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Calculus of Variations

Three-spheres theorem for p-harmonic mappings

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Abstract Let $u = (u^1, \dots, u^n)$ be a p-harmonic mapping in a domain $\Omega \subset \mathbb{R}^n$ for $n \geq 2$. We investigate level sets for compositions of coordinate functions u^i with convex functions satisfying growth conditions and derive the de Giorgi-type estimates. Our main result is the arithmetic three-spheres theorem for coordinate functions of mapping u. The discussion is illustrated by radial p-harmonics.

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1 Introduction

Consider a subharmonic function u in a planar domain and let $M(r)$ denote the maximum of u over a circle $x^2+y^2=r^2$ concentric with two other circles with radii satisfying $r_1 < r < r_2$. Then, the classical Hadamard three-circles theorem asserts that $M(r)$ is a convex function of r, see e.g. Chapter 12 in Protter–Weinberger [19] (Hadamard formulated this result for analytic functions without providing a proof, see Hadamard [13]). Namely it holds that

$$M(r) \leq \frac{\log(r_2/r)}{\log(r_2/r_1)} M(r_1) + \frac{\log(r/r_1)}{\log(r_2/r_1)} M(r_2).$$

The result can be further generalized in the following directions: by considering higher dimensional analogs, by studying equations more general than the Laplace equation and by investigating different types of inequalities involving $M(r)$ or the norms of functions in subject. Indeed, one studies the setting of concentric spheres in $\mathbb{R}^n$, $n \geq 3$ see e.g. Theorem 30

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in [19] or spheres which need not be concentric, see Arakelian–Matevosyan [5]. The three-circles (or the three-spheres) theorem can be extended to the setting of more general elliptic equations, see discussion in [19, Chapter 12], Brummelhuis [7] for a discussion in the setting of second-order linear elliptic equations, Miklyukov–Rasila–Vuorinen [18] for p-harmonic equations, Výborný [22] for quasilinear equations with Lipschitz coefficients; see also Granlund–Marola [12] for studies in the setting of (A, B)-equations of Riccati type and for further references. Finally, instead of the above inequality, one studies estimates involving L^2 or L^∞ norms of solutions to elliptic equations, see e.g. Lin–Nagayasu–Wang [17] and Alessandrini–Rondi–Rosset–Vessella [4]. In the latter publication, among other topics the authors discuss the role of the three-spheres theorems in the studies of the unique continuation problems and ill-posed problems. For related topics and estimates we refer to Colding–De Lellis–Minicozzi [8] and Garofalo–Lin [9].

The main goal of this paper is to prove a variant of three-spheres theorem in the context of coupled elliptic systems of equations represented by p-harmonic systems of equations. According to our best knowledge three-spheres theorems have not yet been studied for systems of equations.

A mapping $u = (u^1, \ldots, u^n) \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$ is called a *p-harmonic mapping* if it is a solution to the following system of equations:

$$\operatorname{div}(|Du|^{p-2}Du) = 0, \quad u = (u^1, \ldots, u^n) : \Omega \subset \mathbb{R}^N \to \mathbb{R}^n, \quad 1 < p < \infty, \tag{1}$$

where Du denotes the Jacobi matrix of u, i.e. $Du = (\nabla u^1, \ldots, \nabla u^n)^T$. Equivalently, this system can be written as follows:

$$\begin{cases} \operatorname{div}(|Du|^{p-2}\nabla u^1) = 0, \\ \quad\vdots \\ \operatorname{div}(|Du|^{p-2}\nabla u^n) = 0. \end{cases} \tag{2}$$

The p-harmonic system of equations is the Euler-Lagrange system of the associated p-Dirichlet energy functional

$$\int_\Omega |Du|^p.$$

In the weak form (2) reads

$$\int_\Omega |Du|^{p-2}\langle \nabla u^i, \nabla \phi^i \rangle = 0 \quad \text{for } i = 1, \ldots, n, \tag{3}$$

where $\phi^i \in C_0^\infty(\Omega)$ are test functions. In what follows we will consider the case of $N = n$. For $p = 2$ the system reduces to the well-known harmonic system of equations (such a system is uncoupled). Therefore, one may view a p-harmonic system as a natural generalization of the harmonic system to the nonlinear setting. If we let the dimension of the target space be $n = 1$, then we retrieve the scalar p-harmonic equation. The above system is strongly coupled by the appearance of the full differential Du. As a consequence many methods of PDEs known in the linear (harmonic) setting fail for $p \neq 2$. This in turn stimulates the development of new methods and new approaches to handle the nonlinear problems.

The p-harmonic systems and their generalizations appear naturally in differential geometry, see e.g. Hardt–Lin [14] or in relation to differential forms and quasiregular maps, see e.g. Bonk–Heinonen [6]. As for the applied sciences, the second order coupled elliptic systems are studied in nonlinear elasticity theory, e.g. Iwaniec–Onninen [15], nonlinear fluid dynamics, as well as in astrophysics or climate sciences and several other areas (see Adamowicz [3] for the list of further references).

We will now introduce notation, describe our approach to the three-spheres estimates and formulate auxiliary results and the main result of the paper. The proofs are presented in the remaining sections.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^2 convex function such that there exists a subinterval $I \subset \mathbb{R}$ with properties:

$$(\phi'(x))^2 \le \phi''(x) \quad \text{for all } x \in I \tag{4}$$

$$\text{either (i) } \phi'(x) > 0 \quad \text{in } I \text{ or (ii) } \phi'(x) < 0 \text{ in } I \tag{5}$$

In what follows we will deal several times with the expression $|Du|/|\nabla u^1|$, its reciprocal and integrals involving it. In particular, we will need some conditions to ensure that

$$\int_{B_r} \left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2} dx < \infty, \tag{6}$$

where B_r is a ball of radius r. The case of $1 < p < 2$ is easy, since then $(\frac{|\nabla u^1|}{|Du|})^{2-p} < 1$ and by employing also continuity of p-harmonic mappings (see e.g. Tolksdorf [20]) we get that the above integral exists and is finite. The case of $p = 2$ is trivial. For $p > 2$ we need, however, to be more careful at critical points of u^1.

First, recall that results due to Tolksdorf [20, Theorem, Formula (1.14)] and Uhlenbeck [21, Theorem 3.2] applied to p-harmonic systems give us, respectively, that

$$\sup_{B_r} |Du| \le \frac{C(p,n)}{r^{\frac{n}{p}}} \left(|B_{2r}|^{\frac{1}{p}} + \|Du\|_{L^p(B_{2r})}\right) \quad \text{for} \quad 1 < p < 2 \tag{7}$$

$$\sup_{B_r} |Du| \le \frac{C(p,n)}{r^{\frac{n}{p}}} \|Du\|_{L^p(B_{2r})} \quad \text{for} \quad p \ge 2. \tag{8}$$

The discrepancy between formulas (7) and (8) is a consequence of different nature of p-harmonic mappings for $1 < p < 2$ (singular case) and $p > 2$ (degenerate elliptic case). In fact (7) holds for $1 < p < \infty$, cf. statement of Theorem and Section 3 in [20]. Nevertheless, the fact that inequality (8) is scale invariant with respect to u and is independent of the size of B_{2r} makes (8) still of interest to us and, therefore, in what follows we will appeal to both estimates. The above discussion allows us to continue estimation at (6) for $p > 2$ in a following way:

$$\int_{B_r} \left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2} dx \le \frac{C(p,n)\|Du\|^{p-2}_{L^p(B_{2r})}}{r^{n(1-\frac{2}{p})}} \int_{B_r} \frac{1}{|\nabla u^1|^{p-2}}. \tag{9}$$

Suppose that a ball B_r is centered at x_0 and that $B_{2r} \subset \Omega$. Furthermore, assume that for $p > 2$

$$|\nabla u^1(x)| \ge |x - x_0|^{\alpha} \quad \text{for } x \in B_r \text{ and some } \alpha < \frac{n}{p-2}. \tag{10}$$

Then, by (9) we have:

$$\int_{B_r} \left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2} dx \le \frac{C(p,n)\|Du\|^{p-2}_{L^p(B_{2r})}}{r^{n(1-\frac{2}{p})}} \int_0^r t^{-\alpha(p-2)+n-1} dt < \infty. \tag{11}$$

For the sake of simplicity and clarity of discussion all the results in the paper are stated for u^1, the first coordinate function of a p-harmonic mapping u. However, the reader should keep in mind that all the presented results hold as well for all coordinate functions u^i, for $i = 1, \ldots, n$ upon the necessary reformulations of results. Denote by

$$M(r) = \sup_{|x-x_0|=r} u^1(x)$$

and

$$m(r) = \inf_{|x-x_0|=r} u^1(x).$$

In Sect. 2 we show the Caccioppoli-type estimates for a composition of a convex function with a coordinate function of a p-harmonic mapping (Lemma 1). According to our best knowledge, such estimates and such an approach to p-harmonic mappings has not been studied in the literature so far.

