$

$$\leq f'(\rho) - \frac{\beta}{\rho}f(\rho) + \frac{y_{m+1}\beta}{\rho(y_{m+1} - sgn(\beta)\rho)}f(\rho)$$

$$= f'(\rho) + \left(\frac{|\beta|}{y_{m+1} - \text{sgn}(\beta)\rho} \right) f(\rho).$$

Hence $0 \leq \left((y_{m+1} - \text{sgn}(\beta)\rho)^{-\beta} f(\rho) \right)'$ and integrating between $s < r$ concludes the proof.
□

With this lemma in hand we can establish the following counterpart to Lemma 2.8.

Lemma 2.17 *Let* $B_R(y) \subset \mathbb{R}^{m+1}$ *with* $y_{m+1} \geq \theta R$ *for* $\theta \geq 2$. *Suppose* $v \in C^2(B_R(y); \mathbb{R}^n) \cap W^{1,2}_\beta(B_R(y); \mathbb{R}^n)$ *satisfies* $\text{div}(x^\beta_{m+1} \nabla v) = 0$ *in* $B_R(y)$. *Then there exists* $C = C(m)$ *such that*

$$\left(\frac{R}{2} \right)^{-(m+1)} \int_{B_{\frac{R}{2}}(y)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \leq \left(1 + \frac{C}{\theta - 1} \right) R^{-(m+1)} \int_{B_R(y)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x.$$

Proof Let $g(r) := \int_{B_r(y)} x^\beta_{m+1} |f|^2 \mathrm{d}x$ where f satisfies the assumptions of Lemma 2.16 and $r \in (0, R)$. Then we have

$$g(r) = \int_0^r \int_{\partial B_\rho(y)} x^\beta_{m+1} |f|^2 \mathrm{d}S(x) \mathrm{d}\rho$$

$$= \int_0^r (y_{m+1} - \text{sgn}(\beta)\rho)^\beta \rho^m (y_{m+1} - \text{sgn}(\beta)\rho)^{-\beta} \rho^{-m} \int_{\partial B_\rho(y)} x^\beta_{m+1} |f|^2 \mathrm{d}S(x) \mathrm{d}\rho$$

$$\leq \int_0^r (y_{m+1} - \text{sgn}(\beta)\rho)^\beta \rho^m \mathrm{d}\rho (y_{m+1} - \text{sgn}(\beta)r)^{-\beta} r^{-m} \int_{\partial B_r(y)} x^\beta_{m+1} |f|^2 \mathrm{d}S(x)$$

$$\leq y^\beta_{m+1} \frac{r}{m+1} (y_{m+1} - \text{sgn}(\beta)r)^{-\beta} \int_{\partial B_r(y)} x^\beta_{m+1} |f|^2 \mathrm{d}S(x). \tag{2.35}$$

Hence

$$0 \leq g'(r) - g(r) y^{-\beta}_{m+1} \frac{m+1}{r} (y_{m+1} - \text{sgn}(\beta)r)^\beta$$

$$= g'(r) - g(r) \frac{m+1}{r} \left(1 - \text{sgn}(\beta) \frac{r}{y_{m+1}} \right)^\beta$$

$$= g'(r) - g(r) \frac{m+1}{r} \left(1 + \beta (1 - \text{sgn}(\beta) \frac{s}{y_{m+1}})^{\beta-1} \text{sgn}(\beta) \frac{-r}{y_{m+1}} \right)$$

$$= g'(r) - g(r) \frac{m+1}{r} + g(r) \frac{m+1}{r} |\beta| \left(1 - \text{sgn}(\beta) \frac{s}{y_{m+1}} \right)^{\beta-1} \frac{r}{y_{m+1}},$$

where $s \in (0, r)$ is such that the Mean Value Theorem holds for the function $r \mapsto (1 - \text{sgn}(\beta) \frac{r}{y_{m+1}})^\beta$. Now recall that $y_{m+1} \geq \theta R \geq \theta r$ for $\theta \geq 2$. We hence find

$$0 \leq g'(r) - g(r) \frac{m+1}{r} + g(r) \frac{m+1}{r} |\beta| \frac{1}{\theta - 1}.$$

It follows that $0 \leq (r^{-(m+1)} r^{\frac{|\beta|(m+1)}{\theta-1}} g(r))'$ and consequently, if $\frac{R}{2} \leq r < R$, we have

$$\left(\frac{R}{2} \right)^{-(m+1)} g \left(\frac{R}{2} \right) \leq 2^{\frac{|\beta|(m+1)}{\theta-1}} r^{-(m+1)} g(r)$$

$$= \left(1 + \frac{|\beta|(m+1)}{\theta - 1} \xi^{\frac{|\beta|(m+1)}{\theta-1} - 1} \right) r^{-(m+1)} g(r)$$

$$\leq \left(1 + \frac{|\beta|(m+1)}{\theta - 1} 2^{|\beta|(m+1)}\right) r^{-(m+1)} g(r) \qquad (2.36)$$

where $\xi \in (1, 2)$ is such that the Mean Value theorem holds for the function $t \mapsto t^{\frac{|\beta|(m+1)}{\theta - 1}}$. Since $\mathrm{div}(x_{m+1}^\beta \nabla v) = 0$, as observed in the proof of Lemma 2.8, we have $\mathrm{div}(x_{m+1}^\beta \nabla |\partial_i v|^2) \geq 0$ for $i = 1, \ldots, m$ and $\mathrm{div}(x_{m+1}^{-\beta} \nabla |x_{m+1}^\beta \partial_{m+1} v|^2) \geq 0$. We apply (2.36) with $f = \partial_i v$ for $i = 1, \ldots, m$, and with $f = x_{m+1}^\beta \partial_{m+1} v$ and $-\beta$ in place of β and combine the results, letting $r \to R^-$ to conclude the proof. $\qquad\square$

2.6 Solutions of the linear degenerate Dirichlet problem

We require further results regarding the following Dirichlet problem: solve

$$\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0 \text{ in } B_R(x_0) \qquad \text{and} \qquad v = \phi \text{ on } \partial B_R(x_0) \qquad (2.37)$$

for a given ϕ, where $x_0 \in \mathbb{R}^{m+1}$, in order to apply a version of the method of harmonic replacement in the proof of Lemma 4.23. A weak solution of (2.37) is a $v \in W_\beta^{1,2}(B_R(x_0); \mathbb{R}^n)$ which weakly satisfies $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_R(x_0)$ with $v - \phi \in W_{\beta,0}^{1,2}(B_R(x_0); \mathbb{R}^n)$. We collect the results we require, which can be found in [19], in the form of a lemma.

Lemma 2.18 *Suppose $\phi \in W_\beta^{1,2}(B_R(x_0); \mathbb{R}^n)$. Then there exists a $v \in W_\beta^{1,2}(B_R(x_0); \mathbb{R}^n)$ which is a weak solution of the Dirichlet problem* (2.37). *Any such solution is unique and continuous in $B_R(x_0)$, if $\phi \in C(\overline{B_R(x_0)}; \mathbb{R}^n)$ then $v(x) \to \phi(z)$ as $x \to z$ for $z \in \partial B_R(x_0)$ and the weak maximum principle*

$$\max_{\overline{B_R(x_0)}} v = \max_{\partial B_R(x_0)} v = \max_{\partial B_R(x_0)} \phi$$

and weak minimum principle

$$\min_{\overline{B_R(x_0)}} v = \min_{\partial B_R(x_0)} v = \min_{\partial B_R(x_0)} \phi$$

both hold, where we take the maximum and minimum component-wise. If $w \in W_\beta^{1,2}(B_R(x_0); \mathbb{R}^n)$ also satisfies $w - \phi \in W_{\beta,0}^{1,2}(B_R(x_0); \mathbb{R}^n)$ then

$$\int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x \leq \int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla w|^2 \mathrm{d}x.$$

Proof Since $|x_{m+1}|^\beta$ is of Muckenhoupt class A_2 it follows from 1.6 of [19] that $|x_{m+1}|^\beta$ is a 2-admissible weight so we may apply the theory of [19]. Aside from the minimising property of v, the assertions of the lemma are consequences of Theorem 3.70, Corollary 6.32, the strong maximum principle 6.5 and lastly 3.17 in [19]. If v is a weak solution of (2.37) for a given ϕ and $w \in W_\beta^{1,2}(B_R(x_0); \mathbb{R}^n)$ with $w - \phi \in W_{\beta,0}^{1,2}(B_R(x_0); \mathbb{R}^n)$ then $w - v \in W_{\beta,0}^{1,2}(B_R(x_0); \mathbb{R}^n)$. Hence, by approximation, we have

$$\int_{B_R(x_0)} |x_{m+1}|^\beta \langle \nabla v, \nabla(w - v) \rangle \mathrm{d}x = 0,$$

so that

$$\int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla w|^2 \mathrm{d}x = \int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x + \int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla(w - v)|^2 \mathrm{d}x$$

which concludes the proof. $\qquad\square$

The uniqueness of solutions to the Dirichlet problem (2.37) implies that solutions with boundary data which are symmetric with respect to $\partial\mathbb{R}^{m+1}_+$ are themselves symmetric. More precisely, we have the following.

Lemma 2.19 *Let $x_0 \in \partial\mathbb{R}^{m+1}_+$ and suppose $v, \phi \in W^{1,2}_\beta(B_R(x_0); \mathbb{R}^n)$ and v is a weak solution of the Dirichlet problem (2.37) with ϕ as boundary data. Let $\phi \in C(\overline{B_R(x_0)}; \mathbb{R}^n)$ and suppose $\phi(x', x_{m+1}) = \phi(x', -x_{m+1})$ for every $(x', x_{m+1}) \in \overline{B_R(x_0)}$. Then $v(x', x_{m+1}) = v(x', -x_{m+1})$ for every $(x', x_{m+1}) \in B_R(x_0)$.*

Proof The continuity of ϕ in $\overline{B_R(x_0)}$, combined with an application of Lemma 2.18, implies that v and, consequently, $\tilde{v}(x', x_{m+1}) := v(x', -x_{m+1})$ are continuous in $\overline{B_R(x_0)}$. We observe that $\tilde{v} \in W^{1,2}_\beta(B_R(x_0); \mathbb{R}^n)$ weakly satisfies $\mathrm{div}(|x_{m+1}|^\beta \nabla \tilde{v}) = 0$ in $B_R(x_0)$ and $\tilde{v}|_{\partial B_R(x_0)} = \phi|_{\partial B_R(x_0)}$ so that $\tilde{v} - \phi \in W^{1,2}_{\beta,0}(B_R(x_0); \mathbb{R}^n)$. Hence v and $\tilde{v}$ solve the same Dirichlet problem; solutions to this problem are unique by Lemma 2.18 and thus $\tilde{v} = v$. □

3 Intrinsic fractional harmonic maps

We assume, translating N if necessary, that $0 \in N$. For reasons discussed in the introduction we always assume $m \geq 2$; when $m \geq 3$ we let $\beta \in (-1, 1)$ and when $m = 2$ we let $\beta \in (-3^{-1}, 1)$. This allows us to apply Lemma 2.3 and we can define

$$\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N) = \left\{ v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n): v(x) \in N \text{ for almost every } x \in \mathbb{R}^{m+1}_+ \right\}.$$

Henceforth, in this section and Sect. 4, we fix an open $\mathcal{O} \subset \partial\mathbb{R}^{m+1}_+$ such that a continuous linear trace operator with respect to $\mathcal{O}$ exists; we do not assume $\mathcal{O}$ is necessarily the whole of $\partial\mathbb{R}^{m+1}_+$. We can, for example, obtain such a trace operator $T: \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n) \to L^p(\mathcal{O}; \mathbb{R}^n)$, where $p = p(\beta) \in (1, 2]$, by combining Lemmata 2.1 and 2.3 with [12, Section 4.3, Theorem 1] whenever $\mathcal{O}$ is contained in the boundary of a Lipschitz $\Omega \subset \mathbb{R}^{m+1}_+$. Define

$$I^\beta(u) = \inf \left\{ E^\beta(v): v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N), Tv = u \right\}$$

for $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N))$. Recall from the introduction that I^β serves as an intrinsic energy for u; it does not depend on the choice of embedding of N into Euclidean space. Moreover, I^β coincides with the square of the fractional Sobolev norm $||u||_{\dot{H}^{\frac{1-\beta}{2}}(\partial\mathbb{R}^{m+1}_+; \mathbb{R}^n)}$ when $N = \mathbb{R}^n$ and $\mathcal{O} = \partial\mathbb{R}^{m+1}_+$.

For every $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N))$, an application of the direct method of the calculus of variations shows that there exists $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$, where T is the trace operator with respect to $\mathcal{O}$, such that $I^\beta(u) = E^\beta(v)$. For a given u, such a v is referred to henceforth as a *minimal harmonic map*. Any minimal harmonic map v is weakly harmonic in $\mathbb{R}^{m+1}_+$ with respect to the metric represented in Euclidean coordinates by $x^\alpha_{m+1}\delta_{ij}$, where $\beta = \frac{\alpha(m-1)}{2}$; the Dirichlet energy on $\mathbb{R}^{m+1}_+$ for this metric is precisely E^β. Such a v therefore satisfies

$$\int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} \left(\langle \psi, A(v)(\nabla v, \nabla v) \rangle - \langle \nabla v, \nabla \psi \rangle \right) \mathrm{d}x = 0 \tag{3.1}$$

for every $\phi \in C_0^\infty(\mathbb{R}_+^{m+1}; \mathbb{R}^n)$, where A is the second fundamental form of N and

$$\langle \nabla v, \nabla \psi \rangle = \sum_{i=1}^{m+1} \langle \partial_i v, \partial_i \psi \rangle \quad \text{and} \quad A(v)(\nabla v, \nabla v) = \sum_{i=1}^{m+1} A(v)(\partial_i v, \partial_i v).$$

Formally, if v is sufficiently regular in $\mathbb{R}_+^{m+1} \cup \mathcal{O}$, we calculate

$$\int_{\mathbb{R}_+^{m+1}} x_{m+1}^\beta \left(\langle \psi, A(v)(\nabla v, \nabla v) \rangle - \langle \nabla v, \nabla \psi \rangle \right) \mathrm{d}x = \int_{\mathcal{O}} \left\langle \left(x_{m+1}^\beta \partial_{m+1} v \right)(x', 0), \phi(x') \right\rangle \mathrm{d}x',$$

for every $\phi \in C_0^\infty(\mathcal{O}; \mathbb{R}^n)$ and any $\psi \in \mathcal{D}_+(\mathbb{R}_+^{m+1}; \mathbb{R}^n)$ with $\psi(x', 0) = \phi(x')$, where $\mathrm{d}x'$ is the Lebesgue measure on $\mathbb{R}^m$. In general, the integral in (3.1) defines a distribution on $\mathcal{O}$ given by

$$\partial_{m+1}^\beta v(\phi) := \int_{\mathbb{R}_+^{m+1}} x_{m+1}^\beta \left(\langle \psi, A(v)(\nabla v, \nabla v) \rangle - \langle \nabla v, \nabla \psi \rangle \right) \mathrm{d}x$$

for $\phi \in C_0^\infty(\mathcal{O}; \mathbb{R}^n)$. This observation allows us, analogously to [30, Proposition 1.1], to identify a superdifferential for I^β. Recall that since N is compact, Theorem 1 in Section 2.12.3 of [39] gives a tubular neighbourhood of N, which has the form $U_\delta(N) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, N) < \delta\}$ for a $\delta = \delta(N) > 0$, and a smooth map $\pi_N : U_\delta(N) \to N$ such that $|\pi_N(y) - y| = \mathrm{dist}(y, N)$ for every $y \in U_\delta(N)$. Using the same method of proof as [30, Proposition 1.1] we deduce the following.

Lemma 3.1 *Let* $u \in T(\dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N))$ *and* $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ *be a minimal harmonic map with* $Tv = u$. *Then for* $\phi \in C_0^\infty(\mathcal{O}; \mathbb{R}^n)$,

$$I^\beta(\pi_N(u + t\phi)) \leq I^\beta(u) - t\partial_{m+1}^\beta v(\phi) + o(|t|) \tag{3.2}$$

as $t \to 0$.

It follows from this proposition that if $\frac{\partial}{\partial t}\big|_{t=0} I^\beta(\pi_N(u + t\phi))$ exists then it is equal to $-\partial_{m+1}^\beta v(\phi)$ where v is any minimal harmonic map with $Tv = u$; this indicates a candidate for the first variation of I^β.

Definition 3.2 Let $\beta \in (-1, 1)$ and $\mathcal{D}_\beta$ be the collection of all $u \in T(\dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N))$ such that there exists a distribution $\lambda_\beta \in \left(C_0^\infty(\mathcal{O}; \mathbb{R}^n) \right)^*$ with $\lambda_\beta = -\partial_{m+1}^\beta v$ for every minimal harmonic map $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ with $Tv = u$. Then we may define a map $\Lambda_\beta : \mathcal{D}_\beta \to \left(C_0^\infty(\mathcal{O}; \mathbb{R}^n) \right)^* : u \mapsto \lambda_\beta = \Lambda_\beta u$.

In [30, Theorem 1.1] Moser showed that Λ_0 is the first variation of I^0. The method of proof of Moser's theorem, applied with the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}^{m+1}$ replaced by $x_{m+1}^\beta \mathrm{d}x$, yields the following.

Lemma 3.3 *If* $u \in \mathcal{D}_\beta$, *then*

$$\frac{\partial}{\partial t}\bigg|_{t=0} I^\beta(\pi_N(u + t\phi)) = \Lambda_\beta u(\phi)$$

for all $\phi \in C_0^\infty(\mathcal{O}; \mathbb{R}^n)$. *If* $u \notin \mathcal{D}_\beta$, *then there exists* $\phi \in C_0^\infty(\mathcal{O}; \mathbb{R}^n)$ *such that the function* $t \mapsto I^\beta(\pi_N(u + t\phi))$ *is not differentiable at 0.*

Consequently, we may define intrinsic fractional harmonic maps as follows.

Definition 3.4 Let $\beta \in (-1, 1)$ and $u \in T(\dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N))$. If $\Lambda_\beta u = 0$ then we say that u is an intrinsic $\frac{1-\beta}{2}$-harmonic map.

As discussed in the introduction, intrinsic $\frac{1-\beta}{2}$-harmonic maps are the boundary values of free boundary harmonic maps from $\mathbb{R}_+^{m+1}$ to N. Such maps may have singularities in $\mathcal{O}$ and we may not expect regularity in general. We consider a smaller class of fractional harmonic maps which locally minimise I^β in order to obtain partial regularity in $\mathcal{O}$.

Definition 3.5 We say that $u \in T(\dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N))$ is an intrinsic locally minimising $\frac{1-\beta}{2}$-harmonic map, or a local minimiser of I^β, if for every compact $K \subset \mathcal{O}$ and every $\tilde{u} \in T(\dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N))$ with $u|_{\mathcal{O}\setminus K} = \tilde{u}|_{\mathcal{O}\setminus K}$ we have $I^\beta(u) \leq I^\beta(\tilde{u})$.

This definition allows us to consider interior regularity for a class of critical points of I^β without explicitly specifying boundary conditions. For example, minimisers of I^β with respect to Dirichlet or free boundary conditions satisfy the definition. In order to deduce regularity results for u, we analyse their minimal harmonic extensions in more detail. To this end, we make the following definition.

Definition 3.6 Let $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$. We say that v is E^β minimising, or energy minimising, in $\mathbb{R}_+^{m+1}$ relative to $\mathcal{O} \subset \partial\mathbb{R}_+^{m+1}$, if for every compact $K \subset \mathbb{R}^{m+1}$ with $K \cap \partial\mathbb{R}_+^{m+1} \subset \mathcal{O}$ and for every $w \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ with $v|_{\mathbb{R}_+^{m+1}\setminus K} = w|_{\mathbb{R}_+^{m+1}\setminus K}$ we have $E^\beta(v) \leq E^\beta(w)$.

Local minimisers of I^β and minimisers of E^β relative to $\mathcal{O}$ are connected as follows.

Lemma 3.7 *Suppose $u \in T(\dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N))$ locally minimises I^β in the sense of Definition* 4.11 *and fix a minimal harmonic map $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ with $Tv = u$. Then v is a minimiser of E^β relative to $\mathcal{O}$.*

Proof Let $K \subset \mathbb{R}^{m+1}$ be compact such that the compact set $K_m := K \cap \partial\mathbb{R}_+^{m+1} \subset \mathcal{O}$ and suppose that $w \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ satisfies $v|_{\mathbb{R}_+^{m+1}\setminus K} = w|_{\mathbb{R}_+^{m+1}\setminus K}$. Define $\tilde{u} = Tw$ and let $\tilde{v}$ be a minimal harmonic map with $T\tilde{v} = \tilde{u}$. Since $\mathcal{O}$ is open in $\partial\mathbb{R}_+^{m+1}$ and $K_m \subset \mathcal{O}$ is compact we have $\text{dist}^m(K_m; \partial\mathcal{O}) > 0$, where dist^m is the distance in $\partial\mathbb{R}_+^{m+1}$. We can therefore choose an open set $\tilde{\mathcal{O}} \subset \mathcal{O}$ with $K_m \subset \tilde{\mathcal{O}} \subset \overline{\tilde{\mathcal{O}}} \subset \mathcal{O}$. Since K_m is closed and $\tilde{\mathcal{O}}$ is open we have $\text{dist}^m(K_m; \partial\tilde{\mathcal{O}}) > 0$ as well. It follows that $\text{dist}(\mathcal{O}\setminus\overline{\tilde{\mathcal{O}}}; K) := \kappa > 0$, where dist is the Euclidean distance in $\overline{\mathbb{R}_+^{m+1}}$. The continuity of the trace operator yields

$$\int_{\mathcal{O}\setminus\tilde{\mathcal{O}}} |u - \tilde{u}|^p \mathrm{d}x = \int_{\mathcal{O}\setminus\tilde{\mathcal{O}}} |T(v - w)|^p \mathrm{d}x \leq C||v - w||_{W^{1,p}((\mathcal{O}\setminus\tilde{\mathcal{O}})\times(0,\kappa);\mathbb{R}^n)}^p = 0,$$

since $v = w$ in $\mathcal{O}\setminus\overline{\tilde{\mathcal{O}}} \times (0, \kappa)$, where p is the number from Lemma 2.1 depending on β. Since v and $\tilde{v}$ are minimal harmonic maps and u is a minimiser of I^β, we have

$$E^\beta(v) = I^\beta(u) \leq I^\beta(\tilde{u}) = E^\beta(\tilde{v}) \leq E^\beta(w)$$

as required. □

As a consequence of the preceding lemma, we can consider the regularity of minimisers of E^β relative to $\mathcal{O}$ on relatively open balls (in the Euclidean topology) centred on $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$ in order to prove regularity of fractional harmonic maps. Our main results, stated and proved in Sect. 4, constitute an ε-regularity and a corresponding partial regularity theorem for minimisers of E^β relative to $\mathcal{O}$ which translate into the following partial regularity result for fractional harmonic maps.

Theorem 3.8 *Fix $\mathcal{O} \subset \partial\mathbb{R}^{m+1}_+$. When $m \geq 3$ let $\beta \in (-1, 1)$ and when $m = 2$ let $\beta \in (-3^{-1}, 1)$. Suppose $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N))$ is a local minimiser of I^β. Then there exists a relatively closed set $\Sigma \subset \mathcal{O}$ with $\mathcal{H}^{m+\beta-1}(\Sigma) = 0$ such that $u \in C^\infty(\mathcal{O}\backslash\Sigma; N)$.*

Proof Fix a minimal harmonic map $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$. As u is a minimiser of I^β, Lemma 3.7 implies v is a minimiser of E^β relative to $\mathcal{O}$. An application of Theorem 4.3 implies the result.   □

4 Partial regularity of minimisers of E^β relative to $\mathcal{O}$

The main result of this section is the following ε-regularity theorem for minimisers of E^β relative to $\mathcal{O}$. To state the theorem and subsequent results we will need the following notation. For a set $\Omega \subset \mathbb{R}^{m+1}_+$ we will sometimes split the boundary $\partial\Omega$ into the (possibly empty) sets $\partial^+\Omega = \partial\Omega \cap \mathbb{R}^{m+1}_+$ and $\partial^0\Omega = \partial\Omega \cap \partial\mathbb{R}^{m+1}_+$. Let $x_0 \in \partial\mathbb{R}^{m+1}_+$ and recall the notation $B^+_R(x_0) = \{x \in \mathbb{R}^{m+1}_+ : |x - x_0| < R\}$.

Theorem 4.1 *If $m \geq 3$, let $\beta \in (-1, 1)$ and if $m = 2$ let $\beta \in (-3^{-1}, 1)$. Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ be a minimiser of E^β relative to $\mathcal{O}$, let $x_0 \in \partial\mathbb{R}^{m+1}_+$ and $B^+_R(x_0)$ be a half-ball with $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$. There exists $\varepsilon = \varepsilon(m, N, \beta) > 0$ such that the following holds. If $R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \leq \varepsilon$ then there is a $\theta = \theta(m, N, \beta) \in (0, 1)$ and a $\gamma = \gamma(m, N, \beta) \in (0, 1)$ such that $v \in C^{0,\gamma}(\overline{B^+_{\theta R}(x_0)}; N)$. Furthermore, for every $l \in \mathbb{N}$ there is a $\theta = \theta(m, N, \beta, l) \in (0, 1)$ and a $\gamma = \gamma(m, N, \beta, l) \in (0, 1)$ such that $D^{\alpha'}v \in C^{0,\gamma}(\overline{B^+_{\theta R}(x_0)}; \mathbb{R}^n)$ for every $\alpha' \in \mathbb{N}^{m+1}_0$ with $|\alpha'| \leq l$ and $\alpha'_{m+1} = 0$.*

Remark 4.2 Henceforth, we assume the conditions on m and β from Theorem 4.1. We have restricted to considering α' with $\alpha'_{m+1} = 0$ as (partial) regularity of these derivatives up to the boundary will yield the desired regularity for fractional harmonic maps stated in Theorem 3.8. The main purpose of the theorem is to provide regularity estimates which are uniform up to $\mathcal{O}$; such estimates do not follow from known theory.

Theorem 4.1, combined with the partial regularity theory for harmonic maps yields the following. We use the notation $\mathcal{H}^t$ to denote the t-dimensional Hausdorff measure, with respect to the Euclidean metric on $\mathbb{R}^{m+1}$, for $t \geq 0$.

Theorem 4.3 *Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ be a minimiser of E^β relative to $\mathcal{O}$. There exists sets $\Sigma_{int} \subset \mathbb{R}^{m+1}_+$ and $\Sigma_{bdry} \subset \mathcal{O}$ such that the following holds. The set Σ_{int} is relatively closed in $\mathbb{R}^{m+1}_+$ and has Hausdorff dimension at most $m - 2$. The set Σ_{bdry} is relatively closed in $\mathcal{O}$ and $\mathcal{H}^{m+\beta-1}(\Sigma_{bdry}) = 0$. The set $\Sigma := \Sigma_{int} \cup \Sigma_{bdry}$ is relatively closed in $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$ and $\mathcal{H}^{m+\beta-1}(\Sigma) = 0$. Furthermore, we have $v \in C^\infty(\mathbb{R}^{m+1}_+\backslash\Sigma_{int}; N)$, $v \in C^{0,1}_{loc}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; N)$ and for every multi-index $\alpha' \in \mathbb{N}^{m+1}$ with $\alpha'_{m+1} = 0$ we have*

$D^{\alpha'}v \in C^{0,1}_{loc}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\setminus\Sigma; \mathbb{R}^n)$ and $\nabla D^{\alpha'}v \in L^\infty_{loc}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\setminus\Sigma; \mathbb{R}^{(m+1)n})$. Finally, for every $\alpha' \in \mathbb{N}^{m+1}_0$ with $\alpha'_{m+1} = 0$, we have $x^\beta_{m+1}\partial_{m+1}D^{\alpha'}v \in C^{0,\gamma}_{loc}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\setminus\Sigma; \mathbb{R}^{(m+1)n})$ for some $\gamma = \gamma(m, N, \beta, \alpha') \in (0, 1)$.

Remark 4.4 The existence and properties of Σ_{int} follows from the theory of Schoen and Uhlenbeck [37].

4.1 Euler–Lagrange and stationary equations for minimisers

Minimisers of E^β relative to $\mathcal{O}$ are critical points of E^β with respect to outer and inner variations, including those which vary their boundary data in $\mathcal{O}$. As a consequence, they satisfy two systems of partial differential equations which we describe presently.

Let $\psi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$, as defined in Sect. 2.1, with $\psi(\cdot, 0) \in C^\infty_0(\mathcal{O}; \mathbb{R}^n)$. For sufficiently small t define an outer variation of v by $v_t = \pi_N(v + t\psi) \in N$, where π_N is the nearest point projection onto N. Critical points of E^β with respect to variations of the form v_t satisfy

$$\int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1}\left(\langle\nabla v, \nabla\psi\rangle - \langle\psi, A(v)(\nabla v, \nabla v)\rangle\right) dx = 0. \tag{4.1}$$

Note that there is a Neumann-type boundary condition implicit in (4.1). In particular, if v is sufficiently smooth in $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$ we have

$$x^\beta_{m+1}\partial_{m+1}v = 0 \text{ in } \mathcal{O}. \tag{4.2}$$

A (weakly) harmonic map satisfying (4.1) is said to be (*weakly*) *harmonic with respect to the Neumann type boundary condition* (4.2).

Define $\Phi_t(x) = x + t\phi(x)$ for $x \in \overline{\mathbb{R}^{m+1}_+}$, where $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^{m+1})$ is such that $\phi(\cdot, 0) \in C^\infty_0(\mathcal{O}; \partial\mathbb{R}^{m+1}_+)$ and $|t|$ is small enough to make Φ_t into a diffeomorphism of $\overline{\mathbb{R}^{m+1}_+}$ with $\Phi_t(\mathcal{O}) \subset \mathcal{O}$. We say $v \in \mathring{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ is a critical point of the Dirichlet energy corresponding to inner variations $v_t := v \circ \Phi_t$, or variations of the independent variable, if v satisfies

$$\int_{\mathbb{R}^{m+1}_+} \sum_{i=1}^{m+1}\sum_{k=1}^{m+1} x^\beta_{m+1}\left(2\left\langle\frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_k}\right\rangle - \delta_{ik}|\nabla v|^2\right)\frac{\partial\phi_k}{\partial x_i}dx$$
$$= \int_{\mathbb{R}^{m+1}_+} \beta x^{\beta-1}_{m+1}\phi_{m+1}|\nabla v|^2 dx \tag{4.3}$$

for every ϕ as above. A weakly harmonic map with respect to the Neumann type boundary condition (4.2) which satisfies (4.3) for every $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^{m+1})$ with $\phi(\cdot, 0) \in C^\infty_0(\mathcal{O}; \partial\mathbb{R}^{m+1}_+)$ is called *weakly stationary harmonic, or stationary harmonic, with respect to the Neumann type boundary condition* (4.2).

4.2 Energy monotonicity

Stationary harmonic maps satisfy a monotonicity formula for an appropriately scaled version of the energy over balls with closure in $\mathbb{R}^{m+1}_+$. This property was proved by Schoen and Uhlenbeck for energy minimisers, see [37, Proposition 2.4], and Price, see the remark after Theorem 1 in [32], for stationary harmonic maps.

As a consequence of (4.3), we show that stationary harmonic maps with respect to the Neumann-type boundary condition (4.2) satisfy a similar monotonicity formula on half-balls $B_\rho^+(y)$ with centre y in $\mathcal{O}$ and which satisfy $\overline{\partial^0 B_\rho^+(y)} = \overline{B^m(y)} \subset \mathcal{O}$. Moreover, we state a version of the formula for balls with closure contained in $\mathbb{R}_+^{m+1}$, giving an explicit expression for the factors that the constants involved depend upon.

Lemma 4.5 *Suppose $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ is a weakly stationary harmonic map with respect to the Neumann-type boundary condition (4.2). Suppose y in $\mathcal{O}$ and consider $B_R^+(y)$ with $\overline{\partial^0 B_R^+(y)} \subset \mathcal{O}$. Then*

$$r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla v|^2 \, dx - s^{1-m-\beta} \int_{B_s^+(y)} x_{m+1}^\beta |\nabla v|^2 \, dx$$

$$= 2 \int_{B_r^+(y) \setminus B_s^+(y)} x_{m+1}^\beta \frac{|(x-y) \cdot \nabla v|^2}{|x-y|^{m+1+\beta}} \, dx$$

whenever $0 \le s \le r \le R$ and therefore $\rho \mapsto \rho^{1-m-\beta} \int_{B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \, dx$ is a non-decreasing function of ρ for $0 < \rho \le R$.

Proof The proof is analogous to that of the monotonicity formula for stationary harmonic maps. We follow [39, Section 2.4] and [29, Lemma 3.3]; we test (4.3) with $\phi(x) = (x - y)\eta(x)$, where $\eta \in C_0^\infty(B_\rho(y))$, which yields

$$(m - 1 + \beta) \int_{\mathbb{R}_+^{m+1}} x_{m+1}^\beta |\nabla v|^2 \, \eta dx + \int_{\mathbb{R}_+^{m+1}} x_{m+1}^\beta (x - y) \cdot \nabla \eta \, |\nabla v|^2 \, dx$$

$$= 2 \int_{\mathbb{R}_+^{m+1}} x_{m+1}^\beta \langle (x - y) \cdot \nabla v, \nabla \eta \cdot \nabla v \rangle \, dx. \tag{4.4}$$

Let $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$ with $\chi(s) \equiv 1$ for $s \ge 1$ and $\chi(s) \equiv 0$ for $s \le \frac{1}{2}$. The smooth functions defined by $\eta_j(x) = \chi(j(\rho - |x - y|))$ are admissible choices for η in (4.4) and $\{\eta_j\}_{j \in \mathbb{N}}$ converges pointwise to the indicator function of $B_\rho^+(y)$. We substitute η_j for η in (4.4) and take the limit as $j \to \infty$, using Lebesgue's Dominated Convergence and Differentiation Theorems, to see that

$$(m - 1 + \beta) \int_{B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \, dx - \rho \int_{\partial^+ B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \, dS(x)$$

$$= -\frac{2}{\rho} \int_{\partial^+ B_\rho^+(y)} x_{m+1}^\beta |(x - y) \cdot \nabla v|^2 \, dS(x)$$

for almost every $\rho > 0$, where dS is the Lebesgue measure on $\partial B_\rho(y)$. Multiplying the above by the factor $-\rho^{-(\beta+m)}$ and bearing in mind that $\frac{d}{d\rho} \int_{B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \, dx = \int_{\partial^+ B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \, dS(x)$ for almost all $\rho > 0$, we find

$$\frac{d}{d\rho} \left(\rho^{1-m-\beta} \int_{B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \, dx \right) = 2 \int_{\partial^+ B_\rho^+(y)} x_{m+1}^\beta \frac{|(x - y) \cdot \nabla v|^2}{|x - y|^{m+1+\beta}} \, dS(x)$$

for almost every $\rho > 0$. Integrating between $0 < s < r$ concludes the proof. $\square$

Remark 4.6 A consequence of Lemma 4.5 is that we can define the *density function*

$$\Theta_v^\beta(y) = \lim_{\rho \to 0^+} \rho^{1-m-\beta} \int_{B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x$$

for every $y \in \mathcal{O}$, analogously to Definition 1 in Section 2.5 of [39]. Using Lemma 4.5 we deduce Θ_v^β is upper semi-continuous in $\mathcal{O}$ for any map v which is weakly stationary harmonic with respect to the Neumann-type boundary condition (4.2).

The following version of the energy monotonicity formula is due to Grosse-Brauckmann [16, Theorem 1]. We do not give a proof, but remark that the explicit form of the constant in the forthcoming formula can be determined using the method of proof of Lemma 4.5.

Lemma 4.7 *Suppose* $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ *is a minimiser of* E^β *relative to* $\mathcal{O}$. *Fix a ball* $B_{\rho_0}(y)$ *with* $\overline{B_{\rho_0}(y)} \subset \mathbb{R}_+^{m+1}$ *for some* $\rho_0 > 0$ *and suppose* r *and* s *satisfy* $0 < s < r < \rho_0$. *Then*

$$e^{rC|\beta|}r^{1-m} \int_{B_r(y)} x_{m+1}^\beta |\nabla v|^2 \, \mathrm{d}x \; - \; e^{sC|\beta|}s^{1-m} \int_{B_s(y)} x_{m+1}^\beta |\nabla v|^2 \, \mathrm{d}x$$

$$\geq 2 \int_{B_r(y) \setminus B_s(y)} x_{m+1}^\beta e^{|x-y||C|\beta|} \frac{|(x-y) \cdot \nabla v|^2}{|x-y|^{m+1}} \mathrm{d}x \tag{4.5}$$

and therefore, for $0 < \rho < \rho_0$, $\rho \mapsto e^{\rho C|\beta|}\rho^{1-m} \int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 \, \mathrm{d}x$ *is a non-decreasing function of* ρ *where* $C = (y_{m+1} - \rho_0)^{-1} = \left(dist(B_{\rho_0}(y), \partial\mathbb{R}_+^{m+1}) \right)^{-1}$.

4.3 A modified lemma of Morrey

In order to prove Theorem 4.21, in analogy with the regularity theory of harmonic maps, we will show that the re-scaled, scale-invariant energies in the monotonicity formulas in Sect. 4.2 decay slightly faster than implied by the Lemmata as the radius decreases. This will permit the application of a well-known lemma of Morrey, see [29, Lemma 2.1] for example, which is used to derive Hölder continuity from sufficiently fast energy decay. We will reduce the hypothesis of this lemma to similar hypothesis for the re-scaled energies from the monotonicity formula. To this end, we introduce a class of ball with closure in $\mathbb{R}_+^{m+1}$ on which the metrics $x_{m+1}^\alpha \delta_{ij}$, discussed in Sect. 3 and corresponding weights x_{m+1}^β are uniformly equivalent to the Euclidean metric and 1 respectively. We also introduce classes of balls and half-balls contained in the interior of a given larger half-ball $B_R^+(x_0)$ for $R > 0$ and $x_0 \in \partial\mathbb{R}_+^{m+1}$.

Define

$$\mathcal{B} = \left\{ B_\rho(y) \subset \mathbb{R}_+^{m+1} : y_{m+1} \geq 2\rho \right\} \quad \text{and} \quad \mathcal{B}_\theta = \left\{ B_\rho(y) \subset \mathbb{R}_+^{m+1} : y_{m+1} \geq \theta\rho \right\}$$

for $\theta \geq 2$. Then $\mathcal{B}_\theta \subset \mathcal{B}$ and $\mathcal{B}_2 = \mathcal{B}$. We further define

$$\mathcal{B}_\theta(x_0, R, r) = \{ B_\rho(y) \subset B_R^+(x_0) : y_{m+1} \geq \theta\rho, \, y \in B_r^+(x_0) \},$$

omitting the subscript θ in the case $\theta = 2$, and let

$$\mathcal{B}^+(x_0, R, r) = \{ B_\rho^+(y) \subset B_R^+(x_0) : y_{m+1} = 0, \, |x_0 - y| < r, \, \rho \leq r \}.$$

Observe that, on any $B_\rho(y) \in \mathcal{B}$, we can choose constants c, C, c_0 and C_0 independently of β such that for every $x \in B_\rho(y)$ and $\beta \in (-1, 1)$ we have

$$cy_{m+1}^\beta \leq x_{m+1}^\beta \leq Cy_{m+1}^\beta \qquad \text{and} \qquad c_0 \leq \frac{\sup_{B_\rho(y)} x_{m+1}^\beta}{\inf_{B_\rho(y)} x_{m+1}^\beta} \leq C_0. \qquad (4.6)$$

Lemma 4.8 *Let $\gamma > 0$, $x_0 \in \partial\mathbb{R}_+^{m+1}$, $a > 0$, $\theta_1 \geq 2$ and $\theta_2 \leq \frac{1}{2}$. Define $\theta = \frac{\theta_2}{2\theta_1}$. Then there exists a constant $C_0 = C_0(m, \gamma, \theta_1, \beta)$ such that if $v \in W_\beta^{1,2}(B_R^+(x_0); \mathbb{R}^n)$ with*

$$r^{-2m} \int_B |x_{m+1}|^{-\beta} \mathrm{d}x \int_B |\nabla v|^2 |x_{m+1}|^\beta \mathrm{d}x \leq ar^\gamma \qquad (4.7)$$

for every $B = B_r^+(y) \in \mathcal{B}^+(x_0, R, \theta_2 R)$ and every $B = B_r(y) \in \mathcal{B}_{\theta_1}(x_0, R, \theta_2 R)$, then for almost every $x_1, x_2 \in B_{\theta R}^+(x_0)$,

$$|v(x_1) - v(x_2)| \leq C_0 a^{\frac{1}{2}} |x_1 - x_2|^{\frac{\gamma}{2}}.$$

Proof Let $B_r(y) \subset B_{2\theta R}(x_0)$ with $y_{m+1} \geq 0$, $y \in B_{\theta R}(x_0)$ and $r \leq \theta R$, for a $\gamma \in (0, 1)$, an $a > 0$ and θ as specified. Such a $B_r(y)$ must satisfy either $B_r(y) \in \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$ or $B_r(y) \notin \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$. We consider these cases in turn and we work with the even reflection of v with respect to $\partial\mathbb{R}_+^{m+1}$, which we do not relabel and which is in $W_\beta^{1,2}(B_R(x_0); \mathbb{R}^n)$.

Suppose $B_r(y) \in \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$ with $r \leq \theta R$. Then $B_r(y) \in \mathcal{B}_{\theta_1}(x_0, R, \theta_2 R)$ and an application of Hölder's inequality and the assumptions of the lemma yields

$$r^{-m} \int_{B_r(y)} |\nabla v| \mathrm{d}x \leq \left(r^{-2m} \int_{B_r(y)} |x_{m+1}|^{-\beta} \mathrm{d}x \int_{B_r(y)} |\nabla v|^2 |x_{m+1}|^\beta \mathrm{d}x \right)^{\frac{1}{2}} \leq a^{\frac{1}{2}} r^{\frac{\gamma}{2}}. \qquad (4.8)$$

Now suppose $B_r(y) \notin \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$ and $r \leq \theta R$. In this case, since $B_r(y) \subset B_{2\theta R}(x_0)$ and $y \in B_{\theta R}(x_0)$ by assumption, we must have $y_{m+1} < \theta_1 r$. Hence $y_{m+1} - r < \zeta r$, where $\zeta \geq 1$ is such that $\theta_1 = \zeta + 1$, and thus $B_r(y) \subset B_{(2+\zeta)r}(y_0)$ where $y_0 = y - (0, y_{m+1})$. We observe that $B_{(2+\zeta)r}^+(y_0) \in \mathcal{B}^+(x_0, R, \theta_2 R)$. Therefore, defining $s = (2 + \zeta)r$ and using the assumptions of the lemma, the symmetry of v and applying Hölder's inequality, we find

$$r^{-m} \int_{B_r(y)} |\nabla v| \mathrm{d}x \leq r^{-m} \int_{B_s(y_0)} |\nabla v| \mathrm{d}x$$

$$= 2r^{-m} \int_{B_s^+(y_0)} |\nabla v| \mathrm{d}x$$

$$\leq 2(2 + \zeta)^m \left(s^{-2m} \int_{B_s^+(y_0)} |x_{m+1}|^{-\beta} \mathrm{d}x \int_{B_s^+(y_0)} |\nabla v|^2 |x_{m+1}|^\beta \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\leq 2(2 + \zeta)^{m+\frac{\gamma}{2}} a^{\frac{1}{2}} r^{\frac{\gamma}{2}}. \qquad (4.9)$$

Since v is even with respect to $\partial\mathbb{R}_+^{m+1}$, we deduce that either (4.8) or (4.9) holds on any $B_r(y) \subset B_{2\theta R}(x_0)$ with $y \in B_{\theta R}(x_0)$ and $r \leq \theta R$. Hence we have established that the hypothesis of the decay lemma of Morrey hold on $B_{2\theta R}(x_0)$, see [29, Lemma 2.1]. An application of this lemma concludes the proof. □

4.4 Interior estimates for Hölder continuity

Using the regularity theory of Schoen and Uhlenbeck [37] and Schoen [38], we show that minimisers $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ of E^β relative to $\mathcal{O}$ essentially satisfy (4.7) in Lemma 4.8, provided the scale-invariant energy $R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x$ is sufficiently small. To this end, we show that the preceding scale-invariant energy $B_R^+(x_0)$ controls the scale-invariant Euclidean energy on a class of ball with closure in $B_R^+(x_0)$. We also recall the relevant theory from [37,38, Sections 1, 2 and 3], stating the results in our context with slightly different notation. Our goal is to prove the following.

Lemma 4.9 *Suppose $v \in \dot{W}_\beta^{1,2}(\overline{\mathbb{R}_+^{m+1}}; N)$ is a minimiser of E^β relative to $\mathcal{O}$. Let $B_R^+(x_0)$ be a half-ball with $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$. There exists an $\varepsilon_0 = \varepsilon_0(m, N) > 0$, a $\theta = \theta(m, N) \geq 2$ and a positive $C = C(m, N)$ such that if*

$$R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \, \mathrm{d}x \leq \varepsilon_0,$$

then

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \mathrm{d}x \leq C \left(\frac{\rho}{r}\right)^\gamma r^{1-m} \int_{B_r(y)} |\nabla v|^2 \mathrm{d}x \tag{4.10}$$

on every $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$ for $0 < \rho \leq r$ and a $\gamma = \gamma(m, N) \in (0, 1)$.

To establish the preceding lemma, we observe the following relationship between the scale-invariant energy on $B_R^+(x_0)$ and the scale-invariant Euclidean energy on a class of ball with closure in $B_R^+(x_0)$.

Lemma 4.10 *Suppose $v \in \dot{W}_\beta^{1,2}(\overline{\mathbb{R}_+^{m+1}}; N)$ is a minimiser of E^β relative to $\mathcal{O}$. Let $B_R^+(x_0)$ be a half-ball with $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$ and suppose $B_\rho(y) \in \mathcal{B}(x_0, R, \frac{R}{3})$. Then there is a constant $C = C(m)$ such that*

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \mathrm{d}x \leq C R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x. \tag{4.11}$$

Proof Notice that any ball $B_\rho(y) \in \mathcal{B}$ satisfies $B_\rho(y) \subset B_{\frac{y_{m+1}}{2}}(y)$ so we can choose the scaling factor $e^{\frac{2|\beta|\rho}{y_{m+1}}}$ in Lemma 4.7. Furthermore, $e^{\frac{2|\beta|\rho}{y_{m+1}}} \leq e$ since $y_{m+1} \geq 2\rho$ and $\beta \in (-1, 1)$. Hence, using (4.6) and applying Lemma 4.7, we find

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \, \mathrm{d}x \leq C y_{m+1}^{-\beta} e^{\frac{2|\beta|\rho}{y_{m+1}}} \rho^{1-m} \int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 \, \mathrm{d}x$$

$$\leq C \left(\frac{y_{m+1}}{2}\right)^{1-m-\beta} \int_{B_{\frac{y_{m+1}}{2}}(y)} x_{m+1}^\beta |\nabla v|^2 \, \mathrm{d}x. \tag{4.12}$$

Let $y = (y_1, \ldots, y_{m+1})$ and $y^+ = (y_1, \ldots, y_m, 0)$. Note that $B_{\frac{y_{m+1}}{2}}(y) \subset B_{\frac{3y_{m+1}}{2}}^+(y^+)$ and, since $B_\rho(y) \in \mathcal{B}\left(x_0, R, \frac{R}{3}\right)$, we have $y \in B_{\frac{R}{3}}^+(x_0)$ and $B_{\frac{3y_{m+1}}{2}}^+(y^+) \subset B_{\frac{R}{2}}^+(y^+) \subset B_R^+(x_0)$. Using these facts we apply Lemma 4.5 to see that

$$\left(\frac{y_{m+1}}{2}\right)^{1-m-\beta} \int_{B_{\frac{y_{m+1}}{2}}(y)} x_{m+1}^\beta |\nabla v|^2 \, \mathrm{d}x \leq C \left(\frac{3y_{m+1}}{2}\right)^{1-m-\beta} \int_{B_{\frac{3y_{m+1}}{2}}^+(y^+)} x_{m+1}^\beta |\nabla v|^2 \, \mathrm{d}x$$

$$\leq C \left(\frac{R}{2}\right)^{1-m-\beta} \int_{B^+_{\frac{R}{2}}(y^+)} x^\beta_{m+1} |\nabla v|^2 \, dx$$

$$\leq C R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \, dx \qquad (4.13)$$

where $C = C(m)$. The combination of (4.12) and (4.13) yields (4.11). □

The maps considered in [37,38] belong to

$$W^{1,2}(\Omega; N) = \left\{ v \in W^{1,2}(\Omega; \mathbb{R}^n) : v(x) \in N \text{ for almost every } x \in B_r(y) \right\},$$

for open $\Omega \subset \mathbb{R}^{m+1}_+$. Consider the compact Riemannian manifold $\overline{B_1(0)}$ with metric $\tilde{g}$. Recall from the introduction that the Dirichlet energy functional on $\overline{B_1(0)}$ is given by

$$E_{\tilde{g}}(v) = \int_{B_1(0)} |\nabla v|^2_{\tilde{g}} \sqrt{\det(\tilde{g})} dx.$$

A minimiser of $E_{\tilde{g}}$ with fixed boundary data is defined in Section 1 of [37] as follows.

Definition 4.11 Any $v \in W^{1,2}(B_1(0); N)$ is an $E_{\tilde{g}}$ *minimising map* if it satisfies $E_{\tilde{g}}(v) \leq E_{\tilde{g}}(w)$ for any $w \in W^{1,2}(B_1(0); N)$ with $v - w \in W^{1,2}_0(\overline{B_1(0)}; \mathbb{R}^n)$.

The metric $\tilde{g}$ is assumed to be of class C^2 on $\overline{B_1(0)}$. For $\Lambda > 0$ denote by $\mathscr{E}_\Lambda$ the class of functionals $E_{\tilde{g}}$ on $\overline{B_1(0)}$ with metric $\tilde{g}$ such that $\tilde{g}_{ij}(0) = \delta_{ij}$ and

$$\sum_{i,j,k} |\partial_k \tilde{g}_{ij}| \leq \Lambda.$$

If v is $E_{\tilde{g}}$-minimising with $E_{\tilde{g}} \in \mathscr{E}_\Lambda$ then we say $v \in \mathscr{H}_\Lambda$.

Schoen and Uhlenbeck [37] proved their ε-regularity theorem for minimisers of functionals of the form $\tilde{E}_{\tilde{g}} + F$, where F gives rise to terms in the Euler–Lagrange equations which are lower order than those coming from the energy. We state the result of their theorem with $F = 0$. The following Lemma is Theorem 3.1 in [37].

Lemma 4.12 *There exists* $\varepsilon = \varepsilon(m, N) > 0$ *such that if* $v \in \mathscr{H}_\Lambda$, $\Lambda \leq \varepsilon$ *and* $\int_{B_1(0)} |\nabla v|^2 \, dx \leq \varepsilon$, *then* v *is Hölder continuous in* $B_{\frac{1}{2}}(0)$ *and*

$$|v(x_1) - v(x_2)| \leq C|x_1 - x_2|^\gamma$$

for constants $C = C(m, N)$ *and* $\gamma = \gamma(m, N) \in (0, 1)$ *and every* $x_1, x_2 \in B_{\frac{1}{2}}(0)$.

It is well known that continuous weakly harmonic maps are smooth, see [22] for example. It is more readily shown that Hölder continuous harmonic maps are smooth; this is the content of the following lemma, which is Lemma 3.1 of [38].

Lemma 4.13 *Consider a ball* $B_r(y) \subset \mathbb{R}^{m+1}_+$ *and suppose* $v \in W^{1,2}(B_r(y); N)$ *is a weakly harmonic map which is Hölder continuous on* $B_r(y)$. *Then* v *is smooth on* $B_r(y)$.

The final lemma we will need is Theorem 2.2 from [38].