Lemma 1 *Let u^1 be the first coordinate function of a p-harmonic mapping u in a domain Ω for $p > 1$. Assume, furthermore, that a convex function $\phi : I \to \mathbb{R}$ satisfies conditions* (4) *and* (5)*. Then the following estimates hold for every ball $\overline{B_r} \subset \Omega$ and $0 < \delta < 1$:*

$$\text{If } 1 < p < 2, \text{ then } \int_{B_{(1-\delta)r}} \left(\frac{|\nabla u^1|}{|Du|}\right)^{2-p} |\nabla \phi(u^1)|^p \leq \frac{c(p,n)}{\delta^p} r^{n-p}. \tag{12}$$

$$\text{If } p \geq 2, \text{ then } \int_{B_{(1-\delta)r}} |\nabla \phi(u^1)|^p \leq \frac{c(p)}{(\delta r)^p} \int_{B_r} \left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2}. \tag{13}$$

Furthermore, for $p > 2$ the growth condition (10) *ensures finitness of the last integral.*

In the next result we study the behavior of $\phi(u^1)$ over the level sets and investigate the de Giorgi type estimates. Such estimates are well-known for solutions of elliptic equations, see e.g. Giusti [10]. In the setting of vector functions and systems of equations such estimates require extra attention and effort. The results of Lemma 2 can be used in further analysis of level sets for coordinate functions of p-harmonic mappings (see Sect. 3 for the proof of Lemma 2).

Let $k \geq 0$. Upon the above notation we define

$$A_{k,r} := \{x \in B_r : \phi(u^1(x)) > k\}.$$

Lemma 2 *Let u^1 be the first coordinate function of a p-harmonic mapping u in a domain $\Omega \subset \mathbb{R}^n$ for $1 < p \leq n$. Assume, furthermore, that conditions* (4) *and* (5) *hold for a convex function $\phi : I \to \mathbb{R}$. Then the following estimates hold for every ball $\overline{B_r} \subset \Omega$ and $0 < \delta < 1$:*

$$\int_{A_{k,(1-\delta)r}} \left(\frac{|\nabla u^1|}{|Du|}\right)^{2-p} |\nabla \phi(u^1)|^p \leq \frac{c(p)}{(\delta r)^p} \int_{A_{k,r}} (\phi(u^1) - k)^p \quad \text{for} \quad 1 < p \leq 2, \tag{14}$$

$$\int_{A_{k,(1-\delta)r}} |\nabla \phi(u^1)|^p \leq \frac{c(p)}{(\delta r)^p} \int_{A_{k,r}} \left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2} (\phi(u^1) - k)^p \quad \text{for} \quad p \geq 2 \text{ and } n > 2. \tag{15}$$

Furthermore, the following inequality holds for $1 < p < 2$:

$$\Big(\sup_{B_{(1-\delta)r}} \phi(u^1)\Big)^p \leq \frac{C^{\frac{n}{p}}}{(\delta r)^n} \int_{B_r} (\phi(u^1))^p + r^{p-n}, \tag{16}$$

where the constant

$$C = C(p, n, c_S)\left[1 + \|\phi'\|^p_{L^\infty(B_r)}\left(r^n + \|Du\|^p_{L^p(B_{2r})}\right)\right]. \tag{17}$$

If, additionally, there exists a positive constant c, such that

$$|\nabla u^1| > c \quad \text{in } B_r, \tag{18}$$

then the following inequality holds for $p > 2$ and $n > 2$:

$$\left(\sup_{B_{(1-\delta)r}} \phi(u^1)\right)^p \leq \frac{C^{\frac{n}{p}}}{(\delta r)^n}\int_{B_r}(\phi(u^1))^p, \tag{19}$$

with constant

$$C = c(p, n, c_S)\left(\frac{\|Du\|_{L^p(B_{2r})}}{cr^{\frac{n}{p}}}\right)^{\frac{n}{p}(p-2)}. \tag{20}$$

Here, c_S stands for the constant in the Sobolev embedding theorem. In the harmonic case $p = 2$, assumption (18) can be neglected and (19) holds with constant $C = c(p, n, c_S)$ in (20).

The main result of this paper is the following version of the arithmetic three-spheres theorem for coordinate functions of p-harmonic mappings. We prove Theorem in Sect. 4 as well as comment on the existence of p-harmonic mappings satisfying assumptions of Theorem.

Theorem (The arithmetic three-spheres theorem) *Let $1 < p \leq n$ and $u = (u^1, \ldots, u^n) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a p-harmonic mapping. Consider three concentric balls centered at $x_0 \in \Omega$ with radii $0 < r_1 < r < r_2$ such that $\overline{B_{2r_2}} \subset \Omega$ and*

$$0 < c \leq \frac{r_1}{r}, \quad \frac{r_1}{r_2} < c < 1, \tag{21}$$

for some fixed c.

If $1 < p < 2$, then let us assume that for a given $\alpha > 0$, the coordinate function u^1 satisfies the following growth condition:

$$|u^1(x) - u^1(x_0)| \geq C|x - x_0|^\alpha \quad \text{for } x \in \Omega \setminus B_{r_2}. \tag{22}$$

If $2 < p < n$, then let us assume instead that there is a positive constant c_1 such that

$$|\nabla u^1| > c_1 \quad \text{in } B_{r_2}. \tag{23}$$

Then there exist a constant C and a radius r_3 such that if $\overline{B_{r_3}} \subset \Omega$, then the following inequalities hold:

$$M(r) \leq CM(r_1) + (1 - C)M(r_3),$$
$$m(r) \geq Cm(r_1) + (1 - C)m(r_3).$$

For $1 < p < 2$ constant C depends on n, p, c_S the constant in the Sobolev embedding theorem, r_1, r_2, c, α and $\|Du\|_{L^p(B_{2r_2})}$, while $r_3 > r_2$ depends on p, n, r_2, α, and $\|Du\|_{L^p(B_{2r_2})}$. For $2 < p \leq n$ constant C depends on n, p, c_S, r_1, r_2, c, c_1 and $\|Du\|_{L^p(B_{2r_2})}$, while $r_3 = r_2$. For $p = 2 < n$ condition (23) is obsolete, $r_3 = r_2$ and constant C depends only on n, p, c_S and c, r_1, r_2.

In the statement of Theorem one may consider local solutions in $\mathbb{R}^n$ of the p-harmonic system (2) instead in the domain Ω. In such a case one may neglect the assumption that $\overline{B_{r_3}} \subset \Omega$.

If $p = 2$, then the p-harmonic system (2) reduces to the uncoupled system of harmonic equations satisfied by coordinate functions u^i for $i = 1, \ldots, n$. In such a case the arithmetic three-spheres theorem holds for each u^i, see Theorem 30 in Protter–Weinberger [19, Chapter 12].

We point out that the case of scalar p-harmonic functions for $1 < p < \infty$ can be handled by the approach very similar to the one for harmonic functions, as one has on the disposal comparison principles and the Harnack-type estimates. Such tools are not known in the setting of coordinate functions of p-harmonic mappings.