Lemma 4.14 *Let* $v \in C^2(B_r(0); N)$ *and* $\tilde{g}$ *be a Riemannian metric on* $B_r(0)$. *Suppose* v *is harmonic with respect to* $\tilde{g}$ *in* $B_r(0)$ *and* $\tilde{g}$ *satisfies* $|\partial_k \tilde{g}_{ij}| \leq \Lambda r^{-1}$ *for* $i, j, k = 1, \ldots, m+1$

and $\Lambda^{-1}(\delta_{ij}) \le (\tilde{g}_{ij}) \le \Lambda(\delta_{ij})$ *in the sense of tensors, where* $\delta_{ij} = 1$ *when* $i = j$ *and* $\delta_{ij} = 0$ *otherwise. Then there exists an* $\varepsilon = \varepsilon(\Lambda, m, N) > 0$ *such that if*

$$r^{1-m} \int_{B_r(0)} |\nabla v|_{\tilde{g}}^2 \, (det(\tilde{g}))^{\frac{1}{2}} \, dx \le \varepsilon$$

then

$$\sup_{B_{\frac{r}{2}}(0)} |\nabla v|_{\tilde{g}}^2 \le Cr^{-(1+m)} \int_{B_r(0)} |\nabla v|_{\tilde{g}}^2 \, (det(\tilde{g}))^{\frac{1}{2}} \, dx \tag{4.14}$$

for a constant $C = C(\Lambda, m, N)$.

Proof of Lemma 4.9 Suppose $R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 dx \le \varepsilon_0$ for an $\varepsilon_0 > 0$ to be chosen small and let ε be the number from Lemma 4.12.

Recall the metric g given in Euclidean coordinates by $g := x_{m+1}^\alpha \delta_{ij}$ and define $\hat{g}$ on $\overline{B_1(0)}$ by

$$\hat{g}_{ij}(x) = \delta_{ij} \left(1 + ry_{m+1}^{-1} x_{m+1}\right)^\alpha = y_{m+1}^{-\alpha} g_{ij}(rx + y). \tag{4.15}$$

The energy corresponding to $\hat{g}$ is

$$E_{\hat{g}}(\hat{v}) = \frac{1}{2} \int_{B_1(0)} \left(1 + ry_{m+1}^{-1} x_{m+1}\right)^\beta |\nabla \hat{v}|^2 \, dx$$

for maps $\hat{v} \in W^{1,2}(B_1(0); \mathbb{R}^n)$. Since $B_r(y) \in B_\theta(x_0, R, \frac{R}{3})$ for a $\theta \ge 2$ to be chosen, we have $B_r(y) \in \mathcal{B}$. Hence, using (4.6) and noting that $\beta = \alpha\left(\frac{m-1}{2}\right) \in (-1, 1)$, we find constants c, C depending only on m such that

$$c \le \left(1 + ry_{m+1}^{-1} x_{m+1}\right)^\beta \le C \qquad \text{and} \qquad c \le \hat{g}_{ij}(x) \le C. \tag{4.16}$$

We note that $\partial_k \hat{g} = 0$ for $k \ne m + 1$ and, again using (4.6), we calculate

$$\left|\partial_{m+1} \left(1 + ry_{m+1}^{-1} x_{m+1}\right)^\alpha\right| = ry_{m+1}^{-1} |\alpha| \left|\left(1 + ry_{m+1}^{-1} x_{m+1}\right)^{\alpha-1}\right| \le Cry_{m+1}^{-1}$$

where C is chosen independently of α. Hence, if we set $\theta = \theta(m, N) \ge \max\left\{2, (m+1)C\varepsilon^{-1}\right\}$ then we conclude that

$$\sum_{i,j,k} |\partial_k \tilde{g}_{ij}| = \sum_{i=1}^{m+1} |\partial_{m+1} \tilde{g}_{ii}| \le \varepsilon. \tag{4.17}$$

We assume the preceding choice of θ henceforth so that (4.17) holds on any $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$.

Define $v_{r,y}(x) = v(rx+y)$ for $x \in B_1(0)$. Lemmata 2.1 and 2.3 imply $\dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N) \hookrightarrow W^{1,2}(B_r(y); N)$ for every $B_r(y)$ with $\overline{B_r(y)} \subset \mathbb{R}_+^{m+1}$, regardless of $\beta \in (-1, 1)$. A change of variables then yields $v_{r,y} \in W^{1,2}(B_1(0); N)$. Furthermore, since v is a minimiser of E^β relative to $\mathcal{O}$ we readily calculate that $v_{r,y}$ is a minimiser of $E_{\hat{g}}$ in the sense of Definition 4.11 on $B_1(0)$, that is, $v_{r,y}$ minimises $E_{\hat{g}}$ among all maps in $W^{1,2}(B_1(0); N)$ with the same boundary values as $v_{r,y}$.

Our considerations so far imply that if $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$, for our preceding choice of θ, then $v_{r,y} \in \mathcal{H}_\varepsilon$. Let C be the constant from Lemma 4.10 and suppose that $\varepsilon_0 \leq C^{-1}\varepsilon$. An application of this lemma, combined with a change of variables, yields

$$\int_{B_1(0)} |\nabla v_{r,y}|^2 \, dx = r^{1-m} \int_{B_r(y)} |\nabla v|^2 \, dx \leq C R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \, dx \leq C\varepsilon_0 \leq \varepsilon.$$

(4.18)

This holds for every $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$. We may therefore apply Lemma 4.12 to $v_{r,y}$ to deduce that it is Hölder continuous in $B_{\frac{1}{2}}(0)$. Re-scaling implies v is Hölder continuous in every $B_{\frac{r}{2}}(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$, or equivalently v is Hölder continuous in every $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$.

Since v is weakly harmonic in $\mathbb{R}^{m+1}_+$ with respect to the metric represented by $x_{m+1}^\alpha \delta_{ij}$ (where $\beta = \alpha\frac{m-1}{2}$), it is weakly harmonic with respect to $x_{m+1}^\alpha \delta_{ij}$ on every $B_r(y)$ with $\overline{B_r(y)} \subset \mathbb{R}^{m+1}_+$. Thus it follows from Lemma 4.13 that v is smooth in each $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$, which holds if and only if each $v_{r,y}$ corresponding to such a $B_r(y)$ is smooth in $B_1(0)$. We further deduce that $v_{r,y}$ is harmonic in $B_1(0)$ with respect to $\hat{g}$ using the chain rule. Moreover, it follows from (4.16) and (4.17) that $\hat{g}$ satisfies the assumptions required of the metric in Lemma 4.14. We combine (4.16) with (4.18) to see that

$$\int_{B_1(0)} \left(1 + ry_{m+1}^{-1}x_{m+1}\right)^\beta |\nabla v_{r,y}|^2 dx \leq C \int_{B_1(0)} |\nabla v_{r,y}|^2 dx \leq C\varepsilon_0. \tag{4.19}$$

In addition to our previous stipulation for ε_0, we further require that $\varepsilon_0 \leq \frac{\varepsilon_1}{C}$ where ε_1 is the number from Lemma 4.14. We apply this lemma, recalling $\beta = \alpha(\frac{m-1}{2})$ and $B_r(y) \in \mathcal{B}$, to see that

$$r^2 \sup_{B_{\frac{r}{2}}(y)} \left(x_{m+1}y_{m+1}^{-1}\right)^\beta |\nabla v|^2 = \sup_{B_{\frac{1}{2}}(0)} \left(1 + ry_{m+1}^{-1}x_{m+1}\right)^\beta |\nabla v_{r,y}|^2$$

$$\leq C \sup_{B_{\frac{1}{2}}(0)} \left(1 + ry_{m+1}^{-1}x_{m+1}\right)^{-\alpha} |\nabla v_{r,y}|^2$$

$$\leq C \int_{B_1(0)} \left(1 + ry_{m+1}^{-1}x_{m+1}\right)^\beta |\nabla v_{r,y}|^2 dx$$

$$= Cr^{1-m} \int_{B_r(y)} \left(x_{m+1}y_{m+1}^{-1}\right)^\beta |\nabla v|^2 dx.$$

As a result, for any $\sigma \in (0, \frac{1}{2}]$ we have

$$(\sigma r)^{1-m} \int_{B_{\sigma r}(y)} \left(x_{m+1}y_{m+1}^{-1}\right)^\beta |\nabla v|^2 dx$$

$$\leq C\sigma^2 r^{1-m} \int_{B_r(y)} \left(x_{m+1}y_{m+1}^{-1}\right)^\beta |\nabla v|^2 dx.$$

Choose σ such that $\sigma^2 \leq \frac{1}{2C}$ and let $\rho \leq r$. Then

$$e^{\frac{|\beta|\sigma r}{y_{m+1}-r}} (\sigma r)^{1-m} \int_{B_{\sigma r}(y)} \left(x_{m+1}y_{m+1}^{-1}\right)^\beta |\nabla v|^2 dx$$

$$\leq \frac{1}{2} e^{\frac{|\beta|r}{y_{m+1}-r}} r^{1-m} \int_{B_r(y)} \left(x_{m+1}y_{m+1}^{-1}\right)^\beta |\nabla v|^2 dx$$

on every $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$. The map $\rho \mapsto e^{\frac{|\beta|\rho}{y_{m+1}-r}} \rho^{1-m} \int_{B_\rho(y)} \left(x_{m+1} y_{m+1}^{-1}\right)^\beta |\nabla v|^2 \mathrm{d}x$
is non-decreasing in ρ for $\rho \leq r$ as a result of Lemma 4.7. Lemma 8.23 of [15] hence implies

$$e^{\frac{|\beta|\rho}{y_{m+1}-r}} \rho^{1-m} \int_{B_\rho(y)} \left(x_{m+1} y_{m+1}^{-1}\right)^\beta |\nabla v|^2 \mathrm{d}x$$

$$\leq C \left(\frac{\rho}{r}\right)^\gamma e^{\frac{|\beta|r}{y_{m+1}-r}} r^{1-m} \int_{B_r(y)} \left(x_{m+1} y_{m+1}^{-1}\right)^\beta |\nabla v|^2 \mathrm{d}x$$

for every $\rho \leq r$ on every $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$. It follows that

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \mathrm{d}x \leq C \left(\frac{\rho}{r}\right)^\gamma r^{1-m} \int_{B_r(y)} |\nabla v|^2 \mathrm{d}x$$

on every $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$ for $0 < \rho \leq r$. This concludes the proof. $\qquad \square$

4.5 A modified lemma of Luckhaus

Here we begin our construction of comparison maps. We prove a partial extension (to the particular case of our degenerate/singular metrics $x_{m+1}^\alpha \delta_{ij}$) of a lemma of Luckhaus, Lemma 3 in [23], as presented in Lemma 1 Section 2.6 of [39].

Let $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ denote the m dimensional unit sphere, centred at the origin and equipped with the metric induced by the Euclidean metric on $\mathbb{R}^{m+1}$. Define $\mathbb{S}_+^m = \mathbb{S}^m \cap \mathbb{R}_+^{m+1}$ with the metric induced from $\mathbb{S}^m$. We let ω denote a point in $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ or $\mathbb{S}_+^m \subset \mathbb{R}_+^{m+1}$ and write $\mathrm{d}\omega$ for the volume element corresponding to the induced metric. Recall the notation $\partial^+ \Omega = \partial\Omega \cap \mathbb{R}_+^{m+1}$ for $\Omega \subset \mathbb{R}^{m+1}$ and $Q_r(y) = \{x \in \mathbb{R}^{m+1} : |x_i - y_i| < r, i = 1, \ldots, m+1\}$ for $y \in \mathbb{R}^{m+1}$. We also write $Q_r^+(y) = Q_r(y) \cap \mathbb{R}_+^{m+1}$ for $y \in \partial\mathbb{R}_+^{m+1}$.

In order to state the modified Luckhaus lemma precisely we introduce the notion of a Sobolev space for functions whose domain is either $\mathbb{S}^m$ or $\mathbb{S}_+^m$.

Definition 4.15 Let $\varepsilon > 0$ and $\rho > 0$. Suppose $S = \rho\mathbb{S}^m$ and $V_\varepsilon = B_{\rho+\varepsilon}(0)\backslash B_{\rho-\varepsilon}(0)$ or $S = \rho\mathbb{S}_+^m$ and $V_\varepsilon = B_{\rho+\varepsilon}^+(0)\backslash B_{\rho-\varepsilon}^+(0)$. An element $v \in L_\beta^2(S; \mathbb{R}^n)$ is said to be in $W_\beta^{1,2}(S; \mathbb{R}^n)$ if the map $v(\rho \frac{x}{|x|}) \in W_\beta^{1,2}(V_\varepsilon; \mathbb{R}^n)$ for some $\varepsilon > 0$. An element $v \in L_\beta^2(S \times [a, b]; \mathbb{R}^n)$, with $a < b$ real numbers, is said to be in $W_\beta^{1,2}(S \times [a, b]; \mathbb{R}^n)$ if the map $v(\rho \frac{x}{|x|}, s) \in W_\beta^{1,2}(V_\varepsilon \times [a, b]; \mathbb{R}^n)$ for some $\varepsilon > 0$. If $N \subset \mathbb{R}^n$ is compact, we say v is in $W_\beta^{1,2}(S; N)$ or $W_\beta^{1,2}(S \times [a, b]; N)$ if v is in $W_\beta^{1,2}(S; \mathbb{R}^n)$ or $W_\beta^{1,2}(S \times [a, b]; \mathbb{R}^n)$ respectively and $v(x) \in N$ for almost every $x \in S$.

We now state our version of the Luckhaus lemma.

Lemma 4.16 Let $m \geq 2$ and $\beta \in (-1, 1)$. Let N be a compact subset of $\mathbb{R}^n$ and suppose $u, v \in W_\beta^{1,2}(\mathbb{S}_+^m; N)$. Then for all $\varepsilon \in (0, 1)$ there is a $w \in W_\beta^{1,2}(\mathbb{S}_+^m \times [0, \varepsilon]; \mathbb{R}^n)$ such that w agrees with u on $\mathbb{S}_+^m \times \{0\}$ and v on $\mathbb{S}_+^m \times \{\varepsilon\}$ in the sense of traces and which satisfies the following. Let $\overline{D}$ be the gradient on $\mathbb{S}_+^m \times [0, \varepsilon]$ and D the gradient on $\mathbb{S}_+^m$. Then $w = w(\omega, s)$ satisfies

$$\int_{\mathbb{S}_+^m \times [0,\varepsilon]} \omega_{m+1}^\beta |\overline{D}w|^2 \mathrm{d}\omega \mathrm{d}s$$
$$\leq C_1 \varepsilon \int_{\mathbb{S}_+^m} \omega_{m+1}^\beta \left(|Du|^2 + |Dv|^2\right) \mathrm{d}\omega + \frac{C_1}{\varepsilon} \int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |u - v|^2 \mathrm{d}\omega \qquad (4.20)$$

where $C_1 = C_1(m, \beta)$. Furthermore, w satisfies

$$dist^2(w(\omega, s), N)$$

$$\leq \frac{C_2}{\varepsilon^{m+\frac{\beta}{2}+\frac{|\beta|}{2}}} \left(\int_{\mathbb{S}_+^m} \omega_{m+1}^\beta \left(|Du|^2 + |Dv|^2 \right) d\omega \right)^{\frac{1}{q}} \left(\int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |u - v|^2 d\omega \right)^{1-\frac{1}{q}} \quad (4.21)$$

$$+ \frac{C_2}{\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}} \int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |u - v|^2 d\omega$$

for almost every $(\omega, s) \in \mathbb{S}_+^m \times [0, \varepsilon]$ where $C_2 = C_2(m, \beta)$ and q satisfies the following. If $\beta \in (-1, 0]$ then (4.21) holds for $q = 2$. If $\beta \in (0, 1)$, for any $p \in (1, \frac{2}{1+\beta})$, there exists $q \in \{2, p\}$ such that (4.21) holds.

Our proof of Lemma 4.16 follows the proof, given in Section 2.12.2 of [39], of Lemma 1 in Section 2.6 of [39].

4.6 Absolute continuity properties of functions in $W_\beta^{1,2}$

We recall the discussion in [39, Section 2.12.1] regarding the absolute continuity properties of $W^{1,p}$ functions, which are inherited by $W_\beta^{1,2}$ functions in view of Lemma 2.1. Let $\mathcal{H}^t$ denote the t-dimensional Hausdorff measure with respect to the Euclidean metric. Consider a rectangle $Q \subset \overline{\mathbb{R}_+^{m+1}}$ of the form $Q = [a_1, b_1] \times \cdots \times [a_{m+1}, b_{m+1}]$ where $a_i < b_i$. Suppose $v \in W_\beta^{1,2}(Q; \mathbb{R}^n)$ with $\beta \in (-1, 1)$. It follows from Lemma 2.1 that if $a_{m+1} > 0$ then $Q \subset \mathbb{R}_+^{m+1}$ and $v|_Q \in W^{1,2}(Q; \mathbb{R}^n)$. Lemma 2.1 also implies that if $a_{m+1} = 0$ then $v|_Q \in W^{1,p}(Q; \mathbb{R}^n)$ for $p = p(\beta)$. Hence, by Lemma 3.1.1 and Theorem 3.1.8 in [28], if $a_{m+1} \geq 0$, we may infer the existence of a representative $\hat{v}$ of v such that, for each $i = 1, \ldots, m+1$, $\hat{v}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{m+1})$ is an absolutely continuous function of x_i for almost all fixed values of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+1}$ with respect to the m dimensional Hausdorff measure $\mathcal{H}^m$ on $[a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_{m+1}, b_{m+1}]$. The classical partial derivatives $\frac{\partial \hat{v}}{\partial x_i}$ agree almost everywhere with the weak derivatives $\frac{\partial v}{\partial x_i}$. Furthermore, for any closed subset N of $\mathbb{R}^n$, if $v(x) \in N$ for almost every x then it is possible to choose $\hat{v}(x) \in N$ for every $x \in \mathbb{R}_+^{m+1}$.

4.7 Proof of Lemma 4.16

Proof of Lemma 4.16 We follow the proof, given in Section 2.12.2 of [39], of Lemma 1 in Section 2.6 of [39]. Throughout, C denotes a constant only depending on m and β.

Suppose $u, v \in W_\beta^{1,2}(\mathbb{S}_+^m; N)$. We reflect u and v evenly in $\partial \mathbb{R}_+^{m+1}$, without relabelling, to get $u, v \in W_\beta^{1,2}(\mathbb{S}^m; N)$ and choose extensions of u and v to $\mathbb{R}^{m+1} \backslash \{0\}$ which are homogeneous of degree zero with respect to the origin. Then we choose representatives of these extensions which satisfy the absolute continuity properties described in Sect. 4.6 on $\overline{Q_1(0)}$. We will denote the representatives of the extensions of u and v by $\hat{u}$ and $\hat{v}$ respectively. Then $\hat{u}(\rho\omega) = \hat{u}(\omega)$, $\hat{v}(\rho\omega) = \hat{v}(\omega)$ for almost every $\rho > 0$ and $\omega \in \mathbb{S}^m$. Moreover, we have the identity $\nabla \hat{u} = |x|^{-1} \nabla \hat{u}(\omega(x)) = Du(\omega(x))$ for $\omega(x) = |x|^{-1}x$, where D is the gradient on $\mathbb{S}_+^m$ and ∇ is the gradient on $\mathbb{R}_+^{m+1}$. We therefore calculate

$$\int_{Q_1^+(0)} x_{m+1}^\beta \left(|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2 \right) dx \leq C \int_{\mathbb{S}_+^m} \omega_{m+1}^\beta \left(|Du|^2 + |Dv|^2 \right) d\omega \quad (4.22)$$

and

$$\int_{Q_1^+(0)} x_{m+1}^\beta |\hat{u} - \hat{v}|^2 dx \le C \int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |u - v|^2 d\omega. \tag{4.23}$$

Let $\varepsilon \in (0, \frac{1}{8})$ and define the closed rectangles $Q_{i,\varepsilon} = [i_1\varepsilon, (i_1 + 1)\varepsilon] \times \cdots \times [i_{m+1}\varepsilon, (i_{m+1} + 1)\varepsilon]$ for $i = (i_1, \dots, i_{m+1}) \in \mathbb{Z}^{m+1}$. Fix $\varepsilon \in (0, \frac{1}{8})$ arbitrarily henceforth. Let F^l denote any l-dimensional face of a $Q_{i,\varepsilon}$. We define

$$\mathcal{Q} = \left\{ Q_{i,\varepsilon} : i \in \mathbb{Z}^{m+1}, Q_{i,\varepsilon} \subset \overline{Q_{\frac{1}{2}}(0)} \right\} \text{ and } \mathcal{F}_i^l = \left\{ F^l \text{ faces of } Q_{i,\varepsilon} \right\}.$$

In addition, we write $x + \mathcal{F}_i^l$ to denote the collection of the translations of all faces in $\mathcal{F}_i^l$ by $x \in \mathbb{R}^{m+1}$.

Consider a non-negative, measurable function $f \colon \overline{Q_1(0)} \to \mathbb{R}$ which is even with respect to the hyperplane $\partial \mathbb{R}_+^{m+1}$. Invoking [39, Section 2.12.2] estimate (3), which is a consequence of Chebyshev's inequality and Fubini's theorem, we see that for every $K \ge 1$ there exists a set $P \subset Q_{0,\varepsilon}$ of measure $|P| \le \frac{C\varepsilon^{m+1}}{K}$, with $C = C(m)$, such that for all $y \in Q_{0,\varepsilon} \backslash P$ and $l \in \{0, \dots, m + 1\}$ we have

$$\varepsilon^{m+1-l} \sum_{\{i : Q_{i,\varepsilon} \in \mathcal{Q}\}} \sum_{y + \mathcal{F}_i^l} \int_{F^l} f d\mathcal{H}^l \le K \int_{Q_1(0)} f dx = 2K \int_{Q_1^+(0)} f dx. \tag{4.24}$$

Since we chose $\hat{u}$ and $\hat{v}$ with the absolute continuity properties described in Sect. 4.6 on $\overline{Q_1(0)}$ it follows that for almost every $x \in Q_{0,\varepsilon}$, with respect to the $m + 1$-dimensional Lebesgue measure, all of the functions $\hat{u}, \hat{v}, \nabla \hat{u}, \nabla \hat{v}$ are $\mathcal{H}^l$ almost everywhere defined on each of the l-dimensional faces of $x + Q_{i,\varepsilon}$ for $Q_{i,\varepsilon} \in \mathcal{Q}$ and $l = 1, \dots, m+1$. Moreover, the gradients of $\hat{u}$ and $\hat{v}$ on any l-dimensional face of $x + Q_{i,\varepsilon}$ coincide $\mathcal{H}^l$ almost everywhere with the tangential parts of $\nabla \hat{u}$ and $\nabla \hat{v}$ respectively. Thus we may choose $x = a \in Q_{0,\varepsilon}$ such that these properties hold and, provided we choose K (depending on m) sufficiently large in (4.24), such that $a_{m+1} \ge \frac{\varepsilon}{2}$ and such that we may apply (4.24) simultaneously for $f(x) = |x_{m+1}|^\beta \tilde{f}(x)$ with $\tilde{f}(x) = |\hat{u}(x) - \hat{v}(x)|^2$ and $\tilde{f}(x) = |\nabla \hat{u}(x)|^2 + |\nabla \hat{v}(x)|^2$ (where ∇ is the gradient on $\mathbb{R}^{m+1}$). In particular, after discarding the integrals in (4.24) taken over any cube faces which do not intersect $\mathbb{R}_+^{m+1}$, we have

$$\varepsilon^{m+1-l} \sum_{\left\{ i : \substack{Q_{i,\varepsilon} \in \mathcal{Q} \\ i_{m+1} \ge -1} \right\}} \sum_{a + \mathcal{F}_i^l} \int_{F^l \cap \mathbb{R}_+^{m+1}} x_{m+1}^\beta \tilde{f} d\mathcal{H}^l \le C \int_{Q_1^+(0)} x_{m+1}^\beta \tilde{f} dx. \tag{4.25}$$

Now we begin the construction of w by defining a map on the one dimensional faces of every $Q \times [0, \varepsilon]$ where $Q := (a + Q_{i,\varepsilon}) \cap \mathbb{R}_+^{m+1}$ with $Q_{i,\varepsilon} \in \mathcal{Q}$ and $i_{m+1} \ge -1$. Let E_j denote a one dimensional face of Q parallel to the jth coordinate axis for $j = 1, \dots, m+1$. Define $w(x, 0) = \hat{u}(x)$ on $Q \times \{0\}$ and $w(x, \varepsilon) = \hat{v}(x)$ on $Q \times \{\varepsilon\}$ and let $w(x, s) = (1 - \frac{s}{\varepsilon})\hat{u}(x) + \frac{s}{\varepsilon}\hat{v}(x)$ for $x \in E_j$ and $s \in [0, \varepsilon]$. Since $\hat{u}(\mathbb{R}_+^{m+1}) \subset N$ by definition, it follows that

$$\text{dist}^2(w(x, s), N) \le \max_{j=1,\dots,m+1} \sup_{E_j} |\hat{u} - \hat{v}|^2 \tag{4.26}$$

for x in the 1-dimensional edges of Q and $s \in [0, \varepsilon]$.

We now estimate $\sup_{E_j} |\hat{u} - \hat{v}|^2$ using the Sobolev embedding theorem for $W_\beta^{1,2}$ along the line segments E_j, which one can deduce analogously to the case for $W^{1,2}$ functions.

Note that in our construction so far, we have discarded any edges $E_j \subset \partial\mathbb{R}_+^{m+1}$. If E_j, $j = 1, \ldots, m+1$, lies in a $Q = (a + Q_{i,\varepsilon}) \cap \mathbb{R}_+^{m+1}$ with $i_{m+1} \geq -1$, with the exception of the case $i_{m+1} = -1$, $j = m+1$ and $\beta \in (0, 1)$, we calculate

$$
\sup_{E_j} |\hat{u} - \hat{v}|^2 \leq \frac{C}{\varepsilon^{\frac{\beta}{2} + \frac{|\beta|}{2}}} \left(\int_{E_j} x_{m+1}^\beta \left(|\partial_j \hat{u}|^2 + |\partial_j \hat{v}|^2 \right) dx_j \right)^{\frac{1}{2}} \left(\int_{E_j} x_{m+1}^\beta |\hat{u} - \hat{v}|^2 dx_j \right)^{\frac{1}{2}}
$$
$$
+ \frac{C}{\varepsilon^{1 + \frac{\beta}{2} + \frac{|\beta|}{2}}} \int_{E_j} x_{m+1}^\beta |\hat{u} - \hat{v}|^2 dx_j. \tag{4.27}
$$

If $\beta > 0$, $i_{m+1} = -1$ and $j = m+1$ then we calculate

$$
\sup_{E_{m+1}} |\hat{u} - \hat{v}|^2
$$
$$
\leq C \left(\int_{E_{m+1}} x_{m+1}^\beta (|\partial_{m+1}\hat{u}|^2 + |\partial_{m+1}\hat{v}|^2) dx_{m+1} \right)^{\frac{1}{p}} \left(\int_{E_{m+1}} x_{m+1}^\beta |\hat{u} - \hat{v}|^2 dx_{m+1} \right)^{1 - \frac{1}{p}}
$$
$$
+ C \varepsilon^{-(1+\beta)} \int_{E_{m+1}} x_{m+1}^\beta |\hat{u} - \hat{v}|^2 dx_{m+1}, \tag{4.28}
$$

for any $p \in (1, \frac{2}{1+\beta})$. The combination of (4.26), (4.27) and (4.28) with (4.25), applied with $l = 1$, yields

$$
\text{dist}^2(w(x, s), N) \leq \frac{C}{\varepsilon^{m + \frac{\beta}{2} + \frac{|\beta|}{2}}} \left(\int_{Q_1^+(0)} x_{m+1}^\beta \left(|\nabla\hat{u}|^2 + |\nabla\hat{v}|^2 \right) dx \right)^{\frac{1}{q}}
$$
$$
\left(\int_{Q_1^+(0)} x_{m+1}^\beta |\hat{u} - \hat{v}|^2 dx \right)^{1 - \frac{1}{q}}
$$
$$
+ \frac{C}{\varepsilon^{m+1 + \frac{\beta}{2} + \frac{|\beta|}{2}}} \int_{Q_1^+(0)} x_{m+1}^\beta |\hat{u} - \hat{v}|^2 dx, \tag{4.29}
$$

where $q \in \{p, 2\}$, for p fixed as above, depends on β.

Next we bound the gradient of w on the product of the 1-dimensional edges of Q with $[0, \varepsilon]$. Let $\overline{\nabla}$ denote the gradient on $E_j \times [0, \varepsilon]$. Recall that $\hat{u}, \hat{v}$ are defined so that the tangential parts of their gradients $\nabla\hat{u}, \nabla\hat{v}$ on $\mathbb{R}_+^{m+1}$ coincide $\mathcal{H}^1$ almost everywhere with their gradients on the edges E_j. It follows that

$$
\sup_{s \in [0,\varepsilon]} |\overline{\nabla} w(x, s)|^2 \leq 8 \left(|\nabla\hat{u}(x)|^2 + |\nabla\hat{v}(x)|^2 \right) + \frac{2}{\varepsilon^2} |\hat{u}(x) - \hat{v}(x)|^2,
$$

for x in any edge E_j, $j = 1, \ldots, m+1$, of Q. Integrating over $E_j \times [0, \varepsilon]$ with respect to $x_{m+1}^\beta dx_j ds$ yields

$$
\int_{E_j \times [0,\varepsilon]} x_{m+1}^\beta |\overline{\nabla} w|^2 dx_j ds \leq 8\varepsilon \int_{E_j} x_{m+1}^\beta (|\nabla\hat{u}|^2 + |\nabla\hat{v}|^2) dx_j + \frac{2}{\varepsilon} \int_{E_j} x_{m+1}^\beta |\hat{u} - \hat{v}|^2 dx_j. \tag{4.30}
$$

Consider again $Q \times [0, \varepsilon]$ for $Q = (a + Q_{i,\varepsilon}) \cap \mathbb{R}_+^{m+1}$ with $Q_{i,\varepsilon} \in \mathcal{Q}$ and $i_{m+1} \geq -1$. Recall that we are excluding cube faces in $\partial\mathbb{R}_+^{m+1}$ from our construction. We use a slightly different procedure to extend w to higher dimensions depending on whether $i_{m+1} = -1$ or $i_{m+1} \geq 0$. Accordingly we introduce some temporary notation for two classes of F^l that we

consider. Let $F_\perp^l$ denote any l-dimensional face of any Q with no edges in the $m+1$ direction and let F_{m+1}^l denote any face of any such Q with edges in the $m+1$ direction. Suppose that $l \geq 2$ and w is already defined with L^2 gradient on every $F_\perp^l$ and $F_\perp^{l-1} \times [0, \varepsilon]$, square $x_{m+1}^\beta d\mathcal{H}^l$-integrable gradient on every F_{m+1}^l and square $x_{m+1}^\beta d\mathcal{H}^{l-1}ds$-integrable gradient on $F_{m+1}^{l-1} \times [0, \varepsilon]$. Furthermore, suppose that $w(x, 0) = \hat{u}(x)$ and $w(x, \varepsilon) = \hat{v}(x)$ for $x \in F_\perp^l$ or $x \in F_{m+1}^l$. These assumptions imply that w is defined $\mathcal{H}^l$ almost everywhere on all the l-dimensional faces of Q for $l \geq 2$. Since $\partial(F_\perp^l \times [0, \varepsilon])$ and $\partial^+(F_{m+1}^l \times [0, \varepsilon])$ are the union of such l-dimensional faces, w is defined $\mathcal{H}^l$ almost everywhere on these sets. If Q is such that $i_{m+1} \geq 0$ then we do not distinguish between $F_\perp^l$ and F_{m+1}^l and extend w to each $F_\perp^l \times [0, \varepsilon]$ and $F_{m+1}^l \times [0, \varepsilon]$ by homogeneous extension of degree zero with respect to $(y, \frac{\varepsilon}{2})$, where y is the centre point of $F_\perp^l$ or F_{m+1}^l. If $i_{m+1} = -1$ then we extend w into $F_\perp^l \times [0, \varepsilon]$ using the same method. We extend w homogeneously of degree 0 from $\partial^+(F_{m+1}^l \times [0, \varepsilon])$ into $F_{m+1}^l \times [0, \varepsilon]$ with respect to the point $(y^+, \frac{\varepsilon}{2})$, where y is the centre point of F_{m+1}^l and $y^+ = y - (0, y_{m+1})$.

Now let F^l denote any l-dimensional face of any Q again. Since the tangential parts of the gradients $\nabla \hat{u}, \nabla \hat{v}$ on $\mathbb{R}^{m+1}$ coincide with the gradients of $\hat{u}$ and $\hat{v}$ on F^l for $\mathcal{H}^l$ almost every $x \in F^l$, using the fact that $\hat{u}$ and $\hat{v}$ are homogeneous of degree zero, we calculate

$$\int_{F^l \times [0, \varepsilon]} x_{m+1}^\beta |\overline{\nabla} w|^2 d\mathcal{H}^l ds \leq C\varepsilon \int_{F^l} x_{m+1}^\beta (|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2) d\mathcal{H}^l$$
$$+ C\varepsilon \sum_{a + \mathcal{F}_i^{l-1}} \int_{F^{l-1} \times [0, \varepsilon]} x_{m+1}^\beta |\overline{\nabla} w|^2 d\mathcal{H}^{l-1} ds, \qquad (4.31)$$

where $\overline{\nabla}$ is the gradient on the set it is integrated over. From (4.31), we inductively deduce that for any $l \in \{2, \dots, m+1\}$ we can extend w to each $F^l \times [0, \varepsilon]$ in $Q \times [0, \varepsilon]$ so that w has an L^2 or $x_{m+1}^\beta d\mathcal{H}^l ds$-integrable gradient $\overline{\nabla} w$ on these faces. Moreover, $\overline{\nabla} w$ satisfies

$$\int_{F^l \times [0, \varepsilon]} x_{m+1}^\beta |\overline{\nabla} w|^2 d\mathcal{H}^l ds \leq C\varepsilon^{l-1} \sum_{a + \mathcal{F}_i^1} \int_{F^1 \times [0, \varepsilon]} x_{m+1}^\beta |\overline{\nabla} w|^2 d\mathcal{H}^1 ds$$
$$+ C \sum_{j=1}^{l} \varepsilon^{l-j+1} \sum_{a + \mathcal{F}_i^j} \int_{F^j} x_{m+1}^\beta (|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2) d\mathcal{H}^j.$$
$$(4.32)$$

So far, we have constructed a map $w = w^{i, \varepsilon}$ on each cube and rectangle $Q = (a + Q_{i, \varepsilon}) \cap \mathbb{R}_+^{m+1}$ such that $Q_{i, \varepsilon} \in \mathcal{Q}$ with $i_{m+1} \geq -1$. It follows from the construction that $w^{(i, \varepsilon)} = w^{(j, \varepsilon)}$ $\mathcal{H}^{l+1}$-almost everywhere on common faces $F^l \times [0, \varepsilon]$ of $(a + Q_{i, \varepsilon}) \cap \mathbb{R}_+^{m+1}$ and $(a + Q_{j, \varepsilon}) \cap \mathbb{R}_+^{m+1}$. Furthermore, for $0 < \varepsilon < \frac{1}{8}$ it follows that

$$Q_{\frac{1}{4}}^+(0) \subset \bigcup_{\left\{i : \substack{Q_{i, \varepsilon} \in \mathcal{Q} \\ i_{m+1} \geq -1}\right\}} a + Q_{i, \varepsilon}.$$

We may therefore define $w \in W_\beta^{1, 2}(Q_{\frac{1}{4}}^+(0) \times [0, \varepsilon]; \mathbb{R}^n)$ by $w|_{(a + Q_{i, \varepsilon}) \cap \mathbb{R}_+^{m+1}}(x, s) = w^{(i, \varepsilon)}(x, s)$ for $s \in [0, \varepsilon]$. Since w is homogeneous of degree 0 on any l-dimensional face of any $Q \times [0, \varepsilon]$ with $l \geq 3$, our inductive procedure preserves (4.29) for all (x, s) in

$Q^+_{\frac{1}{4}}(0) \times [0, \varepsilon]$, with the possible exception of a set P of m-dimensional Hausdorff measure 0. It follows from (4.29) that for $(x, s) \in (Q^+_{\frac{1}{4}}(0) \times [0, \varepsilon]) \backslash P$ we have

$$
\text{dist}^2(w(x, s), N) \le \frac{C}{\varepsilon^{m + \frac{\beta}{2} + \frac{|\beta|}{2}}} \left(\int_{Q^+_1(0)} x^\beta_{m+1} \left(|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2 \right) dx \right)^{\frac{1}{q}}
$$
$$
\left(\int_{Q^+_1(0)} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 dx \right)^{1 - \frac{1}{q}}
$$
$$
+ \frac{C}{\varepsilon^{m + 1 + \frac{\beta}{2} + \frac{|\beta|}{2}}} \int_{Q^+_1(0)} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 dx \tag{4.33}
$$

where $q \in \{2, p\}$ for some fixed $p \in (1, \frac{2}{1+\beta})$. Moreover, we combine (4.30), (4.32) and (4.25) to see that

$$
\int_{Q^+_{\frac{1}{4}}(0) \times [0, \varepsilon]} x^\beta_{m+1} |\overline{\nabla} w|^2 dx ds \le C\varepsilon \int_{Q^+_1(0)} x^\beta_{m+1} \left(|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2 \right) dx
$$
$$
+ \frac{C}{\varepsilon} \int_{Q^+_1(0)} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 dx. \tag{4.34}
$$

The definition of w as required now follows from combining (4.33) and (4.34) with (4.22) and (4.23). The absolute continuity properties, described in Sect. 4.6, of w, viewed as a function defined on a rectangle in polar coordinates, guarantee that for almost every $\rho \in [\frac{1}{8}, \frac{1}{4}]$, w has square $x^\beta_{m+1} d\mathcal{H}^m ds$-integrable gradient on $\partial^+ B^+_\rho(0) \times [0, \varepsilon]$ which coincides $\mathcal{H}^m ds$ almost everywhere with the tangential part of $\overline{\nabla} w$. Using Fubini's theorem and Chebychev's inequality, applied to the map $\rho \mapsto \int_{\partial^+ B^+_\rho(0) \times [0,\varepsilon]} x^\beta_{m+1} |\overline{\nabla} w|^2 d\mathcal{H}^m ds$, we may therefore choose $\rho \in [\frac{1}{8}, \frac{1}{4}]$ such that w has square $x^\beta_{m+1} d\mathcal{H}^m ds$-integrable gradient on $\partial^+ B^+_\rho(0) \times [0, \varepsilon]$ and satisfies

$$
\int_{\partial^+ B^+_\rho(0) \times [0,\varepsilon]} x^\beta_{m+1} |\overline{\nabla} w|^2 d\mathcal{H}^m ds \le C \int_{B^+_{\frac{1}{4}}(0) \times [0,\varepsilon]} x^\beta_{m+1} |\overline{\nabla} w|^2 d\mathcal{H}^{m+1} ds.
$$

We define $\tilde{w}$ on $\mathbb{S}^m_+ \times [0, \varepsilon]$ by $\tilde{w}(\omega, s) = w(\rho\omega, s)$ and observe that this map has the required properties. $\qquad\square$

4.8 Comparison maps

With Lemma 4.16 in hand, we may now construct comparison maps for $W^{1,2}_\beta$ functions which have values in N and are defined on half-balls centred in $\partial \mathbb{R}^{m+1}_+$, provided the re-scaled energy is sufficiently small. We use the notation $\overline{v}_{B^+_\rho(y), \beta} = \left(\int_{B^+_\rho(y)} x^\beta_{m+1} dx \right)^{-1} \int_{B^+_\rho(y)} x^\beta_{m+1} v dx$.

Lemma 4.17 *There exists a $\delta_0 = \delta_0(m, N, \beta) > 0$ such that the following holds. Let $\varepsilon \in (0, 1)$ and $v \in W^{1,2}_\beta(B^+_\rho(y); N)$ with $\rho^{1-m-\beta} \int_{B^+_\rho(y)} x^\beta_{m+1} |\nabla v|^2 dx \le \delta^2_0 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}$, where $y \in \partial \mathbb{R}^{m+1}_+$. Then there is a $\sigma \in (\frac{3\rho}{4}, \rho)$ such that we can find a map $w_\varepsilon \in W^{1,2}_\beta(B^+_\rho(y); N)$ which agrees with v in $B^+_\rho(y) \backslash B^+_\sigma(y)$ and such that*

$$
\sigma^{1-m-\beta} \int_{B^+_\sigma(y)} x^\beta_{m+1} |\nabla w|^2 dx \le C\varepsilon \rho^{1-m-\beta} \int_{B^+_\rho(y)} x^\beta_{m+1} |\nabla v|^2 dx
$$

$$+ \frac{C}{\varepsilon} \rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 dx \quad (4.35)$$

for a constant $C = C(m, \beta)$.

Proof We follow the proof of Corollary 1 in Section 2.7 of [39]. Throughout, C denotes a constant which depends on m and possibly β and we only distinguish different C when necessary. We also assume, without loss of generality, that $\varepsilon \leq \frac{1}{2}$.

Let $\delta_0 > 0$ to be chosen as required and suppose the assumptions of the lemma hold for δ_0. As a consequence of Poincaré Inequality for the A_2 weights $|x_{m+1}|^\beta$ [19], we have

$$\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 dx$$
$$\leq C\rho^{1-m-\beta} \int_{B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 dx \leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \quad (4.36)$$

As discussed in Sect. 4.6, we may work with a representative of v, which we do not relabel, such that $v(B_\rho^+(y)) \subset N$. It follows that

$$\text{dist}^2(\overline{v}_{B_\rho^+(y),\beta}, N) \leq |v(x) - \overline{v}_{B_\rho^+(y),\beta}|^2 \quad (4.37)$$

for every $x \in B_\rho^+(y)$. Integrating (4.37) over $B_\rho^+(y)$ with respect to $x_{m+1}^{\frac{\beta}{2}+\frac{|\beta|}{2}} dx$ and then dividing by $\int_{B_\rho^+(y)} x_{m+1}^{\frac{\beta}{2}+\frac{|\beta|}{2}} dx$ we see that

$$\text{dist}^2(\overline{v}_{B_\rho^+(y),\beta}, N) \leq C\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 dx. \quad (4.38)$$

Combining (4.38) with (4.36) we find

$$\text{dist}^2(\overline{v}_{B_\rho^+(y),\beta}, N) \leq C\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 dx \leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}.$$

Hence, we may choose $\lambda \in N$ such that

$$|\lambda - \overline{v}_{B_\rho^+(y),\beta}|^2 \leq C\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 dx$$
$$\leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \quad (4.39)$$

Using Chebychev's inequality, we choose a $C > 0$ such that there exists $\sigma \in (\frac{3\rho}{4}, \rho)$ such that $\hat{v} \in W_\beta^{1,2}(\mathbb{S}_+^m; N)$, where $\hat{v}(\omega) = v(\sigma\omega + y)$, and such that

$$\int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |D\hat{v}|^2 d\omega \leq C\sigma^{2-m-\beta} \int_{\partial^+ B_\sigma^+(y)} x_{m+1}^\beta |\nabla v|^2 dS(x)$$
$$\leq C\rho^{1-m-\beta} \int_{B_\rho^+(y) \backslash B_{\frac{\rho}{2}}^+(y)} x_{m+1}^\beta |\nabla v|^2 dx$$
$$\leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}} \quad (4.40)$$

where D is the gradient on $\mathbb{S}_+^m$, and

$$\int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |\hat{v} - \overline{v}_{B_\rho^+(y),\beta}|^2 d\omega \leq \sigma^{-m-\beta} \int_{\partial^+ B_\sigma^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 dS(x)$$

$$\leq C\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)\setminus B_{\frac{\rho}{2}}^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}x$$

$$\leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \tag{4.41}$$

We may therefore apply Lemma 4.16 to $\hat{v} \in W_\beta^{1,2}(\mathbb{S}_+^m; N)$ and λ. This yields a $w_0 \colon \mathbb{S}_+^m \times [0, \varepsilon] \to \mathbb{R}^n$ with $w_0 = \hat{v}$ on $\mathbb{S}_+^m \times \{0\}$ and $w_0 = \lambda$ on $\mathbb{S}_+^m \times \{\varepsilon\}$ in the sense of traces. Furthermore, (4.20) yields

$$\int_{\mathbb{S}_+^m \times [0,\varepsilon]} \omega_{m+1}^\beta |\overline{D}w_0|^2 \mathrm{d}\omega \mathrm{d}s \leq C\varepsilon \int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |D\hat{v}|^2 \mathrm{d}\omega$$

$$+ \frac{C}{\varepsilon} \int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |\hat{v} - \lambda|^2 \mathrm{d}\omega, \tag{4.42}$$

where $\overline{D}$ is the gradient on $\mathbb{S}_+^m \times [0, \varepsilon]$ and D is the gradient on $\mathbb{S}_+^m$. In addition, (4.21) implies that

$$\mathrm{dist}^2(w_0(\omega, s), N) \leq \frac{C}{\varepsilon^{m+\frac{\beta}{2}+\frac{|\beta|}{2}}} \left(\int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |D\hat{v}|^2 \mathrm{d}\omega \right)^{\frac{1}{q}} \left(\int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |\hat{v} - \lambda|^2 \mathrm{d}\omega \right)^{1-\frac{1}{q}}$$

$$+ \frac{C}{\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}} \int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |\hat{v} - \lambda|^2 \mathrm{d}\omega \tag{4.43}$$

for every $(\omega, s) \in \mathbb{S}_+^m \times [0, \varepsilon]$, where $q \in (1, 2]$ depends on β. Henceforth we assume that $\delta_0 \leq 1$. Using (4.39) and (4.41) we deduce that

$$\int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |\hat{v} - \lambda|^2 \mathrm{d}\omega \leq 2 \int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |\hat{v} - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}\omega + 2 \int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |\overline{v}_{B_\rho^+(y),\beta} - \lambda|^2 \mathrm{d}\omega$$

$$\leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \tag{4.44}$$

The combination of (4.43) with (4.40) and (4.44) yields

$$\mathrm{dist}(w_0(x, s), N) \leq C\delta_0 \tag{4.45}$$

for every $(\omega, s) \in \mathbb{S}_+^m \times [0, \varepsilon]$.

Choose δ_0, depending on N, m, β, such that $C\delta_0 \leq \hat{\alpha}$ where C is the constant in (4.45) and $\hat{\alpha} > 0$ is sufficiently small to guarantee that the nearest point projection π_N onto N exists and has bounded derivatives in $N_{\hat{\alpha}} = \{x \in \mathbb{R}^n \colon \mathrm{dist}(x, N) \leq \hat{\alpha}\}$. It then follows from (4.45) that we may apply π_N to w_0. Let $\omega \in \mathbb{S}_+^m$ satisfy $\omega = \omega(x) = \frac{x-y}{|x-y|}, r = |x - y|$ and define $w \in W_\beta^{1,2}(B_\rho^+(y); N)$ by

$$w(x) = w(y + r\omega(x)) = \begin{cases} v(y + r\omega(x)) & r \in (\sigma, \rho) \\ \pi_N(w_0(\omega(x), (1 - \frac{r}{\sigma}))) & r \in [(1 - \varepsilon)\sigma, \sigma] \\ \lambda & r \in (0, (1 - \varepsilon)\sigma). \end{cases}$$

Note that w agrees with v in $B_\rho^+(y) \setminus B_\sigma^+(y)$. We then readily calculate that w satisfies (4.35) as required. $\qquad\square$

4.9 Control of the mean squared oscillation

The Euler–Lagrange equations of E^β satisfy the structural conditions $|\mathrm{div}(x_{m+1}^\beta \nabla v)| \leq c_0 x_{m+1}^\beta |\nabla v|^2$, together with the Neumann condition (4.2). For functions satisfying such

conditions, if the re-scaled energy is sufficiently small it is possible to control their mean squared oscillation using the energy as follows.

Lemma 4.18 *For every $\delta > 0$ and every $c_0 > 0$ there exist two constants $\varepsilon = \varepsilon(m, n, \delta, c_0) > 0$ and $\theta = \theta(m, n, \delta, c_0) \in (0, \frac{1}{4}]$ such that the following holds. Let $x_0 \in \partial\mathbb{R}_+^{m+1}$, $R > 0$ and $B_R^+(x_0) \subset \mathbb{R}_+^{m+1}$. Suppose $v \in W_\beta^{1,2}(B_R^+(x_0); \mathbb{R}^n)$ satisfies*

$$\left| \int_{B_R^+(x_0)} x_{m+1}^\beta \langle \nabla v, \nabla \phi \rangle \, dx \right| \leq c_0 \int_{B_R^+(x_0)} x_{m+1}^\beta |\phi| \, |\nabla v|^2 \, dx$$

for every $\phi \in C_0^\infty(B_R(x_0); \mathbb{R}^n)$. If

$$R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 dx \leq \varepsilon$$

then

$$(\theta R)^{-(1+m+\beta)} \int_{B_{\theta R}^+(x_0)} x_{m+1}^\beta |v - \overline{v}_{B_{\theta R}^+(x_0),\beta}|^2 dx \leq \delta R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \, dx.$$

Proof The proof of the lemma is based on a blow-up procedure analogous to that of the proof of Lemma 3.5 in [29] for example.

Observe that the statement of the lemma is invariant under rescaling and translation by any point in $\partial\mathbb{R}_+^{m+1}$; henceforth we assume $R = 1$ and $x_0 = 0$. Suppose, for a contradiction, that there exist $\delta > 0$ and $c_0 > 0$ such that the claim is false. Then for any $\theta \in (0, \frac{1}{4}]$ there is a sequence of maps $(v_k)_{k\in\mathbb{N}}$, with $v_k \in W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n)$ for every k, such that

$$\left| \int_{B_1^+(0)} x_{m+1}^\beta \langle \nabla v_k, \nabla \phi \rangle \, dx \right| \leq c_0 \int_{B_1^+(0)} x_{m+1}^\beta |\phi| |\nabla v_k|^2 dx \tag{4.46}$$

for every $\phi \in C_0^\infty(B_1(0); \mathbb{R}^n)$ and

$$\int_{B_1^+(0)} x_{m+1}^\beta |\nabla v_k|^2 dx := \varepsilon_k \to 0 \text{ as } k \to \infty$$

but

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |v_k - \overline{(v_k)}_{B_\theta^+(0),\beta}|^2 dx > \delta \int_{B_1^+(0)} x_{m+1}^\beta |\nabla v_k|^2 dx = \delta\varepsilon_k. \tag{4.47}$$

Consider the normalised sequence $(w_k)_{k\in\mathbb{N}}$ defined by $w_k = \varepsilon_k^{-\frac{1}{2}}(v_k - \overline{(v_k)}_{B_\theta^+(0),\beta})$. Then $\nabla w_k = \varepsilon_k^{-\frac{1}{2}} \nabla v_k$ and thus

$$\int_{B_1^+(0)} x_{m+1}^\beta |\nabla w_k|^2 dx = 1 \quad \text{and} \quad \overline{(w_k)}_{B_\theta^+(0),\beta} = 0. \tag{4.48}$$

Furthermore, we deduce from (4.47) that

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w_k|^2 \, dx > \delta. \tag{4.49}$$

Using (4.48) and the Poincaré inequality for A_2 weights, we deduce that the sequence $(w_k)_{k\in\mathbb{N}}$ is bounded in $W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n)$. Hence, the Compactness Lemma, Lemma 2.5,

yields a subsequence $(w_{k_j})_{j \in \mathbb{N}}$ which converges weakly in $W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n)$ and strongly in $L_\beta^2(B_1^+(0); \mathbb{R}^n)$ to a $w \in W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n)$.

In view of (4.46) and (4.48) we calculate

$$\left| \int_{B_1^+(0)} x_{m+1}^\beta \langle \nabla w_k, \nabla \phi \rangle \, dx \right| \leq c_0 ||\phi||_{L^\infty(B_1^+(0);\mathbb{R}^n)} \varepsilon_k^{\frac{1}{2}}$$

for every $\phi \in C_0^\infty(B_1(0); \mathbb{R}^n)$. Since $w_{k_j} \rightharpoonup w$ in $W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n)$, it follows that

$$\left| \int_{B_1^+(0)} x_{m+1}^\beta \langle \nabla w, \nabla \phi \rangle dx \right| = \lim_{j \to \infty} \left| \int_{B_1^+(0)} x_{m+1}^\beta \langle \nabla w_{k_j}, \nabla \phi \rangle dx \right|$$

$$\leq c_0 ||\phi||_{L^\infty(B_1^+(0);\mathbb{R}^n)} \lim_{j \to \infty} \varepsilon_{k_j}^{\frac{1}{2}} = 0$$

for every $\phi \in C_0^\infty(B_1(0); \mathbb{R}^n)$. Hence w is a weak solution of the linear Neumann-type problem (2.11) and, in particular, satisfies (2.12) from Lemma 2.7 in $B_1^+(0)$.