For the case $p = n$, Granlund–Marola [12] proved a variant of the three-spheres theorem in the setting of (A, B)-quasilinear equations, in particular for a p-harmonic equation, cf. Theorem 5.4 [12]. However, their approach is based on the existence of the strong maximum (minimum) principle and the Harnack inequality for solutions of the considered (A, B)-equation. Similar results are not known in the setting of coupled p-harmonic mappings $(p \neq 2)$.

2 The proof of Lemma 1

In this section we prove Lemma 1 and then illustrate the discussion by the class of radial p-harmonic mappings.

Proof We begin the proof as in Granlund–Marola [12]. For the sake of simplicity, let us assume that (5) (i) holds, i.e. $\phi'(u^1(x)) > 0$ for all $x \in B_r$. Take a nonnegative function $\xi \in C_0^\infty(B_r)$ and define a test function $\eta(x) = \phi'(u^1(x))^{p-1}\xi^p(x)$ for $x \in B_r$. Then

$$\nabla\eta = (p-1)(\phi'(u^1))^{p-2}\phi''(u^1)\xi^p\nabla u^1 + p\xi^{p-1}(\phi'(u^1))^{p-1}\nabla\xi.$$

We use $\nabla\eta$ in the first equation of p-harmonic system (3) and by using (4) together with (5) we obtain

$$(p-1)\int_{B_r} |Du|^{p-2}|\nabla u^1|^2\phi'(u^1)^p\xi^p \leq p\int_{B_r} |Du|^{p-2}|\nabla u^1|\phi'(u^1)^{p-1}\xi^{p-1}|\nabla\xi|. \quad (24)$$

For some $0 < \epsilon < 1$, whose value will be determined later, we rewrite the right-hand side of (24) and apply the Young inequality:

$$\begin{aligned} p&\int_{B_r}|Du|^{p-2}|\nabla u^1|\phi'(u^1)^{p-1}\xi^{p-1}|\nabla\xi| \\ &= p\int_{B_r}\left(\epsilon|Du|^{(p-2)\frac{p-1}{p}}|\nabla u^1|^{2\frac{p-1}{p}}\phi'(u^1)^{p-1}\xi^{p-1}\right)\left(\frac{1}{\epsilon}|Du|^{\frac{p-2}{p}}|\nabla u^1|^{\frac{2-p}{p}}|\nabla\xi|\right) \\ &\leq (p-1)\epsilon\int_{B_r}|Du|^{p-2}|\nabla u^1|^2\phi'(u^1)^p\xi^p + p\epsilon^{-p}\int_{B_r}\left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2}|\nabla\xi|^p. \quad (25)\end{aligned}$$

In the last inequality we also estimated $\epsilon^{\frac{p}{p-1}} \leq \epsilon$. Now, we use (25) in (24) and by taking e.g. $\epsilon = \frac{1}{2}$ we may include the first integral on the right-hand side of (25) into the left-hand side of (24). In a consequence we arrive at the following estimate:

$$\int_{B_r}|Du|^{p-2}|\nabla u^1|^2\phi'(u^1)^p\xi^p \leq \frac{p2^{p+1}}{p-1}\int_{B_r}\left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2}|\nabla\xi|^p. \quad (26)$$

Denote $c(p) := \frac{p2^{p+1}}{p-1}$.

Case 1: $1 < p < 2$. The left-hand side of the above inequality can be written as follows.

$$\int_{B_r} |Du|^{p-2}|\nabla u^1|^2 \phi'(u^1)^p \xi^p = \int_{B_r} \left(\frac{|\nabla u^1|}{|Du|}\right)^{2-p} |\nabla \phi(u^1)|^p \xi^p. \tag{27}$$

Using (27) and the fact that $|\nabla u^1| \leq |Du|$ in Ω we observe that for $1 < p < 2$ inequality (26) becomes:

$$\int_{B_r} \left(\frac{|\nabla u^1|}{|Du|}\right)^{2-p} |\nabla \phi(u^1)|^p \xi^p \leq c(p) \int_{B_r} |\nabla \xi|^p. \tag{28}$$

Case 2: $p \geq 2$. We have that $|\nabla u^1|^{p-2} \leq |Du|^{p-2}$ in Ω and hence (26) takes the following form.

$$\int_{B_r} |\nabla \phi(u^1)|^p \xi^p \leq c(p) \int_{B_r} \left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2} |\nabla \xi|^p. \tag{29}$$

If we additionally assume (10), then (11) holds and the last integral is finite. In the definition of η we take ξ such that $0 \leq \xi \leq 1$, $\operatorname{supp} \xi \subset B_r$, $\xi \equiv 1$ in $B_{(1-\delta)r}$ and $|\nabla \xi| \leq \frac{c}{\delta r}$ in B_r. Using such ξ in (28) and (29) we arrive at claims (12) and, respectively, (13) of Lemma 1.

If (5) (ii) holds, i.e. $\phi'(u^1(x)) < 0$ for all $x \in B_r$, then as a test function we take $\eta(x) = (-\phi'(u^1(x)))^{p-1}\xi^p(x)$ for $x \in B_r$ and claims of the lemma follow the same way as previously. □

We remark that assertions (28) and (29) of Lemma 1 can be further refined. Namely, the following remark holds by Lemma 1 and the discussion at formulas (7) and (8).

Remark 1 **(A)** Let $1 < p < 2$. Then (7) implies that for $x \in B_r$

$$\frac{1}{|Du(x)|} \geq \frac{1}{\sup_{B_r} |Du|} \geq C(p, n, \operatorname{diam}(\Omega), \|Du\|_{L^p(B_{2r})}) r^{\frac{n}{p}}.$$

In such a case (28) reads:

$$\int_{B_r} \left(|\nabla u^1| r^{\frac{n}{p}}\right)^{2-p} |\nabla \phi(u^1)|^p \xi^p \leq c(p) \int_{B_r} |\nabla \xi|^p.$$

Similarly, for $p > 2$ the Uhlenbeck inequality (8) results in the following estimate.

$$\int_{B_r} |\nabla \phi(u^1)|^p \xi^p \leq C(p, n, \|Du\|_{L^p(B_{2r})}) \int_{B_r} \frac{1}{r^{n(1-\frac{2}{p})}|\nabla u^1|^{p-2}} |\nabla \xi|^p.$$

(B) In fact estimate (26) gives rise to the following inequality for $p > 1$:

$$\int_{B_r} \left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2} |\nabla \phi(u^1)|^p \xi^p \leq \frac{p2^{p+1}}{p-1} \int_{B_r} \left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2} |\nabla \xi|^p.$$

We now turn to considering a special class of p-harmonic mappings, namely radial transformations in the form

$$u(x) = H(|x|)(x_1, \ldots, x_n), \quad \text{for } x = (x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^n,$$

where $H \in C^2(\Omega)$ and $|x|$ stands for the magnitude of x. For such mappings p-harmonic system (2) becomes a nonlinear second order ODE:

$$(p-1)H''(H')^2r^3 + (2p+n-3)(H')^3r^2 + 2(p-1)HH'H''r^2 \\ + (np+3p-4)H(H')^2r + (p+n-2)H^2H''r + (n+1)(p+n-2)H^2H' = 0. \tag{30}$$

The problem of finding suitable examples is the general feature of the p-harmonic world, as we know only few classes of p-harmonic maps and few explicit solutions of the p-harmonic system of equations, namely affine, radial and quasiradial, see e.g. Adamowicz [2,3], Iwaniec–Onninen [15, Part 1] for various applications of radial p-harmonics, and Adamowicz [1, Chapter 2] for the definition of quasiradial p-harmonic mappings.

For the class of radial p-harmonic mappings, Lemma 1 can be refined. Indeed, for radial mappings we can formulate simple conditions for integrability of ratio $|\nabla u^1|/|Du|$ and in turn estimates (12) and (13) reduce to the following result.