We also conclude, using the Compactness Lemma to take limits in (4.48) and (4.49), that

$$\int_{B_1^+(0)} x_{m+1}^\beta |\nabla w|^2 \, dx \leq 1 \quad \text{and} \quad \overline{w}_{B_\theta^+(0),\beta} = 0$$

and

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 \, dx \geq \delta \tag{4.50}$$

respectively. Now, since $\overline{w}_{B_\theta^+(0),\beta} = 0$, the Poincaré inequality yields

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 dx \leq C\theta^{1-m-\beta} \int_{B_\theta^+(0)} x_{m+1}^\beta |\nabla w|^2 dx. \tag{4.51}$$

We apply Lemma 2.7 to w with $\theta \leq \frac{1}{4}$ (so that $2\theta \leq \frac{1}{2}$). This gives a positive constant C (independent of θ) and a $\gamma \in (0, 1)$ such that

$$\theta^{1-m-\beta} \int_{B_\theta^+(0)} x_{m+1}^\beta |\nabla w|^2 dx \leq C(2\theta)^{2\gamma}. \tag{4.52}$$

Combining (4.51) and (4.52) we see that

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 dx \leq C(2\theta)^{2\gamma}. \tag{4.53}$$

This holds for all fixed $\theta \in (0, \frac{1}{4}]$ and we choose $\theta < \frac{1}{2} \left(\frac{\delta}{C} \right)^{\frac{1}{2\gamma}}$ so that (4.53) contradicts (4.50). $\qquad\square$

Remark 4.19 We could have used Lemma 2.8 in place of Lemma 2.7 to the same effect. In using the latter lemma, we observe that Hölder continuity of solutions to the linear Neumann-type problem (2.11) is sufficient to obtain energy decay and consequently Hölder continuity of minimisers of E^β relative to $\mathcal{O}$; we do not need higher regularity for the linear problem at this point.

4.10 Energy decay

We combine our construction of comparison maps in Sect. 4.8 with the improved control of the mean squared oscillation obtained in Sect. 4.9 in order to show that the re-scaled energy decays faster than implied by the boundary monotonicity formula, Lemma 4.5.

Lemma 4.20 *Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ be a minimiser of E^β relative to $\mathcal{O}$. Suppose $B^+_R(x_0)$ is a half-ball with $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$. There exist $\varepsilon_0 = \varepsilon_0(m, N, \beta) > 0$ and $\theta_0 = \theta_0(m, N, \beta) \in (0, \frac{1}{4})$ such that if*

$$R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 dx \leq \varepsilon_0,$$

then

$$(\theta_0 r)^{1-m-\beta} \int_{B^+_{\theta_0 r}(y)} x^\beta_{m+1} |\nabla v|^2 dx \leq \frac{1}{2} r^{1-m-\beta} \int_{B^+_r(y)} x^\beta_{m+1} |\nabla v|^2 dx,$$

for every $B^+_r(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$.

Proof Let $B^+_\rho(y) \subset B^+_r(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$. Then $\rho \leq r \leq \frac{R}{2}$, $y \in \mathcal{O}$ and $|x_0 - y| < \frac{R}{2}$. Suppose v satisfies $R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 dx \leq \varepsilon_0$ for $\varepsilon_0 > 0$ to be chosen. Then for any $\rho \in (0, r]$ the monotonicity formula, Lemma 4.5, yields

$$\rho^{1-m-\beta} \int_{B^+_\rho(y)} x^\beta_{m+1} |\nabla v|^2 dx \leq r^{1-m-\beta} \int_{B^+_r(y)} x^\beta_{m+1} |\nabla v|^2 dx$$

$$\leq \left(\frac{R}{2}\right)^{1-m-\beta} \int_{B^+_{\frac{R}{2}}(y)} x^\beta_{m+1} |\nabla v|^2 dx$$

$$\leq C\varepsilon_0. \tag{4.54}$$

We apply Lemma 4.17 on $B^+_\rho(y) \subset B^+_r(y)$, with $\rho \leq r$ to be chosen later. This gives a δ_0 such that for any $\varepsilon \in (0, 1)$, if

$$\rho^{1-m-\beta} \int_{B^+_\rho(y)} x^\beta_{m+1} |\nabla v|^2 dx \leq \delta^2_0 \varepsilon^{1+m+\frac{\beta}{2}+\frac{|\beta|}{2}} \tag{4.55}$$

then there is a $\sigma \in (\frac{3\rho}{4}, \rho)$ such that we can find a $w_\varepsilon \in W^{1,2}_\beta(B^+_\rho(y); N)$ which agrees with v in $B^+_\rho(y) \backslash B^+_\sigma(y)$ and satisfies

$$\sigma^{1-m-\beta} \int_{B^+_\sigma(y)} x^\beta_{m+1} |\nabla w|^2 dx \leq C\varepsilon \rho^{1-m-\beta} \int_{B^+_\rho(y)} x^\beta_{m+1} |\nabla v|^2 dx$$

$$+ \frac{1}{\varepsilon} C\rho^{-(1+m+\beta)} \int_{B^+_\rho(y)} x^\beta_{m+1} |v - \bar{v}_{B^+_\rho(y),\beta}|^2 dx. \tag{4.56}$$

Assuming (4.55) and consequently (4.56) hold, we make use of the comparison property of w. Since $v = w$ in $B^+_\rho(y) \backslash B^+_\sigma(y)$ we may extend w to an element of $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ by requiring $w = v$ on $\mathbb{R}^{m+1}_+ \backslash B^+_\rho(y)$. As v is a minimiser of E^β relative to $\mathcal{O}$, we deduce that $\int_{B^+_\sigma(y)} x^\beta_{m+1} |\nabla v|^2 dx \leq \int_{B^+_\sigma(y)} x^\beta_{m+1} |\nabla w|^2 dx$. Combining this fact with the monotonicity

formula, Lemma 4.5, and (4.56) gives

$$\left(\frac{3\rho}{4}\right)^{1-m-\beta} \int_{B_{\frac{3\rho}{4}}^+(y)} x_{m+1}^{\beta} |\nabla v|^2 dx \leq C\varepsilon \rho^{1-m-\beta} \int_{B_{\rho}^+(y)} x_{m+1}^{\beta} |\nabla v|^2 dx$$

$$+ \frac{1}{\varepsilon} C\rho^{-(1+m+\beta)} \int_{B_{\rho}^+(y)} x_{m+1}^{\beta} |v - \overline{v}_{B_{\rho}^+(y),\beta}|^2 dx. \tag{4.57}$$

Fix $\varepsilon = \min\{\frac{1}{4}, \frac{1}{4C}\}$, where C is the constant in (4.57) and let $\varepsilon_0 \leq \frac{1}{C}\delta_0^2 \varepsilon^{1+m+\frac{\beta}{2}+\frac{|\beta|}{2}}$ where C is the constant from (4.54). It follows from (4.54) that (4.55) is satisfied and hence, substituting this ε into (4.57), we have

$$\left(\frac{3\rho}{4}\right)^{1-m-\beta} \int_{B_{\frac{3\rho}{4}}^+(y)} x_{m+1}^{\beta} |\nabla v|^2 dx \leq \frac{1}{4} \rho^{1-m-\beta} \int_{B_{\rho}^+(y)} x_{m+1}^{\beta} |\nabla v|^2 dx$$

$$+ \hat{C}\rho^{-(1+m+\beta)} \int_{B_{\rho}^+(y)} x_{m+1}^{\beta} |v - \overline{v}_{B_{\rho}^+(y),\beta}|^2 dx \tag{4.58}$$

for a constant $\hat{C}$ and any $\rho \leq r \leq \frac{R}{2}$. Observe that

$$\left| \int_{B_r^+(y)} x_{m+1}^{\beta} \langle \nabla v, \nabla \phi \rangle \, dx \right| = \left| \int_{B_r^+(y)} x_{m+1}^{\beta} \langle \phi, A(v)(\nabla v, \nabla v) \rangle \, dx \right|$$

$$\leq c_0 \int_{B_r^+(y)} x_{m+1}^{\beta} |\phi| |\nabla v|^2 dx,$$

for every $\phi \in C_0^\infty(B_r(y); \mathbb{R}^n)$ on every $B_r^+(y) \subset B_R^+(x_0)$ where $c_0 = c_0(m, N)$. Hence, we may apply Lemma 4.18 for $\delta = \min\{\frac{1}{4}, \frac{1}{4\hat{C}}\}$ and c_0 as above to obtain a corresponding $\varepsilon_1 > 0$ and $\theta_1 \in (0, \frac{1}{4}]$ such that if $r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^{\beta} |\nabla v|^2 dx \leq \varepsilon_1$ then

$$(\theta_1 r)^{-(1+m+\beta)} \int_{B_{\theta_1 r}^+(y)} x_{m+1}^{\beta} |v - \overline{v}_{B_{\theta_1 r}^+(y),\beta}|^2 dx$$

$$\leq \frac{1}{4\hat{C}} r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^{\beta} |\nabla v|^2 dx. \tag{4.59}$$

Now choose $\varepsilon_0 = \frac{1}{C} \min\{\delta_0^2 \varepsilon^{1+m+\frac{\beta}{2}+\frac{|\beta|}{2}}, \varepsilon_1\}$ where C is the constant from (4.54). It follows that (4.58) and (4.59) hold on any $B_{\rho}^+(y) \subset B_r^+(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$. Thus we may apply (4.58) with $\rho = \theta_1 r$. In turn, assuming this choice of ρ, we combine (4.58) with the monotonicity formula and (4.59) to see that

$$\left(\frac{3\theta_1 r}{4}\right)^{1-m-\beta} \int_{B_{\frac{3\theta_1 r}{4}}^+(y)} x_{m+1}^{\beta} |\nabla v|^2 dx \leq \frac{1}{2} r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^{\beta} |\nabla v|^2 dx.$$

Hence the lemma is proved with the above choice of ε_0 and $\theta_0 = \frac{3\theta_1}{4}$. $\square$

4.11 ε-Regularity as far as Hölder continuity

The culmination of the results in this section so far lead to the following ε-regularity theorem for minimisers of E^β relative to $\mathcal{O}$ which establishes the first part of Theorem 4.1.

Theorem 4.21 *If $m \geq 3$, let $\beta \in (-1, 1)$ and if $m = 2$ let $\beta \in (-3^{-1}, 1)$. Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ be a minimiser of E^β relative to $\mathcal{O}$. Suppose $B^+_R(x_0)$ satisfies $R \leq 1$ and $\partial^0 B^+_R(x_0) \subset \mathcal{O}$. There exists an $\varepsilon = \varepsilon(m, N, \beta) > 0$ and a $\theta = \theta(m, N, \beta) \in (0, 1)$ such that if*

$$R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \leq \varepsilon,$$

then $v \in C^{0,\gamma}(\overline{B^+_{\theta R}(x_0)}; N)$ for some $\gamma = \gamma(m, N, \beta) \in (0, 1)$. In particular,

$$|v(x_1) - v(x_2)| \leq C \left(R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\frac{|x_1 - x_2|}{R} \right)^\gamma \quad (4.60)$$

for every $x_1, x_2 \in B^+_{\theta R}(x_0)$ and a constant $C = C(m, N, \beta)$.

Proof Throughout the proof we adopt the convention that all constants depend only on m, N and β unless stated otherwise. We reinforce this dependence where appropriate.

Let v be a minimiser of E^β relative to $\mathcal{O}$ with $R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \leq \varepsilon$ and set $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$, where ε_0 is the number from Lemma 4.20 and ε_1 is the number from Lemma 4.9.

Observe that the function $\tilde{r} \mapsto \tilde{r}^{1-m-\beta} \int_{B^+_{\tilde{r}}(z)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x$ is non-decreasing on $(0, \frac{R}{2}]$ by the monotonicity formula, Lemma 4.5. Furthermore, the choice of ε allows us to apply Lemma 4.20. We apply this lemma, together with Lemma 8.23 of [15] to deduce that on every $B^+_{\tilde{r}}(z) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$ we have

$$\tilde{r}^{1-m-\beta} \int_{B^+_{\tilde{r}}(z)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \leq C \left(2\frac{\tilde{r}}{R} \right)^{\gamma_0} \left(\frac{R}{2} \right)^{1-m-\beta} \int_{B^+_{\frac{R}{2}}(z)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x$$

$$\leq C \left(\frac{\tilde{r}}{R} \right)^{\gamma_0} R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \quad (4.61)$$

for a constant C and a $\gamma_0 \in (0, 1)$ which depend only on m, N, β.

Our choice of ε also permits the application of Lemma 4.9; this lemma implies that for any $B_r(y) \in \mathcal{B}_{\theta_1}(x_0, R, \frac{R}{3})$, with $\theta_1 \geq 2$ given by the lemma, and any $0 < \rho \leq r$ we have

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \mathrm{d}x \leq C \left(\frac{\rho}{r} \right)^{\gamma_1} r^{1-m} \int_{B_r(y)} |\nabla v|^2 \mathrm{d}x \quad (4.62)$$

for some $\gamma_1 \in (0, 1)$. Since $\theta_1 \geq 2$, for any $B_r(y) \in \mathcal{B}_{\theta_1}(x_0, R, \frac{R}{3})$ we have the inclusions

$$B_r(y) \subset B^+_{\frac{y_{m+1}}{\theta_1}}(y) \subset B^+_{\left(\frac{\theta_1+1}{\theta_1}\right)y_{m+1}}(y^+) \subset B^+_{\frac{3y_{m+1}}{2}}(y^+) \in \mathcal{B}^+\left(x_0, R, \frac{R}{2}\right), \quad (4.63)$$

where $y^+ = y - (0, y_{m+1})$. It follows, applying (4.6), that

$$\left(\frac{y_{m+1}}{\theta_1} \right)^{1-m} \int_{B_{\frac{y_{m+1}}{\theta_1}}(y)} |\nabla v|^2 \mathrm{d}x$$

$$\leq C \left(\frac{(\theta_1 + 1)y_{m+1}}{\theta_1} \right)^{1-m-\beta} \int_{B^+_{\left(\frac{\theta_1+1}{\theta_1}\right)y_{m+1}}(y^+)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x, \tag{4.64}$$

where C depends on m, N, β and θ_1 and thus only on m, N, β. We combine (4.62), applied with $r = \frac{y_{m+1}}{\theta_1}$, with (4.63), (4.64) and (4.61), applied on $B^+_{\tilde{r}}(z)$ with $\tilde{r} = \left(\frac{\theta_1+1}{\theta_1} \right) y_{m+1}$ and $z = y^+$. It follows, after defining $\hat{\gamma} = \min\{\gamma_0, \gamma_1\}$, that

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \, \mathrm{d}x \leq C \left(\frac{\rho}{R} \right)^{\hat{\gamma}} R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x. \tag{4.65}$$

This holds for any $B_\rho(y) \in \mathcal{B}_{\theta_1}(x_0, R, \frac{R}{3})$. Since (4.61) holds on every $B^+_{\tilde{r}}(z) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$ for γ_0, it holds on every $B^+_{\tilde{r}}(z) \in \mathcal{B}^+(x_0, R, \frac{R}{3})$ for $\hat{\gamma}$. We deduce from (4.61) and (4.65) that the hypothesis of Lemma 4.8 are satisfied; applying this lemma concludes the proof. □

Remark 4.22 Once we know that a minimiser of E^β relative to $\mathcal{O}$ is Hölder continuous in some $B^+_R(x_0)$ with $\partial^0 B^+_R(x_0) \subset \mathcal{O}$, known theory for harmonic maps, see Lemma 4.13 which is from [38], implies that v is smooth in $B^+_R(x_0)$. However, this theory does not imply v is smooth up to $\partial^0 B^+_R(x_0)$; henceforth our goal is essentially to prove this fact.

4.12 An L^∞ bound for the gradient

The first step in our proof of higher partial regularity of locally minimising fractional harmonic maps consists of establishing an L^∞ bound for the gradients to solutions of systems of semi-linear equations with growth conditions satisfied by minimisers v of E^β relative to $\mathcal{O}$ and their derivatives $D^{\alpha'} v$ where $\alpha' \in \mathbb{N}_0^{m+1}$ is a multi-index with $\alpha'_{m+1} = 0$. The method of proof is that of harmonic replacement; compare the growth of the average Dirichlet energy of solutions of the semi-linear equations with that of solutions to the linearised system. We follow [38] for example. The monotonicity formulas established in Lemmas 2.8 and 2.17 are a key ingredient of the proof.

We will use the notation $|\Omega|_\beta = \int_\Omega |x_{m+1}|^\beta \mathrm{d}x$ and $|\Omega| = \int_\Omega \mathrm{d}x$ for $\Omega \subset \mathbb{R}^{m+1}$.

Lemma 4.23 *Suppose* $v \in W^{1,2}_\beta(B^+_R(x_0); \mathbb{R}^n) \cap C^{0,\gamma}(\overline{B^+_R(x_0)}; \mathbb{R}^n)$ *where* $B^+_R(x_0)$ *is a half-ball with* $x_0 \in \partial \mathbb{R}^{m+1}_+$, $R \leq 1$ *and* $\gamma \in (0, 1)$. *Suppose* v *satisfies*

$$\int_{B^+_R(x_0)} x^\beta_{m+1} \langle \nabla v, \nabla \psi \rangle \, \mathrm{d}x = \int_{B^+_R(x_0)} x^\beta_{m+1} \langle \psi, G(x, \nabla v) \rangle \, \mathrm{d}x$$

for every $\psi \in C^\infty_0(B_R(x_0); \mathbb{R}^n)$, *where* $G : \mathbb{R}^m \times \mathbb{R}^{(m+1)n}$ *is measurable and* $|G(x, q)| \leq C_1 |q|^2 + C_2$ *for a positive* $C_1 \leq C^*$ *and non-negative* $C_2 \leq C^*$ *for some* $C^* > 0$. *Then* $\nabla v \in L^\infty(B^+_{\frac{R}{3}}(x_0); \mathbb{R}^{(m+1)n})$ *and, in particular, we have*

$$||\nabla v||^2_{L^\infty(B^+_{\frac{R}{3}}(x_0); \mathbb{R}^{(m+1)n})} \leq C_3 \frac{1}{|B^+_R(x_0)|_\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x + C_4 \tilde{C}_2$$

where $C_3 = C_3(m, N, \beta, C^*)$, $C_4 = C_4(m, N, \beta, C^*)$ *and* $\tilde{C}_2 = C_2^{\frac{1}{2}} + C_2$. *In particular, if* $C_2 = 0$ *then* $\tilde{C}_2 = 0$.

Proof Without relabelling, we reflect v evenly across the hyperplane $\partial \mathbb{R}^{m+1}_+$. We observe that $v \in C^{0,\gamma}(\overline{B_R(x_0)}; \mathbb{R}^n) \cap W^{1,2}_\beta(B_R(x_0); \mathbb{R}^n)$ is a weak solution of $\operatorname{div}(|x_{m+1}|^\beta \nabla v) + |x_{m+1}|^\beta \tilde{G} = 0$ in $B_R(x_0)$, where $\tilde{G}$ is measurable and $|\tilde{G}(x,q)| \le C_1 |q|^2 + C_2$. We will derive estimates on classes of $B_\rho(y)$ with $y \in B_{\frac{R}{2}}(x_0)$ and $\rho \le \frac{R}{2}$. We focus initially on an estimate for the average energy on $B_{\frac{\rho}{2}}(y)$ in terms of that on $B_\rho(y)$. Since v is even with respect to $\partial \mathbb{R}^{m+1}_+$ we only need to consider $B_\rho(y)$ with $y_{m+1} \ge 0$. We consider two cases, $y_{m+1} = 0$ and $y_{m+1} > 0$.

Suppose $B_\rho(y)$ is such that $B^+_\rho(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$. An application of Minkowski's inequality, for maps in $L^2_\beta(B_{\frac{\rho}{2}}(y); \mathbb{R}^{n(m+1)})$, yields

$$\left(\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta} \int_{B_{\frac{\rho}{2}}(y)} |x_{m+1}|^\beta |\nabla v|^2 dx \right)^{\frac{1}{2}} \le \left(\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta} \int_{B_{\frac{\rho}{2}}(y)} |x_{m+1}|^\beta |\nabla w|^2 dx \right)^{\frac{1}{2}}$$
$$+ \left(\frac{C}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla(v-w)|^2 dx \right)^{\frac{1}{2}}$$
(4.66)

for any $w \in W^{1,2}_\beta(B_\rho(y); \mathbb{R}^n)$. Let $w \in W^{1,2}_\beta(B_\rho(y); \mathbb{R}^n)$ be the weak solution of $\operatorname{div}(|x_{m+1}|^\beta \nabla w) = 0$ in $B_\rho(y)$ with $w = v$ on $\partial B_\rho(y)$, given by Lemma 2.18. Then w is smooth in $B_\rho(y) \backslash \partial \mathbb{R}^{m+1}_+$ and continuous in $\overline{B_\rho(y)}$. Furthermore, since v is symmetric with respect to $\partial \mathbb{R}^{m+1}_+$, it follows from Lemma 2.19 that w is symmetric with respect to $\partial \mathbb{R}^{m+1}_+$ and, crucially, we are now free to apply Lemma 2.8 to w for every $\beta \in (-1, 1)$.

As $w - v \in C(\overline{B_\rho(y)}; \mathbb{R}^n) \cap W^{1,2}_{\beta,0}(B_\rho(y); \mathbb{R}^n)$ and v satisfies $\operatorname{div}(|x_{m+1}|^\beta \nabla v) + |x_{m+1}|^\beta \tilde{G} = 0$ and w satisfies $\operatorname{div}(|x_{m+1}|^\beta \nabla w) = 0$ weakly in $B_\rho(y)$, we calculate

$$\int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla(v-w)|^2 dx = \int_{B_\rho(y)} |x_{m+1}|^\beta \left\langle v - w, \tilde{G} \right\rangle dx$$
$$\le C_1 \sup_{B_\rho(y)} |v-w| \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 dx$$
$$+ C_2 \sup_{B_\rho(y)} |v-w| \int_{B_\rho(y)} |x_{m+1}|^\beta dx. \quad (4.67)$$

The Hölder continuity of v, together with the weak maximum and minimum principles given by Lemma 2.18 imply

$$\sup_{B_\rho(y)} |v-w| \le C\rho^\gamma. \quad (4.68)$$

Next we use the monotonicity and minimising properties of w to scale its averaged energy. An application of Lemma 2.8, followed by an application of Lemma 2.18 yields

$$\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta} \int_{B_{\frac{\rho}{2}}(y)} |x_{m+1}|^\beta |\nabla w|^2 dx \le \frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla w|^2 dx$$
$$\le \frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 dx. \quad (4.69)$$

Combining (4.66)–(4.69) we see that

$$
\left(\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta} \int_{B_{\frac{\rho}{2}}(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x \right)^{\frac{1}{2}} \leq \left(\frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x \right)^{\frac{1}{2}}
$$

$$
+ C \left(C_1 \rho^\gamma \frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x + C_2 \rho^\gamma \right)^{\frac{1}{2}}.
$$

Define $\tilde{C}_1 = (C_1 + C_2)^{\frac{1}{2}} + C_1$ and $\tilde{C}_2 = C_2^{\frac{1}{2}} + C_2$. We square both sides of the preceding inequality, using Young's inequality ($ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ for $a, b \geq 0$) and the fact that $\gamma \in (0, 1)$ and $\rho \leq R \leq 1$ to see that

$$
\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta} \int_{B_{\frac{\rho}{2}}(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x \leq \frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x
$$

$$
+ C \left(C_1 \rho^\gamma \left(\frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x \right)^2 \right.
$$

$$
\left. + C_2 \rho^\gamma \frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x \right)^{\frac{1}{2}}
$$

$$
+ C \left(C_1 \rho^\gamma \frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x + C_2 \rho^\gamma \right)
$$

$$
\leq \frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x
$$

$$
+ C \left((C_1 + C_2) \rho^\gamma \left(\frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x \right)^2 + C_2 \rho^\gamma \right)^{\frac{1}{2}}
$$

$$
+ C C_1 \rho^\gamma \frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x + C C_2 \rho^\gamma
$$

$$
\leq (1 + C \tilde{C}_1 \rho^{\frac{\gamma}{2}}) \frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x + C \tilde{C}_2 \rho^{\frac{\gamma}{2}}. \tag{4.70}
$$

This holds on every $B_\rho(y)$ with $B_\rho^+(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$.

Now we iterate this estimate on concentric balls. Consider $B_{\frac{R}{2}}(y)$ with $B_{\frac{R}{2}}^+(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$. Let $\rho_k = 2^{-k} \frac{R}{2}$ for $k \in \mathbb{N}_0$. First notice that $\rho_k^{\frac{\gamma}{2}} \leq 2^{-\frac{k\gamma}{2}}$. Hence

$$
\prod_{j=0}^\infty \left(1 + C \tilde{C}_1 \rho_j^{\frac{\gamma}{2}} \right) \leq \prod_{j=0}^\infty \left(1 + C((C^*)^{\frac{1}{2}} + C^*) 2^{-\frac{j\gamma}{2}} \right) \leq \tilde{C} < \infty
$$

where $\tilde{C}$ depends on m, N, β and C^*. It follows from (4.70) that, for every $k \geq 1$, we have

$$
\frac{1}{|B_{\rho_k}(y)|_\beta} \int_{B_{\rho_k}(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x
$$

$$\leq \prod_{j=1}^{k} \left(1 + C\tilde{C}_1 \rho_{k-j}^{\frac{\gamma}{2}}\right) \frac{1}{|B_{\frac{R}{2}}(y)|_\beta} \int_{B_{\frac{R}{2}}(y)} |x_{m+1}|^\beta |\nabla v|^2 dx$$

$$+ C\tilde{C}_2 \sum_{j=1}^{k} \rho_{k-j}^{\frac{\gamma}{2}} \prod_{l=1}^{j-1} \left(1 + C\tilde{C}_1 \rho_{k-l}^{\frac{\gamma}{2}}\right)$$

$$\leq C\tilde{C} \frac{1}{|B_R(x_0)|_\beta} \int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla v|^2 dx + C\tilde{C}_2 \tilde{C} \sum_{j=1}^{k} 2^{\frac{\gamma}{2}(j-k)}$$

$$\leq C\tilde{C} \frac{1}{|B_R^+(x_0)|_\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 dx + C\tilde{C}_2 \tilde{C}. \qquad (4.71)$$

Now we consider estimates on $B_\rho(y)$ with $B_{\frac{y_{m+1}}{2}}(y) \in \mathcal{B}(x_0, R, \frac{R}{3})$; on this class of ball we have

$$B_\rho(y) \subset B_{\frac{y_{m+1}}{2}}(y) \subset B_{\frac{3}{2}y_{m+1}}^+(y^+) \subset B_{\frac{R}{2}}^+(y^+) \in \mathcal{B}^+\left(x_0, R, \frac{R}{2}\right), \qquad (4.72)$$

where $y^+ = y - (0, y_{m+1})$. Let $w \in W_\beta^{1,2}(B_\rho(y); \mathbb{R}^n)$ be the weak solution of $\text{div}(|x_{m+1}|^\beta \nabla w) = 0$ in $B_\rho(y)$ with $w = v$ on $\partial B_\rho(y)$, given by Lemma 2.18 and suppose $\theta \geq 2$ is such that $y_{m+1} \geq \theta\rho$. Then Lemma 2.17 yields

$$\frac{1}{|B_{\frac{\rho}{2}}(y)|} \int_{B_{\frac{\rho}{2}}(y)} x_{m+1}^\beta |\nabla w|^2 dx \leq \left(1 + \frac{C}{\theta - 1}\right) \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} x_{m+1}^\beta |\nabla w|^2 dx.$$

Hence, repeating (4.66)–(4.69) but with $|B_{\frac{\rho}{2}}(y)|_\beta$ replaced by $|B_{\frac{\rho}{2}}(y)|$, we find

$$\left(\frac{1}{|B_{\frac{\rho}{2}}(y)|} \int_{B_{\frac{\rho}{2}}(y)} x_{m+1}^\beta |\nabla v|^2 dx\right)^{\frac{1}{2}} \leq \left((1 + \frac{C}{\theta - 1}) \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 dx\right)^{\frac{1}{2}}$$

$$+ C \left(C_1 \rho^\gamma \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 dx + C_2 \rho^\gamma y_{m+1}^\beta\right)^{\frac{1}{2}}.$$

We square both sides of this inequality analogously to (4.70), noting that $\theta \geq 2$ so $\frac{1}{\theta - 1} \leq 1$, $\gamma \in (0, 1)$ and $\rho \leq R \leq 1$, to see that

$$\frac{1}{|B_{\frac{\rho}{2}}(y)|} \int_{B_{\frac{\rho}{2}}(y)} x_{m+1}^\beta |\nabla v|^2 dx$$

$$\leq \left(1 + C\tilde{C}_1 \rho^{\frac{\gamma}{2}} + \frac{C}{\theta - 1}\right) \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 dx + C\tilde{C}_2 \rho^{\frac{\gamma}{2}} y_{m+1}^\beta. \qquad (4.73)$$

This holds on every $B_\rho(y)$ with $B_{\frac{y_{m+1}}{2}}(y) \in \mathcal{B}(x_0, R, \frac{R}{3})$ and $y_{m+1} \geq \theta\rho$.

We iterate this estimate on concentric balls. Consider $B_{\frac{y_{m+1}}{2}}(y) \in \mathcal{B}(x_0, R, \frac{R}{3})$. Let $\rho_k = 2^{-k} \frac{y_{m+1}}{2}$ for $k \in \mathbb{N}_0$ and observe that $y_{m+1} \geq 2^{k+1} \rho_k$ and $\rho_k^{\frac{\gamma}{2}} \leq 2^{-\frac{k\gamma}{2}}$. Observe

$$\prod_{j=0}^{\infty}\left(1+C\tilde{C}_1\rho_j^{\frac{\gamma}{2}}+\frac{C}{2^{j+1}-1}\right)\le\prod_{j=0}^{\infty}\left(1+C\left((C^*)^{\frac{1}{2}}+C^*\right)2^{-\frac{j\gamma}{2}}+\frac{C}{2^{j+1}-1}\right)$$

$$\le\hat{C}<\infty$$

where $\hat{C}$ depends on m, N, β and C^*. It follows from (4.73) that, for every $k\ge 1$, we have

$$\frac{1}{|B_{\rho_k}(y)|}\int_{B_{\rho_k}(y)}x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x$$

$$\le\prod_{j=1}^{k}\left(1+C\tilde{C}_1\rho_{k-j}^{\frac{\gamma}{2}}+\frac{C}{2^{k-j+1}-1}\right)\frac{1}{|B_{\frac{y_{m+1}}{2}}(y)|}\int_{B_{\frac{y_{m+1}}{2}}(y)}x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x$$

$$+C\tilde{C}_2y_{m+1}^{\beta}\sum_{j=1}^{k}\rho_{k-j}^{\frac{\gamma}{2}}\prod_{l=1}^{j-1}\left(1+C\tilde{C}_1\rho_{k-l}^{\frac{\gamma}{2}}+\frac{C}{2^{k-l+1}-1}\right)$$

$$\le\hat{C}\frac{1}{|B_{\frac{y_{m+1}}{2}}(y)|}\int_{B_{\frac{y_{m+1}}{2}}(y)}x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x+C\tilde{C}_2\hat{C}y_{m+1}^{\beta}. \tag{4.74}$$

Now fix $y\in B_{\frac{R}{3}}^+(x_0)$ which implies $B_{\frac{y_{m+1}}{2}}(y)\in\mathcal{B}(x_0,R,\frac{R}{3})$. We divide (4.74) by y_{m+1}^{β}, let $y^+=y-(0,y_{m+1})$ and combine (4.71), (4.72) and (4.74) to see that

$$\frac{y_{m+1}^{-\beta}}{|B_{\rho_k}(y)|}\int_{B_{\rho_k}(y)}x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x$$

$$\le C\hat{C}\frac{y_{m+1}^{-\beta}}{|B_{\frac{y_{m+1}}{2}}(y)|}\int_{B_{\frac{y_{m+1}}{2}}(y)}x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x+C\tilde{C}_2\hat{C}$$

$$\le C\hat{C}\frac{1}{|B_{\frac{3y_{m+1}}{2}}(y^+)|_{\beta}}\int_{B_{\frac{3y_{m+1}}{2}}(y^+)}|x_{m+1}|^{\beta}|\nabla v|^2\mathrm{d}x+C\tilde{C}_2\hat{C}$$

$$\le C\tilde{C}\hat{C}\frac{1}{|B_R^+(x_0)|_{\beta}}\int_{B_R^+(x_0)}x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x+C(\hat{C}+\hat{C}\tilde{C})\tilde{C}_2.$$

An application of Lebesgue's differentiation theorem concludes the proof. $\qquad\square$

Remark 4.24 A consequence of the preceding lemma is that Hölder continuous weak solutions of $\mathrm{div}(x_{m+1}^{\beta}\nabla v)+x_{m+1}^{\beta}G=0$ in $B_R^+(x_0)$ and $x_{m+1}^{\beta}\partial_{m+1}v=0$ in $\partial^0 B_R^+(x_0)$, with G satisfying the assumptions of the lemma on $B_R^+(x_0)$, are actually Lipschitz continuous on $B_{\frac{R}{3}}^+(x_0)$.

4.13 Existence of higher order derivatives

The existence of higher order derivatives of minimisers of E^{β} relative to $\mathcal{O}$ in directions tangential to $\partial\mathbb{R}_+^{m+1}$ follows using the usual method of difference quotients.

Lemma 4.25 *Fix $l\in\mathbb{N}_0$. Suppose v is a minimiser of E^{β} relative to $\mathcal{O}$ and let $B_R^+(x_0)$ be a half-ball with $\overline{\partial^0 B_R^+(x_0)}\subset\mathcal{O}$. Suppose further that for every multi-index $\alpha'\in\mathbb{N}_0^{m+1}$ with $\alpha'_{m+1}=0$ and $|\alpha'|\le l$ we have $D^{\alpha'}v\in C^{0,\gamma}(\overline{B_R^+(x_0)};\mathbb{R}^n)\cap W_{\beta}^{1,2}(B_R^+(x_0);\mathbb{R}^n)$ for some $\gamma\in(0,1)$ and $\nabla D^{\alpha'}v\in L^{\infty}(B_R^+(x_0);\mathbb{R}^{(m+1)n})$. Then for $\alpha'\in\mathbb{N}_0^{m+1}$ with*

$\alpha'_{m+1} = 0$ *and* $|\alpha'| = l$ *and* $i = 1, \ldots, m$, *the weak derivative* $\nabla \partial_i D^{\alpha'} v$ *exists and* $\nabla \partial_i D^{\alpha'} v \in L^2_\beta(B^+_{\frac{R}{2}}(x_0); \mathbb{R}^{(m+1)n})$.

Proof Without relabelling, we extend A to a smooth section of $T^*\mathbb{R}^n \otimes T^*\mathbb{R}^n \otimes T\mathbb{R}^n$. Fix $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$. Since v is a minimiser of E^β relative to $\mathcal{O}$, the regularity assumptions on v and $D^{\alpha'} v$ imply we may integrate by parts l times in (4.1); for any $\phi \in C_0^\infty(B_R(x_0); \mathbb{R}^n)$ we have

$$\int_{B_R^+(x_0)} x^\beta_{m+1} \langle \nabla D^{\alpha'} v, \nabla \phi \rangle \mathrm{d}x = \int_{B_R^+(x_0)} x^\beta_{m+1} \langle \phi, D^{\alpha'}(A(v)(\nabla v, \nabla v)) \rangle \mathrm{d}x. \tag{4.75}$$

Let $\eta \in C_0^\infty(B_{\frac{3R}{4}}(x_0))$ be a smooth cutoff function such that $\eta \equiv 1$ in $B_{\frac{R}{2}}(x_0)$, $1 \geq \eta \geq 0$ in $B_{\frac{3R}{4}}(x_0) \backslash B_{\frac{R}{2}}(x_0)$ and $|\nabla \eta| \leq \frac{C}{R}$. Furthermore, let $\Delta_i^h D^{\alpha'} v = h^{-1}(D^{\alpha'} v(x + he_i) - D^{\alpha'} v(x))$ be the difference quotient of $D^{\alpha'} v$ and assume $|h| < \frac{R}{4}$. Note that, by approximation, $w = -\Delta_i^{-h}(\eta^2 \Delta_i^h D^{\alpha'} v)$ is an admissible test function for (4.75). We substitute w into (4.75) and apply 'integration by parts' for difference quotients to see that

$$\int_{B_R^+(x_0)} \eta^2 x^\beta_{m+1} |\Delta_i^h \nabla D^{\alpha'} v|^2 \mathrm{d}x = \int_{B_{\frac{3R}{4}}^+(x_0)} \eta^2 x^\beta_{m+1} \langle \Delta_i^h D^{\alpha'} v, \Delta_i^h D^{\alpha'}(A(v)(\nabla v, \nabla v)) \rangle \mathrm{d}x$$
$$- \int_{B_{\frac{3R}{4}}^+(x_0)} 2\eta x^\beta_{m+1} \langle \Delta_i^h \nabla D^{\alpha'} v \cdot \nabla \eta, \Delta_i^h D^{\alpha'} v \rangle \mathrm{d}x. \tag{4.76}$$

We now use Young's inequality, $ab \leq \frac{a^2}{\delta 2} + \delta \frac{b^2}{2}$ for $a, b \geq 0$ and $\delta > 0$, to move all of the terms involving $\Delta_i^h \nabla D^{\alpha'} v$ on the right hand side of (4.76) to the left hand side. We calculate

$$- \int_{B_{\frac{3R}{4}}^+(x_0)} 2\eta x^\beta_{m+1} \langle \Delta_i^h \nabla D^{\alpha'} v \cdot \nabla \eta, \Delta_i^h D^{\alpha'} v \rangle \mathrm{d}x \leq C\delta \int_{B_R^+(x_0)} \eta^2 x^\beta_{m+1} |\Delta_i^h \nabla D^{\alpha'} v|^2 \mathrm{d}x$$
$$+ \frac{C}{\delta} \int_{B_R^+(x_0)} x^\beta_{m+1} |\nabla \eta|^2 |\Delta_i^h D^{\alpha'} v|^2 \mathrm{d}x. \tag{4.77}$$

We need to estimate the term involving $\Delta_i^h D^{\alpha'}(A(v)(\nabla v, \nabla v))$ in a similar fashion. An application of the Mean Value Theorem, noting we are working on $B_{\frac{3R}{4}}^+(x_0)$ and $|h| < \frac{R}{4}$, implies

$$|\Delta_i^h D^{\alpha'}(A(v)(\nabla v, \nabla v))| \leq C_1 |\Delta_i^h \nabla D^{\alpha'} v| + C_2$$

where C_1, C_2 depend on m, N, β and $\|\nabla D^{\tilde{\alpha}'} v\|_{L^\infty(B_R^+(x_0); \mathbb{R}^{(m+1)n})}$ where $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $|\tilde{\alpha}'| \leq |\alpha'|$ and $\tilde{\alpha}'_{m+1} = 0$. Hence, using Young's inequality again, we deduce

$$\int_{B_{\frac{3R}{4}}^+(x_0)} \eta^2 x^\beta_{m+1} \langle \Delta_i^h D^{\alpha'} v, \Delta_i^h D^{\alpha'}(A(v)(\nabla v, \nabla v)) \rangle \mathrm{d}x$$
$$\leq \delta \int_{B_{\frac{3R}{4}}^+(x_0)} \eta^2 x^\beta_{m+1} |\Delta_i^h \nabla D^{\alpha'} v|^2 \mathrm{d}x + C \frac{C_3}{\delta} \int_{B_{\frac{3R}{4}}^+(x_0)} x^\beta_{m+1} \mathrm{d}x, \tag{4.78}$$

where C_3 depends on m, N, β and $||\nabla D^{\tilde{\alpha}'}v||_{L^\infty(B_R^+(x_0);\mathbb{R}^{(m+1)n})}$ where $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $|\tilde{\alpha}'| \leq |\alpha'|$ and $\tilde{\alpha}'_{m+1} = 0$. Choosing δ sufficiently small in (4.77) and (4.78) we combine these inequalities with (4.76). Since $|\nabla \eta| \leq \frac{C}{R}$ and $\eta \equiv 1$ in $B_{\frac{R}{2}}^+(x_0)$, we see that

$$\int_{B_{\frac{R}{2}}^+(x_0)} x_{m+1}^\beta |\Delta_i^h \nabla D^{\alpha'}v|^2 dx \leq C_4(R^{-2}+1)\int_{B_R^+(x_0)} x_{m+1}^\beta dx,$$

where C_4 depends on m, N, β and $||\nabla D^{\tilde{\alpha}'}v||_{L^\infty(B_R^+(x_0);\mathbb{R}^{(m+1)n})}$ where $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $|\tilde{\alpha}'| \leq |\alpha'|$ and $\tilde{\alpha}'_{m+1} = 0$. This bound is independent of h with $|h| < \frac{R}{4}$. Hence by Lemma 2.11 we conclude that the weak derivative $\nabla \partial_i D^{\alpha'}v$ exists and satisfies the above inequality with $\nabla \partial_i D^{\alpha'}v$ in place of $\Delta_i^h \nabla D^{\alpha'}v$. This concludes the proof. $\qquad\square$

4.14 Caccioppoli-type inequality

Here we show that the derivatives of minimisers of E^β relative to $\mathcal{O}$ in directions tangential to $\partial \mathbb{R}_+^{m+1}$ all satisfy essentially the same Caccioppoli-type inequality.

Lemma 4.26 *Fix $l \in \mathbb{N}_0$, let $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ be a minimiser of E^β relative to $\mathcal{O}$ and let $B_R^+(x_0)$ be a half-ball with $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$. Suppose that for every multi-index $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| \leq l$ we have $D^{\alpha'}v \in C^{0,\gamma}(\overline{B_R^+(x_0)}; \mathbb{R}^n) \cap W_\beta^{1,2}(B_R^+(x_0); \mathbb{R}^n)$ for some $\gamma \in (0,1)$ and $\nabla D^{\alpha'}v \in L^\infty(B_R^+(x_0); \mathbb{R}^{(m+1)n})$. Suppose further that for $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$ and some $i \in \{1, \ldots, m\}$, we have $\nabla \partial_i D^{\alpha'}v \in L_\beta^2(B_R^+(x_0); \mathbb{R}^{(m+1)n})$. Let $B_\rho(y) \subset B_R(x_0)$ with $y_{m+1} \geq 0$. For each α' with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$ there are constants $C = C(m, N, \beta)$ and C_1, C_2 which depend on m, N, β and are comprised of polynomial functions, with no constant terms, of $||\nabla D^{\tilde{\alpha}'}v||_{L^\infty(B_\rho(y)\cap\mathbb{R}_+^{m+1};\mathbb{R}^{(m+1)n})}$, where $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $|\tilde{\alpha}'| \leq |\alpha'| = l$ and $\tilde{\alpha}'_{m+1} = 0$, such that*

$$\int_{B_{\frac{\rho}{2}}(y)\cap\mathbb{R}_+^{m+1}} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'}v|^2 dx \leq C\left(C_1 + \frac{1}{\rho^2}\right)\int_{B_\rho(y)\cap\mathbb{R}_+^{m+1}} x_{m+1}^\beta |\partial_i D^{\alpha'}v - \lambda|^2 dx$$

$$+ C_2 \int_{B_\rho(y)\cap\mathbb{R}_+^{m+1}} x_{m+1}^\beta dx \tag{4.79}$$

for any $\lambda \in \mathbb{R}^n$.

Proof Fix α' with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$. Integrating by parts $l+1$ times in (4.1) shows that for every $\psi \in C_0^\infty(B_R(x_0); \mathbb{R}^n)$, we have

$$\int_{B_R^+(x_0)} x_{m+1}^\beta \langle \nabla \partial_i D^{\alpha'}v, \nabla \psi \rangle dx = \int_{B_R^+(x_0)} x_{m+1}^\beta \langle \psi, \partial_i D^{\alpha'}(A(v)(\nabla v, \nabla v)) \rangle dx.$$

Now, by approximation, we may choose $\psi = \eta^2(\partial_i D^{\alpha'}v - \lambda)$ where $\lambda \in \mathbb{R}^n$ is a constant vector and $\eta \in C_0^\infty(B_\rho(y))$ is a cutoff function with $\eta \equiv 1$ in $B_{\frac{\rho}{2}}(y)$, $0 \leq \eta \leq 1$, and $|\nabla \eta| \leq \frac{C}{\rho}$. We calculate

$$\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta^2 |\nabla\partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq \tilde{C}_1 \int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta^2 |\partial_i D^{\alpha'} v - \lambda||\nabla\partial_i D^{\alpha'} v| \mathrm{d}x$$
$$+ \tilde{C}_2 \int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta^2 |\partial_i D^{\alpha'} v - \lambda| \mathrm{d}x$$
$$+ C \int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta |\nabla\eta||\partial_i D^{\alpha'} v - \lambda||\nabla\partial_i D^{\alpha'} v| \mathrm{d}x,$$

$$(4.80)$$

where $C = C(m, N, \beta)$ and $\tilde{C}_1, \tilde{C}_2$ depend on m, N, β and are comprised of polynomial functions, with no constant terms, of $||\nabla D^{\tilde{\alpha}'} v||_{L^\infty(B_\rho(y)\cap\mathbb{R}^{m+1}_+;\mathbb{R}^{(m+1)n})}$ where $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $|\tilde{\alpha}'| \leq l$ and $\tilde{\alpha}'_{m+1} = 0$. We apply Young's inequality, $ab \leq \frac{\delta a^2}{2} + \frac{b^2}{\delta 2}$ for $a, b \geq 0$ and $\delta > 0$, to each term on the right hand side of (4.80). We first apply this inequality to the term corresponding to C, choosing δ sufficiently small depending on C and hence only on m, N, β and recalling $|\nabla\eta| \leq \frac{C}{\rho}$ to see that

$$\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta^2 |\nabla\partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq \tilde{C}_1 \int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta^2 |\partial_i D^{\alpha'} v - \lambda||\nabla\partial_i D^{\alpha'} v| \mathrm{d}x$$
$$+ \tilde{C}_2 \int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta^2 |\partial_i D^{\alpha'} v - \lambda| \mathrm{d}x$$
$$+ \frac{C}{\rho^2} \int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta |\partial_i D^{\alpha'} v - \lambda|^2 \mathrm{d}x \qquad (4.81)$$

where $\tilde{C}_1, \tilde{C}_2$ are possibly different from before but have the same structure and dependence as the constants in (4.80). Applying Young's inequality again, now to the terms in (4.81) corresponding to $\tilde{C}_1, \tilde{C}_2$ concludes the proof. □

4.15 Control of the mean squared oscillation of the derivatives on the boundary

We prove an analogue of Lemma 4.18 for the derivatives of minimisers of E^β relative to $\mathcal{O}$.

Lemma 4.27 *Fix $l \in \mathbb{N}_0$. For every $\delta > 0$ there exist numbers $\varepsilon > 0$, $\tau \in (0, 1)$ and $\theta \in (0, \frac{1}{4}]$ such that the following holds. Suppose $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ is a minimiser of E^β relative to $\mathcal{O}$ with $D^{\alpha'} v \in C^{0,\gamma}(\overline{B^+_R(x_0)}; \mathbb{R}^n) \cap W^{1,2}_\beta(B^+_R(x_0); \mathbb{R}^n)$ for a $\gamma \in (0, 1)$ and $\nabla D^{\alpha'} v \in L^\infty(B^+_R(x_0); \mathbb{R}^{(m+1)n})$ for every multi-index $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| \leq l$, where $B^+_R(x_0) \subset \mathbb{R}^{m+1}_+$ satisfies $R \leq 1$ and $\partial^0 B^+_R(x_0) \subset \mathcal{O}$. Suppose further that for $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$ and some $i \in \{1, \ldots, m\}$, we have $\nabla\partial_i D^{\alpha'} v \in L^2_\beta(B^+_R(x_0); \mathbb{R}^{(m+1)n})$. If $|\alpha'| = l$ and $\alpha'_{m+1} = 0$ and*

$$R^{1-m-\beta} \int_{B^+_R(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \leq \varepsilon^2,$$

then, for every $B^+_r(y) \in \mathcal{B}^+(x_0, R, \tau R)$, either

$$r^{1-m-\beta} \int_{B^+_r(y)} x_{m+1}^\beta |\nabla\partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq \delta \left(\frac{r}{R^{|\alpha'|+2}}\right)^2 \qquad (4.82)$$

or

$$(\theta r)^{-(1+m+\beta)} \int_{B^+_{\theta r}(y)} x_{m+1}^\beta |\partial_i D^{\alpha'} v - \overline{\partial_i D^{\alpha'} v}_{B^+_{\theta r}(y),\beta}|^2 \mathrm{d}x$$

$$\leq \delta r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 dx. \tag{4.83}$$

Proof We use a blow-up argument, analogous in spirit to the argument we used in the proof of Lemma 4.18. First we note that the statement of the lemma is invariant under rescaling and translation by any point in $\partial \mathbb{R}_+^{m+1}$. In particular, suppose the lemma holds for minimisers of E^β relative to $\tilde{\mathcal{O}} \subset \partial \mathbb{R}_+^{m+1}$ whenever $\overline{\partial^0 B_1^+(0)} \subset \tilde{\mathcal{O}}$. If the hypotheses of the lemma hold for minimisers of E^β relative to $\mathcal{O}$ and $B_R^+(x_0)$ satisfies $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$, then applying the lemma to $v_R := v(R \cdot + x_0)$ yields the conclusion of the lemma on $B_R^+(x_0)$.

We now prove the lemma when $R = 1$, $x_0 = 0$ and v is a minimiser of E^β relative to $\mathcal{O}$ and $\overline{\partial^0 B_1^+(0)} \subset \mathcal{O}$. Suppose the statement is false. Then there exists $\delta > 0$ such that, for any fixed $\theta \in (0, \frac{1}{4}]$, we may find a sequence $(v_k)_{k \in \mathbb{N}}$ of minimisers of E^β relative to $\mathcal{O}$ such that the following holds. Each v_k satisfies $D^{\alpha'} v_k \in C^{0,\gamma}(\overline{B_1^+(0)}; \mathbb{R}^n) \cap W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n)$ for a $\gamma \in (0, 1)$ and $\nabla D^{\alpha'} v_k \in L^\infty(B_1^+(0); \mathbb{R}^{(m+1)n})$ for every multi-index $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| \leq l$. For $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$ and a fixed $i = 1, \ldots, m$, each v_k further satisfies $\nabla \partial_i D^{\alpha'} v_k \in L_\beta^2(B_1^+(0); \mathbb{R}^{(m+1)n})$. Moreover, the v_k satisfy

$$\int_{B_1^+(0)} x_{m+1}^\beta |\nabla v_k|^2 \, dx := \varepsilon_k^2 \to 0,$$

and, furthermore, there exists a sequence of numbers $0 < \tau_k \to 0^+$, half-balls $B_{r_k}^+(y_k) \in \mathcal{B}^+(0, 1, \tau_k)$, and numbers $0 < r_k \leq \tau_k \to 0^+$ such that

$$r_k^{1-m-\beta} \int_{B_{r_k}^+(y_k)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v_k|^2 dx > \delta r_k^2 \tag{4.84}$$

and

$$(\theta r_k)^{-(1+m+\beta)} \int_{B_{\theta r_k}^+(y_k)} x_{m+1}^\beta |\partial_i D^{\alpha'} v_k - \overline{(\partial_i D^{\alpha'} v_k)}_{B_{\theta r_k}^+(y_k),\beta}|^2 dx$$

$$> \delta r_k^{1-m-\beta} \int_{B_{r_k}^+(y_k)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v_k|^2 dx \tag{4.85}$$

for α' with $|\alpha'| = l$ and $\alpha'_{m+1} = 0$.