Proposition 1 (Radial Lemma 1) *Let u^1 be the first coordinate function of a radial p-harmonic mapping u in $\Omega \subset \mathbb{R}^n$, $n > 1$ for $p > 1$. Assume that a convex function $\phi : I \to \mathbb{R}$ satisfies conditions* (4) *and* (5)*. Furthermore, assume the following:*

(1) if x is such that $H(x) = 0$, then $H'(x) \neq 0$,

(2) there exist constant $c_\Omega, c'_\Omega > 0$ such that $x_1^2 < c_\Omega(|x|^2 - x_1^2)$ *and* $x_1 > c'_\Omega$ *for all* $x \in \Omega$, (31)

(3) there exists $C > 0$ *such that* $\eta(x) := \dfrac{H'(|x|)}{H(|x|)}|x| < C$ *for all* $x \in \Omega$. (32)

Then the following estimate holds for every ball $\overline{B_r} \subset \Omega$ and $0 < \delta < 1$:

$$\int_{B_{(1-\delta)r}} |\nabla\phi(u^1)|^p \leq \frac{c(\text{diam}(\Omega), p, n, c_\Omega, c'_\Omega)}{\delta^p} r^{n-p}. \tag{33}$$

Conditions (31) and (32) geometrically mean that we require domain Ω to lie inside a cone symmetric about x_1-axis and to consist of points with a positive distance to the hyperplane $\{x_1 = 0\}$.

Proof Let $u(x) = H(|x|)(x_1, \ldots, x_n)$ be a radial p-harmonic mapping in Ω. Denote $r := |x|$ the magnitude of vector x. Then $u^i(x) = H(r)x_i$ for $i = 1, \ldots, n$ and

$$u^i_{x_j}(x) = \begin{cases} H'(r)\frac{x_i x_j}{r}, & \text{for } j \neq i, \\ H'(r)\frac{x_i^2}{r} + H(r), & \text{for } j = i. \end{cases}$$

Hence $|\nabla u^i|^2 = \sum_{j\neq i}(H'\frac{x_i x_j}{r})^2 + (H'\frac{x_i^2}{r} + H)^2 = (H')^2x_i^2 + 2HH'\frac{x_i^2}{r} + H^2$. Upon computing $|\nabla u^i|^2/|\nabla u^1|^2$ and summing over $i = 1, \ldots, n$ we get that

$$g := \frac{|Du|^2}{|\nabla u^1|^2} = \frac{|\nabla u^1|^2 + \ldots + |\nabla u^n|^2}{|\nabla u^1|^2} = \frac{(H')^2r^2 + 2HH'r + nH^2}{(H')^2x_1^2 + 2HH'\frac{x_1^2}{r} + H^2}.$$

Let $x_0 \in B_r$ and consider the following cases.

Case 1: $H(x_0) = 0$. Since, by assumptions we have that $H'(|x_0|) \neq 0$, then $g(x_0) = \frac{r^2}{(x_0)_1^2}$. Under the second part of assumption (31) we obtain that $|g(x_0)| \leq c(\mathrm{diam}(\Omega), c_\Omega)$.

Case 2: $H(x_0) \neq 0$ and $H'(x_0) = 0$. Then $g(x_0) = n$.

Case 3: $H(x_0) \neq 0$ and $H'(x_0) \neq 0$. Then for $\eta := \eta(x_0) = \frac{H'(|x_0|)}{H(|x_0|)}|x_0|$ we have

$$g(x_0) = \frac{\eta\,(\eta + 2) + n}{\frac{|(x_0)_1|^2}{|x_0|^2}\eta\,(\eta + 2) + 1}.$$

Depending on the sign of $\eta(\eta + 2)$ we distinguish two cases:

(a) If $\eta \geq 0$ or $\eta \leq -2$, then $g(x_0) \leq \eta(\eta + 2) + n = (\eta + 1)^2 + n - 1$ and assumption (32) results in $g(x_0) \leq (C + 1)^2 + n - 1$.

(b) If $-2 \leq \eta \leq 0$, then $|g(x_0)| \leq n\frac{|x_0|^2}{||x_0|^2 - |(x_0)_1|^2|} \leq nc'_\Omega$. In the last inequality we also used the first part of assumption (31).

Therefore we conclude that g is bounded by a constant $c(\mathrm{diam}(\Omega), n, c_\Omega, c'_\Omega)$ for all $x \in \overline{B_r}$. Then assertion (33) follows immediately from (12) and (13). □

3 The proof of Lemma 2

The first part of the proof is similar to the one of Lemma 1 (see also Granlund [11, Section 2] and Granlund–Marola [12, Lemma 2.6]). Let $k \geq 0$ and define $\psi(x) = \max\{\phi(u^1(x)) - k, 0\}$ for $x \in B_r$ with for $\overline{B_r} \subset \Omega$. Take $\xi \in C_0^\infty(\Omega)$ and define a test function $\eta(x) = \psi(x)(\phi'(u^1(x)))^{p-1}\xi^p(x)$. Then

$$\begin{aligned}\nabla\eta = {} & \phi'(u^1)(\phi'(u^1))^{p-1}\xi^p\nabla u^1 + (p-1)\psi\phi''(u^1)(\phi'(u^1))^{p-2}\xi^p\nabla u^1 \\ & + p\psi(\phi'(u^1))^{p-1}\xi^{p-1}\nabla\xi.\end{aligned}$$

Using $\nabla\eta$ in the first equation of p-harmonic system (3) we get the following inequality.

$$\begin{aligned}&\int_{B_r}|Du|^{p-2}|\nabla u^1|^2(\phi'(u^1))^p\xi^p + (p-1)\int_{B_r}|Du|^{p-2}|\nabla u^1|^2\psi\phi''(u^1)(\phi'(u^1))^{p-2}\xi^p \\ &\quad \leq p\int_{B_r}|Du|^{p-2}|\nabla u^1||\nabla\xi|\psi(\phi'(u^1))^{p-1}\xi^{p-1}.\end{aligned} \tag{34}$$

We invoke property (4) of function ϕ and use it in the second integral on the left-hand side of the inequality. Since $\psi \geq 0$ in B_r we can drop the aforementioned integral. The Young inequality applied to the integral on the right-hand side gives us the estimate (cf. inequality (25)):

$$\begin{aligned}&p\int_{B_r}|Du|^{p-2}|\nabla u^1||\nabla\xi|\psi(\phi'(u^1))^{p-1}\xi^{p-1} \\ &\quad = p\int_{B_r}\left(|Du|^{(p-2)\frac{p-1}{p}}|\nabla u^1|^{2\frac{p-1}{p}}\phi'(u^1)^{p-1}\xi^{p-1}\right)\left(|Du|^{\frac{p-2}{p}}|\nabla u^1|^{1-2\frac{p-1}{p}}\psi|\nabla\xi|\right) \\ &\quad \leq (p-1)\epsilon\int_{B_r}|Du|^{p-2}|\nabla u^1|^2\phi'(u^1)^p\xi^p + \epsilon^{-p}\int_{B_r}\left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2}\psi^p|\nabla\xi|^p.\end{aligned}$$

Upon choosing small enough value of $0 < \epsilon < 1$, e.g. $\epsilon = \frac{1}{2(p-1)}$ we include the first integral on the right-hand side of the inequality into the integral on the left-hand side of (34) and arrive at the following estimate (cf. (26) in Lemma 1):

$$\int_{B_r} |Du|^{p-2}|\nabla u^1|^2\phi'(u^1)^p\xi^p \le c(p)\int_{B_r}\left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2}\psi^p|\nabla\xi|^p. \tag{35}$$

We choose function ξ so that it satisfies: $\operatorname{supp}\xi \subset B_r$, $0 \le \xi \le 1$, $\xi \equiv 1$ in $B_{(1-\delta)r}$ and $|\nabla\xi| \le \frac{c}{\delta r}$ in B_r. Note also, that by definition $\psi \equiv 0$ in $B_r \setminus A_{k,r}$. This and the choice of ξ in (35) lead us to the following inequality (cf. estimate (26) in Lemma 1):

$$\int_{A_{k,(1-\delta)r}} |Du|^{p-2}|\nabla u^1|^2\phi'(u^1)^p \le \frac{c(p)}{(\delta r)^p}\int_{A_{k,r}}\left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2}\psi^p.$$

The discussion similar to the one for Cases 1 and 2 in Lemma 1 (cf. inequalities (28) and (29)) gives us assertions (14) and (15).

The proofs of supremum estimates (16) (in both cases) require extra attention due to the appearance of expression $g := \left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2}$ under integrals (14) and (15) exploited in derivation of the supremum estimate. Note, that when the p-harmonic system (1) reduces to a single p-harmonic equation, then $|Du| = |\nabla u^1|$, and so $g \equiv 1$. In such a case we retrieve estimates from Lemma 2.6 in Granlund–Marola [12]. Here, we instead follow the method from a book by Giusti [10, Theorem 7.2] and adapt it to the vectorial case.

Using the notation analogous to [10], let $(1-\delta)r \le \sigma r \le \tau r \le r$. At this point the discussion splits into four cases: (1) $1 < p < 2$, (2) $p > 2$ according to estimates (14) and (15), respectively, (3) $p = 2$ and (4) $p = n$.

Case 1: $1 < p < 2$.

Let $\eta \in C_0^\infty(B_{\frac{\sigma+\tau}{2}r})$ such that $\eta \equiv 1$ on $B_{\sigma r}$ and $|\nabla\eta| \le \frac{c}{(\tau-\sigma)r}$. Define $\xi(x) = \eta(x)\psi(x)$ for function ψ as in the first part of the proof. By the Hölder and the Sobolev inequalities we get

$$\int_{A_{k,\sigma r}} \psi^p \le \int_{A_{k,\sigma r}} \xi^p \le \left(\int_{A_{k,\sigma r}} \xi^{\frac{np}{n-p}}\right)^{1-\frac{p}{n}} |A_{k,\sigma r}|^{\frac{p}{n}} \le c_S|A_{k,\tau r}|^{\frac{p}{n}}\int_{A_{k,\sigma r}}|\nabla\xi|^p. \tag{36}$$

Using the definition of ξ we compute that $\nabla\xi = \psi\nabla\eta + \eta\nabla\psi$ and hence

$$\int_{A_{k,\sigma r}} \psi^p \le c_S|A_{k,\tau r}|^{\frac{p}{n}}\left(\int_{A_{k,\frac{\sigma+\tau}{2}r}} |\nabla\phi(u^1)|^p + \frac{1}{(\tau-\sigma)^p r^p}\int_{A_{k,\frac{\sigma+\tau}{2}r}}\psi^p\right). \tag{37}$$

Let $\alpha = \frac{p}{4}(2-p)$ and $\beta = p(1-\frac{p}{4})$. Then, the Young inequality applied with exponents $\frac{2-p}{\alpha} = \frac{4}{p}$ and its conjugate $\left(\frac{2-p}{\alpha}\right)' = \frac{2-p}{2-p-\alpha} = \frac{4}{4-p}$ gives us the following:

$$\begin{aligned}\int_{A_{k,\frac{\sigma+\tau}{2}r}} |\nabla\phi(u^1)|^p &= \int_{A_{k,\frac{\sigma+\tau}{2}r}}\left(|\nabla\phi(u^1)|^{p-\beta}\left(\frac{|\nabla u^1|}{|Du|}\right)^{\alpha}\right)\left(|\phi'(u^1)|^{\beta}|Du|^{\beta}\left(\frac{|\nabla u^1|}{|Du|}\right)^{\beta-\alpha}\right)\\ &\le \frac{p}{4}\int_{A_{k,\frac{\sigma+\tau}{2}r}}\left(\frac{|\nabla u^1|}{|Du|}\right)^{2-p}|\nabla\phi(u^1)|^p\\ &\quad + (1-\frac{p}{4})\int_{A_{k,\frac{\sigma+\tau}{2}r}}\left(\frac{|\nabla u^1|}{|Du|}\right)^{\frac{2p}{4-p}}|\phi'(u^1)|^p|Du|^p. \end{aligned}\tag{38}$$

In order to estimate further (38) for p in the given range we appeal to Tolksdorf's estimate [20] and note that $\left(\frac{|\nabla u^1|}{|Du|}\right)^{\frac{2p}{4-p}} < 1$. Then

$$\int_{A_{k,\frac{\sigma+\tau}{2}r}} \left(\frac{|\nabla u^1|}{|Du|}\right)^{\frac{2p}{4-p}} |\phi'(u^1)|^p |Du|^p$$
$$\leq C(p,n)\, \|\phi'\|^p_{L^\infty(B_{\frac{\sigma+\tau}{2}r})} \left(\frac{2}{(\sigma+\tau)r}\right)^n \left(|B_{(\sigma+\tau)r}| + \|Du\|^p_{L^p(B_{(\sigma+\tau)r})}\right) |A_{k,\frac{\sigma+\tau}{2}r}|.$$

We use this inequality in the second integral on the right-hand side of (38). Moreover, we observe that under assumptions on σ and τ it holds that $\frac{\sigma+\tau}{2} \leq \tau$ and thus $A_{k,\frac{\sigma+\tau}{2}r} \subset A_{k,\tau r}$. We apply estimate (14) with $r := \tau r$ and δ such that $(1-\delta)r\tau := \frac{\sigma+\tau}{2}r$ in the first integral on the right-hand side of (38). In a consequence, estimate (37) takes the following form:

$$\int_{A_{k,\sigma r}} \psi^p \leq C(p,n,c_S)|A_{k,\tau r}|^{\frac{p}{n}} \left(\frac{2^p}{(\tau-\sigma)^p r^p} \int_{A_{k,\tau r}} \psi^p + \|\phi'\|^p_{L^\infty(B_{\frac{\sigma+\tau}{2}r})} \right.$$
$$\left. \times \frac{2^n}{(\sigma+\tau)^n r^n}(r^n + \|Du\|^p_{L^p(B_{2r})})|A_{k,\frac{\sigma+\tau}{2}r}| + \frac{1}{(\tau-\sigma)^p r^p} \int_{A_{k,\frac{\sigma+\tau}{2}r}} \psi^p \right). \qquad (39)$$

If $h < k$, then

$$(k-h)^p |A_{k,\tau r}| \leq \int_{A_{h,\tau r}} (\phi(u^1)-h)^p \quad \text{and} \quad \int_{A_{k,\tau r}} (\phi(u^1)-k)^p \leq \int_{A_{h,\tau r}} (\phi(u^1)-h)^p.$$

Denote

$$C = C(p,n,c_S)\left[1 + \|\phi'\|^p_{L^\infty(B_r)}(r^n + \|Du\|^p_{L^p(B_{2r})})\right].$$

Using these in (39) we obtain the following estimate:

$$\int_{A_{k,\sigma r}} (\phi(u^1)-k)^p \leq \frac{C|A_{k,\tau r}|^{\frac{p}{n}}}{(\sigma-\tau)^p r^p} \left(\int_{A_{h,\tau r}} (\phi(u^1)-h)^p + \frac{(\sigma-\tau)^p r^p}{(\sigma+\tau)^n r^n} |A_{k,\tau r}| \right)$$
$$\leq \frac{C}{(k-h)^{\frac{p^2}{n}}(\sigma-\tau)^p r^p} \left(\int_{A_{h,\tau r}} (\phi(u^1)-h)^p \right)^{1+\frac{p}{n}} \left[1 + \frac{1}{(k-h)^p} \frac{(\sigma-\tau)^p r^p}{(\sigma+\tau)^n r^n} \right]. \quad (40)$$