Since each v_k is a minimiser of E^β relative to $\mathcal{O}$ and $v_k \in W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n) \cap C^{0,\gamma}(\overline{B_1^+(0)}; \mathbb{R}^n)$ we deduce from Lemma 4.23 that each v_k satisfies

$$||\nabla v_k||_{L^\infty(B_{\frac{1}{3}}^+(0); \mathbb{R}^{(m+1)n})}^2 \leq C \int_{B_1^+(0)} x_{m+1}^\beta |\nabla v_k|^2 dx \leq C\varepsilon_k^2 \to 0. \tag{4.86}$$

The assumptions of the lemma guarantee that we may apply Lemma 4.26 on $B_{\frac{1}{6}}^+(0)$ with $\lambda = 0$. We do so and conclude that there are constants $C = C(m, N, \beta)$ and C_1, C_2, which depend on m, N, β and are comprised of a polynomial function, with no constant terms, of $||\nabla v_k||_{L^\infty(B_{\frac{1}{3}}^+(0); \mathbb{R}^{(m+1)n})}$ and consequently satisfy $C_1, C_2 \to 0$ as $k \to \infty$, such that for α' with $|\alpha'| = 1$ and $\alpha'_{m+1} = 0$

$$\int_{B_{\frac{1}{6}}^+(0)} x_{m+1}^\beta |\nabla D^{\alpha'} v_k|^2 dx \leq C(C_1 + 1) \int_{B_{\frac{1}{3}}^+(0)} x_{m+1}^\beta |D^{\alpha'} v_k|^2 dx + C_2$$

$$\leq C\,(C_1 + 1)\,||\nabla v_k||^2_{L^\infty\left(B^+_{\frac{1}{3}}(0);\mathbb{R}^{(m+1)n}\right)} + C_2 \to 0 \qquad (4.87)$$

as $k \to \infty$. If $l \geq 1$, integrating by parts in (4.1), we see that when $|\alpha'| = 1$ and $\alpha'_{m+1} = 0$, $D^{\alpha'} v_k$ satisfies

$$\int_{B^+_{\frac{1}{6}}(0)} x^\beta_{m+1} \langle \nabla D^{\alpha'} v_k, \nabla \psi \rangle \mathrm{d}x = \int_{B^+_{\frac{1}{6}}(0)} x^\beta_{m+1} \langle \psi, G \rangle \mathrm{d}x$$

for every $\psi \in C^\infty_0(B_{\frac{1}{6}}(0); \mathbb{R}^n)$ where, by Young's inequality, $|G(x, q)| \leq C_3 |q|^2 + C_4$ for constants $C_3 = C_3(m, N, \beta)$ and C_4 which depends on m, N, β and is comprised of a polynomial function, with no constant term, of $||\nabla v_k||_{L^\infty(B^+_{\frac{1}{3}}(0);\mathbb{R}^{(m+1)n})}$ and hence $C_4 \to 0$ as $k \to \infty$. Now recall that by assumption $D^{\alpha'} v_k \in C^{0,\gamma}(\overline{B^+_{\frac{1}{6}}(0)}; \mathbb{R}^{(m+1)n}) \cap W^{1,2}_\beta(B^+_{\frac{1}{6}}(0); \mathbb{R}^{(m+1)n})$ for each $\alpha' \in \mathbb{N}^{m+1}$ with $|\alpha'| = 1$ and $\alpha'_{m+1} = 0$. Hence applying Lemma 4.23 again in conjunction with (4.86) and (4.87), we deduce that there exist positive constants $\tilde{C}_3 = \tilde{C}_3(m, N, \beta)$ and $\tilde{C}_4$ which depends on m, N, β and k with $\tilde{C}_4 \to 0$ as $k \to \infty$ such that

$$||\nabla D^{\alpha'} v_k||^2_{L^\infty\left(B^+_{\frac{1}{3}\frac{1}{6}}(0);\mathbb{R}^{(m+1)n}\right)} \leq \tilde{C}_3 \frac{1}{|B^+_{\frac{1}{6}}(0)|_\beta} \int_{B^+_{\frac{1}{6}}(0)} x^\beta_{m+1} |\nabla D^{\alpha'} v_k|^2 \mathrm{d}x + \tilde{C}_4 \to 0$$

as $k \to \infty$. Repeating the preceding process for $D^{\alpha'} v_k$ with $|\alpha'| = 2$, then $|\alpha'| = 3, \ldots, |\alpha'| = l$, we see that

$$||\nabla D^{\alpha'} v_k||^2_{L^\infty(B^+_{\frac{1}{3}6^{-|\alpha'|}}(0);\mathbb{R}^{(m+1)n})} \to 0 \qquad (4.88)$$

as $k \to \infty$ for every α' with $|\alpha'| \leq l$ and $\alpha'_{m+1} = 0$.

Now fix α' with $|\alpha'| = l$ and $\alpha'_{m+1} = 0$. Discarding as may v_k as necessary and re-indexing the resulting sequence we may assume that $2\tau_k \leq 6^{-(l+1)}$ so that $B^+_{r_k}(y_k) \subset B^+_{2r_k}(y_k) \subset B^+_{2\tau_k}(y_k) \in \mathcal{B}^+(0, \frac{1}{3}6^{-l}, 6^{-(l+1)})$ and, in particular, $B^+_{2\tau_k}(y_k) \subset B^+_{\frac{1}{3}6^{-l}}(0)$. Define

$$r_k^{1-m-\beta} \int_{B^+_{r_k}(y_k)} x^\beta_{m+1} |\nabla \partial_i D^{\alpha'} v_k|^2 \mathrm{d}x := \tilde{\varepsilon}_k^2.$$

Note that it is possible to show, combining Lemma 4.26 with $\lambda = 0$ and (4.88), that $\tilde{\varepsilon}_k^2 \to 0$, but this is not required in what follows.

We see from (4.84) and (4.85) that

$$\tilde{\varepsilon}_k^2 > \delta r_k^2 \qquad (4.89)$$

and

$$(\theta r_k)^{-(1+m+\beta)} \int_{B^+_{\theta r_k}(y_k)} x^\beta_{m+1} |\partial_i D^{\alpha'} v_k - \overline{(\partial_i D^{\alpha'} v_k)}_{B^+_{\theta r_k}(y_k),\beta}|^2 \mathrm{d}x > \delta \tilde{\varepsilon}_k^2. \qquad (4.90)$$

Define

$$w_k(x) = \frac{\partial_i D^{\alpha'} v_k(r_k x + y_k) - \overline{(\partial_i D^{\alpha'} v_k)}_{B^+_{\theta r_k}(y_k),\beta}}{\tilde{\varepsilon}_k}.$$

Then

$$\nabla w_k(x) = \frac{r_k}{\tilde{\varepsilon}_k} \nabla \partial_i D^{\alpha'} v_k(r_k x + y_k). \tag{4.91}$$

Hence, using the change of variables $x \mapsto r_k x + y_k$, we find

$$\int_{B_1^+(0)} x_{m+1}^{\beta} |\nabla w_k|^2 \, dx = 1 \quad \text{and} \quad \overline{(w_k)}_{B_\theta^+(0),\beta} = 0. \tag{4.92}$$

Furthermore, after changing variables again, we deduce from (4.90) that

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^{\beta} |w_k|^2 \, dx > \delta. \tag{4.93}$$

The combination of (4.92) and the Poincaré inequality for A_2 weights implies $(w_k)_{k \in \mathbb{N}}$ is bounded in $W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n)$. The the Rellich Compactness lemma, Lemma 2.5, therefore yields a subsequence $(w_{k_j})_{j \in \mathbb{N}}$ which converges weakly in $W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n)$ and strongly in $L_\beta^2(B_1^+(0); \mathbb{R}^n)$ to a $w \in W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n)$.

Now we show that w is a weak solution of the Neumann-type problem (2.11) in $B_1^+(0)$. Let $\phi \in C_0^\infty(B_1(0); \mathbb{R}^n)$ and define $\tilde{\phi} \in C_0^\infty(B_{r_k}(y_k); \mathbb{R}^n)$ by $\tilde{\phi}(z) = \phi\left(\frac{z-y_k}{r_k}\right)$. We observe that $r_k \nabla \tilde{\phi}(z) = \nabla \phi(x)$ where $x \in B_1(0)$ and $z \in B_{r_k}(y_k)$ satisfy $z = r_k x + y_k$. Hence, using the change of variables $x \mapsto r_k x + y_k$ and (4.91), we find

$$\int_{B_1^+(0)} x_{m+1}^{\beta} \langle \nabla w_k, \nabla \phi \rangle \, dx = \frac{r_k}{\tilde{\varepsilon}_k} \int_{B_1^+(0)} x_{m+1}^{\beta} \langle \nabla \partial_i D^{\alpha'} v_k(r_k x + y_k), \nabla \phi(x) \rangle dx$$

$$= \frac{r_k^{-m-\beta}}{\tilde{\varepsilon}_k} \int_{B_{r_k}^+(y_k)} z_{m+1}^{\beta} \langle \nabla \partial_i D^{\alpha'} v_k, \nabla \phi\left(\frac{z-y_k}{r_k}\right) \rangle dz$$

$$= \frac{r_k^{1-m-\beta}}{\tilde{\varepsilon}_k} \int_{B_{r_k}^+(y_k)} z_{m+1}^{\beta} \langle \nabla \partial_i D^{\alpha'} v_k, \nabla \tilde{\phi} \rangle dz. \tag{4.94}$$

As $\tilde{\phi} \in C_0^\infty(B_{r_k}(y_k); \mathbb{R}^n)$, v_k is a minimiser of E^β relative to $\mathcal{O}$ and, in view of (4.89), we have $\frac{r_k^2}{\tilde{\varepsilon}_k^2} < \frac{1}{\delta}$, it follows that

$$\left| \int_{B_{r_k}^+(y_k)} z_{m+1}^{\beta} \langle \nabla \partial_i D^{\alpha'} v_k, \nabla \tilde{\phi} \rangle dz \right| \leq C \int_{B_{r_k}^+(y_k)} z_{m+1}^{\beta} (|\nabla \partial_i D^{\alpha'} v_k| + 1) dz$$

$$= C r_k^{1+m+\beta} \int_{B_1^+(0)} x_{m+1}^{\beta} (|\nabla \partial_i D^{\alpha'} v_k(r_k x + y_k)| + 1) dx$$

$$= C \tilde{\varepsilon}_k r_k^{m+\beta} \int_{B_1^+(0)} x_{m+1}^{\beta} (|\nabla w_k| + \frac{r_k}{\tilde{\varepsilon}_k}) dx$$

$$\leq C \tilde{\varepsilon}_k r_k^{m+\beta} \int_{B_1^+(0)} x_{m+1}^{\beta} (|\nabla w_k| + \delta^{-\frac{1}{2}}) dx, \tag{4.95}$$

where C depends on $m, N, \beta, ||\tilde{\phi}||_{L^\infty(B_{r_k}^+(y_k);\mathbb{R}^n)} = ||\phi||_{L^\infty(B_1^+(0);\mathbb{R}^n)}$ and is comprised of a polynomial, with no constant terms, of $||\nabla D^{\tilde{\alpha}'} v_k||_{L^\infty(B_{\frac{1}{6}-l}^+(0);\mathbb{R}^n)}$ where $|\tilde{\alpha}'| \leq l = |\alpha'|$ and $\tilde{\alpha}'_{m+1} = 0$ and is therefore independent of k in view of (4.88).

We combine (4.92) and (4.95) to see that, for any $\phi \in C_0^\infty(B_1(0); \mathbb{R}^n)$, the weak convergence of w_{k_j} to w in $W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n)$ yields

$$\left| \int_{B_1^+(0)} x_{m+1}^\beta \langle \nabla w, \nabla \phi \rangle \mathrm{d}x \right| = \lim_{j \to \infty} \left| \int_{B_1^+(0)} x_{m+1}^\beta \langle \nabla w_{k_j}, \nabla \phi \rangle \mathrm{d}x \right|$$
$$\leq C \lim_{j \to \infty} r_{k_j} \int_{B_1^+(0)} x_{m+1}^\beta (|\nabla w_{k_j}| + \delta^{-\frac{1}{2}}) \mathrm{d}x$$
$$= 0$$

since $r_{k_j} \to 0$. Hence w is a weak solution of (2.11) in $B_1^+(0)$.

Using the Rellich Compactness Lemma, Lemma 2.5, we take limits in (4.92) and (4.93) to see that

$$\int_{B_1^+(0)} x_{m+1}^\beta |\nabla w|^2 \, \mathrm{d}x \leq 1 \quad \text{and} \quad \overline{w}_{B_\theta^+(0),\beta} = 0 \tag{4.96}$$

and

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 \, \mathrm{d}x \geq \delta \tag{4.97}$$

respectively. Now, in view of (4.96), the Poincaré inequality for A_2 weights yields

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 \mathrm{d}x \leq C\theta^{1-m-\beta} \int_{B_\theta^+(0)} x_{m+1}^\beta |\nabla w|^2 \mathrm{d}x. \tag{4.98}$$

Lastly, since w is a weak solution of (2.11) we may apply Corollary 2.7 to w with $\theta \leq \frac{1}{4}$ (so that $2\theta \leq \frac{1}{2}$). This gives a positive constant C (independent of θ) and a $\gamma \in (0, 1)$ such that

$$\theta^{1-m-\beta} \int_{B_\theta^+(0)} x_{m+1}^\beta |\nabla w|^2 \mathrm{d}x \leq C(2\theta)^{2\gamma}. \tag{4.99}$$

Combining (4.98) and (4.99) we see that

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 \mathrm{d}x \leq C(2\theta)^{2\gamma}. \tag{4.100}$$

This holds for all fixed $\theta \in (0, \frac{1}{4}]$ and we choose $\theta < 2^{-1} \left(\frac{\delta}{C} \right)^{\frac{1}{2\gamma}}$ so that (4.100) contradicts (4.97). Hence the lemma is proved. □

4.16 Control of the mean squared oscillation of the derivatives in the interior

We need a counterpart to Lemma 4.27 which holds on a class of balls with closure contained in the interior of $\mathbb{R}_+^{m+1}$.

Lemma 4.28 *Fix $l \in \mathbb{N}_0$. For every $\delta > 0$ there exist numbers $\varepsilon > 0$, $\tau \in (0, 1)$ and $\theta \in (0, \frac{1}{4}]$ such that the following holds. Suppose $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ is a minimiser of E^β relative to $\mathcal{O}$ with $D^{\alpha'} v \in C^{0,\gamma}(\overline{B_R^+(x_0)}; \mathbb{R}^n) \cap W_\beta^{1,2}(B_R^+(x_0); \mathbb{R}^n)$ for a $\gamma \in (0, 1)$ and $\nabla D^{\alpha'} v \in L^\infty(B_R^+(x_0); \mathbb{R}^{(m+1)n})$ for every multi-index $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| \leq l$, where $B_R^+(x_0) \subset \mathbb{R}_+^{m+1}$ satisfies $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$. Suppose further*

that for $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$ and some $i \in \{1, \dots, m\}$, we have
$\nabla \partial_i D^{\alpha'} v \in L^2_\beta(B_R^+(x_0); \mathbb{R}^{(m+1)n})$. *If $|\alpha'| = l$ and $\alpha'_{m+1} = 0$ and*

$$R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 dx \le \varepsilon^2,$$

then, for every $B_r(y) \in \mathcal{B}_4(x_0, R, \tau R)$, either

$$r^{1-m} \int_{B_r(y)} |\nabla \partial_i D^{\alpha'} v|^2 dx \le \delta \left(\frac{r}{R^{|\alpha'|+2}} \right)^2 \tag{4.101}$$

or

$$(\theta r)^{-(1+m)} \int_{B_{\theta r}(y)} |\partial_i D^{\alpha'} v - \overline{\partial_i D^{\alpha'} v}_{B_{\theta r}(y)}|^2 dx \le \delta r^{1-m} \int_{B_r(y)} |\nabla \partial_i D^{\alpha'} v|^2 dx. \tag{4.102}$$

Proof The method of proof is similar to the proof of Lemma 4.27. We observe that the lemma is invariant under scaling and translation with respect to x_0 in $\partial \mathbb{R}_+^{m+1}$ in the same way as Lemma 4.27. Hence we assume $R = 1$, $x_0 = 0$, v is a minimiser of E^β relative to $\mathcal{O}$ and $\partial^0 B_1^+(0) \subset \mathcal{O}$.

Suppose the statement is false. Then there exists $\delta > 0$ such that, for any fixed $\theta \in (0, \frac{1}{4}]$, we may find a sequence $(v_k)_{k \in \mathbb{N}}$ of minimisers of E^β relative to $\mathcal{O}$ such that the following holds. Each v_k satisfies $D^{\alpha'} v_k \in C^{0,\gamma}(\overline{B_1^+(0)}; \mathbb{R}^n) \cap W_\beta^{1,2}(B_1^+(0); \mathbb{R}^n)$ for a $\gamma \in (0, 1)$ and $\nabla D^{\alpha'} v_k \in L^\infty(B_1^+(0); \mathbb{R}^{(m+1)n})$ for every multi-index $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| \le l$. For $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| = l$ and a fixed $i = 1, \dots, m$, each v_k further satisfies $\nabla \partial_i D^{\alpha'} v_k \in L^2_\beta(B_1^+(0); \mathbb{R}^{(m+1)n})$. The v_k also satisfy

$$\int_{B_1^+(0)} x_{m+1}^\beta |\nabla v_k|^2 \, dx := \varepsilon_k^2 \to 0.$$

There furthermore exists a sequence of numbers $0 < \tau_k \to 0$, balls $B_{r_k}(y_k) \in \mathcal{B}_4(0, 1, \tau_k)$, and numbers $0 < r_k \le \tau_k \to 0$ such that

$$r_k^{1-m} \int_{B_{r_k}(y_k)} |\nabla \partial_i D^{\alpha'} v_k|^2 dx > \delta r_k^2 \tag{4.103}$$

and

$$(\theta r_k)^{-(1+m)} \int_{B_{\theta r_k}(y_k)} |\partial_i D^{\alpha'} v_k - \overline{(\partial_i D^{\alpha'} v_k)}_{B_{\theta r_k}(y_k)}|^2 dx$$
$$> \delta r_k^{1-m} \int_{B_{r_k}(y_k)} |\nabla \partial_i D^{\alpha'} v_k|^2 dx \tag{4.104}$$

for α' with $|\alpha'| = l$ and $\alpha'_{m+1} = 0$.

Since the assumptions of the lemma are the same as the assumptions of Lemma 4.27 we still have (4.88), namely, for every α' with $|\alpha'| \le l$ and $\alpha'_{m+1} = 0$

$$||\nabla D^{\alpha'} v_k||^2_{L^\infty \left(B_{\frac{1}{3}6-|\alpha'|}^+ (0); \mathbb{R}^{(m+1)n} \right)} \to 0. \tag{4.105}$$

Define

$$r_k^{1-m} \int_{B_{r_k}(y_k)} |\nabla \partial_i D^{\alpha'} v_k|^2 dx := \tilde{\varepsilon}_k^2.$$

Fix α' with $|\alpha'| = l$ and $\alpha'_{m+1} = 0$. Discarding as many k as necessary, and re-indexing the resulting sequence to $k \in \mathbb{N}$, we may assume $2\tau_k \leq 6^{-(l+1)}$ so that $B_{r_k}(y_k) \in \mathcal{B}_4(0, \frac{1}{3}6^{-l}, 6^{-(l+1)})$ and, in particular, $B_{r_k}(y_k) \subset B^+_{\frac{1}{3}6^{-l}}(0)$. Note that similarly to in the proof of Lemma 4.27, using Lemma 4.26 with $\lambda = 0$ and (4.105) we can show $\tilde{\varepsilon}_k^2 \to 0$, but this is not used in what follows. Now consider the normalised sequence

$$w_k(x) = \frac{\partial_i D^{\alpha'} v_k(r_k x + y_k) - \overline{(\partial_i D^{\alpha'} v_k)}_{B_{\theta r_k}(y_k)}}{\tilde{\varepsilon}_k}.$$

We have

$$\nabla w_k(x) = \frac{r_k}{\tilde{\varepsilon}_k} \nabla \partial_i D^{\alpha'} v_k (r_k x + y_k). \tag{4.106}$$

Hence, using the change of variables $x \mapsto r_k x + y_k$, we find

$$\int_{B_1(0)} |\nabla w_k|^2 \, dx = 1 \quad \text{and} \quad \overline{(w_k)}_{B_\theta(0)} = 0 \tag{4.107}$$

and, also using (4.104),

$$\theta^{-(1+m)} \int_{B_\theta(0)} |w_k|^2 \, dx > \delta. \tag{4.108}$$

As a result of (4.107) and the Poincaré inequality, we observe $(w_k)_{k \in \mathbb{N}}$ is bounded in $W^{1,2}(B_1(0); \mathbb{R}^n)$. The Rellich Compactness lemma [39, Section 1.3, Lemma 1], thus yields a subsequence $(w_{k_j})_{j \in \mathbb{N}}$ which converges weakly in $W^{1,2}(B_1(0); \mathbb{R}^n)$ and strongly in $L^2(B_1(0); \mathbb{R}^n)$ to some $w \in W^{1,2}(B_1(0); \mathbb{R}^n)$.

Define $f_k(x) = \left(1 + (y_k)^{-1}_{m+1} r_k x_{m+1}\right)^\beta$ for each $k \in \mathbb{N}$. Observe that $a_{k_j} = (y_{k_j})^{-1}_{m+1} r_{k_j} \in [0, 4^{-1}]$ for every j, since each $B_{r_k}(y_k) \in \mathcal{B}_4(0, 1, \tau_k)$. Thus there is a subsequence, which we also index with k_j, which converges to $a \in [0, 4^{-1}]$. Furthermore, $(f_{k_j})_{j \in \mathbb{N}}$ is uniformly bounded and equicontinuous so by the Arzelà–Ascoli theorem, there is a uniformly convergent subsequence which we again index by k_j. Since $f_{k_j}(x) \to f(x) = (1 + ax_{m+1})^\beta$ pointwise, we must also have $f_{k_j} \to f$ uniformly.

Now, for $\phi \in C_0^\infty(B_1(0); \mathbb{R}^n)$, similar calculations to those in the proof of Lemma 4.27 yield

$$\left| \int_{B_1(0)} f_k \langle \nabla w_k, \nabla \phi \rangle \, dx \right| \leq C \|\phi\|_{L^\infty(B_1(0); \mathbb{R}^n)} r_k \int_{B_1(0)} |\nabla w_k| + \delta^{-\frac{1}{2}} dx \to 0 \tag{4.109}$$

as $k \to \infty$. Furthermore, as w_{k_j} converges weakly to w in $W^{1,2}(B_1(0); \mathbb{R}^n)$ and $f_{k_j} \to f$ uniformly, we conclude that

$$\int_{B_1(0)} f \langle \nabla w, \nabla \phi \rangle dx = \lim_{j \to \infty} \int_{B_1(0)} f_{k_j} \langle \nabla w_{k_j}, \nabla \phi \rangle dx = 0. \tag{4.110}$$

Hence w is a weak solution of $\text{div}((1 + ax_{m+1})^\beta \nabla w) = 0$ in $B_1(0)$. By linear elliptic regularity theory, w is smooth in $B_1(0)$. We also conclude by taking limits in (4.107) and (4.108) that

$$\int_{B_1(0)} |\nabla w|^2 \, dx \leq 1 \quad \text{and} \quad \overline{w}_{B_\theta(0)} = 0 \quad \text{and} \quad \theta^{-(1+m)} \int_{B_\theta(0)} |w|^2 \, dx \geq \delta \tag{4.111}$$

respectively using the Rellich Compactness Lemma. Since $|\nabla w|^2$ satisfies a mean value inequality, namely $\sup_{B_\theta(0)} |\nabla w|^2 \le C(m, \beta) \int_{B_1(0)} |\nabla w|^2 dx$ as shown in Theorem 2.1 in section III of [14], we apply the Poincaré inequality and (4.111) to see that

$$\theta^{-(1+m)} \int_{B_\theta(0)} |w|^2 dx \le C\theta^{1-m} \int_{B_\theta(0)} |\nabla w|^2 dx \le C\theta^2 \int_{B_1(0)} |\nabla w|^2 dx \le C\theta^2. \quad (4.112)$$

This holds for all fixed $\theta \in (0, \frac{1}{2}]$ and we choose $\theta < \left(\frac{\delta}{C}\right)^{\frac{1}{2}}$ so that (4.112) contradicts the last statement of (4.111). □

4.17 Higher order ε-regularity

With the preceding theory in hand, we are now in a position to prove our main ε-regularity theorem.

Proof of Theorem 4.1 Observe that the hypothesis of the theorem are invariant with respect to the rescaling $x \mapsto Rx + x_0$. Thus we will assume $R = 1$, $x_0 = 0$ and $\overline{\partial^0 B_1^+(0)} \subset \mathcal{O}$.

We use proof by strong induction. We choose ε to be the number from Theorem 4.21. Then the combination of Theorem 4.21, Lemmata 4.23 and 4.25 yield a $\hat{\theta} = \hat{\theta}(m, N, \beta) \le \frac{1}{2}$ and a $\hat{\gamma} \in (0, 1)$ such that $v \in C^{0,\hat{\gamma}}(\overline{B_{\hat{\theta}}^+(0)}; \mathbb{R}^n) \cap W_\beta^{1,2}(B_{\hat{\theta}}^+(0); \mathbb{R}^n)$, $\nabla v \in L^\infty(B_{\hat{\theta}}^+(0); \mathbb{R}^{(m+1)n})$ and $\partial_i v \in W_\beta^{1,2}(B_{\hat{\theta}}^+(0); \mathbb{R}^n)$ for $i = 1, \ldots, m$. Now fix $l \in \mathbb{N}_0$. The induction hypothesis is that there exists $\tilde{\theta} = \tilde{\theta}(m, N, \beta, l) \le \frac{1}{2}$ and a $\tilde{\gamma} = \tilde{\gamma}(m, N, \beta, l) \in (0, 1)$ such that the following holds. For $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $|\alpha'| \le l$, we have $D^{\alpha'} v \in C^{0,\tilde{\gamma}}(\overline{B_{\tilde{\theta}}^+(0)}; \mathbb{R}^n) \cap W_\beta^{1,2}(B_{\tilde{\theta}}^+(0); \mathbb{R}^n)$ and $\nabla D^{\alpha'} v \in L^\infty(B_{\tilde{\theta}}^+(0); \mathbb{R}^{(m+1)n})$. Furthermore, when $|\alpha'| = l$ and $\alpha'_{m+1} = 0$ we suppose $\nabla \partial_i D^{\alpha'} v \in L_\beta^2(B_{\tilde{\theta}}^+(0); \mathbb{R}^{n(m+1)})$ for $i = 1, \ldots, m$. We have already observed that this is true when $l = 0$. The inductive step will be to show the preceding statement holds, possibly for a different $\tilde{\theta}$ and $\tilde{\gamma}$, for $D^{\alpha'} v$ with $|\alpha'| \le l + 1$ and $\alpha'_{m+1} = 0$. We fix α' with $|\alpha'| = l \ge 0$ and $\alpha'_{m+1} = 0$ henceforth.

Applying Lemma 4.26, we see that

$$\int_{B_{\frac{\rho}{2}}(y) \cap \mathbb{R}_+^{m+1}} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 dx \le C_0 \left(C_1 + \frac{1}{\rho^2}\right) \int_{B_\rho(y) \cap \mathbb{R}_+^{m+1}} x_{m+1}^\beta |\partial_i D^{\alpha'} v - \lambda|^2 dx$$

$$+ C_2 \int_{B_\rho(y) \cap \mathbb{R}_+^{m+1}} x_{m+1}^\beta dx \quad (4.113)$$

for any $B_\rho(y) \subset B_{\tilde{\theta}}(0)$ with $y_{m+1} \ge 0$ and $i = 1, \ldots, m$, where $C_0 = C_0(m, N, \beta)$ and C_1, C_2 depend on m, N, β and are polynomial functions, with no constant terms, of $||\nabla D^{\tilde{\alpha}'} v||_{L^\infty(B_\rho(y) \cap \mathbb{R}_+^{m+1}; \mathbb{R}^{(m+1)n})}$ where $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $|\tilde{\alpha}'| \le l$ and $\tilde{\alpha}'_{m+1} = 0$. We apply Lemmata 4.27 and 4.28, with $\delta = \frac{1}{2} \frac{1}{2^{m+2} C_0}$, to respectively obtain numbers $\varepsilon_1 > 0$, $\tau_1 \in (0, 1)$ and $\theta_1 \in (0, \frac{1}{4}]$ and $\varepsilon_2 > 0$, $\tau_2 \in (0, 1)$ and $\theta_2 \in (0, \frac{1}{4}]$, depending only on δ and hence only on m, N, β, such that if $\tilde{R} \le \tilde{\theta}$ and

$$\tilde{R}^{1-m-\beta} \int_{B_{\tilde{R}}^+(0)} x_{m+1}^\beta |\nabla v|^2 dx \le \min\{\varepsilon_1^2, \varepsilon_2^2\} \quad (4.114)$$

then either (4.82) or (4.83) holds for every $B_{r_1}^+(y) \in \mathcal{B}^+(0, \tilde{R}, \tau_1 \tilde{R})$ and either (4.101) or (4.102) holds for every $B_{r_2}(y) \in \mathcal{B}_4(0, \tilde{R}, \tau_2 \tilde{R})$. It follows from the proof of Theorem 4.21,

bearing in mind $R = 1$ and $x_0 = 0$, that for every $B_r^+(y) \in \mathcal{B}^+(0, 1, \frac{1}{2})$ we have

$$r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \leq C r^{\tilde{\gamma}}$$

for some $\tilde{\gamma} \in (0, 1)$ and some constant $C = C(m, N, \beta)$. In particular, this holds for $y = 0$ and $r \leq \frac{1}{2}$. Hence if $\tilde{R} = \tilde{R}(m, N, \beta, l) = (\min\{\frac{\varepsilon_1^2}{C}, \frac{\varepsilon_2^2}{C}, (\frac{\tilde{\theta}}{2})^{\tilde{\gamma}}\})^{\frac{1}{\tilde{\gamma}}}$ then (4.114) holds on $B_{\tilde{R}}^+(0)$. We have assumed $\tilde{R} \leq \frac{\tilde{\theta}}{2}$ so that we may later apply (4.113) with impunity on any ball or half-ball in $B_{\tilde{R}}^+(0)$.

First we show that (4.7) essentially holds for $\nabla \partial_i D^{\alpha'} v$ on every $B_{r_1}^+(y) \in \mathcal{B}^+(0, \tilde{R}, \tau_1 \tilde{R})$. We know that (4.82) or (4.83) holds on $B_{r_1}^+(y)$. We apply (4.113) with $\lambda = \overline{\partial_i D^{\alpha'} v}_{B_{\theta_1 r_1}^+(y), \beta}$, noting that $|\lambda| \leq ||\nabla D^{\alpha'} v||_{L^\infty(B_{\theta_1 r_1}^+(y); \mathbb{R}^{(m+1)n})}$, to see that

$$\left(\frac{\theta_1 r_1}{2} \right)^{1-m-\beta} \int_{B_{\frac{\theta_1 r_1}{2}}^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x$$

$$\leq C_0 2^{m+\beta-1} (\theta_1 r_1)^{-(1+m+\beta)} \int_{B_{\theta_1 r_1}^+(y)} x_{m+1}^\beta |\partial_i D^{\alpha'} v - \overline{\partial_i D^{\alpha'} v}_{B_{\theta_1 r_1}^+(y), \beta}|^2 \mathrm{d}x + C r_1^2.$$

$$(4.115)$$

Hence, regardless of which of (4.82) or (4.83) holds (bearing in mind our choice of δ above), we have

$$(\sigma_1 r_1)^{1-m-\beta} \int_{B_{\sigma_1 r_1}^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq \frac{1}{2} r_1^{1-m-\beta} \int_{B_{r_1}^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C r_1^2,$$

$$(4.116)$$

where $\sigma_1 = \frac{\theta_1}{2}$ and C depends on $||\nabla D^{\tilde{\alpha}'} v||_{L^\infty(B_{\tilde{\theta}}^+(0); \mathbb{R}^{(m+1)n})}$ where $|\tilde{\alpha}'| \leq l$ with $\tilde{\alpha}'_{m+1} = 0$ and, moreover, may depend on $\tilde{R}, \theta_1, m, N$ and β and hence only on m, N, β, l as $\tilde{R} = \tilde{R}(m, N, \beta, l)$ and $\theta_1 = \theta_1(m, N, \beta)$. This holds for any $B_{r_1}^+(y) \in \mathcal{B}^+(0, \tilde{R}, \tau_1 \tilde{R})$. We may apply (4.116) with r_1 replaced by $\sigma_1^k r_1$ for every $k \in \mathbb{N}$ and iterate to see that

$$\left(\sigma_1^k r_1 \right)^{1-m-\beta} \int_{B_{\sigma_1^k r_1}^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x$$

$$\leq \frac{1}{2^k} r_1^{1-m-\beta} \int_{B_{r_1}^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C \sum_{j=0}^{k-1} 2^{-j} (\sigma_1^{k-1-j} r_1)^2$$

$$\leq \frac{1}{2^k} \left(r_1^{1-m-\beta} \int_{B_{r_1}^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C r_1^2 \right). \quad (4.117)$$

Setting $\gamma_1 = -\frac{\ln 2}{\ln \sigma_1} \in (0, 1)$ we conclude that

$$r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x$$

$$\leq \sigma_1^{1-m-\beta-\gamma_1} \left(\frac{r}{r_1} \right)^{\gamma_1} \left(r_1^{1-m-\beta} \int_{B_{r_1}^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C r_1^2 \right) \quad (4.118)$$

for any $r \leq r_1$, where C depends on the same factors as the constant in (4.116). This holds for any $B_{r_1}^+(y) \in \mathcal{B}^+(0, \tilde{R}, \tau_1 \tilde{R})$.

We want a similar estimate for $r_2^{1-m} \int_{B_{r_2}(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x$ on balls $B_{r_2}(y) \in \mathcal{B}_4(0, \tilde{R}, \tau_2 \tilde{R})$. We calculate the constants in (4.6) from Sect. 4.3 explicitly. Then our choice of δ and a similar argument which lead to (4.118) yields the existence of a $\gamma_2 = \gamma_2(m, N, \beta) \in (0, 1)$ such that for any $B_{r_2}(y) \in \mathcal{B}_4(0, \tilde{R}, \tau_2 \tilde{R})$ and any $r \leq r_2$ we have

$$r^{1-m} \int_{B_r(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq \sigma_2^{1-m-\gamma_2} \left(\frac{r}{r_2}\right)^{\gamma_2} \left(r_2^{1-m} \int_{B_{r_2}(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C r_2^2\right),$$
(4.119)

where $\sigma_2 = \frac{\theta_2}{2}$ and C depends on the same factors as in (4.118). We now use (4.118) and (4.119) to show the hypothesis (4.7) from Lemma 4.8 is satisfied.

Define $\tau = \min\{\frac{\tau_1}{2}, \tau_2\} < \frac{1}{2}$, $\gamma = \min\{\gamma_1, \gamma_2\}$. We apply (4.118) with $r_1 = \tau \tilde{R}$. It follows that for every $B_r^+(y) \in \mathcal{B}^+(0, \tilde{R}, \tau \tilde{R})$ we have $B_{\tau \tilde{R}}^+(y) \in \mathcal{B}^+(0, \tilde{R}, \tau_1 \tilde{R})$ and hence

$$r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x$$

$$\leq C \left(\frac{r}{\tau \tilde{R}}\right)^\gamma \left((\tau \tilde{R})^{1-m-\beta} \int_{B_{\tau \tilde{R}}^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C(\tau \tilde{R})^2\right).$$
(4.120)

Furthermore, applying (4.113) with $\lambda = 0$ and $\rho = 2\tau \tilde{R}$ implies that

$$(\tau \tilde{R})^{1-m-\beta} \int_{B_{\tau \tilde{R}}^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq C(1 + (\tau \tilde{R})^2) \leq C,$$
(4.121)

where C depends on m, N, β, l and $||\nabla D^{\tilde{\alpha}'} v||_{L^\infty(B_{\hat{\theta}}^+(0); \mathbb{R}^{(m+1)n})}$ where $|\tilde{\alpha}'| \leq l$ and $\tilde{\alpha}_{m+1}' = 0$. We combine (4.120) and (4.121) to see that for every $B_r^+(y) \in \mathcal{B}^+(0, \tilde{R}, \tau \tilde{R})$ we have

$$r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x \leq C \left(\frac{r}{\tau \tilde{R}}\right)^\gamma \leq C_3 r^\gamma,$$
(4.122)

where C_3 depends on m, N, β, l and $||\nabla D^{\tilde{\alpha}'} v||_{L^\infty(B_{\hat{\theta}}^+(0); \mathbb{R}^{(m+1)n})}$ where $|\tilde{\alpha}'| \leq l$ and $\tilde{\alpha}_{m+1}' = 0$.

Now let $B_r(y) \in \mathcal{B}_4(0, \tilde{R}, \frac{2\tau}{3} \tilde{R})$. Then $B_r(y) \subset B_{\frac{y_{m+1}}{4}}(y) \subset B_{\frac{3y_{m+1}}{2}}^+(y^+) \subset B_{\tau \tilde{R}}^+(y^+) \in \mathcal{B}^+(0, \tilde{R}, \tau \tilde{R})$, where $y^+ = y - (0, y_{m+1})$. Recalling again (4.6) from Sect. 4.3, we note that

$$\left(\frac{y_{m+1}}{4}\right)^{1-m} \int_{B_{\frac{y_{m+1}}{4}}(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x$$

$$\leq C \left(\frac{3y_{m+1}}{2}\right)^{1-m-\beta} \int_{B_{\frac{3y_{m+1}}{2}}^+(y^+)} x_{m+1}^\beta |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x.$$
(4.123)

Since $y_{m+1} \leq \tau \tilde{R} \leq 1$, applying (4.119) on $B_{\frac{y_{m+1}}{4}}(y) \in \mathcal{B}_4(0, \tilde{R}, \tau_2 \tilde{R})$, using (4.123), and then applying (4.122) gives

$$r^{1-m} \int_{B_r(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x$$

$$\leq C \left(\frac{4r}{y_{m+1}}\right)^{\gamma} \left(\left(\frac{y_{m+1}}{4}\right)^{1-m} \int_{B_{\frac{y_{m+1}}{4}}(y)} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C y_{m+1}^2\right)$$

$$\leq C \left(\frac{r}{y_{m+1}}\right)^{\gamma} \left(C\left(\frac{3y_{m+1}}{2}\right)^{1-m-\beta} \int_{B_{\frac{3y_{m+1}}{2}}^+(y^+)} x_{m+1}^{\beta} |\nabla \partial_i D^{\alpha'} v|^2 \mathrm{d}x + C y_{m+1}^2\right)$$

$$\leq C \left(\frac{r}{y_{m+1}}\right)^{\gamma} \left(C_3 \left(\frac{3y_{m+1}}{2}\right)^{\gamma} + C y_{m+1}^2\right)$$

$$\leq C_4 r^{\gamma} \tag{4.124}$$

where C_4 depends on m, N, β, l and $||\nabla D^{\tilde{\alpha}'} v||_{L^{\infty}(B_{\hat{\theta}}^+(0);\mathbb{R}^{(m+1)n})}$ where $|\tilde{\alpha}'| \leq l$ and $\tilde{\alpha}'_{m+1} = 0$. Together, (4.122) and (4.124) imply (4.7) from Lemma 4.8 holds for $B_r^+(y) \in \mathcal{B}^+(0, \tilde{R}, \frac{2\tau}{3}\tilde{R})$ and $B_r(y) \in \mathcal{B}_4(0, \tilde{R}, \frac{2\tau}{3}\tilde{R})$. Applying this lemma shows that $\partial_i D^{\alpha'} v \in C^{0,\hat{\gamma}}(\overline{B_{\hat{\theta}}^+(0)}; \mathbb{R}^n)$ for some $\hat{\theta}, \hat{\gamma} \in (0,1)$ depending on m, N, β, l. Hence, recalling that the inductive hypothesis implies that $\partial_i D^{\alpha'} v \in W_\beta^{1,2}(B_{\hat{\theta}}^+(0); \mathbb{R}^n)$, we may apply Lemmata 4.23 and 4.25 to respectively imply $\nabla \partial_i D^{\alpha'} v \in L^{\infty}(B_{\frac{\hat{\theta}}{3}}^+(0); \mathbb{R}^{(m+1)n})$ and $\nabla \partial_j \partial_i D^{\alpha'} v \in L_\beta^2(B_{\frac{\hat{\theta}}{6}}^+(0); \mathbb{R}^{n(m+1)})$ for $i, j = 1, \ldots, m$. This completes the inductive step and therefore the proof. □

Remark 4.29 A consequence of the proof is that the number ε in Theorem 4.1 can be taken to be the number from Theorem 4.21.

Theorem 4.1 yields an improvement to Theorem 4.21 for minimisers of E^β relative to $\mathcal{O}$. We are now in a position to prove our partial regularity theorem.

Proof of Theorem 4.3 Aspects of the proof closely follow the proof of Theorem 3.2 in [29]. First, it follows from the theory of Schoen and Uhlenbeck [37] that there exists a set $\Sigma_{\mathrm{int}} \subset \mathbb{R}_+^{m+1}$, with Hausdorff dimension at most $m-2$, such that v is smooth in a neighbourhood of any point in Σ_{int}.

Define
$$\Sigma_{\mathrm{bdry}} = \{y \in \mathcal{O} : \Theta_v^\beta(y) \geq \varepsilon\}$$

where ε is the number given by the Theorem 4.21 and $\Theta_v^\beta(y)$ is the density function defined in Remark 4.6. The upper semi-continuity of Θ_v^β was established in Remark 4.6 which, when combined with the definition of Σ_{bdry}, shows that Σ_{bdry} is relatively closed in $\mathcal{O}$.

We write Σ_{bdry} as a countable union of compact sets of the form $K \cap \Sigma_{\mathrm{bdry}}$, where $K \subset \mathcal{O}$ is compact, and let $\Sigma' \subset \Sigma_{\mathrm{bdry}}$ be such a set. Fix $\delta > 0$ and cover Σ' by a collection of balls $B_{r_i}^m(x_i) \subset \mathcal{O}$ with $\overline{B_{r_i}^m(x_i)} \subset \mathcal{O}$ with $x_i \in \Sigma'$ and $0 < r_i \leq \delta$. The compactness of Σ', combined with Vitali's covering theorem yields a finite subcollection of balls, $B_{r_1}^m(x_1), \ldots, B_{r_I}^m(x_I)$ for some $I \in \mathbb{N}$, of any such cover of Σ', which satisfies $B_{r_i}(x_i) \cap B_{r_j}(x_j) = \emptyset$ for $i \neq j, 1 \leq i, j \leq I$ and $\Sigma' \subset \bigcup_{i=1}^I B_{5r_i}(x_i)$. Using the boundary energy monotonicity formula, Lemma 4.5, we see that

$$\sum_{i=1}^I (10r_i)^{m+\beta-1} \leq \frac{10^{m+\beta-1}}{\varepsilon} \sum_{i=1}^I \int_{B_{r_i}^+(x_i)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x$$

$$\leq \frac{10^{m+\beta-1}}{\varepsilon} \int_{\mathcal{O}\times[0,\delta]} x_{m+1}^{\beta}|\nabla v|^2 \mathrm{d}x.$$

We send $\delta \to 0^+$ and use Lebesgue's Dominated Convergence Theorem to see that $\mathcal{H}^{m-1+\beta}(\Sigma') = 0$ and hence $\mathcal{H}^{m-1+\beta}(\Sigma_{\mathrm{bdry}}) = 0$.

Let $x_0 \in (\mathbb{R}_+^{m+1} \cup \mathcal{O})\backslash\Sigma$. If $x_0 \in \mathbb{R}_+^{m+1}$ then $x_0 \in \mathbb{R}_+^{m+1}\backslash\Sigma_{\mathrm{int}}$ and v is smooth in an open ball centred at x_0 and contained in $\mathbb{R}_+^{m+1}\backslash\Sigma_{\mathrm{int}} \subset (\mathbb{R}_+^{m+1} \cup \mathcal{O})\backslash\Sigma$. If $x_0 \in \mathcal{O}$ then $x_0 \in \mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$ and $\Theta_v^{\beta}(x_0) < \varepsilon$ which, combined with the fact that $\mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$ is open in $\mathcal{O}$, implies there exists an $R > 0$ such that $R^{1-m-\beta}\int_{B_R^+(x_0)} x_{m+1}^{\beta}|\nabla v|^2 \mathrm{d}x \leq \varepsilon$, $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$. Consequently, Theorem 4.21 implies that there are $\theta, \gamma \in (0,1)$ such that $v \in C^{0,\gamma}(\overline{B_{\theta R}^+(x_0)}; N)$. Furthermore, we deduce from (4.61) in the proof of Theorem 4.21 that

$$r^{1-m-\beta} \int_{B_r^+(z)} x_{m+1}^{\beta}|\nabla v|^2 \mathrm{d}x \leq C\left(\frac{r}{R}\right)^{\gamma}\varepsilon$$

on every $B_r^+(z) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$ which shows that $\Theta_v^{\beta}(z) = 0$ for every $z \in \partial^0 B_{\frac{R}{2}}^+(x_0)$. Now setting $\sigma = \min\{\theta, \frac{1}{2}\}$ we see that $\Theta_v^{\beta}(z) = 0$ for $z \in \partial^0 B_{\sigma R}^+(x_0)$ which implies $\partial^0 B_{\sigma R}^+(x_0) \subset \mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$. Furthermore, v is a Hölder continuous weakly harmonic map in any $B_r(y)$ with $\overline{B_r(y)} \subset B_{\sigma R}^+(x_0)$. We apply Lemma 4.13 to see that v is smooth in $B_{\sigma R}^+(x_0)$ and conclude that $B_{\sigma R}^+(x_0) \subset \mathbb{R}_+^{m+1}\backslash\Sigma_{\mathrm{int}}$. Consequently, we have $B_{\sigma R}^+(x_0) \cup \partial^0 B_{\sigma R}^+(x_0) \subset (\mathbb{R}_+^{m+1}\cup\mathcal{O})\backslash\Sigma$. Note that $B_{\sigma R}^+(x_0)\cup\partial^0 B_{\sigma R}^+(x_0)$ is an open ball centred at x_0 in the (Euclidean) topology of $\mathbb{R}_+^{m+1}\cup\mathcal{O}$. Hence Σ is relatively closed in $\mathbb{R}_+^{m+1}\cup\mathcal{O}$. As the Hausdorff dimension of Σ_{int} is at most $m-2 < m-1+\beta$ and $\mathcal{H}^{m-1+\beta}(\Sigma_{\mathrm{bdry}}) = 0$, we deduce that $\mathcal{H}^{m+\beta-1}(\Sigma) = 0$. We also conclude $v \in C_{loc}^{0,\gamma}((\mathbb{R}_+^{m+1}\cup\mathcal{O})\backslash\Sigma; N)$.

Consider $x_0 \in (\mathbb{R}_+^{m+1}\cup\mathcal{O})\backslash\Sigma$ with $x_0 \in \mathcal{O}$. Then, as above, we observe there is an $R > 0$ such that $R^{1-m-\beta}\int_{B_R^+(x_0)} x_{m+1}^{\beta}|\nabla v|^2 \mathrm{d}x \leq \varepsilon$, $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$. In view of Remark 4.29, Theorem 4.1 implies that for every $l \in \mathbb{N}_0$ there exist $\theta, \gamma \in (0,1)$ such that for every $\alpha' \in \mathbb{N}_0^{m+1}$ with $|\alpha'| \leq l$ and $\alpha'_{m+1} = 0$ we have $D^{\alpha'}v \in C^{0,\gamma}(\overline{B_{\theta R}^+(x_0)}; \mathbb{R}^n)$. However, we also know that $(\mathbb{R}_+^{m+1}\cup\mathcal{O})\backslash\Sigma$ is open in $\mathbb{R}_+^{m+1}\cup\mathcal{O}$. Hence there exists $\tilde{R} > 0$ such that $B_{\tilde{R}}^+(x_0)\cup\partial^0 B_{\tilde{R}}^+(x_0) \subset (\mathbb{R}_+^{m+1}\cup\mathcal{O})\backslash\Sigma$. Setting $r = \min\{\theta R, \tilde{R}\}$ we conclude $D^{\alpha'}v \in C^{0,\gamma}(\overline{B_r^+(x_0)}; \mathbb{R}^n)$ and $B_r^+(x_0)\cup\partial^0 B_r^+(x_0) \subset (\mathbb{R}_+^{m+1}\cup\mathcal{O})\backslash\Sigma$. We iteratively apply Lemmata 4.23 and 4.25 to see that for every $\alpha' \in \mathbb{N}_0^{m+1}$ with $|\alpha'| \leq l$ and $\alpha'_{m+1} = 0$ we have $\nabla D^{\alpha'}v \in L^{\infty}(B_{\tilde{r}}^+(x_0); \mathbb{R}^{(m+1)n})$ and $D^{\alpha'}v \in W_{\beta}^{1,2}(B_{\tilde{r}}^+(x_0); \mathbb{R}^n)$ for some $\tilde{r} \leq r$. It follows that $B_{\tilde{r}}^+(x_0)\cup\partial^0 B_{\tilde{r}}^+(x_0) \subset (\mathbb{R}_+^{m+1}\cup\mathcal{O})\backslash\Sigma$. Hence $v \in C_{loc}^{0,1}((\mathbb{R}_+^{m+1}\cup\mathcal{O})\backslash\Sigma; N)$ and for every multi-index $\alpha' \in \mathbb{N}^{m+1}$ with $\alpha'_{m+1} = 0$ we have $D^{\alpha'}v \in C_{loc}^{0,1}((\mathbb{R}_+^{m+1}\cup\mathcal{O})\backslash\Sigma; \mathbb{R}^n)$ and $\nabla D^{\alpha'}v \in L_{loc}^{\infty}((\mathbb{R}_+^{m+1}\cup\mathcal{O})\backslash\Sigma; \mathbb{R}^{(m+1)n})$.

Lastly, for x_0 as in the preceding paragraph, fix $\alpha' \in \mathbb{N}_0^{m+1}$ with $\alpha'_{m+1} = 0$ and $\tilde{r} \leq 1$ such that $B_{\tilde{r}}^+(x_0)\cup\partial^0 B_{\tilde{r}}^+(x_0) \subset (\mathbb{R}_+^{m+1}\cup\mathcal{O})\backslash\Sigma$. Making $\tilde{r}$ smaller if necessary, we may assume that $\nabla D^{\tilde{\alpha}'}v \in L^{\infty}(B_{\tilde{r}}^+(x_0); \mathbb{R}^{(m+1)n})$ and $D^{\tilde{\alpha}'}v \in C^{0,1}(\overline{B_{\tilde{r}}^+(x_0)}; \mathbb{R}^n)$ for every $\tilde{\alpha}' \in \mathbb{N}_0^{m+1}$ with $\tilde{\alpha}'_{m+1} = 0$ and $|\tilde{\alpha}'| \leq |\alpha'| + 2$. We also observe that since $v \in C^{0,1}(\overline{B_{\tilde{r}}^+(x_0)}; N)$ is a Hölder continuous harmonic map, it is smooth in $B_{\tilde{r}}^+(x_0)$ by Lemma 4.13 and so we have $x_{m+1}^{-\beta}\partial_{m+1}(x_{m+1}^{\beta}\partial_{m+1}D^{\alpha'}v) = -(\Delta'D^{\alpha'}v + D^{\alpha'}(A(v)(\nabla v, \nabla v)))$ classi-

cally in $B_{\tilde{r}}^+(x_0)$, where Δ' is the Laplacian with respect to x_i, $i = 1, \ldots, m$. Hence, $x_{m+1}^{-\beta} \partial_{m+1}(x_{m+1}^{\beta} \partial_{m+1} D^{\alpha'} v)$ is bounded in $B_{\tilde{r}}^+(x_0)$. Hence, for every $B_{\rho}^+(y) \in \mathcal{B}^+(x_0, \frac{\tilde{r}}{3}, \frac{\tilde{r}}{6})$ we calculate

$$\rho^{1-m+\beta} \int_{B_{\rho}^+(y)} x_{m+1}^{-\beta} |\nabla \left(x_{m+1}^{\beta} \partial_{m+1} D^{\alpha'} v \right)|^2 \mathrm{d}x \leq C \rho^{2+2\beta}.$$

Moreover, for every $B_{\rho}(y) \in \mathcal{B}(x_0, \frac{\tilde{r}}{3}, \frac{\tilde{r}}{6})$ we calculate

$$\rho^{1-m} \int_{B_{\rho}(y)} |\nabla \left(x_{m+1}^{\beta} \partial_{m+1} D^{\alpha'} v \right)|^2 \mathrm{d}x \leq C \rho^{2-2|\beta|}.$$

An application of Lemma 4.8 concludes the proof. $\square$

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