We are in a position to use the iteration scheme as in Lemma 7.1 in Giusti book [10]. Indeed, for some $d > 0$, to be determined later, let us consider the following quantities:

$$k_i := 2d(1-2^{-i}), \quad \text{for } i = 0,1,\ldots, \qquad k = k_{i+1}, \quad h = k_i, \quad k-h = \frac{d}{2^i}$$
$$\sigma_i := \delta + (1-\delta)2^{-i}, \quad \text{for } i = 0,1,\ldots, \qquad \sigma = \sigma_{i+1}, \qquad \tau = \sigma_i.$$

Hence

$$\tau - \sigma = (1-\delta)2^{-i-1}, \qquad \tau + \sigma = 2\delta + (1-\delta)3 \cdot 2^{-i-1}.$$

Finally, let

$$\phi_i := \int_{A_{k_i,\sigma_i}} (\phi(u^1)-k_i)^p.$$

With this notation inequality (40) reads:

$$\phi_{i+1} \leq \frac{C}{(1-\delta)^p r^p} d^{-\frac{p^2}{n}} (1 + d^{-p} r^{p-n}) 2^{pi(1+p/n)} \phi_i^{1+\frac{p}{n}}, \quad \text{for } i = 0, 1, \dots.$$

The claim of the second part of Lemma 2 for $1 < p < 2$ follows from [10, Lemma 7.1], cf. the proof of Theorem 7.2 in [10]. Indeed, we set that for some $a > 0$,

$$1 + d^{-p} r^{p-n} \leq a.$$

By taking $B = 2^{p(1+p/n)}$, $\alpha = \frac{p}{n}$ and $c = \frac{Ca}{(1-\delta)^p r^p} d^{-\frac{p^2}{n}}$ we verify that the assumption of [10, Lemma 7.1] that $\phi_0 \leq c^{-1/\alpha} B^{-1/(\alpha^2)}$ leads to the following conditions:

$$d^p \geq \frac{(Ca)^{\frac{n}{p}}}{((1-\delta)r)^n} \int_{B_r} (\phi(u^1))^p \quad \text{and} \quad (a-1)d^p \geq r^{p-n},$$

as $A_{0,r} = B_r$. Thus, by taking e.g. $a = 2$ we get that the above inequalities for

$$d^p = \frac{C^{\frac{n}{p}}}{((1-\delta)r)^n} \int_{B_r} (\phi(u^1))^p + r^{p-n}$$

and the claim follows, since by [10, Lemma 7.1] $\lim_{i\to\infty} \phi_i = 0$ and so $\sup_{B_{\delta r}} \phi(u^1) \leq 2d$.

Case 2: $2 < p < n$.

We proceed similarly to the previous case. Upon using estimate (15) in (37) together with the fact that $\frac{|Du|}{|\nabla u^1|}$ is bounded by (8) and our assumptions, we obtain that

$$\int_{A_{k,\sigma r}} (\phi(u^1) - k)^p \leq \frac{C_{\sup}}{((\sigma - \tau)r)^p} \left(\frac{1}{(k-h)^p} \int_{A_{h,\tau r}} (\phi(u^1) - h)^p \right)^{1+\frac{p}{n}}. \tag{41}$$

Constant $C_{\sup} = c(p, n, c_S)(\frac{\|Du\|_{L^p(B_{2r})}}{c} r^{-\frac{n}{p}})^{\frac{n}{p}(p-2)}$. The reasoning similar to the previous case gives us the claim for $2 < p < n$. However, in this case we get the homogeneous estimate (19).

Case 3: $p = 2$.

If $p = 2$ and $n > 2$, then we follow the proof for $2 < p < n$ and since $\left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2} \equiv 1$, assumption (18) and discussion at (38) are not needed and we obtain (19) with constant $C = c(p, n, c_S)$ in (20).

If $p = 2$ and $n = 2$, then in (36) we use a variant of the Sobolev inequality (see e.g. Corollary 1.57 in Malý–Ziemer [16]) and get

$$\int_{A_{k,\sigma r}} \psi^2 \leq \int_{A_{k,\sigma r}} \xi^2 \leq |A_{k,\tau r}| \int_{A_{k,\sigma r}} |\nabla \xi|^2.$$

This leads to the estimate similar to (41) and (19), while the resulting constant C depends on p, n and c_S.

Case 4: $p = n$.

As in the previous case, we use [16, Corollary 1.57] and obtain the following

$$\int_{A_{k,\sigma r}} \psi^n \leq \int_{A_{k,\sigma r}} \xi^n \leq |A_{k,\tau r}| \int_{A_{k,\sigma r}} |\nabla \xi|^n.$$

We follow the proof for the case of $2 < p < n$ and get $C_{\sup} = c(n)\left(\frac{\|Du\|_{L^n(B_{2r})}}{cr}\right)^{n-2}$.

Hence, the proof of Lemma 2 is completed. □

4 The proof of Theorem

In the proof of Theorem we use the doubling property of the Lebesgue measure. This means, that for any ball $B_R \subset \mathbb{R}^n$ it holds that $\mathcal{L}^n(B_{2R}) \le C\mathcal{L}^n(B_R)$, where $C = 2^n$. Below, we also appeal to the $(1, p)$-Poincaré inequality: if $v \in W^{1,p}_{loc}(\Omega)$, then

$$\int_{B_r} |v - v_{B_r}|^p \le Cr^p \int_{B_r} |\nabla v|^p,$$

where v_{B_r} denotes the mean value of v over the ball B_r and C depends on n and p.

Finally, the following auxiliary result is used in the proof of Theorem as well, see Theorem 4.20 in Adamowicz [1] and Appendix A.2 in Adamowicz [3].

Lemma 3 [3, Observation 2] *Let $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ be a p-harmonic mapping in the domain $\Omega \subset \mathbb{R}^n$. If for some u^i, $i = 1, \ldots, n$ there exists $k \in \mathbb{R}$ such that $u^i \le k$ on $\partial\Omega$, then $u^i \le k$ in Ω.*

Proof of Theorem Our approach is based on Theorem 1.3 in Granlund–Marola [12]. However, the fact that now we are in the setting of mappings instead of scalar functions requires new type of the de Giorgi estimates (cf. Lemmas 1 and 2). Moreover, the dependence of those estimates on $\|Du\|_{L^p}$ and $\|\phi'\|_{L^\infty}$ requires additional efforts and caution.

Define the following function (keeping in mind that the exact value of r_3 will be determined later):

$$\phi(t) := -\log\left(\frac{M(r_3) - t + \epsilon}{M(r_3) - M(r_1) + \epsilon}\right) \quad \text{for } t \in (-\infty, M(r_3)), \tag{42}$$

for any but fixed $\epsilon > 0$. It is easy to check that ϕ is C^2 convex and satisfies conditions (4) and (5). Indeed, by computations we see that

$$\phi'(t) = \frac{1}{M(r_3) - t + \epsilon} > 0 \quad \text{and} \quad \phi''(t) = \frac{1}{(M(r_3) - t + \epsilon)^2} \quad \text{in } (-\infty, M(r_3)). \tag{43}$$

Furthermore, since $\phi(u^1) < 0$ on B_{r_1}, it holds that function $\psi := \max\{\phi(u^1), 0\}$ satisfies $\psi \equiv 0$ on B_{r_1}, thus also the mean value of ψ vanishes, $\psi_{B_{r_1}} = 0$. This together with the $(1, p)$-Poincaré inequality and the doubling property of the Lebesgue measure (with doubling constant 2^n) implies

$$\begin{aligned}
\fint_{B_{\frac{r+r_2}{2}}} \psi^p &\le \fint_{B_{\frac{r+r_2}{2}}} \left|\psi - \psi_{B_{(r+r_2)/2}} + \left(\psi_{B_{(r+r_2)/2}} - \psi_{B_{r_1}}\right)\right|^p \\
&\le 2^p \fint_{B_{\frac{r+r_2}{2}}} |\psi - \psi_{B_{(r+r_2)/2}}|^p + 2^{p+n}\left(\fint_{B_{\frac{r+r_2}{2}}} |\psi - \psi_{B_{(r+r_2)/2}}|\right)^p \\
&\le C(p,n)\left(\frac{r+r_2}{2}\right)^p \fint_{B_{\frac{r+r_2}{2}}} |\nabla\psi|^p.
\end{aligned} \tag{44}$$

We consider two cases.
Case 1: $1 < p < 2$. Recall the supremum estimate (16) from Lemma 2:

$$\big(\sup_{B_{(1-\delta)R}} \phi(u^1)\big)^p \le \frac{C_{\sup}}{(\delta R)^n}\int_{B_R}\phi(u^1)^p + R^{p-n},$$

where $C_{\sup}$ is constant in (17), cf. statement of Lemma 2. We apply this estimate with $R = (r+r_2)/2$ and $0 < \delta = (r_2-r)/(r_2+r) < 1$, then use the definition of function ψ and Poincaré-type estimate (44).

$$\begin{aligned}
\big(\sup_{B_r}\phi(u^1)\big)^p &\le C_{\sup}\left(\frac{2}{r_2-r}\right)^n \int_{B_{\frac{r+r_2}{2}}}\phi(u^1)^p + \left(\frac{r+r_2}{2}\right)^{p-n}\\
&\le C_{\sup}\left(\frac{r+r_2}{r_2-r}\right)^n\left(\frac{2}{r+r_2}\right)^n\int_{B_{\frac{r+r_2}{2}}}\psi^p + \left(\frac{r+r_2}{2}\right)^{p-n}\\
&\le C_{\sup}\left(\frac{2cr_2}{cr_2-r_1}\right)^n\left(\frac{r+r_2}{2}\right)^{p-n}\int_{B_{\frac{r+r_2}{2}}}|\nabla\phi(u^1)|^p + \left(\frac{r+r_2}{2}\right)^{p-n}\\
&\le C_{\sup}\left(\frac{2}{c+1}\right)^{p-n}\frac{1}{r_2^{n-p}}\int_{B_{\frac{r+r_2}{2}}}|\nabla\phi(u^1)|^p + \left(\frac{2}{c+1}\right)^{p-n}\frac{1}{r_2^{n-p}}. \qquad (45)
\end{aligned}$$

In the last step we included expression with r_1, r_2, c into the constant $C_{\sup}$. In order to use estimate (12) in (45) we need to use first the Young inequality with exponents $\frac{1}{\alpha} = 1 + \frac{1}{\epsilon}$ and $\beta = 1+\epsilon$, for some $\epsilon \in (0,1)$:

$$\begin{aligned}
\int_{B_{\frac{r+r_2}{2}}}|\nabla\phi(u^1)|^p &= \int_{B_{\frac{r+r_2}{2}}}\left(\frac{|\nabla u^1|}{|Du|}\right)^{(2-p)\alpha}|\nabla\phi(u^1)|^{p\alpha}\left(\frac{|\nabla u^1|}{|Du|}\right)^{(p-2)\alpha}|\nabla\phi(u^1)|^{p(1-\alpha)}\\
&\le \int_{B_{\frac{r+r_2}{2}}}\left(\frac{|\nabla u^1|}{|Du|}\right)^{2-p}|\nabla\phi(u^1)|^p + \int_{B_{\frac{r+r_2}{2}}}\left(\frac{|Du|}{|\nabla u^1|}\right)^{(2-p)\epsilon}|\nabla\phi(u^1)|^p.
\end{aligned} \qquad (46)$$

The second integral on the right-hand side of (46) can be easily estimated as follows:

$$\int_{B_{\frac{r+r_2}{2}}}\left(\frac{|Du|}{|\nabla u^1|}\right)^{(2-p)\epsilon}|\nabla\phi(u^1)|^p \le \|\phi'\|^p_{L^\infty(B_{r_2})}\|Du\|^p_{L^p(B_{r_2})}.$$

We use (12) from Lemma 1 with with r_2 and $\delta = \frac{r_2-r}{2r_2}$ together with properties of radii (21). Then estimate (46) for the p-energy of $\phi(u^1)$ takes the following form:

$$\int_{B_{\frac{r+r_2}{2}}}|\nabla\phi(u^1)|^p \le \left(\frac{2c}{cr_2-r_1}\right)^p r_2^{n-p} + \|\phi'\|^p_{L^\infty(B_{r_2})}\|Du\|^p_{L^p(B_{r_2})}.$$

The expression on the right-hand side is similar to the one in $C_{\sup}$. We use this observation and the last inequality in estimate (45) and obtain the following bound, which we in turn estimate using properties of function ϕ, see (43):

$$
\begin{aligned}
\big(\sup_{B_r}\phi(u^1)\big)^p &\le C_{\sup}\frac{2^{p-n}}{(c+1)^{p-n}}\frac{1}{r_2^{n-p}}\left(C(p,r_1,r_2,c)r_2^{n-p}+\|\phi'\|^p_{L^\infty(B_{r_2})}\|Du\|^p_{L^p(B_{r_2})}\right)\\
&\quad+\frac{2^{p-n}}{(c+1)^{p-n}}\frac{1}{r_2^{n-p}}\\
&\le\frac{C}{r_2^{n-p}}\left\{\left(1+\|\phi'\|^p_{L^\infty(B_{r_2})}\left(r_2^n+\|Du\|^p_{L^p(B_{2r_2})}\right)\right)\left(r_2^{n-p}+\|\phi'\|^p_{L^\infty(B_{r_2})}\|Du\|^p_{L^p(B_{r_2})}+1\right)\right\}\\
&\le C\left(1+\|\phi'\|^p_{L^\infty(B_{r_2})}\left(r_2^n+\|Du\|^p_{L^p(B_{2r_2})}\right)\right)^2\max\{1,r_2^{p-n}\}\\
&\le C\left(1+\frac{1}{|M(r_3)-M(r_2)|^{2p}}\left(r_2^{2n}+\|Du\|^{2p}_{L^p(B_{2r_2})}\right)\right)\max\{1,r_2^{p-n}\}. \qquad (47)
\end{aligned}
$$

Here $C=C(p,n,c,c_S,r_1,r_2)$. By the weak maximum principle in Lemma 3, it holds that $M(r_2)<M(r_3)$ for a non-constant u^1 and some $r_3>r_2$. The continuity of u implies that maxima $M(r_2)$ and $M(r_3)$ are attained at some points $x_3\in S_{r_3}$ and $x_2\in S_{r_2}$. By the mean value theorem and the Tolksdorf estimate we get that

$$|u^1(x_2)-u^1(x_0)|\le\sup_{B_{r_2}}|Du|\,|x_2-x_0|\le C(p,n)(r_2^{\frac{1}{p}}+\|Du\|_{L^p(B_{2r_2})})r_2^{1-\frac{n}{p}}.$$

We now appeal to growth condition (22), to obtain the following estimate:

$$
\begin{aligned}
|M(r_3)-M(r_2)|=|u^1(x_3)-u^1(x_2)|&\ge|u^1(x_3)-u^1(x_0)|-|u^1(x_2)-u^1(x_0)|\\
&\ge Cr_3^{\alpha}-C(p,n)(r_2^{\frac{1}{p}}+\|Du\|_{L^p(B_{2r_2})})r_2^{1-\frac{n}{p}}.
\end{aligned}
$$

We use this inequality on the right-hand side of (47) and notice that by taking sufficiently large r_3, for instance such that

$$Cr_3^{\alpha}\ge(1+r_2^{2n}+\|Du\|^{2p}_{L^p(B_{2r_2})})^{1/(2p)}+C(p,n)(r_2^{\frac{1}{p}}+\|Du\|_{L^p(B_{2r_2})})r_2^{1-\frac{n}{p}}$$

we get that the right-hand side of (47) can now be estimated by $A:=C(p,n,c,c_S,r_1,r_2)\max\{1,r_2^{p-n}\}$. Observe that r_3 depends on n, p, r_2 and $\|Du\|_{L^p(B_{2r_2})}$, but not on r.

Case 2: $2\le p<n$. We start from the estimate similar to (45). Namely, the supremum estimate (19) leads to the following inequality:

$$\big(\sup_{B_r}\phi(u^1)\big)^p\le C_{\sup}\left(\frac{2}{c+1}\right)^{p-n}\frac{1}{r_2^{n-p}}\int_{B_{\frac{r+r_2}{2}}}|\nabla\phi(u^1)|^p.$$

We use Lemma 1 with r_2 and $\delta=\frac{r_2-r}{2r_2}$ together with constant (20) from Lemma 2 and properties of radii (21) to obtain the following inequalities:

$$
\begin{aligned}
\big(\sup_{B_r}\phi(u^1)\big)^p&\le C_{\sup}\left(\frac{2}{c+1}\right)^{p-n}\frac{1}{r_2^{n-p}}\int_{B_{\frac{r+r_2}{2}}}|\nabla\phi(u^1)|^p\\
&\le C_{\sup}\left(\frac{2}{c+1}\right)^{p-n}\frac{1}{r_2^{n-p}}\left(\frac{2}{r_2-r}\right)^p\int_{B_{r_2}}\left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2}
\end{aligned}
$$

$$\leq c(p,n,c_S)\left(\frac{\|Du\|_{L^p(B_{r+r_2})}}{c_1\left(\frac{r+r_2}{2}\right)^{\frac{n}{p}}}\right)^{\frac{n}{p}(p-2)}\left(\frac{2}{r_2-r_1/c}\right)^p\frac{1}{r_2^{n-p}}\left(\frac{\|Du\|_{L^p(B_{2r_2})}}{c_1 r_2^{\frac{n}{p}}}\right)^{\frac{n}{p}(p-2)}$$

$$\leq c(p,n,c_S,c,r_1,r_2)\left(\frac{\|Du\|_{L^p(B_{2r_2})}}{c_1}\right)^{2\frac{n}{p}(p-2)}. \tag{48}$$

Similarly to the previous case, we denote the constant on the right-hand side of the above inequality by A. Observe that for $p=2$ the constant on the right-hand side of (48) depends only on p, n, c_S and c, r_1, r_2 due to (21).

Case 3: $p=n$. We discuss this case separately due to the importance of n-harmonic mappings in nonlinear analysis. As in the previous case we obtain the estimate similar to (45):

$$\left(\sup_{B_r}\phi(u^1)\right)^n \leq C_{\sup}\left(\frac{2}{r_2-r}\right)^n\int_{B_{\frac{r+r_2}{2}}}|\nabla\phi(u^1)|^n.$$

The reasoning analogous to Case 2 gives us that

$$\left(\sup_{B_r}\phi(u^1)\right)^n \leq c(n)\left(\frac{\|Du\|_{L^n(B_{2r_2})}}{c_1}\right)^{2(n-2)}\left(\frac{r_1}{c}+r_2\right)^2\frac{1}{r_2^{n-2}}.$$

As in the previous cases, we denote the constant on the right-hand side of the above inequality by A.

We are now in a position to complete the proof of Theorem. By our assumptions ϕ is strictly increasing and so we have that

$$\log\left(\frac{M(r_3)-M(r)+\epsilon}{M(r_3)-M(r_1)+\epsilon}\right) = -\sup_{B_r}\phi(u^1) \geq -A.$$

Note that in both cases: $1<p<2$ and $p\geq 2$, constant A is independent of ϵ. Upon simplifying this inequality we arrive at the following:

$$M(r) \leq e^{-A}M(r_1)+(1-e^{-A})M(r_3)+(1-e^{-A})\epsilon.$$

Letting $\epsilon \to 0^+$ we reach the first assertion of theorem.

In order to prove the second assertion, we define a function

$$\phi(t) = -\log\left(\frac{t-m(r_3)+\epsilon}{m(r_1)-m(r_3)+\epsilon}\right) \quad \text{for } t\in[m(r_3),\infty). \tag{49}$$

Similarly to the proof of the first assertion, we verify that ϕ is C^2 convex and satisfies conditions (4) and (5). Indeed, by computations we see that

$$\phi'(t) = -\frac{1}{t-m(r_3)+\epsilon} < 0 \quad \text{and} \quad \phi''(t) = \frac{1}{(t-m(r_3)+\epsilon)^2} \quad \text{in } [m(r_3),\infty).$$

As in the case of maxima, we introduce a function $\psi := \max\{\phi(u^1), 0\}$ and show that $\psi \equiv 0$ on B_{r_1}. Then, following the steps of the proof for $M(r)$ we reach conclusion that

$$\log\left(\frac{u^1(x)-m(r_3)+\epsilon}{m(r_1)-m(r_3)+\epsilon}\right) = -\sup_{x\in B_r}\phi(u^1(x)) \geq -A \qquad x\in B_r.$$

Thus,

$$u^1(x) \geq e^{-A}m(r_1)+(1-e^{-A})m(r_3)-(1-e^{-A})\epsilon.$$

The second assertion of the theorem now follows from taking $\epsilon \to 0^+$ and the proof of Theorem is completed. □

Example 1 Let us comment on the existence of p-harmonic mappings satisfying assumptions (22) and (23) of Theorem. In order to do so, we employ radial p-harmonic mappings, cf. Lemma 1 and discussion in Sect. 2. Under the notation of Theorem, let us suppose that $1 < p < 2$ and $u = H(|x|)x$ is a radial p-harmonic mapping in $\Omega \subset \mathbb{R}^n$ such that $H(x_0) = 0$ for $x_0 \in \Omega$. Then (22) reads:

$$|H(|x|)||x_1| \geq C|x|^{\alpha} \quad \text{for } x \in \mathbb{R}^n \setminus B_{r_2}(x_0).$$

For instance, let Ω be such that $\operatorname{dist}(\Omega, \{x \in \mathbb{R}^n : x_1 = 0\}) > c$ and $H(|x|) = |x|^{\frac{2-p-n}{p-1}} + 1$, then computations at (30) reveal that $u = H(|x|)x$ is p-harmonic in Ω and the above condition holds for $\alpha = \left|\frac{2-p-n}{p-1}\right|$, see also Adamowicz [1, Chapter 4.1] for further discussion on radial p-harmonics.

As for $p > 2$ and assumption (23), recall that by the proof of Lemma 1 we have that

$$|\nabla u^1|^2 = (H')^2 x_1^2 + 2HH'\frac{x_1^2}{r} + H^2 = \frac{x_1^2}{r^2}\left(\frac{H'}{H} + 1\right)^2 + 1 - \frac{x_1^2}{r^2} \geq 1 - \frac{x_1^2}{r^2}.$$

From this we infer, that $|\nabla u^1| > c$ follows from $(1 - \frac{x_1^2}{r^2})^{1/2} > c$, which in turn is satisfied e.g. if Ω is contained in cone-type domain $\{x \in \mathbb{R}^n : \frac{x_1^2}{r^2} < 1 - c\}$ provided that $0 < c < 1$.

Remark 2 For $1 < p < 2$, Theorem can be proven in a modified version with radius $r_3 = r_2$ and without imposing the growth condition (22). Namely, for the proof of the first assertion we define function [cf. (42)]:

$$\phi(t) := -\log\left(\frac{M(r_3) - t + 1}{M(r_3) - M(r_1) + 1}\right) \quad \text{for } t \in (-\infty, M(r_3)),$$

and the analogous function for the proof of the second assertion, cf. (49). Then $\|\phi'\|_{L^\infty} < 1$ and estimate (47) simplifies as follows:

$$\left(\sup_{B_r} \phi(u^1)\right)^p \leq C(p, n, c, C_S)\left(1 + r_2^n + \|Du\|^{2p}_{L^p(B_{2r_2})}\right)\max\{1, r_2^{p-n}\}.$$

In such a case no additional growth restriction on u^1 is needed. However, the first assertion of Theorem takes the form:

$$M(r) \leq CM(r_1) + (1 - C)M(r_2) + 1 - C,$$

where $C = C(n, p, c_S, r_1, r_2, c, \|Du\|^{2p}_{L^p(B_{2r_2})})$.

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